Lectures on
Complex Analytic Manifolds

by L. Schwartz

Tata Institute of Fundamental Research, Bombay
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Lecture 1

Introduction

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We shall first give a brief account of the problems we shall be considering.

We wish to define a complex analytic manifold, $V^{(n)}$, of complex dimension n and study how numerous (in a sense to be clarified later) are the holomorphic functions and differential forms on this manifold. If $V^{(n)}$ is compact, like the Riemann sphere, we find by the maximum principle that there are no non-constant holomorphic functions on $V^{(n)}$. On the contrary if $V^{(n)} = C^n$, the space of n complex variables, there are many non-constant holomorphic functions. On a compact complex analytic manifold the problem of the existence of holomorphic functions is trivial, as we have remarked; but not so the problem of holomorphic differential forms. On a compact complex manifold there exist a finite number, say h^p, of linearly independent holomorphic differential forms of degree p. We have to study the relation between these forms and the algebraic cohomology groups of the manifold i.e., the relation between h^p and the p^{th} Betti-number b^p of the manifold. It is necessary not only to study holomorphic functions and forms but also meromorphic functions and meromorphic forms. For instance, it is necessary to go into Cousin's problem which is a generalization of the problem of Mittag-Leffler in the plane. [Mittag-Leffler's problem is to find in the plane meromorphic functions with prescribed polar singularities and po- 2
lar developments]. We may look for the same problem on manifolds and the relation between this problem and the topological properties of the

manifold. In the case of Stein manifolds the problem always admits of a solution as in the case of the complex plane.

We thus observe the great variety of problems which can be examined. The study of these problems can be divided into three fundamental parts:

1) Local study of functions, which is essentially the study of functions on C^n. Immediately afterwards one can pose some general problems for all complex analytic manifolds.

2) The study of compact complex manifolds; in particular, the study of compact Kählerian manifolds.

3) The study of Stein manifolds.

We shall examine some properties of compact Kählerian manifolds in detail. Compact Riemann surfaces will appear as special case of these manifolds. We shall prove the Riemann-Roch theorem in the case of a compact Riemann surface. We shall not concern ourselves with the study of Stein manifolds.

Differentiable Manifolds

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To start with, we shall work with real manifolds and examine later the situation on a complex manifold. We shall look upon a complex analytic manifold of complex dimension n as a $2n$-dimensional real manifold having certain additional properties.

We define an indefinitely differentiable (C^∞) real manifold of dimension N. (We reserve the symbol $\underline{n}$ for the complex dimension). It is first of all a locally compact topological space which is denumerable at infinity (i.e., a countable union of compact sets). On this space we are given a family of 'maps', exactly in the sense of a geographical map. Each map is a homeomorphism of an open set of the space onto an open set in R^N, the N-dimensional Euclidean space. [For instance, if $N = 2$, we can imagine the manifold to be the surface of the earth and the image of a portion of the surface to be a region on the two dimensional page on which the map is drawn]. We require the domains of these maps to

cover the manifold so that every point of the manifold is represented by at least one point in the 'atlas'. We also impose a condition of coherence in the overlap of the originals of two maps. Suppose that the originals of the maps have a non-empty intersection; we get a correspondence in R^N between the images of this intersection if we make correspond to each other the points which are images of the same point of the manifold. We now have a correspondence between two open subsets of R^N and we demand that this correspondence should be indefinitely differentiable i.e., defined by means of N indefinitely differentiable functions of **4**
N variables.

Thus an *N-dimensional C^∞ manifold* is a locally compact space denumerable at infinity for which is given a covering by open sets U_j, and for each U_j a homeomorphism φ_j of U_j onto an open set in R^N such that the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is indefinitely differentiable.

We call the pair (U_j, φ_j) a map, and we say that the family of maps given above defines a differentiable structure on the manifold. We call the family of maps an atlas. If (U, φ) is a map, by composing φ with the coordinate functions on R^N we get N-functions $x_1, \ldots, x_N$ on U; the functions x_i are called the coordinate functions and form the local coordinate system defined by the map (U, φ).

From a theoretical point of view it is better to assume that the atlas we have is a maximal or complete atlas, in the sense that we cannot add more maps to the family still preserving the compatibility conditions on the overlaps. From any atlas we can obtain a unique complete atlas containing the given atlas; we say that two atlases are equivalent or define the same differentiable structure on the manifold if they give rise to the same complete atlas.

We have defined a C^∞ manifold by requiring certain maps from R^N to R^N to be C^∞ maps; it is clear how we should define real analytic or quasi-analytic or k-times differentiable (C^k) manifolds.

We shall be able to put on a C^∞ manifold all the notions in R^N that **5**
are invariant under C^∞ transformations. For instance we have the notion of a C^∞ function on a C^∞ manifold V^N. Let f be a real valued function

defined on V^N. Let (U, φ) be a map; by restriction of f to U we get a function on U; by transportation we get a function on an open subset of R^N, namely the function $f \circ \varphi^{-1}$ on $\varphi(U)$ and we know what it means to say that such a function is a C^∞ function. We define f to be a C^∞ function if and only if for every choice of the map (U, φ) the function $f \circ \varphi^{-1}$ defined on $\varphi(U)$ is a C^∞ function. We are sure that this notion is correct, since the notion of a C^∞ function on R^N is invariant under C^∞ transformation. [We can define similarly C^∞ functions on open subsets of V^N (which are also manifolds!)].

If $x_1, \ldots, x_N$ are coordinate functions corresponding to (U, φ) and $a \in U$ we define $\left(\dfrac{\partial f}{\partial x_1}\right)_a, \ldots, \left(\dfrac{\partial f}{\partial x_N}\right)_a$ to be the partial derivatives of $f \circ \varphi^{-1}$ at $\varphi(a)$.

On a C^k manifold we can define the notion of a C^p function for $p \leq k$.

The space of differentials and the tangent space at a point.

We have now to define the notions of tangent vector and differential of a function at a point of the manifold.

We define the differential of a C^1 function at a point $\underline{a}$ of V^N. For a C^1 function f on R^N the differential at a point $\underline{a}$ is the datum of the
6 system of values $\left(\dfrac{\partial f}{\partial x_1}\right)_a, \ldots, \left(\dfrac{\partial f}{\partial x_N}\right)_a$. We make an abstraction of this in the case of a manifold. Let U be a fixed open neighbourhood containing $\underline{a}$. All C^1 functions defined on this neighbourhood form a vector space, in fact an algebra, denoted by $E_{a,U}$. We say that a function $f \in E_{a,U}$ is stationary at $\underline{a}$ (in the sense of maxima and minima!) if all the first partial derivatives vanish at the point $\underline{a}$. The notion of a function being stationary at $\underline{a}$ has an intrinsic meaning; for if the partial derivatives at $\underline{a}$ vanish in one coordinate system they will do so in any other coordinate system. Let $S_{a,U}$ denote the subspace of functions stationary at a. We define the space of differentials at $\underline{a}$ to be the quotient space $E_{a,U}/S_{a,U}$. If $f \in E_{a,U}$ its differential at $\underline{a}$, $(df)_a$, is defined to be its canonical image in the quotient space. We can prove two trivial properties: its independence of the neighbourhood U chosen and the fact that the space of differentials is of dimension N. If we choose a coordinate system

$(x_1, \ldots, x_N)$ at $\underline{a}$ we have the canonical basis $(dx_1)_a, \ldots, (dx_N)_a$ for the space of differentials; in terms of this basis the differential of a function f has the representation

$$(df)_a = \sum \left(\frac{\partial f}{\partial x_i}\right)_a (dx_i)_a.$$

We now proceed to define the tangent space at $\underline{a}$. If x is a vector in R^N we have the notion of derivation along x. The best way to define this is to define the derivative of a function f along x as $\lim\limits_{t\to 0} \dfrac{f(a+tx) - f(a)}{t}$, if it exists. If x is the unit vector along the x_i-axis we have the partial derivation $\dfrac{\partial}{\partial x_i}$. Thus in R^N a vector defines a derivation. In a mani- 7
fold, on the other hand, we may define a tangent vector as a derivation. We may define a tangent vector as a derivation of the algebra $E_{a,U}$ with values in R^1, i.e., as a linear map $L : E_{a,U} \to R^1$ which has the differentiation property: $L(f \cdot g) = L(f)g(a) + f(a)L(g)$. But there will be some difficulty and we prefer to give a definition which is related directly to the space of differentials defined above. When we consider only C^∞ functions these two definitions coincide.

We define a tangent vector or a derivation at $\underline{a}$ to be a linear function $E_{a,U} \to R'$ which is zero on stationary functions. Once again this definition is independent of U; moreover the tangent space is N-dimensional. If $(x_1, \ldots, x_N)$ is a coordinate system at $\underline{a}$, $\left(\dfrac{\partial}{\partial x_i}\right)_a$ are tangent vectors at $\underline{a}$. These are obviously linearly independent. Let L be any tangent vector at a. We may write any C^1 function f as

$$f = f(a) + \sum (x_i - a_i)\left(\frac{\partial f}{\partial x_i}\right)_a + g$$

where g is stationary at $\underline{a}$ so that we have

$$L(f) = \sum L(x_i)\left(\frac{\partial f}{\partial x_i}\right)_a \quad \text{or} \quad L = \sum L(x_i)\left(\frac{\partial}{\partial x_i}\right)_a$$

Thus corresponding to a coordinate system $(x_1, \ldots, x_N)$ we have the canonical base $\left(\left(\dfrac{\partial}{\partial x_1}\right), \ldots, \left(\dfrac{\partial}{\partial x_N}\right)\right)$ of the tangent space at $\underline{a}$.

Thus at a point $\underline{a}$ on the manifold V^N we have two N-dimensional vector spaces: the tangent space at $\underline{a}$, $T_a(V)$ and the space of differentials
8 at $\underline{a}$, $T^*_a(V)$. They are duals of each other and the duality is given by the scalar product

$$\langle L, (df)_a \rangle = L(f), \quad L \in T_a(V), \quad (df)_a \in T^*_a(V).$$

As it is, we first defined the space of differentials at a point and then the tangent space at that point; as we shall see, it is better to think of the tangent space as the original space and the space of differentials as its dual space.

Lecture 2

C^∞ maps, diffeomorphisms. Effect of a map

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We define a C^∞ map from a C^∞ manifold U of dimension p to a C^∞ manifold V of dimension q (p and q need not be equal). Let Φ be a continuous map of U into V. We say that Φ is a C^∞ map if, for every choice of maps (u, φ) in U and (v, ψ) in V such that $u \cap \Phi^{-1}(v)$ is non empty, the map

$$\psi \circ \Phi \circ \varphi^{-1} : \varphi(u \cap \Phi^{-1}(v)) \to \psi(v)$$

which is a map of an open subset of R^p into one in R^q, is a C^∞ map.

We can define a C^k map of one C^n manifold into another if $k < n$. If U, V, W are C^∞ manifolds and $\Phi; U \to V$ and $\Psi : V \to W$ are C^∞ maps then the map $\Psi \circ \Phi : U \to W$ is also a C^∞ map. A map $\Phi : U \to V$ (U, V C^∞ manifolds) is said to be a C^∞ isomorphism or a diffeomorphism if Φ is $(1, 1)$ and both Φ and Φ' are C^∞ maps.

Let Φ be a C^∞ map of a C^∞ manifold U^p into another C^∞ manifold V^q. Let $\underline{a}$ be a point of U and $b = \Phi(\underline{a})$. We shall now describe how Φ gives rice to a linear map of $T_a(U)$ into $T_b(V)$ and linear map of $T^*_b(V)$ into $T^*_a(U)$. Let us choose an open neighbourhood A of $\underline{a}$ and an open neighbourhood B of b such that $\Phi(A) \subset B$. If f is a C^k function on an open subset of V, $f \circ \Phi$ is also a C^k function on an open subset of U. We call $f \circ \Phi$ the reciprocal image of f with respect to Φ and denote it by $\Phi^{-1}(f)$ or $\Phi^*(f)$. Now let $f \in E_{b,B}$; the restriction of $\Phi^{-1}(f)$ to A 10
belongs to $E_{a;A}$. We thus have a natural linear map $\Phi^{-1} : E_{b,B} \to E_{a,A}$. By this map functions stationary at b go over into functions stationary

at $a : \Phi^{-1} : S_{b,B} \to S_{a,A}$. So Φ induces a linear map of $E_{b,B}/S_{b,B}$ into $E_{a,A}/S_{a,A}$ i.e., a linear map of $T^*_b(V)$ into $T^*_a(U)$. We denote this map also by Φ^{-1}.

We now give this map in terms of coordinate functions $x_1, \ldots, x_p$ at $\underline{a}$ and $y_1, \ldots, y_q$ at $\underline{b}$. Suppose the map is given by $y_j = \Phi_j(x_1, \ldots, x_p)$. Φ_j are C^∞ functions of $(x_1, \ldots, x_p)$. The reciprocal image of the function $f(y_1, \ldots, y_q)$ is the function

$$(x_1, \ldots, x_p) \to f(\Phi_1(x_1, \ldots, x_p), \ldots, \Phi_q(x_1, \ldots, x_p)).$$

Let $(df)_b = \sum\limits_{j=1}^{q} \left(\dfrac{\partial f}{\partial y_j}\right)_b (dy_j)_b$ be a differential at b; then

$$\left(\Phi^{-1}(df)_b\right) = \sum_{i=1}^{q} \left(\frac{\partial f}{\partial y_i}\right)_b \left(\sum_{j=1}^{p} \left(\frac{\partial \Phi_i}{\partial x_j}\right)_a (dx_j)_a\right).$$

In particular,

$$\Phi^{-1}(dy_j)_b = \sum_i \left(\frac{\partial \Phi_j}{\partial x_i}\right)_a (dx_i)_a$$

So in terms of the canonical basis $(dy_1)_b, \ldots, (dy_q)_b$ for $T^*_b(V)$ and the canonical basis $(dx_1)_a, \ldots, (dx_p)_a$ for $T^*_a(U)$ the linear transformation $\Phi^{-1} : T^*_b(V) \to T^*_a(U)$ is given by the Jacobian matrix

$$\left(\left(\frac{\partial \Phi_j}{\partial x_i}\right)_a\right)$$

We have a Jacobian matrix only if we choose coordinate systems at $\underline{a}$ and $\underline{b}$.

11 The map Φ^{-1} goes in the direction opposite to that of the map Φ; we now give a direct transformation. Associated with the linear map $\Phi^{-1} : T^*_b(V) \to T^*_a(U)$ we have the transpose of this map $\Phi : T_a(U) \to T_b(V)$. This mapping is called the differential of the mapping Φ at $\underline{a}$. By the definition of the transpose we have

$$\langle \Phi(L), (df)_b \rangle = \langle L, (d(f \circ \Phi)_a \rangle$$

where $L \in T_a(U)$; that is, we have $\Phi(L)(f) = L(f \circ \Phi)$ and this gives a direct description of the map $\Phi : T_a(U) \to T_b(V)$, because if L is a derivation at $\underline{a}$, the above formula defines $\Phi(L)$ as a linear form on $E_{b;B}$ which is obviously a derivation. We shall hereafter refer to a differential as a tangent co-vector.

Invariance of dimension

We now prove the theorem of invariance of dimension: tow diffeomorphic manifolds have the same dimension. If Φ is a diffeomorphism of U onto V, and $\Phi(a) = b$, $a \in U$, we have the linear maps $\Phi : T_a(U) \to T_b(V)$ and $\Psi : T_b(V) \to T_a(U)$ where Ψ denotes the inverse of the map $\Phi : U \to V$. Since $\Phi \circ \Psi$ = identity and $\Psi \circ \Phi$ = identity, the same relations hold for the associated linear transformations Φ and Ψ. Consequently $T_a(U)$ and $T_b(V)$ are isomorphic; so U and V have the same dimension.

Tensor fields and differential forms

We proceed to examine the question of tensor fields and differential forms. Let $\underline{a}$ be a point on the C^∞ manifold V. An element of $T_a(V)$ **12**
is called a contravariant vector at $\underline{a}$; an element of $T_a^*(V)$ is called a covariant vector at $\underline{a}$. A tensor at $\underline{a}$ of contravariant order p and covariant order q is an element of the tensor product

$$\left(\bigotimes^{p} T_a(V)\right) \otimes \left(\bigotimes^{q} \wedge T_a^*(V)\right).$$

Thus any tensor of any kind that can be defined on a vector space can be put at a point on the manifold.

We call an element of the p th exterior power $\overset{p}{\Lambda}\, T_a(V)$, a p-vector at $\underline{a}$ and an element of $\overset{p}{\Lambda}\, T_a^*(V)$ a p-covector at $\underline{a}$. If $\overset{p}{\omega}$ is a p-covector $\overset{p}{\omega}$ and $\overset{p}{\omega} \wedge \frac{q}{\omega}$ a q-covector $\overset{p}{\omega}\Lambda\underline{q}$ (the exterior product of $\overset{p}{\omega}$ and $\frac{q}{\omega}$) is a $p+q$ covector and we have

$$\overset{p}{\omega}\Lambda\frac{q}{\omega} = (-1)^{pq}\frac{q}{\omega}\Lambda\overset{p}{\omega}.$$

The exterior algebra,

$$T_a^*(V) = \sum_0^N \overset{p}{\Lambda} T_a^*(V), (\overset{0}{\Lambda} T_a^*(V) = R)$$

over $\wedge T_a^*(V)$ will be of particular interest in the sequel.

All this we have at a point of the manifold. Now we consider the whole manifold. A vector field (contravariant) on V is a map which assigns to every point $\underline{a}$ of the manifold V a tangent vector $\theta(a)$ at $\underline{a}$; θ is a map of V into the set of all tangent vectors of V such that $\theta(a) \in T_a(V)$. Similarly a p times contravariant, q times covariant tensor field on V is a map which assigns to every point $\underline{a}$ of V a tensor at $\underline{a}$ of contravariant order p and covariant order q. A scalar field is an ordinary real valued function on V. If we assign to every point $a \in V$ a p-covector $\omega(a)$ at $\underline{a}$ we obtain a differential form ω of degree p on V. $\omega(a)$ is called the value
13 of the differential form at the point $\underline{a}$. Associated with a differentiable function f we have a differential form of degree 1 which assigns to every point $\underline{a}$ the differential of f at a, $(df)_a$.

We state some trivial properties of tensor fields and differential forms. All tensor fields of a particular kind (of contravariant order p and covariant order q) form a vector space. We can add tensors at each point and multiply the tensor at each point by a scalar). The differential forms of degree $p (p = 0, 1, \ldots, N)$ form a vector space. We can multiply a differential form of degree p, $\overset{p}{\omega}$, and a differential form of degree q, $\overset{q}{\omega}$ and obtain a differential form, $\overset{p}{\omega} \Lambda \overset{q}{\omega}$ of degree $p+q$; we have only to take at every point $\underline{a}$ the exterior product $\overset{p}{\omega}(a) \Lambda \overset{q}{\omega}(a)$. Thus the space of all differential forms is an algebra.

Lecture 3

"The Tensor Bundles"

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We have different kinds of tensor's attached at each point of a C^∞ manifold V^N. We shall organize the system of tensors of a specified type into a new manifold.

We first consider the set of all the tangent vectors at all points of V. We denote this set by $T(V)$. We define on $T(V)$ first a topological structure and then a differentiable structure so that $T(V)$ becomes a $2N$-dimensional C^∞-manifold. We do this by means of the fundamental system of maps defining the manifold structure on V. We choose a map (U, φ). Let $x_1, \ldots, x_N$ be the coordinate functions corresponding to this map. Let $a \in U$. In terms of the canonical basis $\left\{\dfrac{\partial}{\partial x_i}\right\}$ a tangent vector L at $\underline{a}$ has the representation

$$L = \sum \xi_j \left(\frac{\partial}{\partial x_j}\right)_a$$

Let $\varphi(a) = (a_1, \ldots, a_N) \in R^N$. We can represent L by the point $(a_1, \ldots, a_N, \xi_1, \ldots, \xi_N)$ in R^{2N}. The set of all tangent vectors at points $a \in U$ are in $(1,1)$ correspondence with the space $\varphi(U) \times R^N$. We carry over the topology in $\varphi(U) \times R^N$ to this set (by requiring the map $L \to (a_1, \ldots, a_N, \xi_1, \ldots, \xi_N)$ to be a homeomorphism). This we do for every map (U, φ). We have now to verify that the topology we introduced is consistent on the overlaps.

Let (W, ψ) be another map and $y_1, \ldots, y_N$ the corresponding coordinate functions. Let $a \in U \cap W$ and $\psi(a) = (b_1, \ldots, b_N)$. Let the com- 15 ponents of L with respect to the canonical basis corresponding to the map (W, ψ) be $(\eta_1, \ldots, \eta_N)$. $b_1, \ldots, b_N$ are C^∞ functions of $(a_1, \ldots, a_N)$. Further

$$\begin{aligned} L &= \sum \eta_i (\frac{\partial}{\partial y_i})_a \\ &= \sum \xi_j \left(\frac{\partial}{\partial x_j}\right)_a \\ &= \sum_j \xi_j \sum_i \left(\frac{\partial y_i}{\partial x_j}\right)_a \left(\frac{\partial}{\partial y_i}\right)_a \end{aligned}$$

so that

$$\eta_i = \sum_j \xi_j \left(\frac{\partial y_i}{\partial x_j}\right)_a$$

So the η are C^∞ functions of $(a_1, \ldots, a_N, \xi_1, \ldots, \xi_N)$. Hence the map

$$(a_1, \ldots, a_N, \xi_1, \ldots, \xi_N) \to (b_1, \ldots, b_N, \eta_1, \ldots, \eta_N)$$

is a C^∞ map; likewise the map

$$(b_1, \ldots, b_N, \eta_1, \ldots, \eta_N) \to (a_1, \ldots, a_N, \xi_1, \ldots, \xi_N)$$

is a C^∞ map. In particular the maps are continuous and this proves that the topology we defined is consistent on the overlaps. Now the above reasoning shows also that we have in fact defined a C^∞ structure on $T(V)$. Of course the atlas we have given is not complete. This is a very special atlas; in fact any map of V gives rise to a map in this atlas.

If we have a C^k-manifold V we can put on $T(V)$ only a $(k-1)$ times differentiable structure; for, the expressions for the η_i in terms of
16 the ξ_i involve the first partial derivatives which are only $(k-1)$ times differentiable. In particular if V is a C^1-manifold $T(V)$ will be only a topological manifold.

The set of all p-times contra variant and q-times covariant tensors on a C^∞ manifold V can be made similarly into a C^∞ manifold

$$\left(\bigotimes^p T(V)\right) \otimes \left(\bigotimes^q T^*(V)\right)$$

We also have the manifold of p-covectors $\overset{p}{\Lambda} T^*(V)$ which is of dimension $N + \binom{N}{P}$. The set of all tangent covectors, not-necessarily homogeneous, (i.e., the union of $\Lambda T_a^*(V)$, $a \in V$) can also be endowed with a C^∞ structure: this manifold, $\Lambda T^*(V)$, is of dimension $N + 2^N$.

All these manifolds built from V have quite a special structure; they have the structure of a vector fibre bundle. We shall not define here precisely the concept of a fibre bundle.

Let us consider, for example, $T(V)$. If $a \in V$, $T_a(V)$ is called the fibre over the point $\underline{a}$. $T(V)$ is the collection of the fibres, each fibre being a vector space of dimension N. Actually $T(V)$ is partitioned into the fibres. $T(V)$ is called the tangent bundle of V. $T(V)$ is the bundle space and V the base space. There are two important maps associated with $T(V)$. One is the canonical projection $\pi : T(V) \to V$ which associates to every tangent vector its origin: if $L \in T_a(V)$, $\pi(L) = a$. (This map is the projection of the bundle onto the base). The other map gives the canonical imbedding of V in $T(V)$; this map $V \to T(V)$ assigns to every point $\underline{a} \in V$ the zero tangent vector at $\underline{a}$. We can now consider V as a submanifold of $T(V)$. (This canonical imbedding is possible in any fibre **17**
bundle in which the fibre is a vector space).

A section of a fibre bundle is a map which associates to every point $\underline{a}$ in the base space a point in the fibre over $\underline{a}$. A section of $T(V)$ is a vector field. A section of $\overset{p}{\Lambda} T^*(V)$ is a differential form of degree p. A tensor field of a definite type is a section of the corresponding tensor bundle.

C^∞ Tensor fields and C^∞ Differential forms

A C^∞ vector field is an indefinitely differentiable section of $T(V)$ i.e. it is a C^∞ map $\theta : V \to T(V)$ such that for $a \in V$, $\theta(a) \in T_a(V)$. A C^∞ differential form is a C^∞ map $\omega : V \to \Lambda T^*(V)$ such that $\omega(a) \in \Lambda T_a^*(V)$ for $a \in V$. A C^∞ tensor field is defined similarly.

It is good to come back to the coordinate system and see how a C^∞ vector field looks. Let θ be a C^∞ vector field. Let (U, φ) be a map of V. Let $a \in U$ and $\varphi(a) = (a_1, \ldots, a_N) \in R^N$. Let the components of $\theta(a)$ in terms of the canonical basis (with respect to (U,φ)) at $\underline{a}$ be $\xi_1,\dots,\xi_N$.

$$\xi_i = \xi_i(a_1,\dots,a_N)$$

are functions of $(a_1,\dots,a_N)$. Thus exactly as in R^N the vector field is given by N-functions of the coordinates of the origin of the vector. If θ is a C^∞ vector field the map

$$(a_1,\dots,a_N) \to (a_1,\dots,a_N,\xi_1,\dots,\xi_N)$$

18 is a C^∞ map and hence the ξ_i's are C^∞ functions of $(a_1,\dots,a_N)$. Conversely if for every choice of the map (U,φ) the ξ_i's are C^∞ functions of $(a_1,\dots,a_N)$, θ is a C^∞ vector field, since the a_i's are always C^∞ functions of $(a_1,\dots,a_N)$.

It will be useful in particular to know how to recognize by means of the coordinate systems whether a given differential form is C^∞. Let us consider, for simplicity, a differential form of degree two, $\overset{2}{\omega}$. Let (U,φ) be a map and $x_1,\dots,x_N$ the corresponding coordinate functions. If $a \in U$, $(dx_1)_a,\dots,(dx_N)_a$ is the canonical basis for $T_a^*(V)$. Once we know the canonical basis for $T_a^*(V)$ we also know the canonical basis for its second exterior power $\overset{2}{\Lambda} T_a^*(V)$. The canonical basis for $\overset{2}{\Lambda} T_a^*(V)$ is

$$\left\{(dx_i)_a\Lambda(dx_j)_a\right\}, i<j.$$

In terms of this basis we write

$$\overset{2}{\omega}(a) = \sum_{i<j} \omega_{ij}(a)\,(dx_i)_a\Lambda(dx_j)_a.$$

$\omega_{ij}(a)$ are functions of $\underline{a}\cdot\overset{2}{\omega}$ is a C^∞ differential form if and only if for every choice of the map (U,φ) the functions ω_{ij} are C^∞ functions.

In terms of the canonical basis $(dx_i)_a$, we can write ω as follows:

$$\omega = \sum_{i<j} \omega_{ij}\, dx_i\Lambda dx_j$$

This is not 'abuse of language'; the expression has a correct meaning. ω_{ij} is a function i.e., a differential form of degree zero. dx_i is the differ-
19 ential of the function x_i i.e., the differential form of degree one which

assigns to the point $\underline{a}$ the differential $(dx_i)_a$ and we have already defined the exterior multiplication of two differential forms.

It is quite remarkable that we can define a manifold structure on the set of all tangent vectors; a priori there was no relation between tangent spaces defined abstractly. Also the notion of a vector varying continuously in a vector space which itself varies is a priori extraordinary.

C^∞ Differential forms

By a form we shall always mean a C^∞ differential form. When we consider C^k differential forms we will state it explicitly. We recall some fundamental properties of the forms: Let $\overset{p}{\omega}$ be a p-form. When we have a map we have a representation of $\overset{p}{\omega}$ in terms of the canonical basis:

$$\overset{p}{\omega} = \sum_{j_1<\ldots<j_p} \omega_{j_1\ldots j_p} dx_{j1} \Lambda \ldots \Lambda dx_{j_p}.$$

This is only a local representation; we do not, in general, have a global representation. The following are the principal properties of the forms.

1. **Differentiability.**

2. **The linear structure:** We can add two p-forms and multiply a form by a scalar. All p-forms form a vector space.

3. **Algebra structure:** We can multiply two forms ω and $\overline{\omega}$ and obtain the form $\omega\Lambda\overline{\omega}$. The multiplication satisfies the anti-commutativity rule.

4. **The reciprocal image of a form:** 20

   This is a very important notion. Let U and V be two C^∞ manifolds, and $\Phi : U \to V$ a C^∞ map. Suppose $\overset{p}{\omega}$ is a differential form of degree p on V. $\overset{p}{\omega}$ gives rise to a differential form of degree p on U, $\Phi^{-1}(\overset{p}{\omega})$, which we call the reciprocal image of ω by Φ. Let $\underline{a} \in U$ and $\Phi(a) = b$. The value of ω at b is a p-covector, $\omega(b)$, at b. $\Phi^{-1}(\overset{p}{\omega})$ is the differential form which assigns to every point

$a \in U$ the p-covector $\Phi^{-1}(\overset{p}{\omega}(b))$. It is seen easily that if ω is a C^∞ form $\Phi^{-1}(\omega)$ is also a C^∞ form.

Any kind of covariant tensor field has a reciprocal image. However it is in general impossible to define the direct image of contravariant tensor field. For it may be that a point $b \in V$ is the image of no point of U or the image of an infinity of points of U. Of course we can define the direct image of a contravariant tensor field when Φ is a diffeomorphism.

One of the reasons for the utility of differential forms is that they have a reciprocal image. Another is the possibility of exterior differentiation.

5. **Exterior differentiation:**

To a given C^k differential form $\overset{p}{\omega}$ of degree p we associate a differential form $d\overset{p}{\omega}$ of degree $p+1$ which is of class C^{k-1}; $d\overset{p}{\omega}$ is called
21 the exterior differential or the coboundary of $\overset{p}{\omega}$. The important point to be noticed is that the exterior differential of a differential form of degree p is of degree $p+1$.

We define the exterior differentiation by axiomatic properties. We restrict our attention to C^∞ forms only. Let $\overset{p}{\mathscr{E}}$ denote the space of all C^∞ p-forms on the manifold (if $p \neq 0, 1, \ldots, N$, $\overset{p}{\mathscr{E}} = 0$). Then $d : \overset{p}{\mathscr{E}} \to \overset{p+1}{\mathscr{E}}$ $(p = 0, 1, \ldots, N)$ is a map which satisfies the following properties:

1) The operation d is purely local: if two forms ω and $\overline{\omega}$ coincide on an open set U, then $d\omega = -d\overline{\omega}$ on U.

2) d is a linear operation:

$$d(\omega + \overline{\omega}) = d\omega + d\overline{\omega}$$
$$d(\lambda\omega) = \lambda d\omega, \lambda \text{ a constant.}$$

3) With respect to the structure of algebra d has the following property:

$$d(\overset{p}{\omega} \wedge \overset{q}{\omega}) = d\overset{p}{\omega} \wedge \overset{q}{\omega} + (-1)^p \overset{p}{\omega} \wedge d\overset{q}{\omega}$$

(The sign $(-1)^p$ in the second term is to be expected as no symbol can pass over a p-form without taking the sign $(-1)^p$).

4) $d^2 = 0$; i.e., $d(d\omega) = 0$.

5) If f is a form of degree zero i.e., a scalar function, then df is the ordinary differential of the function which associates to every point $\underline{a}$ the differential of f at $\underline{a}$.

Lecture 4

Existence and uniqueness of the exterior differentiation. The DGA $\mathscr{E}(V)$.

22

We give a sketch of the proof of the existence and uniqueness of d. First we assume the existence and prove uniqueness. Since d is a local operation it is enough to reason on an open subset U of R^N. Let ω be a p form on U and $x_1, \ldots, x_N$ the coordinate functions in R^N. Then

$$\omega = \sum_{j_1 < \ldots < j_p} \omega_{j_1 \ldots j_p} dx_{j_1} \wedge \ldots \wedge dx_{j_p}.$$

Let d' be an operation satisfying conditions [(1) - (5)]. Since d' is linear it is sufficient to consider the form

$$\omega = \omega_{j_1 \ldots j_p} dx_{j_1} \wedge \ldots \wedge dx_{j_p}$$

By the product rule

$$\begin{aligned} d'\omega &= d'(\omega_{j_1 \ldots j_p} dx_{j_1} \wedge \ldots \wedge dx_{j_p}) \\ &= d'\omega_{j_1 \ldots j_p} dx_{j_1} \wedge \ldots \wedge dx_{j_p} \\ &+ \omega_{j_1 \ldots j_p} d'(dx_{j_1} \wedge \ldots \wedge dx_{j_p}) \end{aligned}$$

By property 5, $df = d'f$ for a function f. So

$$d'(dx_{j_1} \wedge \ldots \wedge dx_{j_p}) = d'(d'x_{j_1} \wedge \ldots \wedge d'x_{j_p}).$$

Using the product rule it is seen by induction that

$$d'(d'x_{j_1} \wedge \ldots \wedge d'x_{j_p})$$

$$= \sum_{i=1}^{p} (-1)^{i-1} d'x_{j_1} \wedge \ldots \wedge d'x_{j_{i-1}} \wedge d'(d'x_{j_i}) \wedge d'x_{j_i+1} \wedge \ldots \wedge d'x_{j_p}$$

23 Since $d'^2 = 0$, this sum is zero. So

$$d'\omega = d\omega_{j_1 \ldots j_p} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_p}.$$

This proves that d' is unique. This formula also shows how we should try to define the operation d to prove the existence. We define d locally by this formula.

Let (U, φ) be a map and $x_1, \ldots, x_N$ the corresponding coordinate functions. Let ω be a p form. In U, ω can be written as

$$\omega = \sum_{j < \ldots < j_p} \omega_{j_1 \ldots j_p} dx_{j_1} \wedge \ldots \wedge dx_{j_p}$$

In U we define

$$d\omega = \sum_{j_1 < \ldots < j_p} d\omega_{j_1 \ldots j_p} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_p}.$$

It can be verified that d has the properties 1 - 5 in U. It follows from this and the uniqueness theorem we proved above that d is defined intrinsically on the whole manifold (If U_i and U_j are of two maps and d_i, d_j the exterior differentiations defined in U_i, U_j respectively by the above formula, then $d_i = d_j$ in $U_i \cap U_j$ by the uniqueness theorem).

Let us consider some examples. In R^3 we have 0, 1, 2 and 3 forms. If f is a zero form,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

24 In R^3 the canonical basis for two forms is usually taken to be $dy \wedge dz$, $dz \wedge dx$ and $dx \wedge dy$. Let $A\,dx + B\,dy + C\,dz$ be a 1-form.

$$\begin{aligned} d(A\,dx + B\,dy + C\,dz) &= \left(\frac{\partial c}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz \\ &+ \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dx \wedge dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy \end{aligned}$$

If $A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$ be a two form, then

$$dA \wedge dy \wedge dz = \frac{\partial A}{\partial x} dx \wedge dy \wedge dz.$$

and

$$d(A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy) = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial Z}\right) dx \wedge dy \wedge dz.$$

The exterior derivatives of 0, 1 and 2 forms correspond respectively to the notions of the gradient of a function, and curl and divergence of a vector field. The formula $d^2 = 0$ corresponds to curl grad = 0 and div *curl* = 0.

We make some remarks on the space $\overset{p}{\mathscr{E}}(V)$ of p forms on V. $\overset{p}{\mathscr{E}}(V)$ is an infinite dimensional vector space. For instance $\overset{0}{\mathscr{E}}(V)$ is infinite dimensional since we can construct a C^∞ function taking prescribed values at arbitrary finite number of points. The space

$$\mathscr{E}(V) = \sum_0^N \overset{p}{\mathscr{E}}(V)$$

is an algebra. (The product of two C^∞ forms is C^∞). We shall later put on $\mathscr{E}(V)$ a topological structure so that it becomes a topological vector **25**
space. $\mathscr{E}(V)$ is a graded algebra: it is decomposed into the homogeneous pieces $\overset{p}{\mathscr{E}}(V)$ and the multiplication obeys the anti-commutativity rule. $\mathscr{E}(V)$ has an internal operation $d : \mathscr{E}(V) \to \mathscr{E}(V)$ by which a homogeneous element of order p is taken into a homogeneous element of order $p + 1$ with the properties $d^2 = 0$ and

$$d(\overset{p}{\omega} \wedge \overset{q}{\omega}) = d\overset{p}{\omega} \wedge \overset{q}{\omega} + (-1)^p \overset{p}{\omega} \wedge d\overset{q}{\omega}.$$

$\mathscr{E}(V)$ has the structure of what is called a graded algebra with a differential operator (DGA).

We consider the behaviour of d with respect to mappings of manifolds. Let $\Phi : U \to V$ be a C^∞ map of the two manifolds U and V. Then we have a mapping $\Phi^{-1} : \mathscr{E}(V) \to \mathscr{E}(U) \cdot \Phi^{-1}$ is a homomorphism of the graded algebra $\mathscr{E}(V)$ into the graded algebra $\mathscr{E}(U)$. We shall now prove that Φ^{-1} is a homomorphism of the DGA's. We have to prove that Φ^{-1} commutes with the coboundary operator d:

$$d\Phi^{-1}(\omega) = \Phi^{-1}(d\omega)$$

where ω is a form on V. (Strictly speaking the 'd's are different). We shall prove this with the minimum possible calculation. Since d and Φ^{-1} have a local character we may assume U and V to be open subsets of Euclidean spaces. Since $\Phi^{-1} : \mathscr{E}(V) \to \mathscr{E}(U)$ is a homomorphism of the algebras it is sufficient to prove the result for a system of elements which generate $\mathscr{E}(V)$. If $x_1, \ldots, x_N$ are the coordinate functions in V, $dx_1, \ldots, dx_N$ and the C^∞ functions on V generate $\mathscr{E}(V)$. So we have
26 only to prove in the case when ω is a 0-form and when ω is a 1-form which is the differential of a function. Let f be a 0-form; the result we wish to prove is just the definition of the reciprocal image:

$$\begin{aligned}\Phi^{-1}(df) &= d(f \circ \Phi) \quad \text{by definition}\\ &= d(\Phi^{-1}(f)).\end{aligned}$$

Let $\omega = df$, f being a function. $d\omega = 0$, so $\Phi^{-1}(d\omega) = 0$. We have to prove that $d\Phi^{-1}(df) = 0$; but we have just proved that

$$\Phi^{-1}(df) = d(\Phi^{-1}(f)), \text{ so that } d\Phi^{-1}(df) = d(d\Phi^{-1}(f)) = 0.$$

Change of Variables.

The reciprocal image of a map gives a good method of obtaining the formula for change of variables. Let us consider for instance polar coordinates in $R^3 : x = r \operatorname{Sin}\theta \operatorname{Cos}\varphi$, $y = r\operatorname{Sin}\theta\operatorname{Sin}\varphi$, $z = r\operatorname{Cos}\varphi$. We wish to express $A\, dx \wedge dy$ in terms of polar coordinates. We have a map $\Phi : R^3 \to R^3 : \Phi(r, \theta, \varphi) = (x, y, z)$. We have to find the reciprocal image of $A\, dx \wedge dy$ with respect to Φ. Since Φ^{-1} preserves products,

$$\begin{aligned}\Phi^{-1}(A\, dx \wedge dy) &= \Phi^{-1}(A)\Phi^{-1}(dx) \wedge \Phi^{-1}(dy)\\ &= A(r\operatorname{Sin}\theta\operatorname{Cos}\varphi, r\operatorname{Sin}\theta\operatorname{Sin}\varphi, r\operatorname{Cos}\varphi)\end{aligned}$$

$$
\begin{aligned}
&= (\operatorname{Sin}\theta \operatorname{Cos}\varphi dr + r \operatorname{Cos}\theta \operatorname{Cos}\varphi d\theta - r \operatorname{Sin}\theta \operatorname{Sin}\varphi d\varphi) \\
&= (\operatorname{Sin}\theta \operatorname{Sin}\varphi dr + r \operatorname{Cos}\theta \operatorname{Sin}\varphi d\theta + r \operatorname{Sin}\theta \operatorname{Cos}\varphi d\varphi) \\
&= A(r^2 \operatorname{Sin}\theta \operatorname{Cos}\theta d\theta \wedge d\varphi - r \operatorname{Sin}^2\theta d\varphi \wedge dr)
\end{aligned}
$$

(We use $dr \wedge dr = 0$, $-dr \wedge d\theta = d\theta \wedge dr$ etc).

Poincaré's Theorem on differential forms in R^N.

We mention here without proof a theorem of Poincaré. We say that a **27**
p-form $\overset{p}{\omega}$ is exact if $\overset{p}{\omega} = d \overset{p-1}{\widetilde{\omega}}$ where $\overset{p-1}{\widetilde{\omega}}$ is a $(p-1)$ form. For $p = 0$, this implies, by convention, that $\overset{p}{\omega} = 0$. Since $d^2 = 0$, a necessary condition for $\overset{p}{\omega}$ to be exact is that $d\overset{p}{\omega} = 0$. Poincaré's theorem is that, in R^N, this condition is also sufficient for $\overset{p}{\omega}$ to be exact, provided $p \geq 1$. Thus a necessary and sufficient condition for a p-form ($p \geq 1$) in R^N to be exact is that its exterior derivative is zero.

Lecture 5

Manifolds with boundary

28

The upper hemisphere (with the rim) is an example of a manifold with boundary. Here the boundary is regular. But in the case of a closed triangle the boundary has singularities at the vertices. We consider only manifolds with good boundary. The manifolds that we have defined earlier are special cases of manifolds with boundary and will be referred to as manifolds without boundary.

An N-dimensional C^∞ manifold with boundary is a locally compact space V^N, countable at infinity, for which is given two kinds of maps with the following properties. A map of the first kind maps an open subset of V homeomorphically onto an open subset of R^N, exactly as in the case of the ordinary manifolds. A map of the second kind maps an open subset of V homeomorphically onto an open subset of the half-space

$$\{(x_1, \ldots, x_N), x_1 \geq 0\}, \text{ in } R^N.$$

The domains of the maps cover V. In the overlaps the maps are related by C^∞ functions. (We calculate the x_1-derivatives at points on the hyperplane $x_1 = 0$ in the positive direction). As in the case of ordinary manifolds we require the family of maps to be complete.

The notions of C^∞ functions, tangent vectors, differential forms etc. can be defined as in the case of manifolds without boundary.

Let $a \in V$. An interior tangent, L, at $\underline{a}$ is defined to be a positive derivation: if $f \geq 0$ is of class C^1 in a neighbourhood of $\underline{a}$ and $f(a) = 0$, 29
then $L(f) \geq 0$ and there exists at least one function f such that $f \geq 0$,

$f(a) = 0$ and $L(f) > 0$. An exterior tangent at $\underline{a}$ is a negative derivation: if $f \geq 0$, $f(a) = 0$ then $L(f) \leq 0$ and there exists at least one f such that $f \geq 0$, $f(a) = 0$ and $L(f) < 0$.

Suppose $\underline{a}$ is a point which is in the interior of the domain of a map of the first kind, say (U, φ). Let f be a function of class C^1 in a neighbourhood of $\underline{a}$ with $f \geq 0$ and $f(a) = 0$. Since f attains the minimum at $\underline{a}$, by considering the function $f \circ \varphi^{-1}$ at $\varphi(a)$ we find that f is stationary at a. So $L(f) = 0$ for any tangent vector L at $\underline{a}$. So at a point which is in the interior of the domain of at least one map of the first kind there are no interior and exterior tangents. On the other hand, let $\underline{a}$ be a point which is mapped by a map of the second kind onto a point in the hyperplane $x_1 = 0$: let (U, φ) be a map of the second kind at a, $(x_1, \ldots, x_N)$ the corresponding coordinate functions and $\varphi(a)$ a point on the hyperplane $x_1 = 0$. Then the tangent vector

$$L = \xi_1 \left(\frac{\partial}{\partial x_1}\right)_a + \cdots + \xi_N \left(\frac{\partial}{\partial x_N}\right)_a$$

is an interior tangent vector at $\underline{a}$ if $\xi_1 > 0$. (and exterior tangent vector if $\xi_1 < 0$). For, if $f \geq 0$ and $f(a) = 0$, $L(f) = \xi_1 \left(\dfrac{\partial f}{\partial x_1}\right)_a$ since the $f \circ \varphi^{-1}$ function is stationary at $\varphi(a)$ on the hyperplane $x_1 = 0$; $L(f) \geq 0$ so if $\xi_1 > 0$ and for the function $f = x_1 (x_1 \geq 0, x_1(a) = 0)$, $L(x_1) = \xi_1 > 0$.

30 It follows that a point which is in the interior of the domain of a map of the first kind is never mapped by a map of the second kind onto a point in the hyperplane $x_1 = 0$ and that a point which is mapped by a map of the second kind onto a point in the hyperplane $x_1 = 0$ is not in the interior of a map of the first kind.

A point which is in the domain of at least one map of the first kind is called an interior point. A point which is mapped by a map of the second kind into a point on the hyperplane $x_1 = 0$ is called a boundary point.

Let $\dot{V}^{N-1}$ denote the set of boundary points of V. $\dot{V}^{N-1}$ is called the boundary of V. $\dot{V}^{N-1}$ is an $N-1$ dimensional manifold without boundary; the maps of $\dot{V}^{N-1}$ are given by the restriction of the maps of the second kind to $\dot{V}^{N-1}$. Moreover the tangent space to $\dot{V}^{N-1}$ at a point $\underline{a} \in \dot{V}^{N-1}$ can be canonically identified with a subspace of the tangent space at $\underline{a}$ to V^N.

Oriented manifolds.

Let E^N be an N-dimensional vector space over real numbers. The space $\overset{N}{\Lambda} E^N$ is one dimensional and is isomorphic to R, but not canonically. To orient E^N is to decide which elements of $\overset{N}{\Lambda} E^N$ should be considered positive and which negative. If $A \neq 0$ and $B \neq 0$ are two elements of $\overset{N}{\Lambda} E^N$, we have $A = aB$, $\underline{a}$ a real number. The non-zero elements of $\overset{N}{\Lambda} E^N$ fall into two classes defined as follows: two elements B and $A = aB$ belong to the same class if $a > 0$ and to opposite classes if $a < 0$. Selecting one of these two classes as the class of positive N-vectors is
called orienting the vector space E^N. A vector space for which a choice 31
of one of the two classes has been made is said to be oriented. If E^N is oriented we may orient E^{*N} in a natural way: $\overset{N}{\Lambda} E^{*N}$ is the dual of $\overset{N}{\Lambda} E^N$; we say that a non-zero element in $\overset{N}{\Lambda} E^{*N}$ is positive if its scalar product with any positive vector in $\overset{N}{\Lambda} E^N$ is positive.

We can orient only vector spaces over ordered fields.

Let V^N be a C^∞ manifold (with or without boundary). By an orientation at a point $\underline{a} \in V^N$ we mean an orientation of the tangent space $T_a(V)$. V^N is said to be oriented if we have chosen at every point of V^N an orientation satisfying the following coherence condition. Let (U, φ) be any connected map (i.e., a map whose domain is connected) and $a \in U$. The differential of the map φ at $\underline{a}$ gives rise to an isomorphism of $T_a(V)$ onto $T_{\varphi(a)}(R^N)$, which can be identified with R^N itself. Since we have oriented $T_a(V)$ this isomorphism induces an orientation on R^N. We require this induced orientation on R^N to be the same for every point $a \in U$.

If a manifold can be oriented it is said to be orientable; otherwise, non-orientable. Not every manifold is orientable. For example the Mobius-band (with or without the boundary) is non-orientable. The dif-

ficulty in non-orientable manifolds is this: Let (U, φ) be a connected map. We can always choose a coherent orientation in U. If (U', φ') is another connected map such that $U \cap U'$ is non-empty we can extend the orientation in U to U'. In non-orientable manifolds it so happens
32 that when we continue the orientation like this along certain paths and come back to U we arrive at the opposite orientation.

We can give a description of the orientation which uses only the maps. Suppose V^N is oriented. Let (U, φ) be a map and $(x_1, \ldots, x_N)$ the coordinate functions of (U, φ). Then this map determines a unique orientation in R^N. If this induced orientation is not the canonical orientation of R^N (canonical orientation in R^N is the orientation for which $e_1 \wedge \ldots \wedge e_N > 0$ where $(e_1, \ldots, e_N)$ is the canonical basis for R^N) the map given by $(x_2, x_1, x_3, \ldots, x_N)$ induces the canonical orientation in R^N. Thus it is possible to cover V^N by the domains of maps which induce on R^N the canonical orientation; if (U, φ) and (U', φ') are two maps which induce on R^N the canonical orientation in R^N the coherence maps $\varphi \circ \varphi'^{-1}$ and $\varphi' \circ \varphi^{-1}$ have a positive Jacobian. Conversely if we have a covering of V^N by the domains of maps all of whose coherence maps have a positive Jacobian, these maps determine an orientation of V^N.

Let V^N be a manifold with boundary and $\dot{V}^{N-1}$ its boundary. If V^N is oriented, $\dot{V}^{N-1}$ is canonically oriented: the $N-1$ vector e_{N-1} tangent to $\dot{V}^{N-1}$ at a point $a \in \dot{V}^{N-1}$ will be said to be positive if, e_1 being an exterior tangent vector to V^N at a, the N vector $e_1 \wedge e_{N-1}$ tangent to V^N at $\underline{a}$ is positive.

We consider only orientable manifolds and assume further that a definite orientation has been chosen for the manifold.

Integration on a Manifold.

33 Let V^N be an oriented manifold (with or without boundary). Let ω be a continuous N-form on V^N which vanishes outside a compact set. With ω we can associate a real number, $\int\limits_V \omega$ called the integral of ω on V possessing the following properties:

1) Let K be a compact set outside of which ω is zero. Let U be an open set containing K. Then

$$\int_U \omega = \int_V \omega.$$

(The orientation on U is the one induced from the orientation on V)

2) If ω and ω' are two continuous N-forms vanishing outside compact sets,

$$\int_V \omega + \omega' = \int_V \omega + \int_V \omega'; \quad \int_V \lambda\omega = \lambda \int_V \omega,\ \lambda$$

a constant.

3) If $\Phi : V^N \to W^N$ is an orientation preserving diffeomorphism of V^N onto W^N and $\overline{\omega}$ a continuous N-form on W^N, then

$$\int_V \overset{-1}{\Phi}\omega = \int_W \omega.$$

4) If V is an open subset of R^N with the canonical orientation of R^N and $\omega = f\ dx_1 \wedge \ldots \wedge dx_N$, f a continuous function vanishing outside a compact set,

$$\int \omega = \int \ldots \int f\ dx_1 \ldots dx_N$$

where the term in the right side denotes the ordinary Riemann integral of f.

It can be proved that these four properties determine the integral uniquely.

Lecture 6

Integration on chains.

Let $\underset{N}{V}$ be an oriented C^∞ manifold without boundary. We shall now define integrals of p-forms on some kind of p-dimensional submanifolds of $\underset{N}{V}$. 34

An elementary p-chain on V is a pair $(\underset{p}{W}, \Phi)$ where $\underset{p}{W}$ is a p-dimensional oriented C^∞ manifold with boundary and $\Phi : \underset{p}{W} \to \underset{N}{V}$ is a C^∞ map which is continuous at infinity (i.e., the inverse image of every compact set of V is a compact set of W). To avoid certain logical difficulties we shall assume that all the manifolds $\underset{p}{W}$ are contained in $R^{\mathscr{N}_0}$. A p-chain on V is a finite linear combination of elementary p-chains with real coefficients. The p-chains evidently form a vector space over R.

The support of a p-form ω is the smallest closed set outside which the form in zero. Thus it is the closure of the set of points $\underline{a}$ such that $\omega(a) \neq 0$. The support of an elementary chain $(\underset{p}{W}, \Phi)$ is the image of the map Φ. The support of an arbitrary chain

$$\Gamma_p = a_1 \Gamma_{p^1} + \cdots + a_k \Gamma_{p^k}$$

(where Γ_{p^i} are distinct elementary chains) is defined to be the union of the supports of Γ_i, $i = 1, 2, \ldots, k$. The support of a chain is always a closed set (since the image of every closed set by a map continuous at infinity is a closed set).

Let $\overset{p}{\omega}$ be a p-form on V and $\Gamma_p = \left(\overset{W}{p}, \Phi\right)$. Let us suppose that the 35 intersection of the supports of $\overset{p}{\omega}$ and Γ_p is compact. Then we define the integral of $\overset{p}{\omega}$ on Γ_p, $\int\limits_{\Gamma_p} \overset{p}{\omega}$, by the formula

$$\int\limits_{\Gamma_p} \overset{p}{\omega} = \int\limits_{\underset{p}{W}} \Phi^{-1} \overset{p}{\omega}.$$

The integral on the right is defined as $\Phi^{-1}(\overset{p}{\omega})$ has compact support. If $\Gamma_p = \sum a_i \Gamma_{p^i}$ ($\Gamma_p i$ elementary chains) is an arbitrary p-chain such that the intersection of the supports of Γ_p and $\overset{p}{\omega}$ is compact define the integral of $\overset{p}{\omega}$ on Γ_p by:

$$\int\limits_{\Gamma_p} \overset{p}{\omega} = \sum a_i \int\limits_{\Gamma_{p^i}} \omega^p$$

Stockes' Formula

Let $(\overset{W}{p}, \Phi)$ be an elementary p-chain. Let $\underset{p-1}{\dot{W}}$ denote the boundary of $\underset{p}{W}$ oriented canonically and $\Phi| \underset{p-1}{\dot{W}}$ the restriction of Φ to $\underset{p-1}{\dot{W}}$. Then $(\underset{p-1}{\dot{W}}, \Phi| \underset{p-1}{\dot{W}})$ is a $p-1$ chain. We define this chain to be the boundary of the p-chain $(\underset{p}{W}, \Phi)$. We define the boundary of an arbitrary p-chain by linearity. We denote the boundary of Γ_p by $b\Gamma_p$. If b is the operator which maps a chain to its boundary then $b^2 = 0$, since the boundary of
36 a manifold with boundary is a manifold without boundary.

Let Γ_p be a p-chain and $\overset{p-1}{\omega}$ a $p-1$ form such that the supports of Γ_p and $\overset{p-1}{\omega}$ have a compact intersection. Stokes' formula asserts that

$$\int\limits_{b\Gamma_p} \overset{p-1}{\omega} = \int\limits_{\Gamma_p} d \overset{p-1}{\omega}.$$

Currents

Currents were introduced by de Rham to put the chains and the forms in the same stock. Currents are generalizations of distributions on R^N. Currents are related to differential forms just the same way distributions are to functions.

Let $\overset{p}{\mathscr{E}}(V)$ denote the space of p-forms on V. We shall introduce a topology in $\overset{p}{\mathscr{E}}(V)$ so that $\overset{p}{\mathscr{E}}(V)$ becomes a topological vector space: We say that the sequence of forms $\{\overset{p}{\omega}_j\}$ tends to zero as $j \to \infty$, if for every map (U, φ) and for every compact set $K \subset U$ each derivative of each coefficient of $\overset{p}{\omega}_j$ [expressed with the help of the coordinate functions $(x_1, \ldots, x_N)$ of the map (U, φ)] tends uniformly to zero on $\varphi(K)$ as $j \to \infty$. We may simply describe this topology as the topology of uniform convergence of the "coefficients" of the forms along with all their derivatives on every compact subset of V. Convergence in the sense described above is called convergence in the sense of $\mathscr{E}$ or in the sense of C^∞.

Let $\overset{p}{\mathscr{D}}$ denote the space of p-forms with compact support. It is difficult to introduce a topology on $\overset{p}{\mathscr{D}}$ adapted to the C^∞ structure. Let $\overset{p}{\mathscr{D}}_K$ **37**
denote the space of p-forms whose supports are contained in the compact set K. We consider $\overset{p}{\mathscr{D}}_K$ as a topological space with the topology induced from $\overset{p}{\mathscr{E}}$.

A current, T, of degree p (or of dimension $N-p$) is a linear form on $\overset{N-p}{\mathscr{D}}$ the restriction of which to every $\overset{N-p}{\mathscr{D}}_K$ (K compact) is continuous. If φ_1 and φ_2 are $N-p$ forms with compact supports

$$T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2)$$
$$T(\lambda\varphi_1) = \lambda T(\varphi_1), \lambda \text{ a constant}$$

if $\varphi_j \in \mathscr{D}^{N-p}$ have their supports in the same compact set K and if $\varphi_j \to 0$ in the sense of C^∞ as $j \to \infty$ then $T(\varphi_j) \to 0$ as $j \to \infty$.

We shall write $\langle T, \varphi \rangle$ instead of $T(\varphi)$.

Lecture 7

Some examples of currents

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1) A C^∞ p-form ω defines a current of degree p. For every $\overset{N-p}{\varphi} \in \overset{N-p}{\mathscr{D}}$ we define
$$\langle \overset{p}{\omega}, \overset{N-p}{\varphi} \rangle = \int\limits_V \overset{p}{\omega} \wedge \overset{N-p}{\varphi}$$
($\overset{p}{\omega} \wedge \overset{N-p}{\varphi}$ has compact support).

We have to verify that if $\{\varphi_j\}$ is a sequence of $N-p$ forms whose supports are contained in the same compact set K and if $\varphi_j \to 0$ in the sense of C^∞ then $\int\limits_V \omega \wedge \varphi_j \to 0$. By using a partition of unity we may assume that K is contained in the domain of a map. Then continuity follows from well-known properties of Riemann integrals on R^N.

If a p-form defines the zero current it can be proved easily that the form itself is zero. (This is a consequence of the existence of numerous $N-p$ forms). We can therefore identify a p-form with the current it gives rise to.

More generally a locally summable p-form defines a current of degree p. A differential form of degree p is said to be locally summable if, for every compact set K contained in the domain U of a map (U, φ), the coefficients of the differential form (expressed in terms of the map (U, φ)) are summable on $\varphi(K)$. If $\overset{p}{\omega}$ is a locally

summable p-form and $\overset{N-p}{\varphi}$ an $N-p$ form with compact support
39 the integral $\int_V \overset{p}{\omega} \wedge \overset{N-p}{\varphi}$ can be defined. Then the scalar product

$$\langle \overset{p}{\omega}, \overset{N-p}{\varphi} \rangle = \int_V \overset{p}{\omega} \wedge \overset{N-p}{\varphi}.$$

defines a current of degree p. It can be proved that if a locally summable p-form defines the zero current the differential form is zero almost every where. (Though we have no notion of a Lebesgue measure on a manifold, the notion of a set of measure zero has an intrinsic meaning. A set on the manifold will be said to be of measure zero if its image by every map has Lebesgue measure zero on R^N). So there is a $(1, 1)$ correspondence between the space of currents of degree p defined by locally summable p-forms and the classes of locally summable p forms, a class being the set of all p-forms almost everywhere equal to the same form.

2) The second example of a current is of quite a different character. An $N-p$ chain Γ_{N-p} defines a current of degree p. We define, for $\varphi \in \overset{N-p}{\mathscr{D}}$

$$\langle \Gamma_{N-p}, \varphi \rangle = \int_{\Gamma_{N-p}} \varphi$$

(Since φ has compact support $\int_{\Gamma_{N-p}} \varphi$ is defined).

In this case it can happen that a chain Γ_{N-p} is not the zero chain, nevertheless the integral of every $N-p$ form on Γ_{N-p} is zero. For this reason we shall consider two chains equivalent if for any $N-p$ form with compact support the integrals of the form on the
40 two chains are equal. There is a $(1, 1)$ correspondence between these equivalence classes and the space of currents of degree p defined by $N-p$ chains.

3) **Currents of Dirac.**

This is the generalization of Dirac's distribution on R^N. Let $\underline{a}$ be a fixed point on V. The Dirac N-current at the point a, $\delta_{(a)}$, is defined by

$$\langle \delta_{(a)}, \varphi \rangle = \varphi(a)$$

where $\underset{N-p}{\varphi}$ is a C^∞ function with compact support. More generally, let X be a fixed $N-p$ vector tangent to the manifold at $\underline{a} \cdot \varphi$ being an $N-p$ form with compact support we define

$$\left\langle \delta^p_{\binom{N-p}{X}}, \overset{N-p}{\varphi} \right\rangle = \left\langle \overset{N-p}{X}, \varphi(a) \right\rangle$$

($\varphi(a)$ is the value of the form at a). The scalar product on the right is given by the duality between $\overset{N-p}{\Lambda} T_a(V)$ and $\overset{N-p}{\Lambda} T_a^*(V)$. This defines a current of degree p. This is called a Dirac current of degree p.

Partition of Unity

Suppose $\{\Omega_i\}$ is an open covering of V. Then there exists a system of C^∞ scalar functions $\{\alpha_i\}$ defined on V such that

(i) $\alpha_i \geq 0$

(ii) support of $\alpha_i \subset \Omega_i$

(iii) $\{\alpha_i\}$ are locally finite i.e., only a finite number of supports of α_i **41**
meet a given compact set. (All functions α_i except a finite number vanish on a given compact set)

(iv) $\sum \alpha_i = 1$ (This sum is finite at every point by condition (iii)).

[If Ω_i are relatively compact, then α_i have compact supports.]

The functions α_i constitute a partition of unity subordinate to the covering $\{\Omega_i\}$ – a partition of the function 1 into non-negative C^∞ functions with small supports.

This theorem on partition of unity proves the existence of numerous non-trivial C^∞ functions and forms on V.

Support of a current

Let T be a current. We say that T is equal to zero in an open set Ω if $\langle T, \varphi\rangle = 0$ for every form φ with compact support contained in Ω.

Suppose Ω_i is a system of open sets and $\underset{i}{\cup}\Omega_i = \Omega$. If a current T is zero in every Ω_i then $T = 0$ in Ω. For, let φ be a form with compact support contained in Ω. Applying the theorem on partition of unity we can decompose the form φ into a finite sum of forms having supports in Ω_i, as the support of φ is compact:

$$\varphi = \sum \alpha_i \varphi = \sum \varphi_i, \ \mathrm{Supp}\, \varphi_i \subset \Omega_i.$$

consequently

$$\langle T, \varphi\rangle = \sum \langle T, \varphi_i\rangle = 0.$$

42 This result shows that there exists a largest open set in which a current T is zero, namely the union of all open sets in which T is zero. (This set may be empty). The complement of this set will be called the support of T. The support of the current defined by a form $(C^\infty)\omega$ coincides with the support of ω; the support of the current defined by a chain is not always identical with the support of the chain.

Main operations on currents

1) **Addition of two currents and multiplication of a current by a scalar:**

   If T_1 and T_2 are two currents of degree p we define $T_1 + T_2$ and λT_1, (λ a constant) by:

$$\langle T_1 + T_2, \varphi\rangle = \langle T_1, \varphi\rangle + \langle T_2, \varphi\rangle$$
$$\langle T, \varphi\rangle = \langle T, \varphi\rangle$$

2) **Multiplication of a current by a form**

   Just as in the case of distributions, we can not multiply two currents. However we can multiply a current by a form. If $\overset{p}{\omega}$ is a p

form and $\overset{q}{\alpha}$ a q form, we have for $\varphi \in \overset{N-p-q}{\mathscr{D}}$

$$\begin{aligned}\langle \omega \wedge \alpha, \varphi \rangle &= \int_V (\omega \wedge \alpha) \wedge \varphi \\ &= \int_V \omega \wedge (\alpha \wedge \varphi) \\ &= \langle \omega, \alpha \wedge \varphi \rangle\end{aligned}$$

and $\langle \omega \wedge \alpha, \varphi \rangle = (-1)^{pq} \langle \alpha \wedge \omega, \varphi \rangle$.

Now for any current $\overset{p}{T}$ of degree p and any q-form $\overset{q}{\alpha}$ we define 43
$\overset{p}{T} \wedge \overset{q}{\alpha}$ by:

$$\langle \overset{p}{T} \wedge \overset{q}{\alpha}, \overset{N-p-q}{\varphi} \rangle = \langle \overset{p}{T}, \overset{q}{\alpha} \wedge \overset{N-p-q}{\varphi} \rangle, \quad \overset{N-p-q}{\varphi} \in \overset{N-p-q}{\mathscr{D}}$$

We define $\overset{q}{\alpha} \wedge \overset{p}{T}$ as $(-1)^{pq} \overset{p}{T} \wedge \overset{q}{\alpha}$.

3) **The coboundary of a current.**

Suppose ω is a p form and φ and $N - p - 1$ form with compact support. Since V is a manifold without boundary Stokes' formula yields

$$\int_V d(\omega \wedge \varphi) = 0$$

But $d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^p \omega \wedge d\varphi$ so that

$$\int_V d\omega \wedge \varphi = (-1)^{p+1} \int_V \omega \wedge d\varphi$$

or

$$\langle d\omega, \varphi \rangle = (-1)^{p+1} \langle \omega, d\varphi \rangle$$

Now for any current T of degree p we define the coboundary dT, which is a current of degree $p + 1$, by:

$$\langle d, T, \varphi \rangle = (-1)^{p+1} \langle T, d\varphi \rangle, \ \varphi \in \overset{N-p-1}{\mathscr{D}}$$

Let $\mathscr{D}'(V)$ denote the direct sum of the spaces of currents of degree $0, 1, \ldots, N \cdot \mathscr{D}'(V)$ is a graded vector space. d is a linear map $d : \mathscr{D}'(V) \to \mathscr{D}'(V)$ which raises the degree of every homogeneous element by 1. Moreover $d^2 = 0$. For, T being a current of degree p we have, for $\varphi \in \overset{N-p-2}{\mathscr{D}}$,

$$\begin{aligned} \langle dd\,T, \varphi\rangle &= (-1)^{p+2}\langle dT, d\varphi\rangle \\ &= (-1)^{p+1}(-1)^{p+2}\langle T, dd\varphi\rangle \\ &= 0. \end{aligned}$$

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Let us consider the coboundary of a current given by a chain Γ_{N-p}

$$\begin{aligned} \langle d\Gamma_{N-p}, \varphi\rangle &= (-1)^{p+1}\langle \Gamma, d\varphi\rangle \\ &= (-1)^{p+1}\int_{\Gamma} d\varphi \\ &= (-1)^{p+1}\int_{\Gamma} \varphi \quad \text{by Stokes' formula,} \\ &= (-1)^{p+1}\langle b\Gamma, \varphi\rangle. \end{aligned}$$

So the coboundary of a current given by a chain is the current given by the boundary of the chain, but for sign.

Thus the coboundary operator of differential forms and the boundary operator of chains appear as particular cases of the coboundary operator of currents.

If T is a current of degree p and α a form, then

$$d(T \wedge \alpha) = dT \wedge \alpha + (-1)^p T \wedge d\alpha.$$

Lecture 8

Currents with compact support

If T is a current of degree p and φ an $N-p$ form with arbitrary support 45
and if the supports of T and φ have compact intersection then $\langle T, \varphi \rangle$ can be defined. In particular if T has compact support $\langle T, \varphi \rangle$ can be defined for any $C^\infty N-p$ form. With this definition of $\langle T, \varphi \rangle$ T becomes a continuous linear functional on $\overset{N-p}{\mathscr{E}}$. (i.e., a current of degree p with compact support can be extended to a continuous linear functional on $\overset{N-p}{\mathscr{E}}$). Conversely a continuous linear functional L on $\overset{N-p}{\mathscr{E}}$ defines a current T of degree p by restriction to $\overset{N-p}{\mathscr{D}}$. It can be easily shown that this current has compact support and that

$$\langle T, \varphi \rangle = L(\varphi) \text{ for every } \varphi \in \mathscr{D}^{N-p}$$

Consequently the space of p-current s with compact supports is identical with the dual space $\overset{p}{\mathscr{E}}{}'$ of $\overset{N-p}{\mathscr{E}}$.

Cohomology spaces of a complex

A complex, E, is a graded vector space with a differential operator of degree 1:

i) E is a vector space (over R) which is the direct sum of sub-spaces E^p where p runs through non-negative (sometimes, all) integers.

ii) E has a coboundary operator: there exists an endomorphism $d : E \to E$ such that $dE^p \subset E^{p+1}$ and $d^2 = 0$.

46 The elements of E^p are called elements of degree p.

An element $\omega \in E$ is said to be a cocycle (or closed) if $d\omega = 0$. An element $\omega \in E$ is said to be a coboundary if there exists an element $\widetilde{\omega}$ such that $d\widetilde{\omega} = \omega$. Let Z denote the vector space of cocycles and B the space of coboundaries. Since $d^2 = 0$, $B \subset Z$. The space $Z/B = H$ (or $H(E)$) is defined to be the cohomology vector space of E. Let Z^p denote the space of cocycles of degree p and B^p the space of coboundaries of degree p. The space H^p is called the pth cohomology vector space of E and the dimension of H^p, b^p, is called the pth Betti number of the complex. We have

$$H = \sum H^p$$

If a complex E is an algebra with respect to a multiplication ($\wedge$) satisfying the conditions.

i) $E^p \wedge E^q \subset E^{p+q}$

ii) $\omega_1 \wedge \omega_2 = (-1)^{pq}\omega_2 \wedge \omega_1$, $\omega_1 \in E^p$, $\omega_2 \in E^q$.

iii) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 - (-1)^p \omega_1 \wedge d\omega_2$, $\omega_1 \in E^p$, $\omega_2 \in E^q$.

is called a differential graded algebra. (D.G.A). If E is a D.G.A., $H(E)$ can be endowed with the structure of an algebra. For Z is a subalgebra of E and B is a two sided ideal of Z. $H(E)$ is known as the cohomology algebra of E.

Cohomology on a Manifold

Associated with a manifold V^N we have a number of complexes and the corresponding cohomology groups.

47 i) $\mathscr{E}(V) = \sum \mathscr{E}^p(V)$, where $\mathscr{E}^p(V)$ is the space of all p forms on V.

ii) $\widetilde{\mathscr{E}}^m(V) = \sum \widetilde{\mathscr{E}}^{p_m}$, where $\widetilde{\mathscr{E}}^{p_m}$ denotes the space of m-times differentiable p-forms, $\overset{p}{\omega}$, for which $d\overset{p}{\omega}$ is also m-times differentiable.

iii) $\mathscr{E}'(V) = \sum \overset{p}{\mathscr{E}}{}'(V)$, where $\overset{p}{\mathscr{E}}{}'(V)$ is the space of currents of degree p with compact support.

iv) $\widetilde{\mathscr{E}}'^{m}(V) = \sum \overset{p}{\widetilde{\mathscr{E}}}{}'^{m}(V)$ where $\overset{p}{\widetilde{\mathscr{E}}}{}'^{m}(V)$ is the dual of $\overset{N-p}{\widetilde{\mathscr{E}}}{}^{m}$

v) $\mathscr{D}(V) = \sum \mathscr{D}^{p}(V)$, where $\mathscr{D}^{p}(V)$ is the space of p forms with compact support.

vi) $\widetilde{\mathscr{D}}^{m}(V) = \sum \overset{p}{\widetilde{D}}{}^{m}(V)$. where $\overset{p}{\widetilde{D}}{}^{m'}(V)$ is the space of m times differentiable forms, ω, of degree p with compact support for which $d\omega$ is also m times differentiable.

vii) $\mathscr{D}'(V) = \sum \overset{p}{\mathscr{D}}{}'(V)$, where $\overset{p}{\mathscr{D}}(V)$ is the space of currents of degree p.

viii) $\widetilde{\mathscr{D}}'^{m}(V) = \overset{p}{\widetilde{D}}{}'^{m}(V)$, where $\overset{p}{\widetilde{\mathscr{D}}}{}'^{m}(V)$ is the dual space of $\overset{N-p}{\widetilde{\mathscr{D}}}{}^{m}(V)$.

Betti-numbers of $\mathscr{E}(R^N)$, $\mathscr{E}(S^N)$ and $\mathscr{E}(T^N)$.

We shall examine the Betti numbers of R^N, the N-sphere S^N, and the N-Torus T^N.

i) R^N. By Poincaré's theorem a closed p-form $\overset{p}{\omega}$ is a coboundary if $p \geq 1$.

So 48

$$b^p(\mathscr{E}(R^N)) = 0 \ \text{ if } \ p \geq 1.$$

If for a zero form f (i.e., a C^∞ function f on R^N) $df = 0$, f is constant on R^N since R^N is connected. By convention $B_0(\mathscr{E}(V))$ (the space of boundaries of degree 0)= 0.

So

$$b^0(\mathscr{E}(R^N)) = 1.$$

ii) N-**sphere** S^N (The set of points on R^{N+1} defined by the equation $x_1^2 + \cdots + x_{N+1}^2 = 1$).

Since S^N is connected $b^0(\mathscr{E}(S^N)) = 1$.

It can be shown that

$$b^p(\mathscr{E}(S^N)) = 0 \text{ for } 1 \leq p \leq N-1$$

and

$$b^N(\mathscr{E}(S^N)) = 1.$$

iii) **Torus** T^N**.** ($T^N = R^N/Z^N$, where Z^N is the group of the integral lattice points in R^N).

It can be proved that

$$b^p(\mathscr{E}(T^N)) = \binom{N}{p}.$$

In this case we can give the complete structure of the cohomology ring. The differential forms $dx_1, \ldots, dx_N$ on R^N define N-differential forms on T^N, which we still denote by $dx_1, \ldots, dx_N$. It turns out that the classes of the forms

$$dx_{i_1} \wedge \ldots \wedge dx_{i_p}, \quad i_1 < \ldots < i_p$$

49 generate the pth cohomology group of T^N.

We have seen that $b^0(\mathscr{E}(R^N)) = 1$ and $b^p(\mathscr{E}(R^N)) = 0$, $p \geq 1$. The Betti numbers of $\mathscr{D}(R^N)$ are not the same as those of $\mathscr{E}(R^N)$. There is a theorem, which we may call Poincaré's theorem for compact supports, which asserts that in R^N a closed p-form with compact support is the coboundary of a $p-1$ form with compact support if $p \leq N-1$ and an N-form $\overset{N}{\omega}$ with compact support is the coboundary of an $N-1$ form with compact support if and only if

$$\int_{R^N} \omega = 0$$

(The integral is defined since $\overset{N}{\omega}$ has compact support). It follows that

$$b^p(\mathscr{D}(R^N)) = 0, \quad 0 \leq p \leq N-1$$

and it can be shown that

$$b^N(\mathscr{D}(R^N)) = 1.$$

This example shows that there are at least two different kinds of cohomologies on a manifold - the cohomology with compact supports and cohomology with arbitrary supports. One part of de Rham's theorem asserts that, of the cohomologies given by

$$\mathscr{E}, \widetilde{\mathscr{E}}^m, \widetilde{\mathscr{D}'}^m, \mathscr{D}'; \mathscr{D}, \widetilde{\mathscr{D}}^m, \widetilde{\mathscr{E}'}^m, \mathscr{E},$$

only two cohomologies are distinct.

Lecture 9

Topology on $\mathscr{E}$, $\mathscr{E}'$, $\mathscr{D}$, $\mathscr{D}'$

50

On $\mathscr{E}'$ we introduce the weak topology. The space $\mathscr{D}$ and $\mathscr{D}'$ are in duality. In each of the spaces $\mathscr{D}$ and $\mathscr{D}'$ we introduce the weak topology (which is a locally convex topology) defined by the other; the topologies on $\mathscr{D}$ and $\mathscr{D}'$ are Hausdorff.

We remark that if F is a topological vector space and F' its dual with the weak topology, then the dual of F' is F.

de Rham's Theorem

The first part of de Rham's theorem.

The first part of de Rham's theorem gives canonical isomorphisms between the cohomology vector spaces of $\mathscr{E}$, $\widetilde{\mathscr{E}}^m$, $\mathscr{D}'$ and $\widetilde{\mathscr{D}'}^m$ and also canonical isomorphisms between the cohomology vector space of $\mathscr{D}$, $\widetilde{\mathscr{D}}^m$, $\mathscr{E}'$ and $\widetilde{\mathscr{E}'}^m$.

For instance, let us consider $\mathscr{E}(V)$ and $\mathscr{D}'(V)$. We define the canonical isomorphism between $H^p(\mathscr{E}(V))$ and $H^p(\mathscr{D}'(V))$. If $\overset{p}{\omega}_1 \in \overset{p}{\mathscr{E}}(V)$ with $d\overset{p}{\omega}_1 = 0$ it determines a closed current, $\overset{p}{\omega}_1$ of degree p (as the coboundary operator for currents is an extension of the coboundary operator for forms). If $\overset{p}{\omega}_1$ and $\overset{p}{\omega}_2$ are cohomologous (i.e., there is a $p-1$ form ${}^{p}\overset{=1}{\widetilde{\omega}}$ with $\overset{p}{\omega}_1 - \overset{p}{\omega}_2 = d\overset{p-1}{\omega}$) the currents $\overset{p}{\omega}_1$ and $\overset{p}{\omega}_2$ are also cohomologous. So we have in fact a linear map of H^p $(\mathscr{E}(V)$ into $H^p(\mathscr{D}'(V)))$. The first part of de Rham's theorem asserts that this mapping is an isomorphism. 51

That this mapping is $(1,1)$ means that if a form considered as a current is the coboundary of a current it is also the coboundary of a differential form. That the map is onto means that in each cohomology class of currents there exists a current defined by a differential form (that is, we have no other cohomology classes of currents than the ones given by closed differential forms).

We call the cohomology spaces given by any one of the complexes $\mathscr{E}, \widetilde{\mathscr{E}}^m, \mathscr{D}', \widetilde{\mathscr{D}'}^m$ the cohomology spaces with arbitrary supports. The cohomology spaces given by any one of the complexes $\mathscr{D}, \widetilde{\mathscr{D}}^m, \mathscr{E}', \widetilde{\mathscr{E}'}^m$ are called the cohomology spaces with compact supports. We shall denote by $H^p(V)$ and $H^p_c(V)$ the pth cohomology spaces with arbitrary supports and compact supports respectively and by b^p and b^p_c the dimensions of H^p and H^p_c.

In this connection we shall give an example of a natural homomorphism which is not an isomorphism in general. We have an obvious linear map from $H^p(\mathscr{D}(V))$ to $H^p(\mathscr{E}(V))$. In general this map is neither $(1,1)$ nor onto. It may happen that a p-form $\overset{p}{\omega}$ with compact support is the coboundary of some $p-1$ form but not the coboundary of a $p-1$ form with compact support (as in the case of R^N, $p = N$); and it may happen that there are cohomology classes of p-forms which contain no forms with compact support.

The second part of de Rham's theorem: the theorem of closure.

In each of the spaces $\mathscr{D}$, $\mathscr{E}$ etc., the space of co-cycles is closed: for if $\omega_j \to \omega$ then $d\omega_j \to d\omega$. The theorem of closure asserts that in each of these spaces the space of coboundaries is also closed.

52 **Some consequences of the theorem of closure**

Let F be a locally convex topological vector space and F' its dual endowed with the weak topology. Suppose G is a linear sub-space of F. Let G° be the subspace of F' orthogonal to G (An element of G^0 is a linear form T on F such that $\langle T, \varphi\rangle = 0$ for every $\varphi \in G$). Let $G^{\circ\circ}$ be the subspace of F orthogonal to G^0. G^{00} is the biorthogonal of G. It is a simple consequence of Hahn-Banach theorem that $G^{00} = \overline{G}$. Similarly the biorthogonal of a sub-space G of F' is $\overline{G}$.

Let further H and G be subspaces of F such that $H \subset G$; suppose H

is closed in F. It can be shown that H^0/G^0 is canonically the topological dual of G/H and conversely. (H^0 and G^0 are subspaces of F' orthogonal to H and G respectively).

These general considerations along with the closure theorem lead to two interesting consequences.

i) **Orthogonality relation**

Let Z be the space of cocycles and B the space of coboundaries in $\mathscr{D}^{N-p}$. Let further Z' be the space of cocycles and B' the space of coboundaries in $\mathscr{D}'^{p}$. It is trivial to see that Z' is the orthogonal space of B. For if $dT = 0$ then

$$\langle T, d\varphi\rangle = \pm\langle dT, \varphi\rangle = 0,$$

conversely if $\langle T, d\varphi\rangle = 0$ for every $\varphi \in \overset{N-p-1}{\mathscr{D}}$,

$$\langle dT, \varphi\rangle = \pm\langle T, d\varphi\rangle = 0$$

for every $\varphi \in \overset{N-p-1}{\mathscr{D}}$ so that $dT = 0$. It is also true that the or- 53
thogonal space of Z is B'. To prove this we first notice that the orthogonal space of B' is Z. For, if $\varphi \in Z$

$$\langle dS, \varphi\rangle = \pm\langle S, d\varphi\rangle$$

as $d\varphi = 0$; conversely if $\langle dS, \varphi\rangle = 0$ for every $S \in \overset{p-1}{\mathscr{D}'}$ then $\langle S, d\varphi\rangle = 0$ for every $S \in \overset{p-1}{\mathscr{D}'}$ so that $d\varphi = 0$. So the biorthogonal of B' is the orthogonal space of Z. But the biorthogonal of B' is the closure of B' and by the closure theorem $\overline{B}' = B'$. So the orthogonal space of Z is B'.

Thus the orthogonal space of the space of cocycles is the space of coboundaries and the orthogonal space of the space of coboundaries is the space of cocycles (in $\mathscr{D}$ and $\mathscr{D}'$ and so on).

ii) **Poincaré's duality theorem**

By the general result on the topological vector spaces stated above it follows that $H^p = Z'/B'$ is canonically the dual of $H_c^{N-p} = Z/B$.

Thus the pth cohomology vector space with arbitrary supports is canonically the (topological) dual of the $(N - p)$ th cohomology vector space with compact supports. This is the Poincaré duality theorem.

Suppose $H^{N-p}(\mathscr{D}(V)) = H_c^{N-p}$ is finite dimensional. Since the topology on H_c^{N-p} is Hausdorff it is the usual topology on $R^{b_c^{N-p}}$. So the topological dual and the algebraic dual of H_c^{N-p} are the same. So H^p is the algebraic dual of H_c^{N-p}. Consequently

$$b^p = b_c^{N-p}.$$

It is to be remarked that H^p and H_c^{N-p} are not canonically isomorphic but only canonically dual.

Lecture 10

Some applications

54

Since a closed 0-form is a function which is constant on each connected component, b^0 is the number of connected components of V. A closed 0-form with compact support is a function which is constant on each compact connected component and zero on each non-compact component. It follows that

b_c^0 = number of compact connected components. By Poincaré's duality theorem we have

$b^N = b_c^0$ = number of compact connected components,

$b_c^N = b^0$ = number of connected components.

Let $\overset{N}{\omega}$ be an N-form with compact support. If $\overset{N}{\omega}$ is to be the coboundary of an $N-1$ form with compact support it is necessary and sufficient that $\langle \overset{N}{\omega}, f \rangle = 0$ for every closed 0-form (orthogonality relations). If V is connected, a closed 0-form is a constant function. So, in case V is connected, for $\overset{N}{\omega}$ to be the coboundary of an $N-1$ form with compact support it is necessary and sufficient that

$$\int_V \overset{N}{\omega} = 0.$$

Similarly we can prove that a necessary and sufficient condition for an N-form $\overset{N}{\omega}$ with arbitrary support to be the coboundary of an $N-1$ 55 form is that integral of $\overset{N}{\omega}$ on every compact connected component of the manifold should be zero.

The third part of de Rham's Theorem

The third part of de Rham's theorem states that if V is compact all the Betti numbers of V are finite.

In the compact case there is no difference between cohomology with compact supports and cohomology with arbitrary supports. The pth and $(N-p)$ th cohomology spaces are canonically the duals of each other and we have the duality relation for the Betti-numbers:

$$b^p = b^{N-p}$$

Riemannian Manifolds

An N-dimensional Euclidean vector space E^N over R is an N dimensional vector space over R with a positive definite bilinear form (or what is the same, a positive definite quadratic form). There is a scalar product (x, y) between any two elements x and y of E^N which is bilinear and which has the properties

$$(x, y) = (y, x), \quad \text{and} \quad (x, x) > 0 \quad \text{for} \quad x \neq 0.$$

Let $e_1, \ldots, e_N$ be a basis of E^N and $g_{ij} = (e_i, e_j)$. If $x = \sum x_i e_i$ and $y = \sum y_i e_i$ are two vectors of E^N we have

$$(x, y) = \sum g_{ij} x_i y_j.$$

A C^∞ manifold V^N is called a C^∞ Riemannian manifold if on each tan-
56 gent space of V^N we have a positive definite bilinear form such that the twice covariant tensor field defined by these bilinear forms is a C^∞ tensor field. Thus at each tangent space of a Riemannian manifold we have a Euclidean structure.

The condition that the tensor field defined by the bilinear forms is a C^∞ tensor field may be expressed in terms of local coordinate systems as follows: for every choice of the local coordinate system $x_1, \ldots, x_N$ the functions

$$g_{ij}(a) = \left(\left(\frac{\partial}{\partial x_i}\right)_a, \left(\frac{\partial}{\partial x j}\right)_a\right)$$

are C^∞ functions in the domain of the coordinate system, the scalar product (,) being given by the bilinear form on $T_a(V)$.

Riemannian structure on an arbitrary C^∞ manifold

We shall now show that we can introduce a C^∞ Riemannian structure on any C^∞ manifold. This result is of importance because the results on a Riemannian manifold which are of a purely topological nature can be proved for any manifold by introducing a Riemannian structure.

Let V be a C^∞ manifold. If (U_i, φ_i) is a map we define a positive definite quadratic form at each $T_a(V)$, $a \in U_i$ by transporting the fundamental quadratic form "$(dx_1^2 + \cdots + dx_N^2)$" at $T_{\varphi(a)}(R^N)$ by means of the isomorphism between $T_a(V)$ and $T_{\varphi(a)}(R^N)$ given by the differential of the map φ at $\underline{a}$. Let $Q_i(a)$ denote the positive definite quadratic form in $T_a(V)$ given by the map (U_i, φ_i). Let $\{\alpha_i\}$ be a partition of unity sub- 57
ordinate to the $\{U_i\}$. For a $\in V$, we define a quadratic form in $T_a(V)$ by: $Q(a) = \sum \alpha_i(a) Q_i(a)$ (the summation being over all U_i containing $\underline{a}$; only a finite number of $\alpha_i(a)$ are different from zero). $Q(a)$ is a positive definite quadratic form as the $Q_i(a)$s are positive definite, $\alpha_i(a) \geq 0$ and at least one $\alpha_i(a) \neq 0$. Since the α_i are locally finite we can find a neighbourhood U of an arbitrary point of V such that

$$Q(a) = \sum \alpha_i(a) Q_i(a) \quad \text{(finite sum) for} \quad a \in U.$$

where $Q_i(a)$ are C^∞ quadratic forms on U. This proves that the quadratic forms $Q(a)$ define a C^∞ Riemannian structure on V.

In the above argument we have made essential use of the positive definiteness of the quadratic form. The same construction would not succeed if we want to construct a C^∞ indefinite metric with prescribed signature; for the sum of two quadratic forms with the same signature may not be a quadratic form with the same signature. In fact, we cannot put on an arbitrary C^∞ manifold a C^∞ indefinite metric with arbitrarily prescribed signature.

Canonical Euclidean structures in $T^*_a(V)$ and $\overset{p}{\Lambda} T^*_a(V)$

Let V^N be a Riemannian manifold. The positive definite quadratic form in $T_a(V)$ defines a *canonical* isomorphism of $T_a(V)$ onto $T^*_a(V)$. For a fixed $y \in T_a$ and any $x \in T_a$, (x, y) is a linear form on T_a. We denote this
58 linear form by $\gamma(y)$. The linear map $\gamma : y \to \gamma(y)$ is an isomorphism of T_a onto T^*_a. We have the relation

$$(x, y) = \langle x, \gamma(y) \rangle$$

connecting the Euclidean structure on T_a and the duality between T_a and T^*_a.

A Euclidean structure in T_a defines a canonical Euclidean structure in $T^*_a(V)$; we simply transport the positive definite quadratic form in T_a to T^*_a by means of the canonical isomorphism γ. [Any linear map of T^*_a to T_a defines a canonical bilinear form in T^*_a; the canonical bilinear form in T^*_a may also be defined as the bilinear form given by the map $\gamma^{-1} : T^*_a \to T_a$. The quadratic forms in T_a and T^*_a are called inverses of each other. If (g_{ij}) is the matrix of the quadratic form in T_a with respect to a basis in T_a the matrix of the canonical quadratic form in T^*_a with respect to the dual base is the matrix $(g_{ij})^{-1}$.

A Euclidean Structure in T_a also defines canonical Euclidean structures in $\overset{p}{\Lambda} T_a$ and $\overset{p}{\Lambda} T^*_a$. Let $e_1, \ldots, e_N$ be an orthonormal basis in T_a (with respect to the quadratic form defining the Euclidean structure). Now, a positive definite quadratic form is uniquely determined if we specify a basis $(x_1, \ldots, x_N)$ as a system of orthonormal basis; the matrix of the quadratic form with respect to this basis is the identity matrix. We take in $\overset{p}{\Lambda} T_a$ the positive definite quadratic form for which the elements $e_{i_1} \wedge \ldots \wedge e_{i_p}$, $i_1 < \ldots < i_p$ form an orthonormal basis. This quadratic
59 form (,) is intrinsic; for we have $(x_1 \wedge \ldots \wedge x_p, y_1 \wedge \ldots \wedge y_p)$

$$\begin{vmatrix} (x_1, y_1) & \ldots & (x_1, y_p) \\ . & \ldots & . \\ (x_p, y_1) & \ldots & (x_p, y_p) \end{vmatrix}$$

for $x_i, y_i \in T_a$ and determinant on the right is intrinsically defined.

The canonical isomorphism $\gamma : T_a \to T_a^*$ has a canonical extension $\gamma : \overset{p}{\Lambda} T_a \to \overset{p}{\Lambda} T_a^*$, which is also an isomorphism. This isomorphism defines the canonical Euclidean structure in $\overset{p}{\Lambda} T_a^*$.

Lecture 11

The star operator

The star operator is defined for a Euclidean *oriented* vector space; this 60
operator associates to every p-vector an $(N - p)$ vector.

Let us consider $T_a^*(V)$ with the canonical Euclidean structure given by the Riemannian structure on V. Let $T_a^*(V)$ be oriented. We first define the star $(*)$ operator on 0 - vectors i.e., scalars. $\overset{N}{\Lambda} T_a^*(V)$ is a one dimensional space in which the class of positive vectors has been chosen. $\overset{N}{\Lambda} T_a^*(V)$ has a canonical Euclidean structure. Let $\tau \in \overset{N}{\Lambda} T_a^*(V)$ be the unique positive vector of length 1. We define $*1 = \tau$. For $\overset{p}{\beta} \in \overset{p}{\Lambda} T_a^*$ we define $*\beta$ as the $(N - p)$ vector which satisfies the relation

$$(\overset{p}{\alpha}, \overset{p}{\beta}) = \overset{p}{\alpha} \wedge (*\beta^p).$$

for every $\overset{p}{\alpha} \in \overset{p}{\Lambda} T_a^*$. There exists one and only one element with this property. We choose an orthonormal basis $e_1, \ldots, e_N$ in T_a^* such that $e_1 \wedge \ldots \wedge e_N > 0$ and define $*$ on the basis elements $e_{i_1} \wedge \ldots \wedge e_{i_p} (i_1 < \ldots < i_p)$ of $\overset{p}{\Lambda} T_a^*$ by:

$$*(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \epsilon e_{k_1} \wedge \ldots \wedge e_{k_{N-p}}$$

where $k_1, \ldots, k_{N-p}$ are the indices complementary to $i_1, \ldots, i_p$ and ϵ is the sign of the permutation

$$\begin{pmatrix} 1 & 2 \ldots\ldots\ldots N \\ i_1 & i_2 \ldots i_p k_1 \ldots k_{n-p} \end{pmatrix}$$

61 we define $*$ on the whole of $\overset{p}{\Lambda} T^*_a$ by linearity. It is immediate that this is the only operation having the property

$$(\overset{p}{\alpha}, \overset{p}{\beta}) = \overset{p}{\alpha} \wedge (*\overset{p}{\beta})$$

We write $*^{-1}\beta = (-1)^{p(N-p)} * \beta$. It is easily verified that

$$* * \overset{p}{\beta} = (-1)^{p(N-p)}\beta.$$

The $*$ operator gives an isomorphism of $\overset{p}{\Lambda} T^*_a$ onto $\overset{N-p}{\Lambda} T^*_a$. This isomorphism carries an orthonormal basis into an orthonormal basis and hence preserves scalar products.

If A and B are two vectors in R^3 with the natural orientation, $*(A \wedge B)$ ($*$ operation with respect to the natural Riemannian structure in R^3) is what is usually called the vector product of A and B. In R^2 the star operation for vector is essentially rotation through an angle $\pi/2$.

The star operator on differential forms

We suppose that V^N is a oriented Riemannian manifold. The $*$ operation is then defined on each $\overset{p}{\Lambda} T^*_a(V)$.

Suppose now that ω is a differential form of degree p. By taking at every point $\underline{a}$ the $N-p$ covector $*\omega(\alpha)$ we get a differential form of degree $N-p$, which we denote by $*\omega$. If ω is a C^∞ p-form $*\omega$ is a $C^\infty(N-p)$ form. In particular we have the N-form $*1$. This N form, denoted by τ, defines the volume element on the Riemannian manifold.

62 If $x_1, \ldots, x_N$ is a local coordinate system for which $dx_1 \wedge \ldots \wedge dx_N > 0$ then

$$\tau = \sqrt{g} \cdot dx_1 \wedge \ldots \wedge dx_N,$$

where $\sqrt{g}$ is the positive square root of the determinant g of the matrix (g_{ij}).

The star operator on differential forms, defined above, gives an isomorphism, called the star isomorphism, between $\overset{p}{\mathscr{E}}$ and $\overset{N-p}{\mathscr{E}}$ and also between $\overset{p}{\mathscr{D}}$ and $\overset{N-p}{\mathscr{D}}$.

The global scalar product of two C^∞-forms

We define the scalar product of two p-forms α and β when the support of either α or β is compact by:

$$(\alpha,\beta) = \int\limits_V (\alpha,\beta)_a \tau$$

($(\alpha,\beta)_a$ is the scalar product of the p-covectors $\alpha(a)$ and $\beta(a)$ at a; τ is the volume element of the Riemannian manifold). We have

$$\begin{aligned}(\alpha,\beta) &= \int\limits_V \alpha \wedge *\beta \\ &= \langle \alpha, *\beta\rangle \\ &= \langle *^{-1}\alpha,\beta\rangle\end{aligned}$$

The $*$ operator on currents

Let T be a current. We define $*T$ by:

$$\langle *T,\varphi\rangle = \langle T, \overset{-1}{*}\varphi\rangle.$$

We then have 63

$$\langle \overset{-1}{*}T,\varphi\rangle = \langle T, *\varphi\rangle.$$

The star operator on currents is the transpose of the operator $\overset{-1}{*}$ defined on forms with compact support.

The Riemannian scalar product of a p-current and a p-form

We define the scalar product of a current T of degree p and a p-form φ by:

$$(T,\varphi) = \langle T, *\varphi\rangle$$

if the support of either T or φ is compact. We shall call this scalar product the Riemannian scalar product between T and φ.

The star coboundary operator ∂

We define the operator ∂ ("del") on currents. If T is a current of degree p we define ∂T, a current of degree $p-1$, by:

$$(\partial T, \overset{p-1}{\varphi}) = (T, d\overset{p-1}{\varphi})$$

∂ is the adjoint of the operator d with respect to the Hilbertian structure defined by the Riemannian scalar product. We have

$$\begin{aligned}
(\partial T^p, \varphi) &= (T, d\varphi) \\
&= \langle \overset{-1}{*} T, d\varphi \rangle \\
&= (-1)^{N-p-1} \langle d \overset{-1}{*} T, \varphi \rangle \\
&= (-1)^{N-p-1} \langle \overset{-1}{*} * d \overset{-1}{*} T, \varphi \rangle \\
&= (-1)^{N-p+1} (* d \overset{-1}{*} T, \varphi)
\end{aligned}$$

64 so that

$$\partial \overset{p}{T} = (-1)^{N-p+1} * d \overset{-1}{*} T$$

and

$$\partial \overset{p}{T} = (-1)^p *^{-1} d * T.$$

From $d^2 = 0$ it follows that $\partial^2 = 0$. The operator ∂ defines a new differential graded structure in $\mathscr{D}'$. The operator is also defined for the spaces $\mathscr{E}$, $\mathscr{E}'$ and $\mathscr{D}$.

We now have a new cohomology, the $*$ cohomology.

This cohomology is not different from the cohomology that we already have. Consider, for instance, $\mathscr{E}$. We define an isomorphism between $\underset{\partial}{H}^{N-p}(\mathscr{E})$ and $H_d^p(\mathscr{E})$. Suppose ω is an $N-p$ form with $\partial\omega = 0$. Then $\pm * d \overset{-1}{*} \omega = 0$. Since $*$ is an isomorphism $d \overset{-1}{*} \omega = 0$ or $d * \omega = 0$ i.e., $*\omega$ is closed. The mapping $\omega \to \omega^*$ induces an isomorphism between $\underset{\partial}{H}^{N-p}(\mathscr{E})$ and $\underset{d}{H}^p(\mathscr{E})$.

The Laplacian Δ

We define the operator Δ by:

$$-\Delta = d\partial + \partial d.$$

This is a differential operator of the second order. Δ preserves degrees. Since $d\partial$ and ∂d are self-adjoint, Δ is self-adjoint:

$$(\Delta T, \varphi) = (T, \Delta\varphi).$$

and Δd commute: $\Delta d = d\partial d = d\Delta$. Δ and ∂ also commute. Δ also commutes with $*$:

$$*\Delta = \Delta * .$$

Lecture 12

The operator Δ on functions

Let $x_1, \dots, x_N$ be a local coordinate system such that $dx_1 \wedge \dots \wedge dx_N > 0$; let $g_{ij} = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ and (g^{ij}) the matrix inverse to the matrix (g_{ij}). Let $\sum \omega_j dx_j$ be a 1-form. We calculate $\partial(\sum_j \omega_j dx_j)$

$$\partial\left(\sum_j \omega_j dx_j\right) = -\overset{-1}{*}\, d * \left(\sum_j \omega_j dx_j\right)$$

Now $*\omega_j dx_j = \omega_j * dx_j$. Suppose

$$*dx_j = \sum_i \widetilde{\omega}_{ij} dx_1 \wedge \dots \wedge d/x_i \wedge \dots \wedge dx_N;$$

from the relation

$$(dx_k, dx_j)\tau = dx_k \wedge *dx_j$$

we obtain

$$g^{kj}\sqrt{g}dx_1 \wedge \dots \wedge dx_N = (-1)^{k-1}\widetilde{\omega}_{kj} dx_1 \wedge \dots \wedge dx_N$$

so that $\widetilde{\omega}_{kj} = (-1)^{k-1} g^{kj}\sqrt{g}$.

$$d\left(*\left(\sum_j \omega_j dx_j\right)\right) = d\left(\sum_{i,j} (-1)^{i-1}\omega_j g^{ij}\sqrt{g} dx_1 \wedge \dots \wedge d/x_i \wedge \dots \wedge dx_N\right)$$

$$= \sum_{i,j} (-1)^{i-1} \frac{\partial}{\partial x_i} (\omega_j g^{ij} \sqrt{g}) dx_i$$
$$\wedge dx_1 \wedge \ldots \wedge d/x_i \wedge \ldots \wedge dx_N$$
$$= \sum_{i,j} \frac{\partial}{\partial x_i} (\omega_j g^{ij} \sqrt{g}) dx_1 \wedge \ldots \wedge dx_N$$
$$= \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} (\omega_j g^{ij} \sqrt{g}) \tau$$

66 For N forms $\overset{-1}{*} = *$. Hence

$$\overset{-1}{*} d * \left(\sum_j \omega_j dx_j \right) = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} (\omega_j g^{ij} \sqrt{g}) \tau$$
$$= \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} (\omega_j g^{ij} \sqrt{g})$$

Finally

$$\partial \left(\sum_j \omega_j dx_j \right) = -\frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x_i} (\omega_j g^{ij} \sqrt{g})$$

We shall now calculate Δu where u is a 0-form. We have

$$\partial u = 0, \Delta u = -\partial du \quad \text{and} \quad du = \sum \frac{\partial u}{\partial x_j} dx_j.$$

By the calculation made above we find that

$$\Delta u = \sum_{i,j} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} \left[g^{ij} \sqrt{g} \frac{\partial u}{\partial x_j} \right].$$

This is the well-known Laplace operator on functions. In particular if $V^N = R^N$ with the natural metric and the natural orientation, the matrix (g_{ij}) is the unit matrix and

$$\Delta u = \sum_i \frac{\partial^2 u}{\partial x_i^2}.$$

The elliptic character of Δ. Harmonic forms

In the case of a partial differential equation of the second order an elliptic operator is usually defined by considering the nature of the quadratic form given by the coefficients of the derivatives of the second order. However it is found more convenient to define an elliptic operator by intrinsic properties of the operator. This definition is valid for systems of **67**
differential equations and also for differential equations of higher order.

A local or differential operator D is defined to be a linear continuous operator on currents ($D : \mathscr{D}' \to \mathscr{D}'$) taking forms into forms[1] and having the local character: DT (T a current) is known in an open set Ω if T is known in Ω. A differential operator D is called an elliptic operator[2] if the following condition is satisfied: If T is a current such that $DT = \alpha$ is a C^∞ form, in an open set Ω then T itself is a C^∞ form in Ω. If D is elliptic every solution of the homogeneous equation $DT = 0$ is a C^∞ form. The operator

$$D = \left(\frac{d}{dx}\right)^m + a_1\left(\frac{d}{dx}\right)^{m-1} + \cdots + a_m, a_i \in \mathscr{E}$$

on the distributions in R is elliptic. In R^2 the operator $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is elliptic while the wave operator $\dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial y^2}$ is not elliptic. The function $\psi(x, y) = f(x + y) + g(x - y)$ where f and g are continuous is a solution of the wave equation (as a distribution), even though the functions may not be differentiable.

We shall admit without proof the important theorem which states that the operator Δ is elliptic.

From the elliptic character of Δ we can deduce that d is elliptic on **68**
$\overset{0}{\mathscr{D}}{}'\partial = 0$ on $\overset{0}{\mathscr{D}}{}'$. So $-\Delta = \partial d$ on $\overset{0}{\mathscr{D}}{}'$. Now if D_1 and D_2 are differential operators and $D_1 D_2$ is elliptic then D_2 is an elliptic operator (but not

[1] Actually this condition is superfluous; it can be proved that it is a consequence of the other conditions.

[2] In current literature such differential operators are referred to as hypoelliptic operators.

necessarily D_1). For let $D_2T = \alpha$ be a C^∞ form; since $D_1D_2T = D_1\alpha$, $D_1\alpha$ is a C^∞ form and D_1D_2 is elliptic it follows that T is a C^∞ form. Since $\Delta = -d\partial$ is elliptic, d is elliptic on $\overset{0}{\mathscr{D}}'$.

A form ω which satisfies the equation $\Delta\omega = 0$ is called a harmonic form.

Compact Riemannian manifolds

We shall assume henceforth that V is a compact, oriented Riemannian manifold.

If V is compact every harmonic form is closed and $*$ closed. (This result is false when V is not compact; for example, a closed 0-form in R^N is a constant while there exist non-constant harmonic functions). For let ω be a harmonic form

$$\begin{aligned}(\Delta\omega, \omega) &= (d\partial\omega, \omega) + (\partial d\omega, \omega)\\ &= (\partial\omega, \partial\omega) + (d\omega, d\omega).\end{aligned}$$

Since $(\partial\omega, \partial\omega) \geq 0$, $(d\omega, d\omega) \geq 0$ and $(\Delta\omega, \omega) = 0$ it follows that $(\partial\omega, \partial\omega) = 0$ and $(d\omega, d\omega) = 0$. But if f is a continuous non-negative function such that $\int\limits_V f\tau = 0$ then $f \equiv 0$. Since the Riemannian scalar product is positive definite it follows that

$$d\omega = 0 \quad \text{and} \quad \partial\omega = 0.$$

The Hilbert space of square summable forms

69 We now define the Hilbert space $\mathscr{H}^p$ of square summable differential forms of degree p.

A form ω is said to be measurable if its coefficients are measurable on every map.

An element of $\mathscr{H}$ is a class, a class consisting of all measurable forms which are equal almost every-where to a form ω for which $(\omega, \omega) = \int\limits_V (\omega, \omega)_a\tau$ is finite. If ω and $\widetilde{\omega}$ are elements of $\mathscr{H}$ then $(\omega, \widetilde{\omega})$ is defined and this defines a positive definite scalar product in $\mathscr{H}$. We can prove that $\mathscr{H}$ is complete with respect to the norm given by the scalar product. So $\mathscr{H}$ is a Hilbert space.

Lecture 13

Intrinsic characterization of $\mathscr{H}$

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$\mathscr{H}$, considered as a topological vector space, is intrinsically attached to the manifold i.e., $\mathscr{H}$ is independent of the Riemannian metric. We shall now give an intrinsic characterization of $\mathscr{H}$.

Suppose U is the domain of a local coordinate system $x_1, \ldots, x_N$ and K a compact set contained in U. Suppose ω is a measurable p form such that $(\omega, \omega) < \infty$. Suppose

$$\omega = \sum \omega_I dx_I$$

on U. (I is a system of p indices written in the increasing order). Let further

$$(\omega, \omega)_a = \sum g^{IJ}(a)\omega_I(a)\omega_J(a).$$

If $m(a)$ is the smallest eigen value of the matrix $(g^{IJ}(a))$, $m(a)$ is a continuous function in K and hence has a lower bound m_1 in K; $m_1 > 0$, since $(g^{IJ}(a))$ are positive definite. Since,

$$\sum_{I,J} g^{IJ}(a)\omega_I(a)\omega_J(a) \geq m(a) \sum_{I,J} [\omega_I(a)]^2$$

it follows that

$$(\omega, \omega) \geq \int\limits_K \left(\sum_{I,J} g^{IJ}\omega_I\omega_J \right) \sqrt{g} dx_1 \ldots dx_N$$

$$\geq Cm_1 \int_K \sum |\omega_I|^2 dx_1 \dots dx_N$$

where $C \geq 0$ is the lower bound of $\sqrt{g}$ in K. Thus, if $(\omega, \omega) < \infty$,

$$\int_K \sum |\omega_I|^2 dx_1 \dots dx_N < \infty.$$

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Conversely, suppose ω is a measurable p-form such that for every compact K contained in the domain U of a map

$$\int_K \sum |\omega_I|^2 dx_1 \dots dx_N < \infty;$$

then $(\omega, \omega) < \infty$. We choose a finite covering of the manifold by domains U_λ of maps and a partition of unity $\{\alpha_\lambda\}$ subordinate to the covering U_λ. Let K_λ be the support of α_λ. Then, with the obvious notation,

$$\begin{aligned}
(\omega, \omega) &= \int_V (\omega, \omega)_a \tau \\
&= \sum_\lambda \int_V (\omega, \omega)_a \alpha_\lambda \tau \\
&= \sum_\lambda \int_{U_\lambda} (\omega, \omega)_a \alpha_\lambda \tau \\
&= \sum_\lambda \int_{U_\lambda} g^{IJ}(a)^{(\lambda)} \omega_I^{(\lambda)} \omega_J^{(\lambda)} \alpha_\lambda \sqrt{g}^{(\lambda)} dx_1^{(\lambda)} \dots dx_N^{(\lambda)}.
\end{aligned}$$

Let $M_\lambda(a)$ be the greatest eigenvalue of $(g^{IJ}(a)^{(\lambda)})$. The functions M_λ, α_λ and $\sqrt{g}^{(\lambda)}$ are bounded in K_λ so that

$$\begin{aligned}
(\omega, \omega) &\leq \sum c_\lambda \int_{K_\lambda} \left(\sum |\omega_I^{(\lambda)}|^2 \right) dx_1^{(\lambda)} \dots dx_N^{(\lambda)}, C_\lambda \text{ a constant,} \\
&< \infty.
\end{aligned}$$

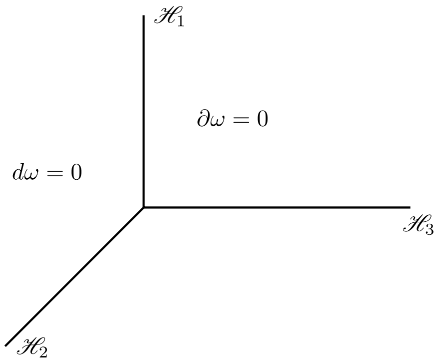
72 Thus the elements of $\mathscr{H}$ can be characterized as classes, a class being the set of all measurable forms almost everywhere equal to a form whose coefficients are square summable on every compact set contained in the domain of a map.

A simple application of the Fisher-Riesz theorem would now show that $\mathscr{H}$ is complete.

It may be remarked convergence in $\mathscr{H}$ implies convergence in $\mathscr{D}'$.

Decomposition of $\mathscr{H}$

We shall now decompose the space $\mathscr{H}$ into the direct sum of three fundamental spaces which are mutually orthogonal. Let $\mathscr{H}_1$ be the subspace of the elements $\omega \in \mathscr{H}$ such that $\Delta\omega = 0$ (in the sense of currents). It follows from the elliptic character of Δ that $\mathscr{H}_1$ is exactly the space of harmonic forms. Since d and ∂ are continuous operators on currents, $\mathscr{H}_1$ is closed. In fact we shall see later that $\mathscr{H}_1$ is finite dimensional.



We define $\mathscr{H}_2$ to be the space of elements $\omega \in \mathscr{H}$ such that $(\omega, *\varphi) =$ 73
0 for every $\varphi \in \mathscr{D}$ with $\partial\varphi = 0$. $\mathscr{H}_2$ is a closed subspace because of the continuity of the scalar product. We shall now give another interpretation of the space $\mathscr{H}_2$. If $\omega \in \mathscr{H}_2$ then $\langle \omega, *\varphi \rangle = 0$ for every $*\varphi$ such that $d(*\varphi) = 0$ or ω is orthogonal to all $N - p$ forms which are closed. By the orthogonality theorem ω is the coboundary of a current. Now let $\widetilde{\mathscr{H}}$ be the subspace of elements ω of $\mathscr{H}$ for which $d\omega \in \mathscr{H}$ (the $\mathscr{H}$'s have

different degrees). Then d operates on $\widetilde{\mathscr{H}}$ and this gives a cohomology. There is a natural homomorphism of $H(\widetilde{\mathscr{H}})$ into $H(\mathscr{D}')$. Another de Rham's theorem states that this homomorphism is an isomorphism onto. This implies, in particular, that if $\omega \in \mathscr{H}$ is the coboundary of a current it is also the coboundary of an element $\widetilde{\omega} \in \widetilde{\mathscr{H}}$. So $\mathscr{H}_2$ is the space of forms of $\mathscr{H}$ which are coboundaries of forms of $\mathscr{H}$:

$$\mathscr{H}_2 = d\mathscr{H} \cap \mathscr{H}.$$

We define $\mathscr{H}_3$ to be the space of elements $\omega \in \mathscr{H}$ such that $(\omega, \varphi) = 0$ for every $\varphi \in \mathscr{D}$ with $d\varphi = 0$. $\mathscr{H}_3$ is closed. If $\omega \in \mathscr{H}_3$, $*\omega \in \mathscr{H}_2$ and therefore $= \partial\widetilde{\omega}$, $\widetilde{\omega} \in \mathscr{H}$. So $\mathscr{H}_3$ is the space of forms of $\mathscr{H}$ which are star coboundaries of forms of $\mathscr{H}$:

$$\mathscr{H}_3 = \partial\mathscr{H} \cap \mathscr{H}.$$

74 We shall prove that $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ are mutually orthogonal . Suppose $\alpha \in \mathscr{H}_1$ and $\beta \in \mathscr{H}_2$. Then

$$\begin{aligned}(\alpha, \beta) &= (\alpha, d\widetilde{\omega}), \widetilde{\omega} \in \mathscr{H}\\ &= (\partial\alpha, \widetilde{\omega}) \text{ since } \alpha \text{ is a } C^\infty \text{ form}\\ &= 0 \quad \text{as } \partial\alpha = 0.\end{aligned}$$

Similarly $\mathscr{H}_1$ and $\mathscr{H}_3$ are orthogonal. To prove that $\mathscr{H}_2$ and $\mathscr{H}_3$ are orthogonal we shall first prove that

$$\mathscr{H}_2 = \overline{d\mathscr{D}} \quad \text{and} \quad \mathscr{H}_3 = \overline{\partial\mathscr{D}}$$

Evidently $d\mathscr{D} \subset \mathscr{H}_2$ and as $\mathscr{H}_2$ is closed $\overline{d\mathscr{D}} \subset \mathscr{H}_2$. If $\mathscr{H}_2 \neq \overline{d\mathscr{D}}$ as $\mathscr{H}_2$ is a closed subspace we can find a vector $\lambda \in \mathscr{H}_2$, $\lambda \neq 0$ such that λ is orthogonal to $d\mathscr{D}$ or $(\lambda, d\varphi) = 0$ for every $\varphi \in \mathscr{D}$ or $(\partial\lambda, \varphi) = 0$ for every $\varphi \in \mathscr{D}$. This implies that $\partial\lambda = 0$. Already $d\lambda = 0$ since λ is derived. So $\lambda \in \mathscr{H}_1$ and $\lambda \in \mathscr{H}_2$. As $\mathscr{H}_1$ and $\mathscr{H}_2$ are orthogonal $\lambda = 0$; this proves $\mathscr{H}_2 = \overline{d\mathscr{D}}$. Similarly $\mathscr{H}_3 = \overline{\partial\mathscr{D}}$. If $\alpha, \beta \in \mathscr{D}$

$$(d\alpha, \partial\beta) = (dd\alpha, \beta) = 0$$

i.e., $d\mathscr{D}$ and $\partial\mathscr{D}$ are orthogonal. Since $\overline{d\mathscr{D}} = \mathscr{H}_2$ and $\overline{\partial\mathscr{D}} = \mathscr{H}_3$ it follows that $\mathscr{H}_2$ and $\mathscr{H}_3$ are orthogonal.

We next show that $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ span $\mathscr{H}$. Suppose ω is orthogonal to $\mathscr{H}_2$ and $\mathscr{H}_3$; then $(\omega, d\alpha) = 0$ for every $\alpha \in \mathscr{D}$ or $(\partial\omega, \alpha) = 0$ for every $\alpha \in \mathscr{D}$ or $\partial\omega = 0$; similarly $d\omega = 0$. Hence $\omega \in \mathscr{H}_1$. This proves that

$$\mathscr{H} = \mathscr{H}_1 + \mathscr{H}_2 + \mathscr{H}_3.$$

We denote by π_1, π_2 and π_3 the projections on $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ respec- **75**
tively.

$\mathscr{H}_1 + \mathscr{H}_2$ is the orthogonal of $\mathscr{H}_3$ or $\partial\mathscr{D}$, so $\mathscr{H}_1 + \mathscr{H}_2$ is the space of forms ω with $d\omega = 0$ (in the sense of the currents). $\mathscr{H}_1 + \mathscr{H}_3$ is the space of forms ω with $\partial\omega = 0$ and $\mathscr{H}_2 + \mathscr{H}_3$ is the space of forms orthogonal to the space of harmonic forms.

Cohomology and harmonic forms. Hodge's theorem

In every cohomology class of C^∞ forms there exists one and only one harmonic form. Let ω be a representative of the cohomology class; $\omega \in \mathscr{H}_1 + \mathscr{H}_2$. $\pi_1\omega$ is a harmonic form and $\pi_2\omega$ which is the coboundary of a current is the coboundary of a form by de Rham's theorem. As

$$\omega - \pi_1\omega = \pi_2\omega$$

ω and $\pi_1\omega$ are cohomologous. So $\pi_1\omega$ is a harmonic form belonging to the class of ω. If ω_1 and ω_2 are two harmonic forms in the same cohomology class, $\omega_1 - \omega_2 \in \mathscr{H}_1 \cap \mathscr{H}_2 = 0$ or $\omega_1 = \omega_2$. Thus we have, in fact, a canonical isomorphism between the cohomology space of C^∞ forms and the space of harmonic forms. The dimension of the space of harmonic forms of degree p is the pth Betti number of the manifold which is finite by the third part of de Rham's theorem.

Suppose $\omega \in \mathscr{H}_2$; then $\omega = d\widetilde{\omega}$, $\widetilde{\omega} \in \mathscr{H}$. There is one and only **76**
one choice of the primitive $\widetilde{\omega}$ which belongs to $\mathscr{H}_3$. For, if $\omega = d\theta$, $\theta \in \mathscr{H}$ put $\widetilde{\omega} = \pi_3\theta$; then $\omega = d\widetilde{\omega}$. If $\omega = d\widetilde{\omega}_1 = d\widetilde{\omega}_2$, $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \mathscr{H}_3$, $d(\widetilde{\omega}_1 - \widetilde{\omega}_2) = 0$, which implies that $\widetilde{\omega}_1 - \widetilde{\omega}_2 \in \mathscr{H}_3 \cap \mathscr{H}_2$ so that $\widetilde{\omega}_1 = \widetilde{\omega}_2$.

Similarly any element of $\mathscr{H}_3$ is the star coboundary of one and only one element of $\mathscr{H}_2$. In both the cases the choice of the unique primitive is linear but not yet continuous.

Now let $\omega \in \mathscr{H}_2$; $\omega = d\theta$, $\theta \in \mathscr{H}_3$. But $\theta = \partial\widetilde{\omega}$, $\widetilde{\omega} \in \mathscr{H}_2$. Consequently for every

$$\omega \in \mathscr{H}_2, \omega = d\partial\widetilde{\omega}, \widetilde{\omega} \in \mathscr{H}_2.$$

This $\widetilde{\omega}$ is unique. Every element ω of $\mathscr{H}_2$ is thus expressed in one and only one way in the form

$$\omega = d\partial\widetilde{\omega}, \widetilde{\omega} \in \mathscr{H}_2.$$

However, for $\omega \in \mathscr{H}_2$, $\partial d\omega = 0$ so that $d\partial = -\Delta$. So any form $\omega \in \mathscr{H}_2$ can be written uniquely in the form

$$\omega = -\Delta\widetilde{\omega}, \widetilde{\omega} \in \mathscr{H}_2.$$

Similarly every form in $\mathscr{H}_3$ is uniquely expressible as the Laplacian of an element of $\mathscr{H}_3$. As a consequence every form orthogonal to all the harmonic forms is the Laplacian of one and only one element of $\mathscr{H}_2 + \mathscr{H}_3$. Conversely if ω is the Laplacian of an element, then it belongs to $\mathscr{H}_2 + \mathscr{H}_3$.

77 We have thus completely solved the equations $dX = A$, $\partial X = A$ and $\Delta\, X = A$ in $\mathscr{H}$. The equation $dX = A$ is solvable if $A \in \mathscr{H}_2$. If $A \in \mathscr{H}_2$ there exists a unique solution $X \in \mathscr{H}_3$. Any general solution is given by $X = X_0 +$ closed form. The equation $\partial X = A$ is solvable if $A \in \mathscr{H}_3$ and there exists a unique solution $X_0 \in \mathscr{H}_2$. A general solution is obtained by $X = X_0 +$ closed form. The equation $\Delta X = A$ is solvable if $A \in \mathscr{H}_2 + \mathscr{H}_3$; there exists a unique solution orthogonal to the space of harmonic forms and a general solution is obtained by adding to this particular solution any harmonic form. These results constitute Hodge's theorem.

Lecture 14

Green's Operator G

Suppose $A \in \mathscr{H}_2 + \mathscr{H}_3$. We know that there exists a unique $X \in \mathscr{H}_2 + \mathscr{H}_3$ 78
such that $-\Delta X = A$. We write $X = GA$; we then have an operator $G : A \to GA$ on $\mathscr{H}_2 + \mathscr{H}_3$ with the property $-\Delta G = I$ on $H_2 + \mathscr{H}_3$. (I is the identity operator). If $\alpha \in \mathscr{H}_2 + \mathscr{H}_3$ and $\Delta\alpha \in \mathscr{H}_2 + \mathscr{H}_3$ we have $-G\Delta\alpha = \alpha$. We extend G to the whole space $\mathscr{H}$ by putting $G\alpha = 0$ for $\alpha \in \mathscr{H}_1$. $G : \mathscr{H} \to \mathscr{H}$ is an operator which is zero on $\mathscr{H}_1$ and which leaves the spaces $\mathscr{H}_2$ and $\mathscr{H}_3$ invariant. We shall prove a little later that G is continuous. G is called the Green's operator.

Let $\omega \in \mathscr{H}$. We can write ω uniquely as

$$\omega = \pi_1\omega + \widetilde{\omega}, \widetilde{\omega} \in \mathscr{H}_2 + \mathscr{H}_3.$$

But $\widetilde{\omega} = -\Delta G\widetilde{\omega} = -\Delta G\omega$ as $\widetilde{\omega} \in \mathscr{H}_2 + \mathscr{H}_3$ and $G = 0$ on $\mathscr{H}_1$. So

$$\omega = \pi_1\omega - \Delta G\omega.$$

Thus we have the formula

$$\begin{aligned} I &= \pi_1 - \Delta G \\ &= \pi_1 + d\partial + \partial dG. \end{aligned}$$

Since $I = \pi_1 + \pi_2 + \pi_3$, it follows that $\pi_2 = d\partial G$ and $\pi_3 = \partial dG$.

Decomposition of $\mathscr{D}$

$\mathscr{H}$ has certain fundamental defects; d, ∂ and Δ do not operate on $\mathscr{H}$. 79
But these operators operate on $\mathscr{D}$. We now consider $\mathscr{D}$.

G operates on $\mathscr{D}$. For let ω be a C^∞ form. $\omega = \pi_1\omega - \Delta G\omega$ or $\Delta G\omega = \pi_1\omega - \omega$. Since $\pi_1\omega - \omega$ is a C^∞ form it follows from the elliptic character of Δ that $G\omega$ is a C^∞ form. π_1 evidently operates on $\mathscr{D}$. Since $\pi_2 = d\partial G$ and $\pi_3 = \partial dG$, π_2 and π_3 also operate on $\mathscr{D}$.

Let

$$\mathscr{D}_1 = \mathscr{D} \cap \mathscr{H}_1, \mathscr{D}_2 = \mathscr{D} \cap \mathscr{H}_2, \mathscr{D}_3 = \mathscr{D} \cap \mathscr{H}_3$$

$\mathscr{D}_1$ is the space of all harmonic forms. $\mathscr{D}_2 = d\mathscr{D}$ and $\mathscr{D}_3 = \partial\mathscr{D}$ by de Rham's theorem. $\mathscr{D}_1$, $\mathscr{D}_2$ and $\mathscr{D}_3$ re closed subspaces of $\mathscr{D}$. ($\mathscr{D} = \mathscr{E}$ has a genuine topology). The linear maps π_1, π_2 and π_3 from $\mathscr{D}$ onto the spaces $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$ respectively have the following properties:

$$\begin{aligned} \pi_i\pi_j &= 0 \ \text{ for } \ i \neq j \\ \pi_i^2 &= \pi_i \\ I &= \pi_1 + \pi_2 + \pi_3, \end{aligned}$$

Consequently $\mathscr{D}$ is the direct sum of the closed subspaces $\mathscr{D}_1$,
80 $\mathscr{D}_2$ and $\mathscr{D}_3$. For any element $\omega \in \mathscr{D}$ we have the decomposition formula:

$$\omega = \pi_1\omega + d\partial G\omega + \partial dG\omega.$$

Continuity of G

Let F be a Fréchet space i.e., a complete topological vector space with a denumerable basis of neighbourhoods of 0. Banach's closed graph theorem states that if a linear map $G : F \to F$ is discontinuous then there exists a sequence of elements $\varphi_j \to 0$ such that $G\varphi_j$ tends to a non-zero element θ. So in order to prove that a linear map G of a Fréchet space into itself is continuous it is sufficient to show that $\varphi_j \to 0$ and $G\varphi_j \to \theta$ together imply that $\theta = 0$. [We can give an example of a normed vector space in which the closed graph theorem is not true. Let E be the space of polynomials in the closed interval $(0, 1)$ with the topology of uniform convergence in $(0,1)$ (Norm $f = \text{Max}_{x\in(0,1)} |f(x)|$). The operator $\dfrac{d}{dx}$ is a discontinuous operator on this space, as a sequence of polynomials may tend uniformly to zero in $(0,1)$ while their derivatives may not. However the closed graph theorem is not true for this operator; for if a sequence of polynomials $P_j \to 0$ uniformly in $(0,1)$ and $\dfrac{dP_j}{dx} \to \theta$ uniformly in $(0,1)$, we must have $\theta = 0$. Here the space is not complete. In fact the completion of E in the norm defined above is the space of continuous functions in $(0,1)$ (Weierstrass approximation theorem) and $\dfrac{d}{dx}$ can not be extended to this space].

We shall now use the closed graph theorem to prove the continuity of G in $\mathscr{D}$. $\mathscr{D}$ is a Fréchet space. (In general $\mathscr{E}$ is a Fréchet space; 81
here since the manifold is compact $\mathscr{D} = \mathscr{E}$). Let $\{\varphi_j\}$ be a sequence of elements of $\mathscr{D}$ such that $\varphi_j \to 0$ and $G\varphi_j \to \theta$; we have to show that $\theta = 0$. Write

$$\varphi_j = \pi_1\varphi_j - \Delta G\varphi_j$$

π_1 is continuous in $\mathscr{D}$. For if $\omega \in \mathscr{D}$

$$\pi_1\omega = \sum_k (\omega, \theta_k)\theta_k$$

where θ_k is an orthonormal base for $\mathscr{H}_1$. If $\varphi_j \to \varphi$ in $\mathscr{D}(\varphi_j, \theta_k) \to (\varphi, \theta_k)$. Therefore if $\varphi_j \to 0$ in $\mathscr{D}$, $\pi_1\varphi_j \to 0$ in $\mathscr{D}$. Now since $G\varphi_j \to \theta$ and Δ is continuous $\Delta G\varphi_j \to \Delta\theta$. So $\Delta\theta = 0$ or $\theta \in \mathscr{H}_1$; already $\theta \in \mathscr{H}_2 + \mathscr{H}_3$; consequently $\theta = 0$.

Similarly it can be proved that G is continuous in $\mathscr{H}$.

Self-adjointness of G

We shall now show that G is self-adjoint in $\mathscr{H}$;

$$(G\varphi, \psi) = (\varphi, G\psi), \varphi, \psi \in \mathscr{H}$$

If we put $G\varphi = \alpha$ and $G\psi = \beta$ we have

$$\psi = \pi_1\psi - \Delta\beta \text{ and } \varphi = \pi_1\varphi - \Delta\alpha$$

so that

$$\begin{aligned}(G\varphi,\psi) &= (\alpha,\psi)\\ &= (\alpha,\pi_1\psi) - (\alpha,\Delta\beta)\end{aligned}$$

82 and

$$\begin{aligned}(\varphi,G\psi) &= (\varphi,\beta)\\ &= (\pi_1\varphi,\beta) - (\Delta\alpha,\beta)\end{aligned}$$

Since $(\alpha,\pi_1\psi) = (\pi_1\varphi,\beta) = 0$ ($\alpha,\beta \in \mathscr{H}_2 + \mathscr{H}_3$ while $\pi_1\psi$, $\pi_1\varphi \in \mathscr{H}_1$) we have only to prove that $(\alpha,\Delta\beta) = (\Delta\alpha,\beta)$. But this is evident when φ or ψ (and hence α or β) is a C^∞ form. So we have the relation $(G\varphi,\psi) = (\varphi,G\psi)$ when φ or ψ is a C^∞ form. Since $\mathscr{D}$ is dense in $\mathscr{H}$ and G is continuous we obtain

$$(G\varphi,\psi) = (\varphi,G\psi), \varphi,\psi \in \mathscr{H}.$$

G is hermitian positive $(G\varphi,\varphi) \geq 0$ and $(G\varphi,\varphi) = 0$ if and only if φ is harmonic. For if $G\varphi = \alpha$

$$\begin{aligned}(G\varphi,\varphi) &= (\alpha,\pi_1\varphi - \Delta\alpha)\\ &= (\alpha,-\Delta\alpha)\\ &\geq 0\end{aligned}$$

and $(G\varphi,\varphi) = 0$ if and only if $\Delta\alpha = 0$ or $\Delta\varphi = 0$.

Lecture 15

Decomposition of $\mathscr{D}'$

We shall now extend the operator G to $\mathscr{D}'$. For a current T we define 83
GT by:

$$(GT, \varphi) = (T, G\varphi).$$

(This definition is consistent) Since G is continuous on $\mathscr{D}$, it is easy to verify that GT is continuous on $\mathscr{D}$. For if $\varphi_j \to 0$ in $\mathscr{D}$, $G\varphi_j \to 0$ and $(GT, \varphi_j) = (T, G\varphi_j) \to 0$. The operator G is continuous on $\mathscr{D}'$ endowed with the weak topology. For if $T_j \to 0$ in the weak topology, for a fixed $\varphi \in \mathscr{D}$ $(GT_j, \varphi) = (T_j, G\varphi) \to 0$. We define the operators π_1, π_2 and π_3 on $\mathscr{D}'$ by:

$$\pi_1 = I + \Delta G$$
$$\pi_2 = d\partial G, \pi_3 = \partial d G.$$

The operators π_1, π_2 and π_3 verify the relations:

$$\pi_i \pi_j = 0 \text{ for } i \neq j$$
$$\pi_i^2 = \pi_i$$
$$I = \pi_1 + \pi_2 + \pi_3.$$

$\mathscr{D}'$ is the direct sum of $\mathscr{D}'_1$ (the space of harmonic forms), $\mathscr{D}'_2 = d\mathscr{D}'$ and $\mathscr{D}'_3 = \partial\mathscr{D}'$. For a current T we have the decomposition formula

$$T = \pi_1 T + d\partial GT + \partial dGT.$$

Commutativity of an operator with Δ, π_1 and G

84 If an operator $A : \mathscr{D} \to \mathscr{D}$ commutes with Δ it also commutes with π_1 and G. If $A\Delta\omega = \Delta A\omega$ we have to show that $\pi_1 A\omega = A\pi_1\omega$ and $GA\omega = AG\omega$. If $\pi_1 A\omega = \widetilde{\omega}$, $\widetilde{\omega}$ is characterised by the properties:

i) $\Delta\widetilde{\omega} = 0$

ii) $A\omega - \widetilde{\omega} = \Delta\eta$ for some $\eta \in \mathscr{D}$.

We shall verify that $\widetilde{\omega} = A\pi_1\omega$ also possesses these properties.

$$\Delta\widetilde{\omega} = \Delta A\pi_1\omega = A\Delta\pi_1\omega = 0$$

as $\pi_1\omega$ is a harmonic form;

$$\begin{aligned} A\omega - \widetilde{\omega} &= A\omega - A\pi_1\omega \\ &= A(\omega - \pi_1\omega) \\ &= A(\Delta\eta') \\ &= \Delta(A\eta'). \end{aligned}$$

So $\pi_1 A\omega = A\pi_1\omega \cdot \omega_1 = GA\omega$ is characterised by:

i) $-\Delta\omega_1 = A\omega - \pi_1 A\omega$

ii) $\pi_1\omega_1 = 0$

We shall show that $\omega_1 = AG\omega$ has these properties.

$$\begin{aligned} -\Delta AG\omega &= -A\Delta G\omega \\ &= A(\omega - \pi_1\omega) \\ &= (A\omega - \pi_1 A\omega) \text{ (as } A \text{ and } \pi_1 \text{ commute)} \end{aligned}$$

85 and

$$\begin{aligned} \pi_1 AG\omega &= A\pi_1 G\omega \\ &= 0. \end{aligned}$$

The operator Δ commutes with each of the operators d, ∂, Δ, $*$, π_1, π_2, π_3, G. Consequently π_1 and G also commute with each of these operators.

Operators on currents as operators on cohomology spaces

Suppose A is an operator on currents with the following properties:

i) for every cohomology class $\overset{0}{\alpha}$ there exists at least one element of the class such that $A\alpha$ is closed.

ii) If α is a coboundary and $A\alpha$ is closed, then $A\alpha$ is a coboundary. Then we can define A on the cohomology vector spaces intrinsically: for each cohomology class $\overset{0}{\alpha}$ we choose a representative α such that $A\alpha$ is closed and map $\overset{0}{\alpha}$ onto the cohomology class determined by $A\alpha$. Of course, this definition makes no use of any particular metric.

If there exists at least one Riemannian metric on the manifold for which A and Δ commute then the conditions i) and ii) are verified. In a cohomology class $\overset{0}{\alpha}$ we choose the harmonic form α. Then $A\alpha$ is closed, in fact, harmonic. $\Delta A\alpha = A\Delta\alpha = 0$. To verify the second condition we notice that a necessary and sufficient condition for a closed element ω to be a coboundary is that $\pi_1\omega = 0$. If α is a coboundary and $A\,\alpha$ is closed,

$$\begin{aligned} \pi_1 A\alpha &= A\pi_1\omega \text{ (as } A \text{ and } \pi_1 \text{ commute)} \\ &= 0. \end{aligned}$$

So A operates *intrinsically* on the cohomology spaces. **86**

[We may simply identify the cohomology space with the space of harmonic forms and let A operate on the space of harmonic forms. (A operates on the space of harmonic forms as A and Δ commute).]

Complex differential forms on a manifold

Let V^N be a C^∞ manifold. Just as we considered real valued C^∞ functions on V^N we may also consider complex-valued C^∞ functions on V^N;

a complex valued C^∞ function is of the form $\varphi = f + ig$ where f and g are real valued C^∞ functions.

At a point $\underline{a}$ of V^N we can define the space of differentials $T^*_a(V^N)$ with respect to the complex valued functions, just the same way we did in the case of real valued functions. The space $T^*_a(V^N)$ is of complex dimension N. (Henceforth, $T^*_a(V^N)$ will always denote the space of complex differentials at $\underline{a}$. However $T_a(V^N)$ will only denote the real tangent space at $\underline{a}$). Now if F is a vector space over the reals, the space $F + iF$ is called the complexification of F; the complex dimension of $F + iF$ is equal to the real dimension of F. We shall always consider $T^*_a(V^N)$ as the complexification of the space of real differentials at $\underline{a}$. Similarly the space $\overset{p}{\Lambda} T^*_a(V)$ will be considered as the complexification
87 of the real tangent p-covectors at $\underline{a}$. Thus an element $\omega \in T^*_a(V)$ has the canonical decomposition $\omega = \omega_1 + i\omega_2$ where ω_1 and ω_2 are real tangent p-covectors at $\underline{a}$.

If a complex vector space is obtained as the complexification of a real vector space, we have the notion of complex conjugate in this space. For if $G = F + iF$ is the complexification of the real vector space F any element $\omega \in G$ has the canonical decomposition $\omega = \omega_1 + i\omega_2$ where $\omega_1, \omega_2 \in F$; the complex conjugate of ω is the element $\omega_1 - i\omega_2$.

The manifold $\overset{p}{\Lambda} T^*_N(V)$ of all $\overset{p}{\Lambda} T^*_a(V)$ is a C^∞ manifold with real dimension $N + 2\binom{N}{p}$.

We extend the operators d, ∂, Δ to the complex differential forms by linearity $d(\omega_1 + i\omega_2) = d\omega_1 + id\omega_2$ and a scalar product in the space of real differentials at $\underline{a}$ canonically to a Hermitian scalar product in its complexification i.e., in $T^*_a(V)$. We extend $*$ by anti-linearity:

$$*(\omega_1 + i\omega_2) = *\omega_1 - i * \omega_2$$

so that the relation

$$(\alpha, \beta)\tau = \alpha \Lambda * \beta$$

is preserved. The space of (complex) square summable forms becomes a Hilbert space over complex numbers.

Lecture 16

Real vector spaces with a J-Structure

Suppose $G^{(n)}$ is a vector space over the complex numbers. G can also be considered as a vector space over the real numbers. If $e_1, \ldots, e_n$ is a basis of G^n over C, then $e_1, \ldots, e_n$, $ie_1, \ldots, ie_n$ is a basis of G over R. The multiplication by i is a linear transformation of G, considered as a vector space over R, whose square is $-I$, where I is the identity map. The vectors e and ie are dependent when G is considered as a vector space over C but are independent over R. To avoid confusion we shall introduce the notion of a real vector space with a J-structure. **88**

A $2n$-dimensional real vector space G along with a linear transformation $J : G \to G$ with $J^2 = -I$ will be a called a real vector space with a J-structure.

(A real vector space G with a J-structure can be considered as a vector space over complex numbers by defining

$$(\alpha + i\beta)e = \alpha e + \beta Je$$

where α and β are real numbers and e is an element of G).

Let G be a real vector space with a J-structure and $G + iG$ its complexification; the operator J is extended canonically to $G + iG$ by defining:

$$J(X + iY) = JX + iJY, X, Y \in G$$

J is a linear transformation on the complex vector space $G + iG$. (We **89**
notice that the operations J and multiplication by i are different on $G +$

iG; if x is a vector in G, Jx is a vector in G while ix is a vector in iG). The operator J on $G + iG$ has the eigen values i and $-i$.

J is an isomorphism of $G + iG$ onto itself. Canonically we have an isomorphism of the dual space of $G + iG$, $(G + iG)^*$, onto itself contragradient to J (this isomorphism is the inverse of the transpose of J). We denote this operator on the dual space also by J. If $\alpha \in G + iG$ and $\beta \in (G + iG)^*$ then

$$\langle \alpha, \beta \rangle = \langle J\alpha, J\beta \rangle$$
$$\langle J\alpha, \beta \rangle = \langle \alpha, J^{-1}\beta \rangle = -\langle \alpha, J\beta \rangle.$$

J is also defined canonically on the exterior products of $(G + iG)^*$. On the p-th exterior product we have

$$J^2 = (-1)^p I.$$

The operators $\dfrac{\partial}{\partial z_j}$ and $\dfrac{\partial}{\partial \bar{z}_j}$

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

90 We have

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial z_j} + \frac{\partial}{\partial \bar{z}_j}$$
$$\frac{\partial}{\partial y_j} = i\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial \bar{z}_j}\right)$$

The reason for such a definition is as follows. If we take an analytic function of $2n$ real variables $x_1, y_1, \ldots, x_n, y_n$ it can be extended to an

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[0]We now define the operators $\dfrac{\partial}{\partial z_j}$ and $\dfrac{\partial}{\partial \bar{z}_j}$ on differentiable functions on $C^n/(z_1, \ldots, z_n)$, $z_j = x_j + iy_j$ are the coordinate functions on C^n). A priori they do not make sense. We define

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),$$

analytic function of $2n$ complex variables, which we still denote by x_1, $y_1, \ldots, x_n, y_n$. Consider

$$z_j = x_j + iy_j, \bar{z}_j = x_j - iy_j$$

as $2n$-independent complex variables (Here $\bar{z}_j$ does not mean the complex conjugate of z_j; this is so only if x_j and y_j are real). By changing the variables x_j, y_j to z_j, $\bar{z}_j$ we get the above expressions for the partial derivatives $\dfrac{\partial}{\partial z_j}$ and $\dfrac{\partial}{\partial \bar{z}_j}$. Thus these relations are true for analytic functions of real variables prolonged into the complex field. So we take these as general definitions.

We have

$$\frac{\partial}{\partial z_j}(z_k) = \delta_{jk}, \frac{\partial}{\partial \bar{z}_j}(z_j) = 0.$$

$\dfrac{\partial}{\partial_{z_j}}$ $\dfrac{\partial}{\partial \bar{z}_j}$ are the complex derivations. $\dfrac{\partial f}{\partial z_j}$ and $\dfrac{\partial f}{\partial \bar{z}_j}$ are defined for any **91**
differentiable function f.

Suppose f is a C^∞ function on C^n, We have

$$df - \sum_j \frac{\partial f}{\partial x_j} dx_j + \sum_j \frac{\partial f}{\partial y_j} dy_j.$$

But this may be written as

$$\mathrm{df} = \sum_j \frac{\partial f}{\partial z_j} dz_j + \sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j$$

(Here $\dfrac{\partial f}{\partial z_j}$ and $\dfrac{\partial f}{\partial \bar{z}_j}$ are the complex derivatives of f defined above. dz_j and $d\bar{z}_j$ are the differentials of the functions z_j and $\bar{z}_j$). Thus we have an expression for df as though z_j and $\bar{z}_j$ were independent variables. Similarly the formula for the coboundary of a differential form continues to hold as though z_j and $\bar{z}_j$ were independent variables.

Holomorphic functions on C^n

If f is a complex valued function defined on an open subset Ω of C' we say that f is holomorphic in Ω if

$$\lim_{\zeta\to 0} \frac{f(z+\zeta)-f(z)}{\zeta},$$

exists at every point z of Ω. We know that if f is holomorphic it satisfies the Cauchy rule; the Cauchy rule can be written as $\dfrac{\partial f}{\partial \bar{z}} = 0$. Conversely, if f is C' and $\dfrac{\partial f}{\partial \bar{z}} = 0$, f is holomorphic. So a holomorphic function
92 of one complex variable may be defined as a C^1 function of x and y for which $\dfrac{\partial f}{\partial \bar{z}} = 0$. (We may say that a holomorphic function is independent of $\bar{z}$). For functions of several variables we adopt a similar definition. We say that a complex valued function f defined on an open subset of C^n is holomorphic if f is a C^∞ function with respect to the $2n$ real coordinates and

$$\frac{\partial f}{\partial \bar{z}_j} = 0 (j = 1, 2, \ldots, n).$$

Transformation formulae

Suppose we have a diffeomorphism

$$(z_1, \ldots, z_n) \to (\zeta_1, \ldots, \zeta_n)$$

between two open subsets of C^n given by the functions $\zeta_k = \zeta_k(x_1, y_1, \ldots, x_n, y_n)$, $k = 1, \ldots, n$. We then have the following formulae:

$$d\zeta_k = \sum_j \left(\frac{\partial \zeta_k}{\partial z_j} dz_j + \frac{\partial \zeta_k}{\partial \bar{z}_j} d\bar{z}_j \right)$$

$$d\bar{\zeta}_k = \sum_j \left(\frac{\partial \bar{\zeta}_k}{\partial z_j} dz_j + \frac{\partial \bar{\zeta}_k}{\partial \bar{z}_j} d\bar{z}_j \right).$$

In particular if the ζ_i are holomorphic functions of $z_1, \ldots, z_n$ we have

$$d\zeta_k = \sum_j \frac{\partial \zeta_k}{\partial z_j} dz_j$$

(Here $\dfrac{\partial \zeta k}{\partial z_j}$ is the ordinary partial derivative of ζ_k with respect to z_j de- **93**
fined in the theory of holomorphic functions; and thus our notation is coherent). In this case we also have

$$d\bar{\zeta}_k = \sum_j \left(\overline{\frac{\partial \zeta_k}{\partial z_j}}\right) d\bar{z}_j$$

$$\frac{\partial}{\partial z_j} = \sum_k \frac{\partial \zeta_k}{\partial z_j} \frac{\partial}{\partial \zeta_k}$$

$$\frac{\partial}{\partial \bar{z}_j} = \sum_k \frac{\partial \bar{\zeta}_k}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{\zeta} k}.$$

Canonical complex structure on R^{2n}

Let $(x_1, \ldots, x_{2n})$ be the coordinate functions on R^{2n}. We shall identify R^{2n} with C^n by the map

$$(x_1, \ldots, x_{2n}) \to (z_1, \ldots, z_n)$$

where

$$z_j = x_{2j-1} + ix_{2j} (j = 1, 2, \ldots, n)$$

We thus have a canonical complex structure on R^{2n}. If $(e_1, \ldots, e_{2n})$ is the canonical basis for R^{2n} then the canonical complex structure on R^{2n} is given by the J operator defined by:

$$Je_{2j-1} = e_{2j}, Je_{2j} = -e_{2j-1} (j = 1, \ldots, n)$$

Let U be an open subset of R^{2n} and $\Phi : U \to R^{2n}$ be a C^∞ map. **94**
In terms of the canonical complex coordinates on R^{2n} we may give this map by:

$$(z_1, \ldots, z_n) \to (\zeta_1, \ldots, \zeta_n)$$

We say that the map Φ is complex analytic (with respect to the canonical complex structure on R^{2n}) if $\zeta_1, \ldots, \zeta_n$ are holomorphic functions of $(z_1, \ldots, z_n)$.

Complex analytic manifolds

A complex analytic manifold, $V^{(n)}$, of complex dimension n is a C^∞ manifold of real dimension $2n$ with an atlas $\{(U_i, \varphi_i)\}$ (which is incomplete with respect to the C^∞ structure) having the following property: for any two maps (U_i, φ_i) and (U_j, φ_j) of the atlas, the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is complex analytic (with respect to the canonical complex structure on R^{2n}). We assume that the atlas is complete with respect to the complex analytic structure.

Some examples of complex analytic manifolds

i) C^n. The simplest example of a complex analytic manifold is C^n itself.

ii) The Riemann sphere S^2. Consider S^2 as the one point compactification of $C^1 : S^2 = C^1 \cup \infty$. Take for one map the identity map of C^1. For the second map take the map ζ defined by

$$\zeta(z) = 1/z, \quad z \neq \infty$$
$$\zeta(\infty) = 0$$

95 in the complementary set of 0. The intersection of these two maps is the complement of the points 0 and ∞ and here $\zeta = 1/z$ is a holomorphic function of z.

It is known that the only spheres on which we may have a complex analytic structure are S^2 and S^6. On S^2 we have a complex analytic structure. For S^6 we do not know.

iii) **The complex projective space** PC^n

The right generalization of the Riemann sphere is the n-dimensional complex projective space, PC^n. The n-dimensional complex projective space is defined as follows. We take C^{n+1} and omit 0. In $C^{n+1} - (0)$ we introduce an equivalence relation two points $z = (z_1, \ldots, z_{n+1})$ and $z' = (z'_1, \ldots, z'_{n+1})$ are equivalent if $z'_i = \lambda z_i (i = 1, \ldots, n+1)(z' = \lambda z)$ for some $\lambda \neq 0$. The quotient space of $C^{n+1} - (0)$ by this relation (with the quotient topology) is the n-dimensional complex projective space, $PC^n \cdot PC^n$ is a compact complex analytic manifold of complex dimension n. We introduce complex analytic coordinate systems in PC^n as follows. For a fixed i consider the set of points $\underline{a}$ in PC^n whose representatives in C^{n+1} are of the form $(z_1, \ldots, z_{n+1})$, $z_i \neq 0$.

Then the mapping **96**

$$a \to \left(\frac{z_1}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{j+1}}{z_i}, \ldots, \frac{z_{n+1}}{z_i}\right).$$

gives a map. The maps obtained for $i = 1, 2, \ldots, n+1$ cover PC^n and are related by holomorphic functions on the overlaps.

iv) The complex torus.

Lecture 17

The operator J

We now pass on to some intrinsic properties of a complex analytic man- **97**
ifold $V^{(n)}$ of complex dimension n. The (real) tangent space to V at a, $T_a(V)$, is a vector space of dimension $2n$ over R. We shall now introduce on $T_a(V)$ an intrinsic J-structure. Take a map at $\underline{a}$ into R^{2n}. This map gives an isomorphism between $T_a(V)$ and R^{2n}. In R^{2n} we have a J corresponding to the canonical complex structure in R^{2n}; by the canonical isomorphism between $T_a(V)$ and R^{2n} (given by the map) we also have a J in $T_a(V)$. We shall now prove that this J in $T_a(V)$ is intrinsic. If $J_{z_1,\ldots,z_n}$ and $\mathscr{J}_{\zeta_1,\ldots,\zeta_n}$ are the J operators on $T_a(V)$ corresponding to the local coordinates $(z_1,\ldots,z_n)$ and $(\zeta_1,\ldots,\zeta_n)$ we have to prove that J and $\mathscr{Y}\,\mathscr{J}$ are the same. To prove this we consider the complexification of $T_a(V)$, $T_a(V)+iT_a(V)$. We extend J and $\mathscr{J}$ to $T_a(V)+iT_a(V)$; J and $\mathscr{J}$ are operators in $T_a(V)+iT_a(V)$ with eigenvalues $\pm i$. To prove that J and $\mathscr{J}$ are the same it is sufficient to show that the corresponding eigen spaces are the same. From the relations

$$J\left(\frac{\partial}{\partial z_k}\right)=i\frac{\partial}{\partial z_k},\quad J\left(\frac{\partial}{\partial \bar{z}_k}\right)=-i\frac{\partial}{\partial \bar{z}_k}$$

it follows that for J the eigenspace corresponding to the eigenvalue i is the space spanned by

$$\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}$$

and the eigen-space corresponding to $-i$ is the space spanned by **98**

$$\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n};$$

for $\mathscr{J}$ the corresponding spaces are spanned by

$$\frac{\partial}{\partial \zeta_1}, \ldots, \frac{\partial}{\partial \zeta_n} \quad \text{and} \quad \frac{\partial}{\partial \bar{\zeta}_1}, \ldots, \frac{\partial}{\partial \bar{\zeta}_n},$$

respectively. Since the maps are related on the overlaps by holomorphic functions we have the relations

$$\frac{\partial}{\partial z_j} = \sum_k \frac{\partial \zeta_k}{\partial z_j} \frac{\partial}{\partial \zeta_k},$$

$$\frac{\partial}{\partial \zeta_j} = \sum_k \frac{\partial z_k}{\partial \zeta_j} \frac{\partial}{\partial z_k},$$

$$\frac{\partial}{\partial \bar{z}_j} = \sum_k \frac{\partial \bar{\zeta}_k}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{\zeta}_k},$$

$$\frac{\partial}{\partial \bar{\zeta}_j} = \sum_k \frac{\partial \bar{z}_k}{\partial \bar{\zeta}_j} \frac{\partial}{\partial \bar{z}_k};$$

these relations show that the eigen space of J and $\mathscr{J}$ corresponding to the eigen values i and $-i$ are the same.

J operates on the space of tangent covectors real or complex; we
99 have the relations $Jdz_k = -idz_k$ $Jd\bar{z}_k = id\bar{z}_k$. J also operates on the space of tangent p-covectors (real or complex) at $\underline{a}$; consequently J operates on the space differential forms. In fact J is a real operator i.e., J takes real differential forms into real differential forms. For forms of degree P we have

$$J^2 = (-1)^p I$$

Bigradation for differential forms

A differential form ω is said to be of bidegree or of type (p, q) if in every map, ω has the form

$$\omega = \sum_{\substack{j_1 < \ldots < j_p \\ k_1 < \ldots < k_q}} \omega_{j_1 \ldots j_p k_1 \ldots k_q} dz_{j_1} \wedge \ldots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \ldots \wedge d\bar{z}_{k_q}.$$

This definition is correct since if two maps overlap and a form defined in the overlap be of type (p, q) in one of the maps it is also of type (p, q) in the other map. (This follows easily from the fact that the maps are related by holomorphic functions on the overlaps). p is called the z degree and q the $\bar{z}$ degree of ω.

Any differential form ω of total degree r can be uniquely written in the form

$$\omega = \sum_{p+q=r} \overset{(p,q)}{\widetilde{\omega}}$$

where $\overset{(p,q)}{\widetilde{\omega}}$ is a form of bidegree (p, q).

Holomorphic functions and forms

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A (complex valued) function f on $V^{(n)}$ is said to be holomorphic if f is holomorphic on every map; this definition is correct as a holomorphic function of n holomorphic functions on C^n is again holomorphic. A holomorphic differential form of degree p is a form of type $(p, 0)$ whose coefficients in every map are holomorphic.

The operators d_z and $d_{\bar{z}}$

Suppose ω is a form of bidegree (p, q). A priori $d\omega$ is the sum of forms of all bidgree (r, s) with $r + s = p + q + 1$. However we shall show that $d\omega$ is the sum of a form of bidegree $(p + 1, q)$ and a form of bidegree $(p, q + 1)$. Let

$$\omega = \sum_{J,K} \omega_{J,K} dz_J \wedge d\bar{z}_K$$

in a map. $J = (j_1, \ldots, j_p)$, $K = (k_1, \ldots, k_q)$ denote a system of indices in

$$\begin{aligned} d\omega &= \sum d\omega_{J,K} dz_J \wedge d\bar{z}_K \\ &= \sum \frac{\partial \omega_{J,K}}{\partial z_1} dz_1 \wedge dz_J \wedge d\bar{z}_K \end{aligned}$$

$$+\sum \frac{\partial \omega_{J,K}}{\partial \bar{z}_1} d\bar{z}_1 \wedge dz_J \wedge d\bar{z}_J$$

Here the first form is of bidegree $(p+1, q)$ and the second of bidegree
101 $(p, q+1)$; therefore these two forms have an intrinsic meaning. Thus

$$d\overset{(p,q)}{\omega} = \overset{(p+1q)}{\alpha} + \overset{(p,q+1)}{\beta}$$

intrinsically, where α and β are forms of bidegree $(p+1, q)$ and $(p, q+1)$ respectively. We now define the operators d_z and $d_{\bar{z}}$ by:

$$d_z\omega = \alpha, \quad d_{\bar{z}}\omega = \beta.$$

We have

$$d\omega = d_2\omega + d_{\bar{z}}\omega.$$

We observe that, in a map, $d_z\omega$ involves only the partial derivatives with respect to z while $d_{\bar{z}}\omega$ involves only the partial derivatives with respect to $\bar{z} \cdot d_z$ increases the degree corresponding to z by one while $d_{\bar{z}}$ increases the degree corresponding to $\bar{z}$ by one.

We extend the operators d_z and $d_{\bar{z}}$ to all forms by linearity.

We shall now consider some properties of d_z and $d_{\bar{z}} \cdot d_z$ and $d_{\bar{z}}$ are complex operators (If ω is real $d_z\omega$ is complex). d_z is of type $(1, 0)$ and $d_{\bar{z}}$ is of type $(0, 1)$. [An operator is said to be of type (r, s) if it takes a form of bigradation (p, q) into a form of bigradation $(p+r, q+s)$]. These operators are local operators; they are linear. If ω is a form of degree r we have the formulae:

$$d_z(\omega \wedge \widetilde{\omega}) = d_z\omega \wedge \widetilde{\omega} + (-1)^r \omega \wedge d_z\widetilde{\omega},$$
$$d_{\bar{z}}(\omega \wedge \widetilde{\omega}) = d_{\bar{z}}\omega \wedge \widetilde{\omega} + (-1)^r \omega \wedge d_{\bar{z}}\widetilde{\omega}.$$

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It is enough to prove this for homogeneous forms. We take $d(\omega \wedge \widetilde{\omega})$ and decompose it:

$$\begin{aligned} d(\overset{p,q}{\omega} \wedge \overset{s,t}{\widetilde{\omega}}) &= d\omega \wedge \widetilde{\omega} + (-1)^{p+q}\omega \wedge d\widetilde{\omega} \\ &= (d_z\omega \wedge \widetilde{\omega} + (-1)^{p+q}\omega \wedge d_z\widetilde{\omega}) \end{aligned}$$

$$+ (d_{\bar{z}}\omega \wedge \widetilde{\omega} + (-1)^{p+q}\omega \wedge d_{\bar{z}}\widetilde{\omega});$$

the first term is of bidegree $(p + s + 1, q + t)$ while the second is of bidegree $(p + s, q + t + 1)$ and this proves the result.

Further we have the relations

$$d_z d_z = 0, d_{\bar{z}} d_{\bar{z}} = 0 \quad \text{and} \quad d_z d_{\bar{z}} + d_{\bar{z}} d_z = 0.$$

For, from the relations

$$(d_z + d_{\bar{z}})(d_z + d_{\bar{z}}) = d^2 = 0$$

we obtain

$$d_z d_z + (d_z d_{\bar{z}} + d_{\bar{z}} d_z) + d_{\bar{z}} d_{\bar{z}} = 0.$$

To conclude the above relations from this we have only to observe that for any form $\overset{p,q}{\omega}$ the forms $d_z d_z \omega$, $(d_z d_{\bar{z}}\omega + d_{\bar{z}} d_z \omega)$ and $d_{\bar{z}} d_{\bar{z}}\omega$ are of different bidegrees namely of the bidegrees $(p+2, q)$, $(p+1, q+1)$ and $(p, q+2)$ respectively.

z and $\bar{z}$ cohomologies

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We have now two new coboundary operators d_z and $d_{\bar{z}}$ and hence two new cohomologies, z and $\bar{z}$ cohomologies. We shall confine our attention to the $\bar{z}$ cohomology; this will give all the information about holomorphic forms. We shall see that the construction of holomorphic forms depends on the $\bar{z}$ cohomology. We shall denote by $H^{p,q}_{\bar{z}}(\mathscr{E}(V))$ the space $Z^{p,q}/B^{p,q}$ where $Z^{p,q}$ and $B^{p,q}$ are the subspaces of $\mathscr{E}^{p,q}(V)$ (the space of forms of bidegree (p, q)) consisting of $\bar{z}$ cocycles and $\bar{z}$ coboundaries respectively.

Similarly we have the space $H^{p,q}_{\bar{z}}(\mathscr{D}(V))$.

Intrinsic characterization of holomorphic forms

Let ω be a C^∞ form of type $(p, 0)$; a necessary and sufficient condition for ω to be holomorphic is that

$$d_{\bar{z}}\omega = 0.$$

Let

$$\omega = \sum_K \omega_K dz_K$$

in a map.

$$d_{\bar{z}}\omega = \sum_{K,1} \frac{\partial \omega_K}{\partial \bar{z}_1} d\bar{z}_1 \wedge dz_K$$

In ω is holomorphic, ω_K are holomorphic; hence $\dfrac{\partial \omega_K}{\partial \bar{z}_1} = 0$ and $d_{\bar{z}}\omega = 0$. If $d_{\bar{z}}\omega = 0$

$$\sum_{K,1} \frac{\partial \omega_K}{\partial \bar{z}_1} d\bar{z}_1 \wedge dz_K = 0;$$

104 in the sum on the left side all the terms $d\bar{z}_1 \wedge dz_K$ are different (i.e., if $(K, 1) \neq (K', l')$, $d\bar{z}_1 \wedge dz_K \neq d\bar{z}_{l'} \wedge dz_{K'}$). Consequently $\dfrac{\partial \omega_K}{\partial \bar{z}_1} = 0$. This proves that the ω_K are all holomorphic.

Holomorphic forms and $\bar{z}$ cohomology

Let us now consider the space $H^{p,0}_{\bar{z}}(\mathscr{E}(V)) \cdot H^{p,0}_{\bar{z}}(\mathscr{E}(V))$ is just the space of $(p, 0)$ forms which are $\bar{z}$ cocycles (since a $(p, 0)$ from which is $\bar{z}$ coboundary is trivial). i.e., $H^{p,0}_{\bar{z}}(\mathscr{E}(V))$ is the space of holomorphic p-forms. Similarly $H^{p,0}_{\bar{z}}(\mathscr{D}(V))$ is the space of holomorphic forms with compact support. A holomorphic form with compact support is zero on each non-compact connected component.

Consider the space $\mathscr{E}^{\cdot,q} = \sum\limits_p \mathscr{E}^{p,q}$ of C^∞ forms of $\bar{z}$ degree q. $\sum\limits_q \mathscr{E}^{\cdot,q}$ is a complex with the differential operator $d_{\bar{z}}$. This gives rise to the cohomology groups $H^{\cdot,q}_{\bar{z}}$. $H^{\cdot,0}_{\bar{z}}$ is the space holomorphic forms.

Lecture 18

The canonical orientation of a complex manifold

We shall now show that a complex analytic manifold $V^{(n)}$ (considered as a $2n$-dimensional real manifold) is orientable and has a canonical orientation. The orientation on V^{2n} is determined by the maps giving the complex analytic structure on V^{2n}; we have to verify that two such maps, considered as real coordinate systems, have a positive Jacobian on the overlaps. To prove this, let $(z_1, \ldots, z_n) \to (\zeta_1, \ldots, \zeta_n)$ be a holomorphic map of C^n to C^n. Let D be the Jacobian of $\zeta_1, \ldots, \zeta_n$ with respect to $z_1, \ldots, z_n$; let further $z_i = x_i + iy_i$, $\zeta_i = \xi_i + i\eta_i$ and J the Jacobian of the functions $(\xi_1, \eta_1, \ldots, \xi_n, \eta_n)$ with respect to $(x_1, y_1, \ldots, x_n, y_n)$. We shall prove that $J = |D|^2$. Now $d\zeta_i = d\xi_i + \mathrm{i d}\,\eta_i$ and $d\overline{\zeta}_i = d\xi_i - \mathrm{i d}\,\eta_i$ so that 105

$$d\xi_i \wedge d\eta_i = -\frac{1}{2i} d\zeta \wedge d\overline{\zeta}_i;$$

Similarly

$$dx_i \wedge dy_i = \frac{-1}{2i} dz_i \wedge d\overline{z}_i i_1$$

We have

$$\begin{aligned} J = \frac{d\xi_1 \wedge d\eta_1 \wedge \ldots \wedge d\xi_n \wedge d\eta_n}{dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n} &= \frac{d\zeta_1 \wedge d\overline{\zeta}_1 \wedge \ldots \wedge d\zeta_n \wedge d\overline{\zeta}_n}{dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_n} \\ &= \frac{d\zeta_1 \wedge \ldots \wedge d\zeta_n \wedge d\overline{\zeta}_1 \wedge \ldots \wedge d\overline{\zeta}_n}{dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n} \\ &= D\overline{D} = |D|^2 \end{aligned}$$

as 106

$$\frac{d\zeta_1 \wedge \ldots \wedge d\zeta_n}{dz_1 \wedge \ldots \wedge dz_n} = D, \quad \frac{d\overline{\zeta}_1 \wedge \ldots \wedge d\overline{\zeta}_n}{d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n} = \overline{D}.$$

In the case of our maps $D \neq 0$ and therefore $J > 0$.

Currents

We shall now define bigradation for currents and the operators J, d_z and $d_{\bar{z}}$ on currents.

A current T is said to be of bidegree (p, q) if

$$\langle T, \overset{r,s}{\varphi} \rangle = 0$$

whenever $(p, q) \neq (n-r, n-s)$. ($\overset{r,s}{\varphi}$ is a form of bidegree (r, s)). A current $\overset{p,q}{T}$ of bidegree (p, q) can be considered as a continuous linear functional on $\overset{n-p,n-q}{\mathscr{D}}$, the space of forms of bidegree $(n-p, n-q)$ with compact support.

We define the operator J on currents by:

$$\langle JT, \varphi \rangle = \langle T, J^{-1}\varphi \rangle.$$

We do this because, when ω is a form we have

$$\langle J\omega, \varphi \rangle = \langle \omega, J^{-1}\varphi \rangle.$$

Writing $J^{-1}\varphi = \psi$, we have to prove that

$$\int J\omega \wedge J\psi = \int \omega \wedge \psi \quad \text{or} \quad \int J(\omega \wedge \psi) = \int \omega \wedge \psi$$

107 but

$$J(\omega \wedge \psi) = \omega \wedge \psi$$

since $\omega \wedge \psi$ is a form of degree $2n$ and $J = I$ on a form of degree $2n$. (In general

$$J\overset{p,q}{\omega} = (-i)^p i^q \omega = i^{q-p}\omega).$$

We define d_z and $d_{\bar{z}}$ on currents by:

$$\langle d_z \overset{p,q}{T}, \overset{n-p-1,n-q}{\varphi} \rangle = (-1)^{p+q+1} \langle T, d_{\bar{z}}\varphi \rangle$$
$$\langle d_{\bar{z}} \overset{p,q}{T}, \overset{n-p,n-q-1}{\varphi} \rangle = (-1)^{p+q+1} \langle T, d_{\bar{z}}\varphi \rangle.$$

These relations are true for forms. In fact when ω is a form of total degree p we have

$$\int_V d_z \omega \wedge \varphi = (-1)^{p+1} \int_V \omega \wedge d_z \varphi.$$

For, φ is of type $(n-1, n)$ so that

$$d_{\bar{z}}(\omega \wedge \varphi) = 0 \quad \text{and} \quad d_z(\omega \wedge \varphi) = d(\omega \wedge \varphi)$$

By Stokes' formula $\int_V d(\omega \wedge \varphi) = 0$; hence $\int_V d_z(\omega \wedge \varphi) = 0$ or

$$\int_V (d_z \omega \wedge \varphi) + (-1)^p \omega \wedge d_z \varphi) = 0).$$

We now have some more cohomologies and there is need to prove **108**
some kind of de Rham's theorems. It can be proved that the $\bar{z}$ cohomologies of $\mathscr{D}'$ and $\mathscr{E}$ are the same and those of $\mathscr{D}$ and $\mathscr{E}'$ are the same.

Ellipticity of the system $\partial/\partial\bar{z}k$

We shall show that the system $\dfrac{\partial}{\partial \bar{z}_k}$ in C^n is elliptic i.e., if

$$\frac{\partial T}{\partial \bar{z}_k} = \alpha_k \; (k = 1, 2, \ldots, n)$$

where T is a current and α_k are C^∞ forms then T is a C^∞ form. We have

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i \frac{\partial}{\partial y_k}\right)$$

$$\frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)$$
$$\frac{\partial}{\partial z_k}\frac{\partial}{\partial \bar{z}_k} = \frac{1}{4}\left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2}\right)$$

so that

$$\sum_k \frac{\partial}{\partial z_k}\frac{\partial}{\partial \bar{z}_k} = \frac{\Delta}{4}$$

where Δ is the usual Laplacian in R^{2n}.

Since $\dfrac{\partial T}{\partial \bar{z}_k} = \alpha_k \in \mathscr{E}$, $\Delta T \in \mathscr{E}$. By the elliptic character of Δ, T is C^∞.

109 It follows in particular that a distribution T on C^n which satisfies the Cauchy relations

$$\frac{\partial T}{\partial \bar{z}_k} = 0\,(k = 1, 2, \ldots, n)$$

is a holomorphic function. Similarly if a current T of bidegree $(p, 0)$ on C^n satisfies the system of partial differential equations

$$\frac{\partial T}{\partial \bar{z}_k} = 0\,(k = 1, 2, \ldots, n)$$

then T is a holomorphic form of degree p.

Ellipticity of $d_{\bar{z}}$ on $\mathscr{D}^{0'}$

Let $V^{(n)}$ be a complex analytic manifold. If T is a current of degree *zero* and $d_{\bar{z}}T = \alpha$ is a C^∞ form, then T is a C^∞ function. For, in a map, the relation $d_{\bar{z}}T = \alpha$ implies

$$\frac{\partial T}{\partial \bar{z}_k} = \alpha_k$$

where α_k are C^∞ functions. By the result proved earlier T is a C^∞ function.

As a consequence we obtain that if $\overset{p,0}{T}$ is a current of bidegree $(p, 0)$ such that $d_{\bar{z}}\overset{p,q}{T} = 0$ then $\overset{p,0}{T}$ is a holomorphic form.

J-Hermitian forms

Let G^{2n} be a vector space over R with a J-structure. A positive definite J-Hermitian form on G is a map H of $G \times G$ into the complex numbers with the following properties: 110

1) H is R-bilinear.

2) $H(JX, Y) = -H(X, JY) = iH(X, Y)$ for X, $Y \in G$ (Hence $H(JX, JY) = H(X, Y)$).

3) $H(X, X) > 0$ for $X \neq 0$.

The real part of the positive definite Hermitian form, $(X, Y) = Rl\, H(X, Y)$, defines a Euclidean structure in G. Since $H(X, Y)$ is invariant under J, (X, Y) is also invariant under J:

$$(JX, JY) = (X, Y)$$

Since $(JX, X) = 0$ the vectors X and JX are orthogonal with respect to the Euclidean structure. From the relation $H(JX, Y) = iH(X, Y)$, we have

$$(JX, Y) = -\operatorname{Im} H(X, Y).$$

Hence

$$\begin{aligned} H(X, Y) &= (X, Y) + i \operatorname{Im} H(X, Y) \\ &= (X, Y) - i(JX, Y) \\ &= (X, Y) + i(X, JY). \end{aligned}$$

This shows that H is completely determined by its real part. If (X, Y) is a positive definite quadratic form on G which is invariant under J this 111
determines a positive definite J-Hermitian form on G if we set

$$H(X, Y) = (X, Y) + i(X, JY).$$

Since $(JX, X) = 0$, ($)(JX, Y)$ is an anti-symmetric bilinear form on G. This defines an element Ω of $\overset{2}{\wedge} G^*$ (G^* is the dual of G) by the formula

$$\langle \Omega, X \wedge Y \rangle = (JX, Y), X, Y \in G.$$

G can be considered (canonically) as a vector space over complex numbers, as we have already remarked. If $(u_1, \ldots, u_n)$, $(v_1, \ldots, v_n)$ are the coordinates of the vectors U and V of G with respect to a complex basis $(e_1, \ldots, e_n)$.,

$$H(U, V) = \sum_{j,k} g_{jk} u_j \bar{v}_k, g_{jk} = \bar{g}_{kj}, (g_{ij}) > 0.$$

Let us compute Ω in terms of this basis. Ω is given by

$$\begin{aligned} \langle \Omega, U \wedge V \rangle &= -\frac{H(U, V) - H(V, U)}{2i} \\ &= -\frac{H(U, V) - \overline{H(U, V)}}{2i} \\ &= -\frac{1}{2i} \sum_{j,k} g_{jk} (u_j \bar{v}_k - \bar{u}_k v_j). \end{aligned}$$

If $(e_1^*, \ldots, e_n^*)$ is the dual basis of $(e_1, \ldots, e_n)$

$$\langle e_j^* \wedge \bar{e}_k^*, U \wedge V \rangle = \begin{vmatrix} u_j, \bar{u}_k \\ v_j, \bar{v}_k \end{vmatrix}$$

112 so that

$$\Omega = -\frac{1}{2i} \sum g_{jk} e_j^* \wedge \bar{e}_k^*.$$

If we choose an orthonormal basis $(e_1, \ldots, e_n)$ for the hermitian form,

$$\Omega = -\frac{1}{2i} \sum e_j^* \wedge \bar{e}_k^*.$$

Hermitial Manifolds

Let $V^{(n)}$ be a complex analytic manifold. At each tangent space $T_a(V)$ we have a canonical J-structure. $V^{(n)}$ will be called a Hermitian manifold if on each $T_a(V)$ we have a positive definite J-Hermitial form such that the twice covariant tensor field defined by these forms is a C^∞ tensor field.

On every complex analytic manifold we can put a Hermitian structure, just the same way we introduced a Riemannian structure on a C^∞ manifold.

In a Hermitian manifold $V^{(n)}$ the real part of the J-Hermitian form on each $T_a(V)$ gives rise to a Riemannian structure on the manifold; the imaginary part gives rise to a real C^∞ differential form Ω of bidegree $(1, 1)$.

If in a coordinate system $(z_1, \ldots, z_n)$ the Hermitian form is given by

$$\sum g_{jk} dz_j d\bar{z}_k,$$

then in this map, **113**

$$\Omega = -\frac{1}{2i} \sum g_{jk} dz_j \wedge d\bar{z}_k.$$

If τ is the volume element associated with the Riemannian structure we have the relation $\Omega^n = n!\tau$. For, if $(z_1, \ldots, z_n)$, $z_j = x_j + iy_j$, is a local coordinate system at $a \in V$ in which the Hermitian form at $\underline{a}$ is given by $\sum (dz_k)_a \overline{(dz_k)_a}$, we have,

$$\begin{aligned}
\Omega_a &= -\frac{1}{2i} \sum_k (dz_k)_a \wedge (d\bar{z}_k)_a \\
&= \sum (dx_k)_a \wedge (dy_k)_a; \\
\Omega_a^n &= (dx_1)_a \wedge (dy_1)_a \wedge \ldots \wedge (dx_n)_a \wedge (dy_n)_a \wedge n! \\
&= n!\tau_a.
\end{aligned}$$

Kahlerian Manifolds

A Hermitian manifold is called a Kahlerian manifold if $d\Omega = 0$.

There exist manifolds with an infinity of Hermitian structures but with no Kahlerian structure.

For a compact Kahlerian manifold V,

$$b^{2p} \geq 1.$$

To prove this we notice that

$$\int_V \Omega^n = \int_V n!\tau > 0.$$

114 Ω^n is a cocycle but not a coboundary, since $\int_V \Omega^n \neq 0$. It follows that the forms $\Omega^p (1 \leq p \leq n)$, which are cocycles, are not coboundaries. Hence the $2p$-th cohomology groups are different form zero or $b^{2p} \geq 1$.

Since $b^4(S^6) = 0$, if S^6 is a complex analytic manifold it is not Kahlerian.

Every complex analytic manifold of complex dimension 1 is Kahlerian because and 2-form is closed.

Lecture 19

Some more operators

115

Let

$$\widetilde{d} = -id_z + id_{\overline{z}}$$

Then

$$\widetilde{d} = JdJ^{-1}$$

For, if $\overset{p,q}{\omega}$ is a form of bidegree (p, q), $J\omega = i^{q-p}\omega$ and $J^{-1}\omega = i^{p-q}\omega$ so that

$$\begin{aligned} Jd_z J^{-1}\omega &= i^{q-(p+1)} i^{p-q} d_z \omega \\ &= -id_z \omega; \end{aligned}$$

thus

$$Jd_z J^{-1} = -id_z$$

and similarly

$$Jd_{\overline{z}} J^{-1} = id_{\overline{z}}$$

so that

$$\widetilde{d} = JdJ^{-1}.$$

$\widetilde{d}$ is the transform of d by the automorphism J. Since J and d are real operators (i.e., take real forms into real forms) $\widetilde{d}$ is a real operator. Now we can decompose the operators d_z and $d_{\overline{z}}$ into real and imaginary parts as follows:

$$d_z = \frac{1}{2}(d + i\widetilde{d})$$

$$d_{\overline{z}} = \frac{1}{2}(d - i\widetilde{d}).$$

We have evidently $\widetilde{d}\widetilde{d} = 0$ and $d\widetilde{d} + \widetilde{d}d = 0$.

The operators ∂_z and $\partial\overline{z}$ are the adjoints (with respect to the Riemannian structure) of the operators d_z and $d_{\overline{z}}$ respectively:

$$(d_z\alpha, \beta) = (\alpha, \partial_z\beta)$$
$$(d_{\overline{z}}\alpha, \beta) = (\alpha, \partial_{\overline{z}}\beta)$$

116 the scalar product being the global Riemannian scalar product. ∂_z is an operator of type $(-1, 0)$ while $\partial_{\overline{z}}$ is an operator of type $(0, -1)$. If $\widetilde{\partial}$ is the adjoint of $\widetilde{d}$

$$\widetilde{\partial} = i\partial_z - i\partial_{\overline{z}};$$

we have

$$\partial_z = \frac{1}{2}(\partial - i\widetilde{\partial})$$
$$\partial_{\overline{z}} = \frac{1}{2}(\partial + i\widetilde{\partial}).$$

The following relations are easily verified:

$$\partial_z\partial_z = 0, \partial_{\overline{z}}\partial_{\overline{z}} = 0, \partial_z\partial_{\overline{z}} + \partial_{\overline{z}}\partial_z = 0$$
$$\partial\partial = 0, \widetilde{\partial}\widetilde{\partial} = 0, \partial\widetilde{\partial} + \widetilde{\partial}\partial = 0.$$

We now introduce two more operators, L and Λ. L is simply multiplication by $\Omega : L\omega = \Omega \wedge \omega$.

L is an operator of type $(1, 1)$. Λ is the adjoint of L. We have

$$\Lambda = *^{-1}L * .$$

For

$$\begin{aligned}(\Lambda\alpha, \beta) &= (\alpha, L\beta)\\ &= \int^{-1} * \overline{\alpha \wedge L\beta}\end{aligned}$$

$$
\begin{aligned}
&= \int^{-1} \overline{*\alpha \wedge \Omega \wedge \beta} \\
&= \int \overline{\Omega \wedge \overset{-1}{*}\alpha \wedge \beta} \\
&= \int \overline{(L \overset{-1}{*} \alpha) \wedge \beta} \\
&= \int \overline{\overset{-1}{*}[*L \overset{-1}{*} \alpha] \wedge \beta} \\
&= (*L \overset{-1}{*} \alpha, \beta) \\
&= (\overset{-1}{*} L * \alpha, \beta)
\end{aligned}
$$

We have used above the fact that Ω is a real form of degree 2. **117**

Commutativity relations in a Kahlerian manifold

We shall now consider the commutativity properties of the operators on a Kahlerian manifold.

L commutes with the operators d, $\widetilde{d}$, d_z and $d_{\overline{z}}$; we denote this by

$$\sqrt[L]{d, \widetilde{d}, d_z, d_{\overline{z}}}$$

For,

$$d(\Omega \wedge \omega) = \Omega \wedge d\omega \ \text{ as } \ d\Omega = 0;$$

since

$$d_z\Omega + d_{\overline{z}}\Omega = 0$$

and $d_z\Omega$ and $d_{\overline{z}}\Omega$ are forms of bidegree $(2,1)$ and $(1,2)$ we have $d_z\Omega = 0$ and $d_{\overline{z}}\Omega = 0$; it follows that

$$
\begin{aligned}
d_z(\Omega \wedge \omega) &= \Omega \wedge d_z\omega \\
d_{\overline{z}}(\Omega \wedge \omega) &= \Omega \wedge d_{\overline{z}}\omega
\end{aligned}
$$

By taking the adjoints we find that **118**

$$\sqrt[\Lambda]{\partial, \widetilde{\partial}, \partial_z, \partial_{\overline{z}}}$$

L does not commute with ∂, $\widetilde{\partial}$, ∂_z and $\partial_{\overline{z}}$. Λ does not commute with d, $\widetilde{d}$, d_z and $d\overline{z}$.

$$L\partial\omega = \Omega \wedge \partial\omega, \quad \partial L\omega = \partial(\Omega \wedge \omega)$$

and we have no rule for the ∂ of a product so that it is not possible to compare $L\partial$ and ∂L directly. We have the following formula which gives the defect of commutativity of d and Λ: writing

$$[\Lambda, d] = \Lambda d - d\Lambda \text{ we have}$$
$$[\Lambda, d] = -\widetilde{\partial}$$

(Consequently Λ and d do not commute). This formula follows from the following formulae:

$$[\Lambda, d_z] = i\partial_{\overline{z}}$$
$$[\Lambda, d_{\overline{z}}] = -i\partial_z$$

which we shall prove in the next lecture. From these relations we have at once

$$[\Lambda, \widetilde{d}] = \partial.$$

119 By taking the adjoints we find that

$$[L, \partial] = \widetilde{d}$$
$$[L, \partial_z] = \mathrm{id}_{\overline{z}}$$
$$[L, \partial_{\overline{z}}] = -\,\mathrm{id}_z$$
$$[L, \widetilde{\partial}] = -d.$$

We shall derive some important formulae from the above formulae.

$$\begin{aligned} d\widetilde{\partial} &= d(d\Lambda - \Lambda d) \\ &= -d\Lambda d \\ \widetilde{\partial} d &= (d\Lambda - \Lambda d)d \\ &= d\Lambda d. \end{aligned}$$

Adding we find that

$$d\widetilde{\partial} + \widetilde{\partial} d = 0$$

i.e., d and $\widetilde{\partial}$ anti-commute (In a Riemannian structure d and ∂ have no commutativity relation; in the case of a Kählerian manifold d and $\widetilde{\partial}$ anti-commute). We have further the following formulae:

$$\begin{aligned}
d\widetilde{d} + \widetilde{d} d &= 0\\
\partial\widetilde{\partial} + \widetilde{\partial}\partial &= 0\\
d\widetilde{\partial} + \widetilde{\partial} d &= 0\\
\partial\widetilde{d} + \widetilde{d}\partial &= 0\\
d_z d_{\bar{z}} + d_{\bar{z}} d_z &= 0\\
\partial_z \partial_{\bar{z}} + \partial_{\bar{z}} \partial_z &= 0\\
d_z \partial_{\bar{z}} + \partial_{\bar{z}} d_z &= 0\\
\partial_z d_{\bar{z}} + d_{\bar{z}} \partial_z &= 0.
\end{aligned}$$

We now consider Δ. **120**

$$\begin{aligned}
d\partial &= (d_z + d_{\bar{z}})(\partial_z + \partial_{\bar{z}})\\
&= d_z\partial_z + d_{\bar{z}}\partial_{\bar{z}} + d_z\partial_{\bar{z}} + d_{\bar{z}}\partial_z;\\
\partial d &= \partial_z d_z + \partial_{\bar{z}} d_{\bar{z}} + \partial_{\bar{z}} d_z + \partial_z d_{\bar{z}}.
\end{aligned}$$

By addition,

$$-\Delta = -\Delta_z - \Delta_{\bar{z}},$$

where

$$\begin{aligned}
-\Delta_z &= d_z\partial_z + \partial_z d_z\\
-\Delta_{\bar{z}} &= d_{\bar{z}}\partial_{\bar{z}} + \partial_{\bar{z}} d_{\bar{z}}.
\end{aligned}$$

Since Δ_z and $\Delta_{\bar{z}}$ are pure operators (i.e., operators of type $(0,0)$), Δ is also a pure operator; in other words, Δ does not change the bigradation. If

$$\widetilde{\Delta} = \widetilde{d}\widetilde{\partial} + \widetilde{\partial}\widetilde{d}$$

then

$$-\widetilde{\Delta} = -\Delta_z - \Delta_{\bar{z}}$$

so that

$$\Delta = \widetilde{\Delta}.$$

121 However,

$$\widetilde{\Delta} = J\Delta J^{-1}.$$

So

$$\Delta = J\Delta J^{-1}$$

i.e., J and Δ commute.

Moreover,

$$\begin{aligned} d_z\partial_z &= \frac{1}{2}(d + i\widetilde{d}) \cdot \frac{1}{2}(\partial - i\widetilde{\partial}) \\ &= \frac{1}{4}[d\partial + \widetilde{d}\widetilde{\partial} + i\widetilde{d}\partial - id\widetilde{\partial}] \end{aligned}$$

and

$$\partial_z d_z = \frac{1}{4}[\partial d + \widetilde{\partial}\widetilde{d} + i\partial\widetilde{d} - i\widetilde{\partial}d]$$

so that

$$-\Delta_z = -\frac{\Delta}{4} - \frac{\widetilde{\Delta}}{4}.$$

Similarly

$$-\Delta_{\overline{z}} = -\frac{\Delta}{4} - \frac{\widetilde{\Delta}}{4}.$$

Consequently

$$\Delta_z = \Delta_{\overline{z}}$$

and

$$\boxed{\Delta = 2\Delta_z = 2\Delta_{\overline{z}} = \widetilde{\Delta}}$$

This formula shows that Δ can be obtained in terms of d_z and $\partial_{\overline{z}}$ alone or in terms $d_{\overline{z}}$ and $\partial_{\overline{z}}$ alone.

We shall now see that Δ commutes with all the operators we have introduced. Let $P^{r,s}$ be the projection of the space of forms on the space
122 of forms of bidegree (r, s). ($P^{r,s}$ maps a form on its homogeneous com-

ponent of bidegree (r, s)). Since Δ is a pure operator, Δ commutes with $P^{r,s}$. Δ commutes with L.

$$\begin{aligned} d\partial L - Ld\partial &= d\partial L - dL\partial + dL\partial - Ld\partial \\ &= d[\partial, L] + [d, L]\partial \\ &= -d\widetilde{d} \end{aligned}$$

(d and L commute as Ω is a closed form of even degree). Similarly

$$[\partial d, L] = \widetilde{-d}d$$

and hence

$$[-\Delta, L] = 0.$$

Since Δ and L commute, Δ and Λ also commute. From the relation

$$\Delta(\Omega \wedge \omega) = \Omega \wedge \Delta\omega$$

we see that if a form ω is harmonic the form $\Omega \wedge \omega$ is also harmonic. In particular the form Ω itself is harmonic. In fact Ω is $*$ closed (it is already closed). To prove this we observe that

$$[L, \partial] = \widetilde{d}$$

or

$$\Omega \wedge \partial\omega - \partial(\Omega \wedge \omega) = \widetilde{d}\omega$$

(In general we do not have a formula for the ∂ of a product of two forms; however this formula gives an expression for the ∂ of the product of a differential form by Ω). Taking $\omega = 1$, we find that $\partial\Omega = 0$. Thus Ω is **123**
closed with respect to d, ∂d_z, $d_{\bar{z}}$, ∂_z and $\partial_{\bar{z}}$.

Since $\widetilde{d} = JdJ^{-1}$, Δ commutes with $\widetilde{d}$; hence commutes with $\widetilde{\partial}$. Since Δ commutes with d, $\widetilde{d}$, ∂ and $\widetilde{\partial}$, Δ commutes with d_z, $d_{\bar{z}}$, ∂_z and $\partial_{\bar{z}}$. That Δ commutes with d_z and $d_{\bar{z}}$ is very important, as this result connects harmonic forms with z and $\bar{z}$ cohomologies.

Thus in a Kählerian manifold Δ commutes with all the operators, in particular with d_z and $d_{\bar{z}}$.

In the case of a compact Kählerian manifold we have a decomposition of $\mathscr{D}$ (and $\mathscr{D}'$) as the direct sum of the space of harmonic forms, the space of z (or $\bar{z}$) coboundaries and the space of z (resp $\bar{z}$) star coboundaries. For we have the decomposition

$$I = \pi_1 + \Delta G;$$

replacing Δ by the new expressions we have

$$\begin{aligned}
I &= \pi_1 + \tilde{d}\tilde{\partial}G + \tilde{\partial}\tilde{d}G,\\
I &= \pi_1 + 2d_z\partial_z G + 2\partial_z d_z G,\\
I &= \pi_1 + 2d_{\bar{z}}\partial_{\bar{z}} G + 2\partial_{\bar{z}} d_{\bar{z}} G.
\end{aligned}$$

These formulae will have important consequences in connection with the z and $\bar{z}$ cohomologies.

$$\Delta, G, \pi_1$$
$$*, d, \partial, \pi_1, G, \ldots, J, P^{r,s}, L, \Lambda, \tilde{d}, \tilde{\partial}, d_z, d_{\bar{z}}, \partial_z, \partial_{\bar{z}}.$$

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Holomorphic forms on a Kählerian manifold

We have seen that every holomorphic form on C^n is harmonic. In an arbitrary Hermitian manifold it is not true that every holomorphic form is harmonic. However in a Kahlerian manifold every holomorphic form is harmonic. To prove this we use the fact that

$$\Delta = 2\Delta_{\bar{z}}, \Delta_{\bar{z}} = d_{\bar{z}}\partial_{\bar{z}} + \partial_{\bar{z}}d_{\bar{z}}.$$

If a form $\overset{p,0}{\omega}$ is holomorphic, then $d_{\bar{z}}\omega = 0$ and therefore $\partial_{\bar{z}}d_{\bar{z}}\omega = 0$; since $\partial_{\bar{z}}$ decreases the $\bar{z}$ degree by 1, $\partial_{\bar{z}}\omega = 0$ and hence $d_{\bar{z}}\partial_{\bar{z}}\omega = 0$. Therefore $\Delta\omega = 0$.

Lecture 20

Proof of the formula $[\Delta, d_z] = i\partial_{\overline{z}}$

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To prove the bracket relation

$$[\Lambda, d_z] = i\partial_{\overline{z}}$$

in the case of a Kählerian manifold, we first verify this relation in the case of C^n with the canonical Kählerian metric

$$\sum_{k=1}^{n} dz_k d\overline{z}_k.$$

In this case

$$\Omega = -\frac{1}{2i}\sum_k dz_k \wedge d\overline{z}_k.$$

The real part of the Hermitian form is given by

$$\sum_k (dx_k^2 + dy_k^2).$$

The Euclidean structure given by the metric on the real tangent space at a point $\underline{a}$ induces a Euclidean structure on the real co-tangent space at $\underline{a}$. We extend this Euclidean structure to a Hermitian structure in the complex co-tangent vector space. The $2n$ vectors $dx_1, dy_1, \ldots, dx_n$, dy_n form an orthonormal basis for the real cotangent vector space; the vectors dz_1, $d\overline{z}_1, \ldots, dz_n$, $d\overline{z}_n$, form an orthogonal basis for the complex co-tangent vector space; each of these vectors is of length $\sqrt{2}$ as $dz_k =$ 126

$dx_k + idy_k$ and $d\bar{z}_k = dx_k - idy_k$. The vector $dz_J \wedge d\bar{z}_K$ is of length $\sqrt{2^{j+k}}$ where j and k are the number of elements in J and K.

We shall now introduce some elementary operators in C^n and express the operators Λ, d_z, ∂_z etc. in terms of these operators. The operator

$$e_k = dz_k \Lambda$$

operates on forms by multiplying every form on the left by $dz_k \cdot e_k$ is an operator of the type $(1, 0)$. $\bar{e}_k$ is the operator

$$d\bar{z}_k \Lambda.$$

$\bar{e}_k$ is an operator of the type $(0, 1)$. i_k and $\bar{i}_k$ are defined to be the adjoints of e_k and $\bar{e}_k$ respectively. i_k is an operator of the type $(-1, 0)$ while $\bar{i}_k$ is an operator of the type $(0, -1)$. We shall prove that the linear operator i_k is given by the formula

$$\begin{aligned} &i_k[\omega dz_J \wedge d\bar{z}_K] = 0 \\ &i_k[\omega \wedge dz_k \wedge dz_{J'} \wedge d\bar{z}_K] = 2\omega \wedge dz_{J'} \wedge d\bar{z}_K \end{aligned}$$

(J' is a set of indices without the index k). (this amounts essentially to the suppression of dz_k). We shall verify that the operator defined by these formulae is the adjoint of e_k. For any two forms α and β we shall verify that

$$(e_k\alpha, \beta)_a = (\alpha, i_k\beta)_a.$$

127 It is sufficient to verify these for the elementary forms i.e., to verify that

$$(e_k\alpha' dz_J \wedge d\bar{z}_K, \beta' dz_L \wedge d\bar{z}_M)_a = (\alpha' dz_J \wedge d\bar{z}_K, i_k\beta' dz_L \wedge d\bar{z}_M)_a$$

or

$$(e_k dz_J \wedge d\bar{z}_K, dz_L \wedge d\bar{z}_M)_a = (dz_J \wedge d\bar{z}_K, i_k dz_L \wedge d\bar{z}_M)_a.$$

The elements $\{dZ_J \wedge d\bar{z}_K\}$ are orthogonal. The right and left sides both vanish except when $L = k + J$ and $M = K$. In this case both the sides are equal to 2^{1+r+s} where r and s are the number of indices in J and K respectively.

We shall now introduce two more elementary operators:

$$\partial_k = \frac{\partial}{\partial z_k}, \bar{\partial}_k = \frac{\partial}{\partial \bar{z}_k}.$$

We can prove that the adjoint of ∂_k is $-\bar{\partial}_k$ and the adjoint of $\bar{\partial}_k$ is $-\partial_k$ in a similar way. We can express all the operators in the Kählerian structure by means of these operators. We have

$$L = -\frac{1}{2i}\sum e_k\bar{e}_k$$

$$\Lambda = \frac{1}{2i}\sum \bar{i}_k i_k \quad \text{(taking the adjoint).}$$

$$d_z = \sum \partial_k e_k = \sum e_k\partial_k (e_k \text{ and } \partial_k \text{ commute})$$

$$d_{\bar{z}} = \sum \bar{\partial}_k\bar{e}_k$$

$$\partial_{\bar{z}} = -\sum \partial_k\bar{i}_k.$$

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Now we can prove the bracket relation. We have

$$\Lambda d_z = \sum_{k,l} \frac{1}{2i}\bar{i}_k i_k \partial_l e_l,$$

$$d_z\Lambda = \sum_{k,l} \frac{1}{2i}\partial_1 e_1 \bar{i}_k i_k.$$

Since ∂_1 commutes with i_k and $\bar{i}_k$,

$$\Lambda d_z = \sum_{k,l} \frac{1}{2i}\partial_1\bar{i}_k i_k e_1.$$

e_1 and i_k do not commute. For $k \neq 1$ $i_k e_1 = -e_1 i_k$ so that

$$\sum_{\substack{k,1\\k\neq 1}} \frac{1}{2i}\partial_1\bar{i}_k i_k e_1 = \sum_{\substack{k,1\\k=\neq 1}} -\frac{1}{2i}\partial_1\bar{i}_k e_1 i_k$$

$$= \sum_{\substack{k,1\\k\neq 1}} \frac{1}{2i}\partial_1 e_1\bar{i}_k i_k.$$

Now 129

$$i_k e_k + e_k i_k = 2$$

(as $i_k e_k$ is zero for a term which contains dz_k as factor while $e_k i_k$ is zero in the contrary case). So

$$\begin{aligned}\frac{1}{2i}\partial_k \bar{i}_k i_k e_k &= -\frac{1}{2i}\partial_k \bar{i}_k e_k i_k + \frac{2}{2i}\partial_k i_k\\ &= \frac{1}{2i}\partial_k e_k \bar{i}_k i_k - i\partial_k i_k.\end{aligned}$$

Consequently,

$$\begin{aligned}\Lambda d_z &= \sum_{k,1}\frac{1}{2i}\partial_1 \bar{i}_k i_k e_1\\ &= \sum_{k,1}\frac{1}{2i}\partial_1 e_1 \bar{i}_k i_k - i\sum \partial_k \bar{i}_k\\ &= d_z\Lambda + i\partial_{\bar{z}}\end{aligned}$$

which proves the bracket relation.

We shall now derive the bracket relation in the case of an arbitrary Kählerian manifold by using a theorem of differential geometry.

Suppose we have two C^∞ tensor fields $\oplus$ and $\oplus'$ of the same kind on a C^∞ manifold V. We shall say that $\oplus$, $\oplus'$ coincide upto the order m at
130 a point $\underline{a} \in V$ if the coefficients as well as the partial derivatives upto order m of the coefficients of $\oplus$ and $\oplus'$ coincide at $\underline{a}$. This has an intrinsic meaning: for if the property is true for one map at $\underline{a}$ it is true for any other map at $\underline{a}$. Let now V be a Riemannian manifold with the field of positive-definite quadratic forms Q. Let $\underline{a}$ be a point of V. According to a theorem of Riemannian geometry we can find another C^∞ field Q' of positive definite quadratic forms defined in a neighbourhood of $\underline{a}$ such that Q and Q' coincide upto the first order at $\underline{a}$ and such that Q' gives the Euclidean structure in a neighbourhood of $\underline{a}$ (i.e., there exists a map in which

$$Q' = \sum \delta_{ij} dx_i dx_j$$

where δ_{ij} is the Kronecker Symbol). Q' is said to be an osculating Euclidean structure for Q at $\underline{a}$.

We now consider the analogous question for Hermitian manifolds. Suppose a C^∞ manifold V^{2n} has two different complex structures, (1) and (2). We shall say that the complex structures (1) and (2) coincide at $\underline{a} \in V^{2n}$ upto order m if for every function f holomorphic in (1) the field $(d_{\overline{z}})_a f$ is zero upto order m at $\underline{a}$. Equivalently, we may say that the structures (1) and (2) coincide upto order m at $\underline{a}$ if the intrinsic J-operators J_1 and J_2 coincide upto order m on the tangent space at $\underline{a}$. Let $V^{(n)}$ be a Hermitian manifold. Let H denote the field of Hermitian forms giving the Hermitian structure and Ω the associated exterior 2-form. Let **131**
$\underline{a}$ be a point of $V^{(n)}$. Is it possible to find a Hermitian structure, H' on V (with the same underlying C^∞ structure as $V^{(n)}$) such that

i) the two analytic structures coincide upto order 1 at $\underline{a}$

ii) the fields of Hermitian forms H and H' coincide upto order 1 at $\underline{a}$.

iii) H' is the canonical hermitian structure on C^n for some map at $\underline{a}$?. There is one trivial necessary condition. If Ω' is the exterior 2-form corresponding to H', Ω and Ω' coincide upto order 1 at $\underline{a}$ so that $d\Omega$ and $d\Omega'$ coincide at $\underline{a}$; but $d\Omega' = 0$ at $\underline{a}$. Therefore, $d\Omega(a) = 0$ is a necessary condition. This condition can also be proved to be sufficient; this is the difficult part. In particular in a Kählerian manifold there exists an osculating Hermitian structure at every point.

Let $V^{(n)}$ be a Kahlerian manifold. At $\underline{a} \in V^{(n)}$ we choose an osculating Hermitian structure. The operators J corresponding to the two structures coincide at $\underline{a}$ up to order 1. The operators $\widetilde{d} = JdJ^{-1}$ coincide at $\underline{a}$. Since d_z is a linear combination of d and $\widetilde{d}$ the operators d_z coincide at a; similarly the operators $d_{\overline{z}}$ coincide at $\underline{a}$. The operators Λ coincide upto order 1 at $\underline{a}$ and the operators $[\Lambda, d_z]$ coincide at $\underline{a}$. The operators $i\partial_{\overline{z}}$ also coincide at $\underline{a}$. Since we have proved the relation

$$[\Lambda, d_z] = i\partial_{\overline{z}}$$

for the canonical Hermitian structure in C^n the same relation holds **132** also for any Kählerian manifold.

The relation

$$[\Lambda, d_{\bar{z}}] = -i\partial_z$$

is proved similarly.

Lecture 21

Compact Manifolds with a Kählerian structure

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Let V be a compact complex analytic manifold. We shall assume that there exists a Kählerian metric on V. From this assumption we shall derive some intrinsic properties of V i.e., properties of V which depend only on the complex analytic structure of V but not on any particular Kählerian metric.

Let $H^{(r,s)}$ denote the quotient space of the space of the d-closed forms of bidegree (r, s), by the space of forms of bidegree (r, s) which are coboundaries (not necessarily of homogeneous forms). Then the pth cohomology space H^p is the direct sum of the spaces $H^{(r,s)}, r + s = p$. To prove this we observe that, as $P^{(r,s)}$ commutes with Δ, $P^{(r,s)}$ operates canonically (i.e., independent of Kählerian metric) on H^p (see lecture 15). To define this map, we choose in each cohomology class the harmonic form, ω, belonging to this class; since Δ and $P^{(r,s)}$ commute, each homogeneous component of the harmonic form is harmonic and hence closed. Thus the homogeneous components of the harmonic form define cohomology classes and $P^{(r,s)}$ is the map which maps the cohomology class of ω into the cohomology class determined by the (r, s) component of ω. If $p^{r,s}H$ is the image of H by $P^{(r,s)}$, $P^{(r,s)}H$ may be identified with the space of harmonic forms of bidegree (r, s) and

$$H^p = \sum P^{(r,s)}H.$$

But $P^{(r,s)}H$ is just the space $H^{(r,s)}$ defined above and hence 134

$$H^p = \sum_{r+s=p} H^{(r,s)}$$

This decomposition is intrinsic; but we have used harmonic forms to prove this.

If we define the double Betti number $b^{(r,s)}$ to be the dimension of the space $H^{(r,s)}$ we have

$$b^p = \sum_{r+s=p} b^{(r,s)}$$

Since the mapping $\omega \to \overline{\omega}$, which assigns to every form its complex conjugate, induces an isomorphism of $H^{(r,s)}$ onto $H^{(s,r)}$, we find that $b^{(r,s)} = b^{(s,r)}$; so, when p is odd b^p is the sum of numbers which are pairwise equal and hence b^p is even. So the Betti numbers for odd dimensions are even. Moreover $H^{(n-r,n-s)}$ is dual to the space $H^{(r,s)}$ so that $b^{(n-r,n-s)} = b^{(r,s)}$. Thus we have

$$b^{(r,s)} = b^{(s,r)} = b^{(n-r,n-s)} = b^{(n-s,n-r)}$$

We can also prove that b^p is even for odd p by introducing a canonical complex structure on the pth cohomology space $H^{p(R)}$ formed from the *real* forms alone. [H^p is the complexification of $H^{p(R)}$ and the com-
135 plex dimension of H^p is equal to the real dimension of $H^{p(R)}$]. Since J commutes with Δ, J operates canonically on $H^{p(R)}$. Since p is odd $J^2 = -I$ and J gives a complex structure on $H^{p(R)}$. So the (real) dimension of $H^{p(R)}$ is even; and hence b^p is even.

The next result on the Betti-numbers is the following:

$$b^{(r,s)} \geq b^{(r-1,s-1)} \text{ if } r+s \leq n+1$$

(From this it follows at once that $b^p \geq b^{p-2}$ if $p \leq n+1$). To prove this we need the following result: the map

$$\Omega : \overset{r-1,s-1}{\Lambda} T_a^*(V) \to \overset{r,s}{\Lambda} T_a^*(V)$$

(multiplication by Ω) is one to one (i.e., $\Omega \wedge \omega = 0$ if and only if $\omega = 0$) provided $r + s \leq n + 1$. Since L commutes with Δ, L gives a map of the space of harmonic forms of bidegree $(r-1, s-1)$ into the space of harmonic forms of bidegree (r, s); by the algebraic result stated above this map is one to one if $r + s \leq n + 1$; consequently $b^{(r-1,s-1)} \leq b^{r,s}$, if $r + s \leq n + 1$.

The map $\Omega : \overset{r-1,s-1}{\Lambda} \to \overset{r,s}{\Lambda}$

We shall now prove that the map $\Omega : \overset{r-1,s-1}{\Lambda} \to \overset{r,s}{\Lambda}$ is one to one for $r+s \leq n+1$. Since Ω is an operator of type $(1, 1)$ it is sufficient to prove that

$$\Omega : \overset{q-2}{\Lambda} \to \overset{q}{\Lambda}$$

is one to one for $q \leq n + 1$. This would follow if we prove that the map 136

$$\Omega^p : \overset{n-p}{\Lambda} \to \overset{n+p}{\Lambda}$$

is one to one for $p \geq 1$. For, if $q \leq n + 1$, $q - 2 = n - p$, for some $p \geq 1$ and

$$\Omega \wedge \omega = 0 \Rightarrow \Omega^p \wedge \omega = 0 \Rightarrow \omega = 0$$

since

$$\Omega^p : \overset{n-p}{\Lambda} \to \overset{n+p}{\Lambda}$$

is one to one. Since $\overset{n-p}{\Lambda}$ and $\overset{n+p}{\Lambda}$ are of the same dimension it is enough to show that the map Ω^p is onto. Let $(x_1, y_1, \ldots, x_n, y_n)$ be a basis for the space of real differentials at $\underline{a}$ such that

$$\Omega = \sum x_i \wedge y_i$$

Put $x_i \wedge y_i = \alpha_i$. The elements of $\overset{n+p}{\Lambda}$ are generated by elements of the form $x_A \wedge y_B$, where the set of indices A and B have at least p indices in common. Consequently it is sufficient to prove that these elements $\omega = x_A \wedge y_B$ are divisible by Ω^p. We may assume that

$$\omega = \alpha_1 \wedge \ldots \wedge \alpha_{p+s} \wedge x_C \wedge y_D$$

where the indices C and D have no elements in common. Since the transformation $x_k \to y_k$, $y_k \to -x_k$ for indices k in D does not affect Ω we may assume that ω is of the form

$$\alpha_1 \wedge \ldots \wedge \alpha_{p+s} \wedge x_{p+s+1} \wedge \ldots \wedge x_{n-s}.$$

Since 137

$$(p+s)!\alpha_1 \wedge \ldots \wedge \alpha_{p+s} = (\alpha_1 + \cdots + \alpha_{p+s})^{p+s}$$

(the exponent represents power with respect to the exterior product), we have

$$\begin{aligned}\frac{\omega}{(p+s)!} &= (\alpha_1 + \cdots + \alpha_{p+s})^{p+s} \wedge x_{p+s+1} \wedge \ldots \wedge x_{n-s} \\ &= (\alpha_1 + \cdots + \alpha_p + \alpha_{p+1} + \cdots + \alpha_{p+s} + \cdots + \alpha_{n-s})^{p+s} \\ &\qquad \wedge x_{p+s+1} \wedge \ldots \wedge x_{n-s}\end{aligned}$$

If we put $\gamma = \alpha_{n-s+1} + \cdots + \alpha_n$

$$\begin{aligned}\frac{\omega}{(p+s)!} &= (\Omega - \gamma)^{p+s} \wedge x_{p+s+1} \wedge \ldots \wedge x_{n-s} \\ &= [\Omega^{p+s} - (p \underset{1}{+} s)\Omega^{p+s-1}\gamma + \cdots + (-1)^s \binom{p+s}{s} \Omega^p \gamma^s]. \\ &\quad . \wedge x_{p+s+1} \wedge \ldots \wedge x_{n-s}\end{aligned}$$

as $\gamma^{s+1} = 0$. The left side containing Ω^p as a factor.

The space $H^{(p,0)}$

We shall now show that the space $H^{(p,0)}$ is just the space of holomorphic differential forms of degree p. A closed differential form of degree $(p, 0)$ is z closed (by homogeneity) and hence holomorphic; and a holomorphic form (of degree p) is harmonic and hence closed. On the other hand, since a holomorphic differential form is harmonic, a closed dif-
138 ferential form of bidegree $(p, 0)$ cannot be a coboundary unless it is the zero form.

From this we derive at once a majorant (in terms of the pth Betti-number) for the number of linearly independent holomorphic p-forms. Since

$$b^p = b^{(p,0)} + \cdots + b^{(0,p)} \quad \text{and} \quad b^{(p,0)} = b^{(0,p)}$$

we have

$$2b^{(p,0)} \leq b^p (\text{for } p \neq 0)$$

For differential forms of degree 1 we have

$$\begin{aligned} b^1 &= b^{(1,0)} + b^{(0,1)} \\ &= 2b^{(1,0)} \end{aligned}$$

i.e., the dimension of the space of holomorphic differential forms of degree 1 is equal to half the first Betti number.

Compact Riemann Surfaces

A complex analytic manifold of complex dimension 1 is usually called a Riemann surface. We can always introduce a Kählerian metric on a Riemann surface. Let $V^{(1)}$ be a compact, connected Riemann surface; $b^2 = b^0 = 1$. Let $b^1 = 2g$. The number g (= half the first Betti number) is called the genus of the Riemann surface. The number of linearly independent holomorphic forms of degree 1 is equal to the genus of the surface, by what we have seen. Since there are $2g$ independent 1-cycles and g independent holomorphic 1-forms the periods of a holomorphic 1-form cannot be prescribed arbitrarily on a basis of 1-cycles. However it can be proved that there exists a unique holomorphic differential form **139**
with prescribed real parts of the periods.

The Riemann sphere, S^2, is of genus zero. So there are no holomorphic differential forms of degree 1 apart from the 0-form. Of course, this can be proved directly. Let ω be a holomorphic 1-form on S^2. ω can be written as $f(z)dz$ in the plane, where $f(z)$ is an entire function in the plane. Using the map given by $1/z$ at ∞ we find that $f(1/z)$. $1/z^2$ should be holomorphic at the origin.

If

$$f(z) = \sum a_n z^n \quad \text{then} \quad f(1/z)1/z^2 = \sum \frac{a_n}{z^{n+2}}$$

so that $f(1/z)$. $1/z^2$ has a pole at the origin unless $f \equiv 0$.

Next we consider a torus with the complex structure induced from C^1. Here the genus is 1. So the differential dz (which is well defined on the torus) is, but for a constant multiple, the only holomorphic 1-form on the torus.

All other compact Riemann surfaces can be considered as the quotient spaces of the unit circle by certain Fuchsian groups.

Lecture 22

The identity between d, z and $\bar{z}$ cohomologies: de Rham theorem (Compact Kählerian manifolds)

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We shall now prove that the cohomology spaces defined by the operators d, d_z, $d_{\bar{z}}$ and $\widetilde{d}$ are canonically isomorphic. That the ordinary cohomology and the $\bar{z}$ cohomology are the same will be of importance in the study of holomorphic and meromorphic forms on a compact Kählerian manifold.

We shall give the proof in the case of the ordinary cohomology and the $\bar{z}$ cohomology. Let H be a cohomology space with respect to d and $H_{\bar{z}}$ the corresponding $\bar{z}$ cohomology space. We have a canonical mapping from H to $H_{\bar{z}}$. In each cohomology class $\overset{0}{\omega}$ (with respect to d) we choose a form, ω, which is $\bar{z}$ closed and map the class $\overset{0}{\omega}$ to the $\bar{z}$ cohomology class determined by ω. In each d-cohomology class such a form exists, the harmonic form belonging to the class. We have to verify that if a form, α, which is a d-coboundary is $\bar{z}$ closed then α is a $\bar{z}$ coboundary. Since α is a d-coboundary $\pi\alpha_1 = 0$. By the decomposition formula

$$\begin{aligned} \alpha &= \pi_1\alpha + 2d_{\bar{z}}\partial_{\bar{z}}G\alpha + 2\partial_{\bar{z}}d_{\bar{z}}G\alpha \\ &= 2d_{\bar{z}}\partial_{\bar{z}}\alpha \end{aligned}$$

so that ω is a $\bar{z}$ coboundary. The mapping so defined is actually an isomorphism. For the space $H_{\bar{z}}$ can be identified with the space of harmonic forms and we know that the space H also can be identified with 141 the space of harmonic forms. We have thus a canonical isomorphism between d and $\bar{z}$ cohomologies, which is independent of the Kählerian metric.

Thus the $\bar{z}$ cohomology for a compact complex analytic manifold with a Kählerian metric is just the ordinary cohomology. The first and third parts of de Rham's theorem for $\bar{z}$ cohomology for such manifolds follow immediately. The projections $\pi_1, \pi_{2,\bar{z}} = 2d_{\bar{z}} - \partial_{\bar{z}} G, \pi_{3,\bar{z}}$ are continuous and the image spaces corresponding to these projections are closed because they are kernels. This proves the second part of de Rham's theorem for $\bar{z}$ cohomology.

de Rham theorems for $\bar{z}$ cohomology of an arbitrary complex analytic manifold

The first part of de Rham's theorem for $\bar{z}$ cohomology in the case of an arbitrary manifold was proved only recently. The third part of de Rham's theorem is also true: the $\bar{z}$ Betti-numbers are finite for a compact manifold. However it is not true in general that the spaces of $\bar{z}$ coboundaries are closed. On the other hand if we assume that all the Betti-numbers are finite this theorem can be restored.

The complex projective space

Let PC^n denote the complex projective space of n dimensions PC^n is a compact Kählerian manifold (see the appendix). Algebraic manifolds
142 imbedded without singularities in PC^n are also compact Kählerian manifolds. (In general it is true that a complex analytic manifold regularly imbedded in a Kählerian manifold is a Kählerian manifold). For PC^n we have $b^p = 1$ for even p and $b^p = 0$ for odd p. The form Ω^k (Ω is the exterior 2-form associated with the Kählerian metric) gives an element $\neq 0$ of the pth cohomology space.

In this connection we shall state another part of de Rham's theorem (arbitrary C^∞ manifold, d-cohomology). This part of de Rham theorem states that each cohomology class of currents contains a cycle (closed chain) and that a cycle which is the coboundary of a current is the boundary of a chain. We shall also make use of de Rham's correspondence between cohomology of forms and homology of chains in which exterior products of cohomology classes correspond to algebraic intersections of homology classes.

Let PC^{n-1} be a hyperplane of $PC^n \cdot PC^{n-1}$ is a cycle. PC^{n-1} determines an element ($\neq 0$) of the 2nd cohomology class (of currents). So Ω is homologous (in the sense of the currents) to $k \cdot PC^{n-1}$ where k is a real number: $\Omega \approx k \cdot PC^{n-1}$. Actually $k > 0$. For,

$$\int\limits_{PC^n} \Omega^n = \langle \Omega, \Omega^{n-1} \rangle = k \int\limits_{PC^{n-1}} \Omega^{n-1}$$

Since PC^n and PC^{n-1} are Kähler manifolds (with the associated exterior 2-form Ω),

$$\int\limits_{PC^n} \Omega^n > 0 \quad \text{and} \quad \int\limits_{PC^{n-1}} \Omega^{n-1} > 0 \quad \text{so that} \quad k > 0.$$

Considering n hyperplanes in general position (whose intersection is a point PC^0) we find that 143

$$\begin{aligned} \Omega^n &\approx k^n PC^0 \\ &\approx k^n x \ \text{ point (as a chain or current).} \end{aligned}$$

Therefore

$$\int\limits_{PC^n} \Omega^n = k^n \langle PC^0, 1 \rangle = k^n.$$

So

$$k = \sqrt[n]{\int\limits_{PC^n} \Omega^n}.$$

If we choose the Kählerian metric whose associated 2-form is

$$\Omega' = \Omega / \sqrt[n]{\int\limits_{PC^n} \Omega^n}$$

we have $\Omega'^p \approx PC^{n-p}$. In other words if the associated 2-form Ω of a Kählerian metric satisfies the relation

$$\int_{PC^n} \Omega^n = 1$$

then

$$\Omega^p \approx PC^{n-p}.$$

The volume of PC^n with respect to the volume element given by such a metric is $n!$.

The Betti numbers of PC^n verify, of course, all the properties of the Betti numbers of a compact Kählerian manifold. We have

$$b^{0,0} = b^{1,1} = \ldots = b^{n,n} = 1.$$

144 and all the other double Betti-numbers are zero. In particular $b^{p,0} = 0$ for $p \neq 0$. So on the complex projective space there are no holomorphic differentials of any degree except the degree zero (in which case the holomorphic forms are constant functions).

Example of a compact complex manifold which is not Kählerian

Let us consider in C^n the shell between the two spheres of radii 1 and $2 : (1 \leq |z| \leq 2)$. Identify the two points on the spheres which are on the same radius.

Let us denote by V the space obtained after this identification. Let G denote the properly discontinuous group of analytic automorphisms of $(C^n - 0)$ consisting of the homothetic transformations.

$$(z_1, \ldots, z_n) \to (2^k z_1, \ldots, 2^k z_n)$$

k running over all integers, positive, negative or zero. V is a fundamental domain for this group. Since we can always introduce a complex analytic structure in the quotient space obtained from a properly discontinuous group, V can be endowed with a natural analytic structure. V is compact. V is not Kählerian for $n > 1$. It can be proved that, for $n > 1$,

$$\begin{aligned} b^0 &= b^1 = 1, \\ b^i &= 0 \text{ for } 2 \leq i \leq 2(n-1). \\ b^{2n-1} &= 1, b^{2n} = 1. \end{aligned}$$

For $n \geq 3$, the Betti-numbers b^p are not greater than 1 for even p. For 145
$n \geq 2$ odd dimensional Betti numbers are not even.

If $P(z_1, \ldots, z_n)$ and $Q(z_1, \ldots, z_n)$ are homogeneous polynomials of the same degree $P(z_1, \ldots, z_n)/Q(z_1, \ldots, z_n)$ defines a meromorphic function on V. In this connection we may remark that there exist compact complex analytic manifolds which admit of no non-constant meromorphic function. Examples are provided by certain torii.

Lecture 23

Cousin's Problem

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Cousin's problem for meromorphic functions in the complex plane is Mittag-Leffler's problem. Mittag-Leffler's problem in the plane is as follows: Given a discrete set of points in the plane and polar developments at each of these points, construct a function meromorphic in the whole plane having the given points as poles and the given developments as the polar developments. We know that this problem always admits of a solution. Let a_i be the given points and

$$P_i(1/z-a_i) = \frac{C_{i,1}}{(z-a_i)} + \frac{C_{i,2}}{(z-a_i)} + \cdots + \frac{C_i, p_i}{(z-a_i)^{p_i}}$$

be the polar development at $\underline{a_i}$. Then we can find polynomial $Q_i(z)$ such that the series

$$\sum (P_i\left(\frac{1}{z-a_i}\right) - Q_i(z))$$

coverges absolutely and uniformly on every compact set not containing anyone of the $\underline{a_i}$. The limit function gives a solution of the problem. The solution is unique upto an additive entire function. The indeterminacy in the problem is thus an entire function.

Let us consider the corresponding problem on the Riemann sphere. We have a finite number of points $\underline{a_i}$ and polar developments $P_i\left(\dfrac{1}{z-a_i}\right)$ at $\underline{a_i} \neq \infty$ and a polar development $P(z)$ (a polynomial without constant term) at ∞. Then the function

$$P_i(1/z-a_i) + P(z)$$

147 is meromorphic on the Riemann sphere and has the prescribed development at the given points. The solution is unique upto an additive constant. The method of construction of the solution corresponds to the classical decomposition of a rational function into partial fractions.

We want to consider similar problems on complex analytic manifolds. We shall first define a meromorphic function.

Suppose we consider in C^n a 'function' $\varphi = \dfrac{f(z_1, \ldots, z_n)}{g(z_1, \ldots, z_n)}$ which is the quotient of two holomorphic functions f and g. The variety of poles is the set of zeros of $g(z_1, \ldots, z_n)$. The variety of poles is not, in general, a true manifold; if all the partial derivatives of $\underline{g}$ vanish at a point this point is a singular point for this variety. (For example the variety defined by $z_1^2 + z_2^2 + z_3^2 = 0$ has a singularity at the origin). We shall call the set of zeros of an analytic function an analytic subset. At points where the analytic subsets defined by the zeros of the functions f and g intersect the quotient f/g is indeterminate. (In the case $n = 1$, if f and g are coprime at a point $\underline{a}$ it is impossible to have $f(a) = 0$ and $g(a) = 0$ simultaneously. However for $n \geq 2$ this is possible; for example the functions z_1 and z_2 are coprime at the origin and vanish simultaneously
148 at the origin). So the function f/g is defined only in the complement of certain analytic subsets. It can be shown that an analytic subset is a set of measure zero. It is not good to define a meromorphic function on $V^{(n)}$ to be the quotient of two holomorphic functions on $V^{(n)}$; as then the only meromorphic functions on a compact complex analytic manifold would be constants! These considerations lead to the following definition of a meromorphic function on a complex analytic manifold.

A meromorphic function, φ, on a complex analytic manifold is a complex valued function defined almost everywhere on the manifold such that for every point $\underline{a}$ of the manifold there exists a neighbourhood U_a of $\underline{a}$ such that, in U_a, φ is almost everywhere equal to the quotient of two holomorphic functions defined everywhere on U_a.

A meromorphic form of degree p is a field of covectors defined almost everywhere on the manifold such that every point has a neighbourhood in which the form is almost everywhere equal to the quotient of a holomorphic form of degree p by a holomorphic function.

Suppose we are given an open covering (U_i) of the manifold and in

every U_i a meromorphic differential form $\overset{p}{\omega}_i$ of degree p such that the form $\omega_{ij} = \omega_i - \omega_j$ is almost everywhere in $U_i \cap U_j$ equal to a holomorphic form defined everywhere in $U_{ij} = U_i \cap U_j$. These constitute a Cousin's datum. Cousin' s problem is to find a meromorphic form $\overset{p}{\omega}$ on the whole manifold such that $\omega - \omega_i$ is a holomorphic form in U_i (i.e., $\omega - \omega_i$) is almost everywhere equal to a holomorphic form defined everywhere in U_i). The indeterminacy of the problem is evidently **149**
an additive holomorphic form (defined over the whole manifold). ω_i are "singular parts" of ω in U_i. Cousin's datum for the Mittag-Leffler's problem is the following: With each a_i we associate an open set U_i such that $U_i \cap U_j$ is empty for $i \neq j$. In U_i we take for ω_i the meromorphic function given by the polar development. In the complementary set U_0 of the points a_i we take the function $\omega_0 \equiv 0$. All these together constitute the Cousin datum for the Mittag-Leffler problem.

We now wish to formulate the Cousin datum and problem in terms of currents. If $n = 1$, a meromorphic function (or a form) can be considered as a current. In C^1 the function $\dfrac{1}{(z-a)}$ defines a current in the usual manner. But the function $\dfrac{1}{(z-a)k}$, $k > 1$, is not summable in any neighbourhood of $\underline{a}$. However if we take the Cauchy principal value, $\dfrac{1}{(z-a)k}$ defines a current: for every $\varphi \in \mathscr{D}$

$$\langle \frac{1}{(z-a)^k}, \varphi \rangle = \lim_{\epsilon \to 0} \iint\limits_{|z-a| \geq \epsilon} \frac{\varphi dx\, dy}{(z-a)k}$$

This enables us to consider meromorphic functions or forms as currents when $n = 1$; considering meromorphic functions and forms as currents in this way leads to good solutions of problems on a compact Riemann surface. However, it becomes difficult to associate canonically a current with an arbitrary meromorphic function in higher dimensions. If the analytic subset defined by the singularities is a true manifold it is possible to associate canonically a current with the meromorphic function. In the **150**
case of a general meromorphic function it has not yet been possible to associate canonically a current with the meromorphic function.

We shall introduce currents in the problem in some other way. We can find currents $\overset{p,0}{T_i}$ (of bidegree $(p,0)$) in U_i such that $T_i - T_j = \omega_{ij}$ (as a current) in U_{ij}. The indeterminacy in the choice of the T_i is a current defined on the whole manifold. If S is a current on the whole manifold $T'_i = T_i + S$ also possess the same property: $T'_i - T'_j = \omega_{ij}$ in U_{ij}, conversely if T_i and T'_i are two systems of currents such that T_i and T'_i are defined in U_i and

$$\begin{aligned} T_i - T_j &= \omega_{ij} \quad \text{in} \quad U_{ij},\\ T'_i - T'_j &= \omega_{ij} \quad \text{in} \quad U_{ij}, \end{aligned}$$

there exists a current S on the whole manifold such that $T'_i = T_i + S$. We define the current S by 'piecing' together the currents S_i defined in U_i by $S_i = T'_i - T_i$. The currents S_i define a single current on the whole manifold as we have, in U_{ij},

$$\begin{aligned} S_i - S_j &= (T'_i - T_i) - (T'_j - T_j)\\ &= (T'_i - T'_j) - (T_i - T_j)\\ &= \omega_{ij} - \omega_{ij}\\ &= 0 \end{aligned}$$

151 To find the currents T_i we proceed as follows. We choose a partition of unity $\{\alpha_i\}$ subordinate to the covering system $\{U_i\}$. We put

$$T_i = \sum \alpha_k \omega_{ik}$$

(the summation being over all k for which $U_i \cap U_k$ is non-empty) where $\alpha_k\omega_{ik}$ is the C^∞ form defined in U_i as:

$$\alpha_k\omega_{ik} = \begin{cases} \alpha_k\omega_{ik}, & \text{in } U_i \cap U_k\\ 0 \text{ in } U_i \cap \text{ (complement of support of } \alpha_k) \end{cases}$$

The definition of $\alpha_k\omega_{ik}$ is consistent, as

$$\alpha_k\omega_{ik} = 0 \text{ on } U_i \cap U_k \cap \text{ (complement of support of } \alpha_k).$$

T_i is a C^∞ form in U_i. Now

$$T_i - T_j = \sum \alpha_k(\omega_{ik} - \omega_{jk}) \text{ in } U_{ij}.$$

$\alpha_k(\omega_{ik} - \omega_{jk})$ is zero outside $U_{ijk} = U_i \cap U_j \cap U_k$.

In U_{ijk} we have the relation

$$\omega_{ij} + \omega_{jk} + \omega_{ki} = 0$$

and, in U_{ik}, the relation

$$\omega_{ik} + \omega_{ki} = 0$$

so that **152**

$$\begin{aligned} T_i - T_j &= \sum_k \alpha_k(\omega_{ik} - \omega_{jk}) \\ &= \sum_k \alpha_k \omega_{ij} \\ &= \left(\sum_k \alpha_k\right) \omega_{ij} \\ &= \omega_{ij}, \text{ in } U_{ij} \text{ as } \sum_k \alpha_k = 1. \end{aligned}$$

This result is a particular case of a more general one. Suppose we have an open covering $\{U_i\}$ and a system of currents $\widetilde{\omega}_{ij}$ defined in $U_{ij} = U_i \cap U_j$ such that

$$\widetilde{\omega}_{ij} + \widetilde{\omega}_{jk} + \widetilde{\omega}_{ki} = 0 \text{ in } U_i \cap U_j \cap U_k$$
$$\text{and} \quad \widetilde{\omega}_{ij} + \widetilde{\omega}_{ji} = 0 \text{ in } U_{ij}.$$

Then it is possible to find a system of currents T_i in U_i such that

$$T_i - T_j = \widetilde{\omega}_{ij} \quad \text{in} \quad U_{ij}$$

and T_i is given explicitly by

$$T_i = \sum \alpha_k \widetilde{\omega}_{ik}$$

where α_k is a partition of unity subordinate to the covering U_i. If $\widetilde{\omega}_{ij}$ are C^∞ forms, T_i also can be chosen to be C^∞ forms.

[One can also consider the following problem which is more general: given an open covering $\{U_i\}$ and a system of holomorphic forms ω_{ij} in U_{ij} such that 153

$$\omega_{ij} + \omega_{jk} + \omega_{ki} = 0 \quad \text{in} \quad U_{ijk}$$

and

$$\omega_{ij} + \omega_{ji} = 0 \quad \text{in} \quad U_{ij},$$

to find holomorphic forms h_i in U_i such that

$$h_i - h_j = \omega_{ij} \quad \text{in} \quad U_{ij}]$$

By means of the currents T_i we define a current of bidegree $(p, 1)$ defined on the whole manifold. We put $R_i = d_{\overline{z}} T_i$ in U_i. In U_{ij},

$$R_i - R_j = d_{\overline{z}}(T_i - T_j) = d_{\overline{z}}\omega_{ij} = 0$$

(as ω_{ij} is holomorphic). So the currents R_i define a single current defined on the whole manifold, which we denote by $\overset{p,1}{R}$. Since the currents R_i are $\overline{z}$ closed, the current $\overset{p,1}{R}$ is also closed. ($\overset{p,1}{R}$ is locally a coboundary but need not be a coboundary in the large). If we replace $\{T_i\}$ by a system $\{T'_i\}$ having the same properties as T_i, $T'_i = T_i + S$, S a current defined on the whole manifold, the 'R' corresponding to T'_i would be $\overset{p,1}{R} + d_{\overline{z}} S$. So we can associate canonically with the Cousin datum a whole $\overline{z}$ cohomology class of bidegree $(p, 1)$. This $\overline{z}$ cohomology class is the 'obstruction' to the solution of Cousin's problem.

154 We shall prove that in order that Cousin's problem be solvable it is necessary and sufficient that the $\overline{z}$ cohomology class associated with the Cousin datum is the zero class. Suppose Cousin's problem is solvable and let ω be a solution. Let further $\omega - \omega_i = h_i$, h_i holomorphic in U_i. Take $T_i = -h_i$. In U_{ij},

$$T_i - T_j = -h_i + h_j$$

$$
\begin{aligned}
&= -(\omega - \omega_i) + (\omega - \omega_j)\\
&= \omega_i - \omega_j\\
&= \omega_{ij}.\\
d_z - T_i &= -d_{\bar{z}} h_i = 0
\end{aligned}
$$

So the current of bidegree $(p, 1)$ associated with the system T_i is zero, or the $\bar{z}$ cohomology class associated with the Cousin datum is the zero class. Conversely if the associated $\bar{z}$ cohomology class is the zero class, Cousin's problem is solvable. In this case we can find T_i such that the associated $\overset{p,1}{R}$ is zero (we may have to adjust the S suitably). That is, we find T_i such that

$$T_i - T_j = \omega_{ij} \text{ in } U_{ij} \qquad \text{and } d_{\bar{z}} T_i = 0.$$

Then by the ellipticity of $d_{\bar{z}}$ (on $\overset{0}{\mathscr{D}'}$), T_i is a holomorphic form h_i. Then a solution is given by the form $\omega = \omega_i - h_i$. In U_{ij}, $\omega_i - h_i = \omega_j - h_j$ so that ω is a meromorphic form well-defined on the manifold.

Let $\overset{p,1}{R}$ be the current defined by: $\overset{p,1}{R} = d_{\bar{z}} T_i$ in U_i. If $R = d_{\bar{z}} S$, a **155**
solution of the Cousin's problem is given by the form ω:

$$\omega = \omega_i + S - T_i \quad \text{in} \quad U_i.$$

In a compact Kählerian manifold d and $\bar{z}$ cohomologies coincide and with a Cousin datum we have an ordinary cohomology class.

As an example let us consider the Riemann sphere. Here $b' = 0$; so $b^{0,1} = 0$. So Cousin's problem for meromorphic functions is solvable for any Cousin datum.

Lecture 24

Cousin's Problem on C^n

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It has been proved recently that for C^n (the complex n-space) $H_{\overline{z}}^{(p,q)} = 0$ for $q > 0$. So Cousin's problem for meromorphic forms is solvable in C^n for any Cousin datum.

Cousin's problem on a compact Kählerian manifold: pseudo-solution

We can give a more explicit solution in the case of a compact Kählerian manifold. A necessary and sufficient condition for $\overset{p,1}{R}$ to be a $\overline{z}$ coboundary is that $\pi_1 R = 0$. (This gives a system of $b^{p,1}$ linear conditions for R). Now

$$\begin{aligned} R &= \pi_1 R + 2d_{\overline{z}}\partial_{\overline{z}}GR + 2\partial_{\overline{z}}d_{\overline{z}}GR \\ &= \pi_1 R + 2d_{\overline{z}}\partial_{\overline{z}}GR. \end{aligned}$$

We can choose $S = 2\partial_{\overline{z}}GR$.

Then

$$\omega = \omega_i + 2\partial_{\overline{z}}GR - T_i \quad \text{in} \quad U_i$$

is a solution of the Cousin's problem. Even in the case when Cousin's problem has no solution, ω defined as above makes sense. Even if T_i are currents, $(2\partial_{\overline{z}}GR - T_i)$ is a C^∞ form; for

$$d_{\overline{z}}(2\partial_{\overline{z}}GR - T_i) = 2d_{\overline{z}}\partial_{\overline{z}}GR - d_{\overline{z}}T_i$$

$$= (2d_{\bar{z}}\partial_{\bar{z}}GR - R)$$
$$= -\pi_1 R.$$

157 $\pi_1 R$ is a C^∞ form and $(2\partial_{\bar{z}}GR - T_i)$ is a current of bidegree $(p, 0)$; by the ellipticity of $d_{\bar{z}}$, $(2\partial_{\bar{z}}GR - T_i)$ is a C^∞ form. ω is thus a sum of a meromorphic form and a C^∞ form. When Cousin's problem is solvable $(2\partial_{\bar{z}}GR - T_i)$ is a holomorphic form and ω is a solution of Cousin's problem. In any case, we call ω a pseudo-solution. When Cousin's problem is solvable, the pseudo-solution is actually a solution.

Cousin's problem on PC^n

For $p \neq 1$ Cousin's problem has a solution for any Cousin datum, since $b^{p,1} = 0$ for $p \neq 1$. The solution is unique if $p \geq 2$ and is determined upto an additive constant when $p = 0$, as holomorphic differentials of degree $p \geq 1$ are zero and those of degree zero are constants. For $p = 1$, for Cousin's problem to be solvable it is necessary and sufficient that R be orthogonal to all closed C^∞ forms of bidegree $(n-1, n-1)$. But Ω^{n-1} is a generator of the cohomology classes of bidegree $(n-1, n-1)$. So the condition is

$$\langle R, \Omega^{n-1} \rangle = 0.$$

When the solution exists, the solution is unique.

Cousin's problem on a compact Riemann Surface

We shall now consider Cousin's problem on a compact Riemann surface, which we assume to be connected. We shall first consider the Cousin's problem for meromorphic differential forms of degree 1. For Cousin's problem to be solvable, it is necessary and sufficient that $\overset{1,1}{R}$
158 ($\overset{1,1}{R}$ is a current of the associated cohomology class) be orthogonal to all closed zero forms i.e., constants: or

$$\langle \overset{1,1}{R}, 1 \rangle = 0.$$

We shall now interpret this condition in terms of residues. Let ω be a meromorphic differential form of degree 1. We define the residue of ω at a pole $\underline{a}$ of ω, denoted by $\operatorname{Res}_a \omega$, as follows: We choose a local coordinate system (z) at $\underline{a} \cdot \omega = f(z)dz$, where $f(z)$ is a meromorphic function of z. Then $\operatorname{Res}_a \omega$ is defined to be the residue of $f(z)dz$ at $z(a)$. This definition is intrinsic. If we choose a regular curve C, contained in the domain of a map, winding around $\underline{a}$ once in the positive sense

$$\begin{aligned}\operatorname{Res}_a(f(z)dz) &= \frac{1}{2\pi i}\int\limits_C f(z)dz\\ &= \frac{1}{2\pi i}\int\limits_C \omega.\end{aligned}$$

Since the Riemann surface has no boundary, the sum of the residues of a meromorphic differential of degree 1 is zero. This gives a trivial necessary condition on the Cousin datum, for Cousin's problem to be solvable. We shall see that this condition is also sufficient.

Let $a_1, \ldots, a_m$ be a finite number points given on the Riemann surface. At each $\underline{a}$ we have a map (W_a, φ_a) such that $W_a \cap W_b$ is empty for $a \neq b$. Let U_a be a neighbourhood of $\underline{a}$ such that $\overline{U}_a \subset W_a$ and $\varphi_a(\overline{U}_a)$ is a closed disc. In each W_a we are given a meromorphic form of degree 159
1, ω_a, having a pole only at $\underline{a}$. Let U_0 be the complement of the set of points $a_1, \ldots, a_m$. In U_0 we put $\omega_0 \equiv 0$. With this Cousin datum we associate currents T_a in U_a and T_0 in U_0 defined as follows:

$$\begin{aligned}T_a &= 0 \text{ in } U_a\\ T_0 &= -\sum \widetilde{\omega}_a \text{ in } U_0\end{aligned}$$

where

$$\widetilde{\omega}_a = \begin{cases}\omega_a \text{ in } \overline{U}_a\\ 0 \text{ outside } \overline{U}_a.\end{cases}$$

The form $\widetilde{\omega}_a$ has discontinuities along the boundary of U_a; but $\widetilde{\omega}_a$ is locally summable in U_0 and defines a current in U_0. In $U_a \cap U_0$,

$$T_a - T_0 = \widetilde{\omega}_a = \omega_a$$

The current

$$R = \begin{cases} d_{\bar{z}} T_a & \text{in} \;\; U_a \\ d_{\bar{z}} T_0 & \text{in} \;\; U_0 \end{cases}$$

is the "obstruction". $R = 0$ in U_a. Since T_0 is of bidegree $(1, 0)$, $d_{\bar{z}} T_0 = dT_0 =$. If φ is a C^∞ function with compact support *in* U_0 (i.e., if the support of φ does not contain the points a_i),

$$\begin{aligned} \langle d_{\bar{z}} T_0, \varphi \rangle &= \langle \overset{1,0}{d} T_0, \varphi \rangle \\ &= \langle T_0, d\varphi \rangle \\ &= \langle - \sum \widetilde{\omega}_a, d\varphi \rangle \\ &= - \sum_a \iint\limits_{U_a} \omega_a \wedge d\varphi \\ &= \sum_a \iint\limits_{U_a} d(\omega_a \varphi) \\ &= \sum_a \int\limits_{bU_a} \omega_a \varphi. \end{aligned}$$

160 Consequently

$$R = \sum (bU_a) \wedge \omega_a$$

in U_0 and this relation is also true in U_a. [ω_a is a C^∞ form in a neighbourhood of the support of bU_a. So multiplication of bU_a by ω_a is possible]. Now the necessary and sufficient condition for Cousin's problem to have a solution is

$$\langle R, 1 \rangle = 0$$

or

$$\sum_a \int\limits_{bU_a} \omega_a = 0$$

i.e.,

$$\sum_a \operatorname{Res}_a \omega_a = 0.$$

In particular if we take ω_a without residues a solution of the problem always exists.

The indeterminacy in the solution is a holomorphic differential form of degree 1. Thus in this problem we have one condition of compatibility and g degrees of indeterminacy. 161

If we choose a Kählerian metric on the Riemann surface, an explicit solution is given by

$$\omega = \begin{cases} \omega_a + 2\partial_{\bar{z}} GR \text{ in } U_a \\ 2\partial_{\bar{z}} GR + \sum \widetilde{\omega}_a \text{ in } U_0. \end{cases}$$

We may write

$$\omega = \sum_a \widetilde{\omega}_a + 2\partial_{\bar{z}} G[\sum_a \{bU_a \wedge \omega_a\}]$$

as $\omega_a = \widetilde{\omega}_a$ in U_a. This expression has always a meaning and gives a solution of the Cousin's problem when the problem is solvable. The general solution of Cousin's problem is given by

$$\omega = \sum_a \widetilde{\omega}_a + 2\partial_{\bar{z}} G[\sum_a (bU_a \wedge \omega_a)] + h$$

where h is a holomorphic differential form of degree 1 on the Riemann surface.

Cousin's problem for meromorphic functions (Compact Riemann Surface)

Here in each U_a we have a meromorphic function f_a instead of a meromorphic differential form. In U_0 we take the function $f_0 = 0$. Let $\tilde{f}_a$ be the function defined on the surface by

$$\widetilde{f}_a = \begin{cases} f_a \text{ in } \overline{U}_a \\ 0 \text{ outside } \overline{U}_a \end{cases}$$

We consider the currents $T_a = 0$ in U_a and $T_a = -\sum \widetilde{f}_a$ in U_0. Let 162

$\overset{0,1}{R}$ be the associated current of bidegree $(0,1)$. If $\overset{1,0}{\varphi}$ is a C^{∞} form with compact support in U_0

$$
\begin{aligned}
\langle d_{\bar{z}} T_0, \overset{1,0}{\varphi} \rangle &= -\langle T, d_{\bar{z}} \varphi \rangle \\
&= \langle \sum_{a} \widetilde{f_a}, d_{\bar{z}} \varphi \rangle \\
&= \sum_{a} \iint\limits_{U_a} d_{\bar{z}} (f_a \varphi) \\
&= \sum \iint\limits_{U_a} d(f_a \varphi) \\
&= \sum \int\limits_{bU_a} f_a \varphi.
\end{aligned}
$$

Consequently,

$$\overset{0,1}{R} = \sum_{a} (bU_a) f_a.$$

A necessary and sufficient condition for this Cousin's problem to have a solution is that R be orthogonal (with respect to $\langle\ ,\ \rangle$) to all harmonic forms of bidegree $(1,0)$ i.e., R be orthogonal to all holomorphic forms of degree 1 or

$$\sum \int\limits_{bU_a} f_a h = 0$$

for every holomorphic form h of degree 1.

For Cousin's problem for meromorphic functions we have g independent compatibility conditions and 1 degree of indeterminacy while
163 for Cousin's problem for meromorphic forms we had one compatibility condition and g degress of indeterminacy. This suggests a sort of duality between Cousin's problems for meromorphic functions and forms. This duality will be made precise in the theorem of Riemann-Roch.

The above results prove the existence of lots of meromorphic functions and forms on a compact Riemann surface.

Lecture 25

Some applications

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Before proving the Riemann-Roch theorem we shall make some applications of the existence theorems proved above.

We shall first prove that every compact connected Riemann Surface, V, of genus zero is analytically homeomorphic to the Riemann sphere, S^2. Since the genus of V is zero, there are no compatibility conditions on any Cousin's problem for meromorphic functions. So we can construct a meromorphic function, f, having a simple pole at a point $\underline{a}$ and regular elsewhere. Every meromorphic function on a compact (connected) Riemann surface assumes every value in S^2 the same number of times. So f assumes every value in S^2 exactly once and maps V conformally onto S^2.

In the case of a torus ($g = 1$) there is no distinction between meromorphic functions and meromorphic forms of degree 1 (because of the existence of dz). Evidently, the problem of finding an elliptic function with prescribed periods and singularities is a problem of finding a meromorphic function or a meromorphic differential of degree 1 with prescribed singularities on a torus. Therefore our theorem yields the existence of elliptic functions for which singularities are prescribed in the fundamental parallelogram with the restriction that the sum of the residues at the singularities is zero. The function is determined uniquely upto an additive constant by the singularities. In particular if we prescribe the principal part $1/z^2$ at the origin and choose the constant term 165
in the Laurent development at the origin to be zero, we obtain the Weier-

strass $\mathscr{P}$-function.

The Riemann-Roch Theorem

Let $V^{(1)}$ be a compact connected Riemann Surface of genus g. A divisor D on V is a formal linear combination of points of V with integer coefficients such that all but a finite number of coefficients are zero:

$$D = \sum \alpha_p p - \sum \beta_q q;$$

p, q are points of V, α_p, β_q integers, $\alpha_p > 0, \beta_q > 0$. The degree of the divisor D is defined to be the integer $(\sum \alpha_p - \sum \beta_q)$. We write $A = \sum \alpha_p$, $B = \sum \beta_q$. Meromorphic function f on V is said to be a multiple of the divisor D if at every point $\underline{p}$ f has a zero of order $\geq \alpha_p$ and at every point q, f has a pole of order $\leq \beta_q$. For example in C^1 a function f is a multiple of the divisor D if and only if

$$f = h \prod (z-p)^{\alpha_p} (z-q)^{-\beta} q$$

where h is an entire function. To find a meromorphic function which is a multiple of a given divisor is to find a meromorphic function for which the maximum number of poles with the maximum order at each pole and the minimum number of zeros with the minimum order at each zero are
166 prescribed. Since the order of a zero or a pole of a meromorphic differential form of degree 1 has an invariant meaning we can similarly define what it means to say that a differential form is a multiple of D. (The order is defined by means of a map). Let $\overset{0}{m}(D)$ denote the dimension of the vector space of the meromorphic function which are multiples of D and $\overset{1}{m}(D)$ that of the vector space of the meromorphic differential forms which are multiples of D.

The Riemann-Roch theorem asserts that

$$\boxed{\overset{0}{m}(-D) - \overset{1}{m}(D) = d - g + 1}$$

(d is the degree of the divisor D; $-D$ is the inverse of D).

To prove the Riemann-Roch theorem let us first consider *the meromorphic functions which are multiples of* $-D$. The singular part of such a function at a point $\underline{p}$ can be written in a map as

$$f_p = \sum_{j \leq \alpha_p} C_{j,p} 1/z_p^j$$

(z_p is the function defining the map). The singular parts f_p give a Cousin's datum. If we denote by $[\frac{1}{z_p^j}]$ a pseudo-solution (which depends on the choice of U_i and the metric) associated with the Cousin datum $\frac{1}{z_p^j}$, a pseudo-solution associated with the Cousin datum f_p is

$$\omega = \sum_{p, j \leq \alpha_p} C_{j,p} [\frac{1}{z_p^j}] + C$$

and this gives the general (true) solution when the Cousin problem has a solution. **167**

As conditions for the Cousin datum to be compatible we obtain

$$-\sum_{p, j \leq \alpha_p} C_{j,p} \operatorname{Res}_p(h_\nu / z_p^j) = 0, (\nu = 1, 2, \ldots, g)$$

where $\{h_\nu\}$ is a basis for the space of holomorphic forms of degree 1. (The minus sign on the left side is introduced for later convenience). To write the condition that the solution has a zero of order $\geq \beta_q$ at q it is sufficient to write that the product of ω by dz_q/z_q^k has zero residue at q for $k \leq \beta_q$. This residue exists if ω is a meromorphic form, otherwise it is defined by

$$\frac{1}{2i\pi} \int\limits_{bU_q} \omega \frac{dz_q}{z_q^k}$$

So we obtain the equations

$$\sum_{p, j \leq \alpha_p} C_{j,p} \frac{1}{2i\pi} \int\limits_{bU_q} \frac{dz_q}{z_q^k} [\frac{1}{z_p^j}] + \frac{C}{2i\pi} \int\limits_{bU_q} \frac{dz_q}{z_q^k} = 0$$

for every q and for every $k \leq \beta_q$. We have here $A + 1$ unknowns, $C_{j,p}$ and C, and $g + B$ equations:

$$g(\nu = 1, 2, \ldots, g) \cdot B(q, k \leq \beta_q).$$
$$\left(A = \sum \alpha_p B = \sum \beta_q\right)$$

168 Next let us consider *the meromorphic forms of degree* 1 *which are multiples of D.* The singular part of such a differential in a map at q can be written as

$$\sum_{q,k \leq \beta_q} d_{k,q} \frac{dz_q}{z_q^k}$$

A general pseudo-solution of the associated Cousin problem is

$$\sum_{q,k \leq \beta_q} d_{k,q} \left[\frac{dz}{z_q^k}\right] + \sum e_\nu h_\nu$$

If we write down the condition for the solution of the Cousin problem to exist, we obtain

$$\sum_{q,k \leq \beta_q} d_{k,q} \operatorname{Res}_q \left[\frac{dz_q}{z_q^k}\right] = 0.$$

That the solution should have a zero of order $\geq \alpha_p$ at $\underline{p}$ gives the conditions

$$-\left\{\sum_{q,k} d_{k,q} \frac{1}{2i\pi} \int_{bU_p} \frac{1}{z_p^j} \left[\frac{dz_q}{z_q^k}\right] + \sum e_\nu \operatorname{Res}_p(h_\nu / z_p^j)\right\} = 0.$$

In this case we have $B + g$ unknowns, $d_{k,q}$ and e_ν, and $A + 1$ equations (compatibility condition and $A(p, j \leq \alpha_p)$ equations). We shall now show that the systems of equations

$$(I) \begin{cases} -\sum C_{j,p} \frac{1}{2\pi i} \operatorname{Res}_p(h_\nu / z_p^j) = 0 \, (\nu = 1, \ldots, g), \\ \sum\limits_{p,j} C_{j,p} \frac{1}{2\pi i} \int\limits_{bU_q} \frac{dz_q}{z_p^k} \left[\frac{1}{z_p^j}\right] \frac{C}{2\pi i} \int\limits_{bU_q} \frac{dz_q}{z_q^k} = 0 \end{cases}$$

169 and

$$(II)\begin{cases}\sum d_{k,q}\,\mathrm{Res}_q\left[\frac{dz_q}{z_q^k}\right]=0\\ -\left\{\sum\limits_{q,k\le\beta_q} d_{k,q}\frac{1}{2i\pi}\int\limits_{bU_p}\frac{1}{z_p^j}\left[\frac{dz_q}{z_q^k}\right]+\sum e_\nu\,\mathrm{Res}_p\left(\frac{h_\nu}{z_p^j}\right)\right\}=0\end{cases}$$

are transposes of each other. We have only to verify (the rest of the verification being trivial) that

$$-\int\limits_{bU_p}\frac{1}{z_p^j}\left[\frac{dz_q}{z_q^k}\right]=\int\limits_{bU_q}\frac{dz_q}{z_q^k}\left[\frac{1}{z_p^j}\right]$$

Now,

$$\left[\frac{1}{z_p^j}\right]=\frac{\widetilde{1}}{z_p^j}+2\partial_{\overline{z}}G\left(\frac{bU_p}{z_p^j}\right)$$

$$\left[\frac{dz_q}{z_q^k}\right]=\frac{\widetilde{dz}}{z_q^k}+2\partial_{\overline{z}}G\left(\frac{bU_q\wedge dz_q}{z_q^k}\right)$$

(Refer to the last lecture).

Since $\dfrac{\widetilde{1}}{z_p^j}=0$ on bU_q and $\dfrac{\widetilde{dz}}{z_p^k}=0$ on bU_p is remains only to verify 170
that

$$-\int\limits_{bU_p}\frac{1}{z_p^j}2\partial_{\overline{z}}G\frac{(bU_q\wedge dz_q)}{z_q^k}$$

$$=\int\limits_{bU_q}\frac{dz_q}{zq^k}2\partial_{\overline{z}}G\left(bU_p\frac{1}{z_p^j}\right)$$

or

$$-\langle(bU_p)\frac{1}{z_p^j},2\partial_{\overline{z}}G\left(bU_q\wedge\frac{dz_q}{z_q^k}\right)\rangle$$

$$=\langle 2\partial_{\overline{z}}G\left(bU_p\cdot\frac{1}{z_p^j}\right),bU_q\wedge\frac{dz_q}{z_q^k}\rangle.$$

If we put

$$S = bU_p \cdot \frac{1}{z_p^j} \quad \text{and} \quad T = bU_q \wedge \frac{dz_q}{z_q^k}.$$

We have to verify that

$$-\langle \overset{1}{S}, \partial_{\overline{z}} \overset{1}{G} T \rangle = \langle \overset{2}{T}, \partial_{\overline{z}} \overset{0}{G} S \rangle$$

Write

$$2\partial_{\overline{z}} GS = U, \quad 2\partial_{\overline{z}} GT = V$$

We have then

$$S = \pi_1 S + d_{\overline{z}} U, T = \pi_1 T + d_{\overline{z}} V.$$

(as S and T are $\overline{z}$ closed). So we have to verify that

$$-\langle \pi_1 S + d_{\overline{z}} U, V \rangle = \langle U, \pi_1 T + d_{\overline{z}} V \rangle.$$

171 Now $\langle \pi_1 S, V \rangle = 0$ as $\pi_1 S$ is harmonic and V is a $\partial_{\overline{z}}$ coboundary. Similarly $\langle U, \pi_1 T \rangle = 0$. It remains to show that

$$-\langle d_{\overline{z}} U, V \rangle = \langle d_{\overline{z}} V, U \rangle.$$

We define the set of singularities of a current T to be the smallest closed subset of the manifold in the (open) complement of which T is an indefinitely differentiable form (such a set exists, for, if T, is a form in a family of open subsets then it is a form in their union). Now let S and T be two currents whose sets of singularities have no common point; then it is possible to give a meaning to $\langle S, T \rangle$. It is possible to find decompositions.

$$S = S_1 + S_2, T = T_1 + T_2$$

where S_2 and T_2 are C^∞ forms and S_1 and T_1 are currents whose supports have no common point. (For example, let $1 = \alpha + \beta$ be a partition of unity subordinate to $\complement F$, $\complement G$ where F and G are the supports of S and T respectively. We may take $S_1 = \beta S$, $S_2 = \alpha S$, $T_1 = T_2 = T$). *By definition* we put:

$$\langle S, T \rangle = \langle S_1, T_2 \rangle + \langle S_2, T_1 \rangle + \langle S_2, T_2 \rangle;$$

each bracket on the right side has a meaning since S_2 and T_2 are C^∞ forms ($\langle S_1, T_1\rangle$ is not written; we take it by definition to be zero which is natural because the supports of S and T have no common point). This definition is correct provided we prove i) it is independent of the choice **172**
of the choice of the decompositions of S and T and ii) it gives the usual result if S or T is a form. Suppose we have another decomposition S'_i, $T'_i (i = 1, 2)$. It is sufficient to prove that the decompositions S_i, T_i and S_i, T'_i give the same result; for then the decompositions S_i, T'_i and S'_i, T'_i will also give the same result for an analogous reason. We have to show that

$$\langle S_1, T'_2 - T_2\rangle + \langle S_2, T'_1 - T_1\rangle + \langle S_2, T'_2 - T_2\rangle$$

is zero. The last two brackets can be added by linearity because S_2 is a C^∞ form and their sum is $\langle S_2, 0\rangle = 0$. It remains to prove that $\langle S_1, T'_2 - T_2\rangle = 0$. We shall add to $\langle S_1, T'_2 - T_2\rangle$ the expression $\langle S_1, T'_1 - T_1\rangle$ which has a meaning because $T'_1 - T_1$ is a form ($(T'_1 - T_1) + (T'_2 - T_2)$ is the 0 form and $T'_2 - T_2$ is a form) and which is equal to zero since the supports of S_1 and $T'_1 - T_1$ have no common point. But now the sum $\langle S_1, T'_1 - T_1\rangle + \langle S_1, T'_2 - T_2\rangle$ is equal, by linearity, to $\langle S, 0\rangle = 0$. So $\langle S_1, T'_2 - T_2\rangle = 0$. This proves (i). (ii) is trivial because if S is a form we may take the decomposition $S = 0 + S$; $T = T + 0$. Now we have proved the existence of the expression when the sets of singularities of S and T have no common point. (We did not pay any attention to the question of supports, assuming the manifold to be compact; if the manifold is not compact, we have to assume further that the intersection of the supports of S and T is compact). That the sets of singularities of S and T have **173**
no common point can be expressed in the following way: every point of the manifold has an open neighbourhood in which one at least of the currents S and T (not necessarily the same one) is a form. In this case the properties of differentiation such as the formula

$$\langle d_{\bar{z}} S, T\rangle = (-1)^{p+1} \langle S, d_{\bar{z}} T\rangle$$

(S is of total degree p) remain obviously valid as is seen immediately by means of a decomposition S_i, T_i of S, T. In our case since the sets of singularities of U and V have no common point

$$\begin{aligned} & -\langle d_{\bar z}U, V\rangle = \langle d_{\bar z}V, U\rangle \\ \text{or} \qquad & -\langle S, \partial_{\bar z}GT\rangle = \langle T, \partial_{\bar z}GS\rangle \end{aligned}$$

Thus the systems (I) and (II) are transposes of each other. Two systems which are transposes of each other have the same rank ρ, ρ = number of unknowns - degree of indeterminacy. So we have

$$(A+1) - \overset{0}{m}(-D) = (B+g) - \overset{1}{m}(D)$$

or

$$\begin{aligned} \overset{0}{m}(-D) - \overset{1}{m}(D) &= (A-B) - g + 1 \\ &= d - g + 1. \end{aligned}$$

This completes the proof of the Riemann-Roch theorem.

Let D be a positive divisor, $D = \sum \alpha_p p$, $\alpha_p \geq 0$. All the differentials of degree 1 which are multiples of D are holomorphic. Now an holomorphic differential form of degree 1 has exactly $2g-2$ zeros. (This can
174 be proved by using Poincaré's theorem on vector fields or deduced from the Riemann Roch theorem). Consequently if $\sum \alpha_p > 2g-2$, $\overset{1}{m}(D) = 0$ and

$$\overset{0}{m}(-D) = d - g + 1 \; (d > 2g-2)$$

i.e., if we give a sufficient number of poles the g conditions of compatibility are all independent.

Appendix

Kählerian structure on the complex projective space

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The unit sphere S^{2n+1} in C^n can be considered in a natural way as a fibre bundle over PC^n with circles as fibres (if z is a point of S^{2n+1} the points $e^{i\theta}z$, θ real, will constitute the fibre through z).

We shall first determine how J operates on the tangent spaces of PC^n. Let $\pi : S^{2n+1} \to PC^n$ denote the projection map. We shall denote the differential of π also by π. The differential map π maps the tangent bundle of S^{2n+1} onto that of PC^n. Let $X = \pi U$, U tangent to S^{2n+1}, be a vector tangent to PC^n, say at a. J is uniquely determined by:

$$\langle JX, df\rangle = i\langle X, df\rangle$$
$$\langle JX, \overline{df}\rangle = -i\langle X, \overline{df}\rangle$$

for every f holomorphic in a neighbourhood of $\underline{a}$. If $z_{n+1} \neq 0$, z_k/z_{n+1} is a local coordinate system; we may suppose that $\underline{a}$ belongs to the domain of this coordinate system. J is uniquely determined by:

$$\langle JX, d(z_k/z_{n+1})\rangle = i\langle X, d(z_k/z_{n+1})\rangle$$
$$\langle JX, \overline{d(z_k/z_{n+1})}\rangle = -i\langle X, \overline{d(z_k/z_{n+1})}\rangle.$$

$(u_1, \ldots, u_n)$ being the coordinates of U, let (iU) denote the vector whose coordinates are $(iu_1, \ldots, iu_n)$. The vector $\pi(iU)$ has the properties 176

$$\langle \pi(iU), d(z_k/z_{n+1})\rangle = \langle iU, d(z_k/z_{n+1})\rangle = i\langle X, d(z_k/z_{n+1})\rangle;$$

and $\langle \pi(iU), \overline{d(z_k/z_{n+1})}\rangle = -i\langle X, \overline{d(z_k/z_{n+1})}\rangle$.

Therefore

$$\pi(iU) = JX$$

So, to find JX we multiply U by i and take its image by π.

Consider now the form

$$\omega = \sum \bar{z}_k dz_k$$

of bidegree $(1, 0)$ in C^{n+1}. Put

$$H = \sum dz_k \overline{dz_k} - \omega\overline{\omega}.$$

This Hermitian form on C^{n+1} induces on S^{2n+1} a semi-definite Hermitian form. To prove that this form is semi-definite on S^{2n+1}, let $U = (u_1, \ldots, u_n)$ be a vector tangent to S^{2n+1} at $z \in S^{2n+1}$, $z = (z_1, \ldots, z_n)$. By Schwarz's inequality

$$|\sum \bar{z}_k u_k| \leq |\bar{z}|\, |U| = |U|$$

(since z is on the unit sphere). Similarly

$$|\sum z_k \bar{u}_k| \leq |U|.$$

So

$$H(U, U) = \sum u_k \bar{u}_k - \left(\sum \bar{z}_k \bar{u}_k\right)\left(\sum z_k \bar{u}_k\right) \geq 0.$$

Moreover $H(U, U) = 0$ if and only if $u_k = \mu z_k$, μ complex. Actually $\mu = i\lambda$, where λ is real. For, since z is on S^{2n+1} and U is tangent to the unit sphere at z

$$\sum (\bar{z}_k u_k + z_k \bar{u}_k) = 0$$

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$$\left(\sum z_k \bar{z}_k\right)(\mu + \overline{\mu}) = 0 \quad \text{or} \quad Rl\mu = 0.$$

Thus $H(U, U) = 0$ if and only if $U = i\lambda z$, λ real i.e., if and only if U is tangent to the fibre, that is if $\pi(U) = 0$.

If U is tangent to the fibre at z and $V = (v_1, \ldots, v_n)$ is any vector tangent to S^{2n+1} at z, $H(U, V) = 0$. For by Schwarz's inequality

$$|H(U, V)| \leq \sqrt{H(U, U) H(V, V)} = 0$$

Moreover H is invariant under the operations $z \to e^{i\theta}z$, θ real. These two facts imply that H defines quadratic forms $\widetilde{H}$ on tangent spaces of PC^n (if X and Y are tangent to PC^n at $\underline{a}$ we choose vectors U and V tangent to S^{2n+1} at some point in $\pi^{-1}(a)$ such that $\pi U = X$ and $\pi V = Y$ and define $\widetilde{H}(X, Y) = H(U, V)$. The two properties proved above imply that this definition defines $\widetilde{H}$ invariantly).

$\widetilde{H}$ are J Hermitian forms.

i) Evidently $\widetilde{H}$ is R-bilinear

ii) $\widetilde{H}(JX, Y) = -\widetilde{H}(X, JY) = i\widetilde{H}(X, Y)$

For,

$$\widetilde{H}(JX, Y) = H(iU, V) = i\widetilde{H}(X, Y)$$

as we have seen that $JX = \pi(iU)$. Similarly

$$\widetilde{H}(X, JY) = -i\widetilde{H}(X, Y)$$

iii) $\tilde{H}(X, X) > 0$ for $X \neq 0$. This follows from the fact that $H(U, U) =$ **178**
0 if and only if U is tangent to the fibre. It remains to prove that the exterior form $\widetilde{\Omega}$ associated with $\widetilde{H}$ is closed. If Ω is the form associated with H,

$$\begin{aligned} -2i\Omega &= \sum dz_k \wedge \overline{dz_k} - \omega \wedge \overline{\omega} \\ &= -d\omega - \omega \wedge \overline{\omega} \end{aligned}$$

$\Omega = \pi^{-1}\widetilde{\Omega}$. But on S^{2n+1}, $\overline{\omega} = -\omega$; for, the relation

$$\sum \bar{z}_k z_k = 1$$

yields

$$\sum (\bar{d}z_k z_k + \bar{z}_k dz) = 0$$

Consequently $-2i\Omega = d\omega$; so Ω is closed.

Now $d\Omega$ is the reciprocal image of $d\widetilde{\Omega}$; therefore $\widetilde{\Omega}$ is also closed. This proves that PC^n is Kählerian.

It may be remarked that even though Ω is a coboundary, $\widetilde{\Omega}$ is not a coboundary, (as PC^n is Kählerian!); ω is not a reciprocal image.

Another method to prove that PC^n is Kählerian would be to consider PC^n as symmetric space with respect to the unitary group, U^{n+1}. Since U^{n+1} is compact we may construct an invariant Hermitian metric by the averaging process. The associated 2-form Ω will be an invariant form. But in a symmetric space any invariant form is closed.

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179 We do not give here an exhaustive bibliography on the subject. We mention some articles and books which may help the reader of these notes to obtain some additional information on the subject or which contain other methods of exposition.

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