Lectures on
The Riemann Zeta–Function

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Tata Institute of Fundamental Research, Bombay
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The aim of these lectures is to provide an introduction to the theory of the Riemann Zeta-function for students who might later want to do research on the subject. The Prime Number Theorem, Hardy's theorem on the Zeros of $\zeta(s)$, and Hamburger's theorem are the principal results proved here. The exposition is self-contained, and required a preliminary knowledge of only the elements of function theory.
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Theorem 1. If $D$ is a domain bounded by a contour $C$ for which Cauchy’s theorem is valid, and $f$ is continuous on $C$ regular in $D$, then “$|f| \leq M$ on $C$” implies “$|f| \leq M$ in $D$”, and if $|f| = M$ in $D$, then $f$ is a constant.

Proof. (a) Let $z_o \in D$, $n$ a positive integer. Then

$$|f(z_o)|^n = \left| \frac{1}{2\pi i} \int_C \frac{(f(z))^n dz}{z - z_o} \right|$$

$$\leq \frac{l_c \cdot M^n}{2\pi \delta},$$

where $l_c$ = length of $C$, $\delta$ = distance of $z_o$ from $C$. As $n \to \infty$

$$|f(z)| \leq M.$$  

(b) If $|f(z_o)| = M$, then $f$ is a constant. For, applying Cauchy’s integral formula to $\frac{d}{dz} |(f(z))^n|$, we get

$$|n(f(z_o))^{n-1} \cdot f'(z_o)| = \left| \frac{1}{2\pi i} \int_C \frac{f^n dz}{(z - z_o)^2} \right|$$

$$\leq \frac{l_c M^n}{2\pi \delta^2},$$
so that
\[ |f'(z_0)| \leq \frac{I_e M}{2\pi \delta^2} \cdot \frac{1}{n} \to 0, \text{ as } n \to \infty \]

Hence
\[ |f'(z_0)| = 0. \]

(c) If \( |f(z_0)| = M \), and \( |f'(z_0)| = 0 \), then \( f''(z_0) = 0 \), for
\[
\frac{d^2}{dz^2} [ |f(z)|^n ] = n(n-1)|f(z)|^{n-2}(f'(z))^2 + n|f(z)|^{n-1}f''(z). 
\]

At \( z_0 \) we have
\[
\frac{d^2}{dz^2} [ |f(z)|^n ]_{z=z_0} = nf^{n-1}(Z_0)f''(z_0),
\]
so that
\[
|nf^{n-1}f''(z_0)| = \left| \frac{2!}{2\pi i} \int_C \frac{|f(z)|^n dz}{(z-z_0)^3} \right| \leq \frac{2!l_e}{2\pi \delta^3} M^n,
\]
and letting \( n \to \infty \), we see that \( f''(z_0) = 0 \). By a similar reasoning we prove that all derivatives of \( f \) vanish at \( z_0 \) (an arbitrary point of \( D \)). Thus \( f \) is a constant.

\[ \square \]

**Remark.** The above proof is due to Landau [12, p.105]. We shall now show that the restrictions on the nature of the boundary \( C \) postulated in the above theorem can be dispensed with.

**Theorem 2.** If \( f \) is regular in a domain \( D \) and is not a constant, and \( M = \max_{z \in D} |f| \), then
\[ |f(z_0)| < M, \quad z_0 \in D. \]

For the proof of this theorem we need a
1. The Maximum Principle

**Lemma.** If $f$ is regular in $|z - z_0| \leq r$, $r > 0$, then

$$|f(z_0)| \leq M_r,$$

where $M_r = \max_{|z - z_0| = r} |f(z)|$, and $|f(z_0)| = M_r$ only if $f(z)$ is constant for $|z - z_0| = r$.

**Proof.** On using Cauchy’s integral formula, we get

$$f(z_0) = \frac{1}{2\pi i} \int_{|z - z_0| = r} \frac{f(z)}{z - z_0} \, dz$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta. \quad (1)$$

Hence

$$|f(z_0)| \leq M_r.$$

Further, if there is a point $\zeta$ (such that) $|\zeta - z_0| = r$, and $|f(\zeta)| < M_r$, then by continuity, there exists a neighbourhood of $\zeta$, on the circle $|z - z_0| = r$, in which $|f(z)| \leq M_r - \varepsilon$, $\varepsilon > 0$ and we should have $|f(z_0)| < M_r$; so that “$|f(z_0)| = M_r$” implies “$|f(z)| = M_r$ everywhere on $|z - z_0| = r$”. That is $|f(z_0 + re^{i\theta})| = |f(z_0)|$ for $0 \leq \theta \leq 2\pi$, or

$$f(z_0 + re^{i\theta}) = f(z_0)e^{i\varphi}, \quad 0 \leq \varphi \leq 2\pi$$

On substituting this in (1), we get

$$1 = \frac{1}{2\pi} \int_0^{2\pi} \cos \varphi \, d\theta$$

Since $\varphi$ is a continuous function of $\theta$ and $\cos \varphi \leq 1$, we get $\cos \varphi \cdot 1$ for all $\theta$, i.e. $\varphi = 0$, hence $f(z)$ is a constant. \qed

**Remarks.** The Lemma proves the maximum principle in the case of a circular domain. An alternative proof of the lemma is given below [7, Bd 1, p.117].
Aliter. If \( f(z_0 + re^{i\theta}) = \phi(\theta) \) (complex valued) then

\[
f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta
\]

Now, if \( a \) and \( b \) are two complex numbers, \( |a| \leq M, |b| \leq M \) and \( a \neq b \), then \( |a + b| < 2M \). Hence if \( \phi(\theta) \) is not constant for \( 0 \leq \theta < 2\pi \), then there exist two points \( \theta_1, \theta_2 \) such that

\[
|\phi(\theta_1) + \phi(\theta_2)| = 2M_r - 2\varepsilon, \text{ say, where } \varepsilon > 0
\]

On the other hand, by regularity,

\[
\begin{align*}
|\phi(\theta_1 + t) - \phi(\theta_1)| &< \varepsilon/2, \text{ for } 0 < t < \delta \\
|\phi(\theta_2 + t) - \phi(\theta_2)| &< \varepsilon/2, \text{ for } 0 < t < \delta
\end{align*}
\]

Hence

\[
|\phi(\theta_1 + t) + \phi(\theta_2 + t)| < 2M_r - \varepsilon, \text{ for } 0 < t < \delta.
\]

Therefore

\[
\int_0^{2\pi} \phi(\theta) d\theta = \int_0^{\theta_1} + \int_{\theta_1}^{\theta_1 + \delta} + \int_{\theta_1 + \delta}^{\theta_2} + \int_{\theta_2}^{\theta_2 + \delta} + \int_{\theta_2 + \delta}^{2\pi}
\]

\[
= \left[ \int_0^{\theta_1} + \int_{\theta_1}^{\theta_2} + \int_{\theta_1 + \delta}^{\theta_2 + \delta} \right] \phi(\theta) d\theta + \int_0^{\delta} [\phi(\theta_1 + t) + \phi(\theta_2 + t)] dt
\]

Hence

\[
| \int_0^{2\pi} \phi(\theta) d\theta | \leq M_r(2\pi - 2\delta) + (2M_r - \varepsilon)\delta = 2\pi M_r - \varepsilon\delta
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta < M_r
\]
5 **Proof of Theorem 2.** Let $z_o \in D$. Consider the set $G_1$ of points $z$ such that $f(z) \neq f(z_o)$. This set is not empty, since $f$ is non-constant. It is a proper subset of $D$, since $z_o \in D$, $z_o \notin G_1$. It is open, because $f$ is continuous in $D$. Now $D$ contains at least one boundary point of $G_1$. For if it did not, $G'_1 \cap D$ would be open too, and $D = (G_1 \cup G'_1)D$ would be disconnected. Let $z_1$ be the boundary point of $G_1$ such that $z_1 \in D$. Then $z_1 \notin G_1$, since $G_1$ is open. Therefore $f(z_1) = f(z_o)$. Since $z_1 \in BdG_1$ and $z_1 \in D$, we can choose a point $z_2 \in G_1$ such that the neighbourhood of $z_1$ defined by $|z - z_1| \leq |z_2 - z_1|$ lies entirely in $D$. However, $f(z_2) \neq f(z_1)$, since $f(z_2) \neq f(z_o)$, and $f(z_o) = f(z_1)$. Therefore $f(z)$ is not constant on $|z - z_1| = r$, (see (1) on page 3) for, if it were, then $f(z_2) = f(z_1)$. Hence if $M' = \max_{|z-z_1|\leq|z_2-z_1|}|f(z)|$, then

$$|f(z_1)| < M' \leq M = \max_{z \in D}|f(z)|$$

i.e. $|f(z_o)| < M$

Remarks. The above proof of Theorem 2 [7, Bd I, p.134] does not make use of the principle of analytic continuation which will of course provide an immediate alternative proof once the Lemma is established.

**Theorem 3.** If $f$ is regular in a bounded domain $D$, and continuous in $\bar{D}$, then $f$ attains its maximum at some point in $Bd D$, unless $f$ is a constant.

Since $D$ is bounded, $\bar{D}$ is also bounded. And a continuous function on a compact set attains its maximum, and by Theorem 2 this maximum cannot be attained at an interior point of $D$. Note that the continuity of $|f|$ is used.

**Theorem 4.** If $f$ is regular in a bounded domain $D$, is non-constant, and if for every sequence $\{z_n\}$, $z_n \in D$, which converges to a point $\xi \in Bd D$, we have

$$\lim_{n \to \infty} \sup_{z \in D} |f(z_n)| \leq M,$$

then $|f(z)| < M$ for all $z \in D$. [17] p.111]
Proof. D is an $F_{\sigma}$, for define sets $C_n$ by the property: $C_n$ consists of all $z$ such that $|z| \leq n$ and such that there exists an open circular neighbourhood of radius $1/n$ with centre $z$, which is properly contained in $D$. Then $C_n \subset C_{n+1}$, $n = 1, 2, \ldots$; and $C_n$ is compact.

Define

$$\max_{z \in C_n} |f(z)| \equiv m_n.$$ 

By Theorem 2 there exists a $z_n \in \text{Bd} \ C_n$ such that $|f(z_n)| = m_n$. The sequence $\{m_n\}$ is monotone increasing, by the previous results; and the sequence $\{z_n\}$ is bounded, so that a subsequence $\{z_{n_p}\}$ converges to a limit $\xi \in \text{Bd} \ D$. Hence

$$\lim_{p \to \infty} |f(z_{n_p})| \leq M$$

i.e. $$\lim_{p \to \infty} m_{n_p} \leq M$$

or $$m_n < M$$ for all $n$

i.e. $$|f(z)| < M, \quad z \in D.$$ 

N.B. That $\xi \in \text{Bd} \ D$ can be seen as follows. If $\xi \in D$, then there exists an $N_0$ such that $\xi \in C_{N_0} \subset D$, and $|f(\xi)| < m_{N_0} \leq \lim m_n$, whereas we have $f(z_{n_p}) \to f(\xi)$, so that $|f(z_{n_p})| \to |f(\xi)|$, or $\lim m_n = |f(\xi)|$.

Corollaries. (1) If $f$ is regular in a bounded domain $D$ and continuous in $D$, then by considering $e^{f(z)}$ and $e^{-f(z)}$ instead of $f(z)$, it can be seen that the real and imaginary parts of $f$ attain their maxima on $\text{Bd} \ D$.

(2) If $f$ is an entire function different from a constant, then

$$M(r) \equiv \text{l.u.b.} |f(z)|, \quad \text{and}$$

$$A(r) \equiv \text{l.u.b. Re}\{f(z)\}$$

are strictly increasing functions of $r$. Since $f(z)$ is bounded if $M(r)$ is, for a non-constant function $f$, $M(r) \to \infty$ with $r$. 

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1. The Maximum Principle
(3) If \( f \) is regular in \( D \), and \( f \neq 0 \) in \( D \), then \( f \) cannot attain a minimum in \( D \).

Further if \( f \) is regular in \( D \) and continuous in \( \bar{D} \), and \( f \neq 0 \) in \( \bar{D} \), and \( m = \min_{z \in \partial D} |f(z)| \), then \( |f(z)| > m \) for all \( z \in D \) unless \( f \) is a constant. [We see that \( m > 0 \) since \( f \neq 0 \), and apply the maximum principle to \( 1/|f| \)].

(4) If \( f \) is regular and \( \neq 0 \) in \( D \) and continuous in \( \bar{D} \), and \( |f| \) is a constant \( \gamma \) on \( \partial D \), then \( f = \gamma \) in \( D \). For, obviously \( \gamma \neq 0 \), and, by Corollary 3, \( |f(z)| \geq \gamma \) for \( z \in D \), se that \( f \) attains its maximum in \( D \), and hence \( f = \gamma \).

(5) If \( f \) is regular in \( D \) and continuous in \( \bar{D} \), and \( |f| \) is a constant on \( \partial \bar{D} \), then \( f \) is either a constant or has a zero in \( D \).

(6) Definition: If \( f \) is regular in \( D \), then \( \alpha \) is said to be a boundary value of \( f \) at \( \xi \in \partial D \), if for every sequence \( z_n \to \xi \in \partial D \), \( z_n \in D \), we have

\[
\lim_{n \to \infty} f(z_n) = \alpha
\]

(a) It follows from Theorem 4 that if \( f \) is regular and non-constant in \( D \), and has boundary-values \( \{\alpha\} \), and \( |\alpha| \leq H \) for each \( \alpha \), then \( |f(z)| < H, z \in D \). Further, if \( M' = \max_{\alpha \in \partial \bar{D}} |\alpha| \) then \( M' = M \equiv \max_{z \in \bar{D}} |f(z)| \).

For, \( |\alpha| \) is a boundary value of \( |f| \), so that \( |\alpha| \leq M \), hence \( M' \leq M \). On the other hand, there exists a sequence \( \{z_n\} \), \( z_n \in D \), such that \( |f(z_n)| \to M \). We may suppose that \( z_n \to z_o \) (for, in any case, there exists a subsequence \( \{z_{n_p}\} \) with the limit \( z_o \), say, and one can then operate with the subsequence). Now \( z_o \notin D \), for otherwise by continuity \( f(z_o) = \lim_{n \to \infty} f(z_n) = M \), which contradicts the maximum principle. Hence \( z_o \in \partial D \), and \( M \) is the boundary-value of \( |f| \) at \( z_o \). Therefore \( M \leq M' \), hence \( M = M' \).

(b) Let \( f \) be regular and non-constant in \( D \), and have boundary-values \( \{\alpha\} \) on \( \partial D \). Let \( m = \min_{z \in \bar{D}} |f|; m' = \min_{\alpha \in \partial \bar{D}} |\alpha| > 0 \).
1. The Maximum Principle

Then, if \( f \neq 0 \) in \( D \), by Corollary (6)a, we get

\[
\max_{z \in D} \frac{1}{|f(z)|} = \frac{1}{m'} = \frac{1}{m};
\]

so that \(|f| > m = m'\) in \( D \). Thus if there exists a point \( z_0 \in D \) such that \(|f(z_0)| \leq m'\), then \( f \) must have a zero in \( D \), so that \( m' > 0 = m \). We therefore conclude that, if \( m' > 0 \), then \( m' = m \) if and only if \( f \neq 0 \) in \( D \) [6, Bd. I, p.135].
Lecture 2

The Phragmen-Lindelof principle

We shall first prove a crude version of the Phragmen-Lindelof theorem and then obtain a refined variant of it. The results may be viewed as extensions of the maximum-modulus principle to infinite strips.

**Theorem 1.** [6, p.168] We suppose that

1. \( f \) is regular in the strip \( \alpha < x < \beta \); \( f \) is continuous in \( \alpha \leq x \leq \beta \)
2. \( |f| \leq M \) on \( x = \alpha \) and \( x = \beta \)
3. \( f \) is bounded in \( \alpha \leq x \leq \beta \)

Then, \( |f| \leq M \) in \( \alpha < x < \beta \); and \( |f| = M \) in \( \alpha < x < \beta \) only if \( f \) is a constant.

**Proof.** (i) If \( f(x+iy) \to O \) as \( y \to \pm \infty \) uniformly in \( x, \alpha \leq x \leq \beta \), then the proof is simple. Choose a rectangle \( \alpha \leq x \leq \beta, |y| \leq \eta \) with \( \eta \) sufficiently large to imply \( |f(x \pm i\eta)| \leq M \) for \( \alpha \leq x \leq \beta \). Then, by the maximum-modulus principle, for any \( z_o \) in the interior of the rectangle,

\[ |f(z_o)| \leq M. \]
(ii) If $|f(x + iy)| \not\to 0$, as $y \to \pm \infty$, consider the modified function $f_n(z) = f(z)e^{z^2/n}$. Now

$$|f_n(z)| \to 0 \text{ as } y \to \pm \infty \text{ uniformly in } x,$$

and

$$|f_n(z)| \leq Me^{z^2/n}$$

on the boundary of the strip, for a suitable constant $c$.

Hence

$$|f_n(z_0)| \leq Me^{z_0^2/n}$$

for an interior point $z_0$ of the rectangle, and letting $n \to \infty$, we get

$$|f(z_0)| \leq M.$$

If $|f(z_0)| = M$, then $f$ is a constant. For, if $f$ were not a constant, in the neighbourhood of $z_0$ we would have points $z$, by the maximum-modulus principle, such that $|f(z)| > M$, which is impossible. \hfill \Box

Theorem 1 can be restated as:

**Theorem 1’**: Suppose that

(i) $f$ is continuous and bounded in $\alpha \leq x \leq \beta$

(ii) $|f(\alpha + iy)| \leq M_1$, $|f(\beta + iy)| \leq M_2$ for all $y$

Then

$$|f(x_0 + iy_0)| \leq M_1^{L(x_0)}M_2^{1-L(x_0)}$$

for $\alpha < x_0 < \beta$, $|y_0| < \infty$, where $L(t)$ is a linear function of $t$ which takes the value 1 at $\alpha$ and 0 at $\beta$. If equality occurs, then

$$f(z) = cM_1^{L(z)}M_2^{1-L(z)}; \quad |c| = 1.$$

**Proof.** Consider

$$f_1(z) = \frac{f(z)}{M_1^{L(z)}M_2^{1-L(z)}}$$

and apply Theorem 1 to $f_1(z)$.

More generally we have \[12\] p.107] \hfill \Box
Theorem 2. Suppose that

(i) $f$ is regular in an open half-strip $D$ defined by:
\[ z = x + iy, \quad \alpha < x < \beta, \quad y > \eta \]

(ii) $\lim_{z \to \xi} |f| \leq M$, for $\xi \in \text{Bd} \ D$

(iii) $f = O \left[ \exp \left\{ \frac{\sigma e^{\theta \pi |z|}}{\pi - \alpha} \right\} \right], \quad \theta < D$

uniformly in $D$.

Then, $|f| \leq M$ in $D$; and $|f| < M$ in $D$ unless $f$ is a constant.

Proof. Without loss of generality we can choose
\[ \alpha = -\frac{\pi}{2}, \quad \beta = +\pi/2, \quad \eta = 0 \]

Set
\[ g(z) = f(z). \exp(-\sigma e^{-ikz}) \equiv f(z). (\omega(z))^\sigma, \quad \text{say}, \]

where $\sigma > 0$, $\theta < k < 1$. Then, as $y \to +\infty$,
\[ g = O \left[ \exp \left\{ e^{\theta |z|} - \sigma e^{ky} \cos \frac{k\pi}{2} \right\} \right] \]

uniformly in $x$, and
\[ e^{\theta |z|} - \sigma e^{ky} \cos \frac{k\pi}{2} \leq e^{\theta (y + \pi/2)} - \sigma e^{ky} \cos \frac{k\pi}{2} \]
\[ \to -\infty, \quad \text{as} \quad y \to \infty, \]

uniformly in $x$, since $\theta < k$. Hence
\[ |g| \to 0 \quad \text{uniformly as} \quad y \to \infty, \]

so that
\[ |g| \leq M, \quad \text{for} \quad y > y', \quad \alpha < x < \beta. \]
Let \( z_o \in D \). We can then choose \( H \) so that
\[
|g(z)| \leq M
\]
for \( y = H, \alpha < x < \beta \), and we may also suppose that \( H > y_0 = \text{im}(z_o) \).

Consider now the rectangle defined by \( \alpha < x < \beta \), \( y = 0 \), \( y = H \). Since \( |\omega\sigma| \leq 1 \) we have
\[
\lim_{z \to \xi \text{ in } D} |g| \leq M
\]
for every point \( \xi \) on the boundary of this rectangle. Hence, by the maximum-modulus principle,
\[
|g(z_o)| < M, \text{ unless } g \text{ is a constant; or } \ |f(z_o)| < M|e^{\sigma e^{-ikz_o}}|
\]
Letting \( \sigma \to 0 \), we get
\[
|f(z_o)| < M.
\]

\( \square \)

**Remarks.** The choice of \( \omega \) in the above proof is suggested by the critical case \( \theta = 1 \) of the theorem when the result is no longer true. For take
\[
\theta = 1, \quad \alpha = -\pi/2, \quad \beta = \pi/2, \quad \eta = 0, \quad f = e^{-iz}
\]
Then
\[
|e^{-iz}| = e^{\text{Re}(-iz)} = e^{\text{Re}(e^{-iz})} = e^{\cos x}
\]
So
\[
|f| = 1 \text{ on } x = \pm\pi/2, \quad \text{and } f = e^{\sigma} \text{ on } x = 0.
\]

**Corollary 1.** If \( f \) is regular in \( D \), continuous on \( \text{Bd } D \), \( |f| \leq M \) on \( \text{Bd } D \), and
\[
f = O(e^{\frac{j \pi}{\theta m}}), \quad \theta < 1,
\]
then \( |f| \leq M \) in \( D \).
Corollary 2. Suppose that hypotheses (i) and (iii) of Theorem 2 hold; suppose further that $f$ is continuous on $\partial D$,

$$f = O(y^a) \text{ on } x = \alpha, \quad f = O(y^b) \text{ on } x = \beta.$$ 

Then $f = O(y^c)$ on $x = \gamma$, uniformly in $\gamma$, where $c = p\gamma + q$, and $px + q$ is the linear function which equals $a$ at $x = \alpha$ and $b$ at $x = \beta$.

Proof. Let $\eta > 0$. Define $\varphi(z) = f(z)\psi(z)$, where $\psi$ is the single-valued branch of $(-zi)^{-(px+q)}$ defined in $D$, and apply Theorem 2 to $\varphi$. We first observe that $\varphi$ is regular in $D$ and continuous in $\bar{D}$. Next

$$|\psi(z)| = |(y - ix)^{-(px+q)} - i\eta y| = |y - ix|^{-(px+q)} \exp(py \log y - ix) \left[ 1 + O \left( \frac{1}{y} \right) \right]$$

$$= y^{-(px+q)} \left[ 1 + O \left( \frac{1}{y} \right) \right] \exp \left( py \left( -x + O \left( \frac{1}{y^2} \right) \right) \right)$$

$$= y^{-(px+q)} e^{-px} \left[ 1 + O \left( \frac{1}{y} \right) \right],$$

the $O$'s being uniform in the $x$'s.

Thus $\varphi = O(1)$ on $x = \alpha$ and $x = \beta$. Since $\psi = O(y^k) = O(|z|^k)$, we see that condition (iii) of Theorem 2 holds for $\varphi$ if $\theta$ is suitably chosen. Hence $\varphi = O(1)$ uniformly in the strip, which proves the corollary. □

Theorem 3. [12, p. 108] Suppose that conditions (i) and (iii) of Theorem 2 hold. Suppose further that $f$ is continuous in $\partial D$, and

$$\lim_{y \to \infty} |f| \leq M \text{ on } x = \alpha, \quad x = \beta$$

Then

$$\lim_{y \to \infty} |f| \leq M \text{ uniformly in } \alpha \leq x \leq \beta.$$ 

Proof. $f$ is bounded in $\bar{D}$, by the previous theorem. Let $\eta \geq 0, \sum > 0$. Let $H = H(\epsilon)$ be the ordinate beyond which $|f| < M + \epsilon$ on $x = \alpha, x = \beta$. 

Let $h > 0$ be a constant. Then
\[ \left| \frac{z}{z + hi} \right| < 1 \]
in the strip. Choose $h = h(H)$ so large that
\[ \left| f(z) \cdot \frac{z}{z + hi} \right| < M + \varepsilon \text{ on } y = H \]
(possible because $\left| \frac{f \cdot z}{z + hi} \right| \leq |f| < M$). Then the function
\[ g(z) = f \cdot \frac{z}{z + hi} \]
satisfies the conditions of Theorem 2 (with $M + \varepsilon$ in place of $M$) in the
strip above $y = H$. Thus
\[ |g| \leq M + \varepsilon \text{ in this strip} \]
and so $\lim_{y \to \infty} |g| \leq M + \varepsilon$ uniformly.

That is,
\[ \lim_{y \to \infty} |f| \leq M + \varepsilon, \]

since $\frac{z}{z + hi} \to 1$ uniformly in $x$. Hence
\[ \lim_{y \to \infty} |f| \leq M. \]

\[ \square \]

Corollary 1. If conditions (i) and (iii) of Theorem 2 hold, if $f$ is continuous on $\partial D$, and if
\[ \lim_{y \to \infty} |f| \leq \begin{cases} a & \text{on } x = \alpha, \\ b & \text{on } x = \beta, \end{cases} \]

where $a \neq 0$, $b \neq 0$, then
\[ \lim_{y \to \infty} |f| \leq e^{px+q}, \]

uniformly, $p$ and $q$ being so chosen that $e^{px+q} = a$ for $x = \alpha$ and $b$ for $x = \beta$. 

2. The Phragmen-Lindelöf principle

Proof. Apply Theorem 3 to \( g = f e^{-pz+q} \).

Corollary 2. If \( f = O(1) \) on \( x = \alpha \), \( f = o(1) \) on \( x = \beta \), then \( f = o(1) \) on \( x = \gamma \), \( \alpha < \gamma \leq \beta \).

[if conditions (i) and (iii) of Theorem 2 are satisfied].

Proof. Take \( b = \varepsilon \) in Corollary 1 and note that \( e^{pz+q} \to 0 \) as \( \varepsilon \to 0 \) for fixed \( \gamma \), provided \( p \) and \( q \) are chosen as specified in Corollary 1.
Lecture 3

Schwarz’s Lemma

**Theorem 1.** Let \( f \) be regular in \( |z| < 1 \), \( f(o) = 0 \).

If, for \( |z| < 1 \), we have \( |f(z)| \leq 1 \), then

\[
|f(z)| \leq |z|, \quad \text{for} \quad |z| < 1.
\]

Here, equality holds only if \( f(z) \equiv cz \) and \( |c| = 1 \). We further have \( |f'(o)| \leq 1 \).

**Proof.** \( \frac{f(z)}{z} \) is regular for \( |z| < 1 \). Given \( o < r < 1 \) choose \( \rho \) such that \( r < \rho < \frac{1}{2} \); then since \( |f(z)| \leq 1 \) for \( |z| = \rho \), it follows by the maximum modulus principle that

\[
\left| \frac{f(z)}{z} \right| \leq \frac{1}{\rho},
\]

also for \( |z| = r \). Since the L.H.S is independent of \( \rho \to 1 \), we let \( \rho \) and obtain \( |f(z)| \leq |z| \), for \( |z| < 1 \).

If for \( z_0 \), \( (|z_0| < 1) \), we have \( |f(z_0)| = |z_0| \), then \( \left| \frac{f(z_0)}{z_0} \right| = 1 \), (by the maximum principle applied to \( \frac{f(z)}{z} \)) hence \( f = cz \), \( |c| = 1 \).

Since \( f(z) = f'(o)z + f''(o)z^2 + \cdots \) in a neighbourhood of the origin, and since \( \left| \frac{f(z)}{z} \right| \leq 1 \) in \( z \to 1 \), we get \( |f'(o)| \leq 1 \). More generally, we have \( \square \).
Theorem 1': Let $f$ be regular and $|f| \leq M$ in $|z| < R$, and $f(o) = 0$. Then

$$|f(z)| \leq \frac{M|z|}{R}, \quad |z| < R.$$  

In particular, this holds if $f$ is regular in $|z| < R$, and continuous in $|z| \leq R$, and $|f| \leq M$ on $|z| = R$.

Theorem 2 (Carathéodory's Inequality). \[12, p.112\] Suppose that

(i) $f$ is regular in $|z| < R$ $f$ is non-constant

(ii) $f(o) = 0$

(iii) Re $f \leq \cup$ in $|z| < R$. (Thus $\cup > 0$, since $f$ is not a constant).

Then,

$$|f| \leq \frac{2 \cup \rho}{R - \rho}, \text{ for } |z| = \rho < R.$$  

Proof. Consider the function

$$\omega(z) = \frac{f(z)}{f(z) - 2 \cup}.$$  

We have Re{$f - 2 \cup$} $\leq -\cup < 0$, so that $\omega$ is regular in $|z| < R$.

If $f = u + iv$, we get

$$|\omega| = \sqrt{\frac{u^2 + v^2}{(2 \cup -u)^2 + v^2}}$$  

so that

$$|\omega| \leq 1, \text{ since } 2 \cup -u \geq |u|.$$  

But $\omega(o) = o$, since $f(o) = o$. Hence by Theorem[1]

$$|\omega(z)| \leq \frac{|z|}{R}, \quad |z| < R.$$  

But $f = -\frac{2 \cup \omega}{1 - \omega}$. Now take $|z| = \rho < R$.

Then, we have,
3. Schwarz’s Lemma

\[ |f| \leq \frac{2 \cup |\omega|}{1 - |\omega|} \leq \frac{2 \cup \rho}{R - \rho} \]

N.B. If \( |f(z_o)| = \frac{2 \cup \rho}{R - \rho} \), for \( |z_o| < R \), then

\[ f(z) = \frac{2U \cdot c z}{1 - cz}, \quad |c| = 1. \]

More generally, we have \( \square \)

**Theorem 2′**: Let \( f \) be regular in \( |z| < R \); \( f \) non-constant

\[ f(o) = a_o = \beta + i \gamma \]

\( \text{Re} f \leq \cup (|z| < R) \), so that \( \cup \geq \beta \)

Then

\[ |f(z) - a_o| \leq \frac{2(\cup - \beta)\rho}{R - \rho} \]

for \( |z| = \rho < R \)

**Remark**. If \( f \) is a constant, then Theorem 2 is trivial. We shall now prove Borel’s inequality which is sharper than Theorem 2.

**Theorem 3** (Borel’s Inequality). [12, p.114] Let \( f(z) = \sum_{n=0}^\infty a_n z^n \), be regular in \( |z| < 1 \). Let \( \text{Re} f \leq \cup \), \( f(o) = a_o = \beta + i \gamma \).

Then

\[ |a_n| \leq 2(\cup - \beta) \equiv 2 \cup_1, \text{ say, } n > 0. \]

**Proof**. We shall first prove that \( |a_1| < \cup_1 \), and then for general \( a_n \).

(i) Let \( f_1 \equiv \sum_{n=1}^\infty a_n z^n \). Then \( \text{Re} f_1 \leq \cup_1 \).

For \( |z| = \rho < 1 \), we have, by Theorem 2

\[ |f_1| \leq \frac{2 \cup_1 \rho}{1 - \rho} \]

i.e. \( \left| \frac{f_1(z)}{z} \right| \leq \frac{2 \cup_1}{1 - \rho} \)
3. Schwarz’s Lemma

Now letting \( z \to 0 \), we get
\[
|a_1| \leq 2\mathcal{U}_1
\]

(ii) Define \( \omega = e^{2\pi i/k} \), \( k \) a +ve integer.

Then
\[
\sum_{v=1}^{k} \omega^v = o \begin{cases} 
& \text{if } m \neq 0 \\
& \text{& if } m \neq \text{ a multiple of } k \\
= k \begin{cases} 
& \text{if } m = 0 \\
& \text{or if } m = \text{ a multiple of } k.
\end{cases}
\end{cases}
\]

We have
\[
\frac{1}{k} \sum_{r=0}^{k-1} f_1(\omega^r z) = \sum_{n=1}^{\infty} a_{nk} z^n = g_1(z^k) \equiv g_1(\zeta), \text{ say.}
\]

The series for \( g_1 \) is convergent for \( |z| < 1 \), and so for \( |\zeta| < 1 \). Hence \( g_1 \) is regular for \( |\zeta| < 1 \). Since \( \text{Re} g_1 \leq \mathcal{U}_1 \), we have
\[
| \text{coefficient of } \zeta | \leq 2\mathcal{U}_1 \text{ by the first part of the proof i.e. } |a_4| \leq 2\mathcal{U}_1.
\]

**Corollary 1.** Let \( f \) be regular in \( |z| < R \), and \( \text{Re}\{f(z) - f(0)\} \leq \mathcal{U}_1 \). Then
\[
|f'(z)| \leq \frac{2 \mathcal{U}_1 R}{(R - \rho)^2}, \quad |z| = \rho < R
\]

**Proof.** Suppose \( R = 1 \). Then
\[
|f'(z)| \leq \sum_{n=1}^{\infty} n|a_n| \rho^{n-1} \leq 2\mathcal{U}_1 \sum_{n=1}^{\infty} n\rho^{n-1}
\]

This argument can be extended to the \( n^{\text{th}} \) derivative. \( \square \)

**Corollary 2.** If \( f \) is regular for every finite \( z \), and
\[
e^{f(z)} = O(e^{\mathcal{U}_1}), \text{ as } |z| \to \infty,
\]

then \( f(z) \) is a polynomial of degree \( \leq k \).
3. Schwarz’s Lemma

Proof. Let \( f(z) = \sum a_n z^n \). For large \( R \), and a fixed \( \zeta, |\zeta| < 1 \), we have

\[ \text{Re}\{f(R\zeta)\} < cR^k. \]

By Theorem 3, we have

\[ |a_n R^n| \leq 2(2R^k - \text{Re} a_0), \quad n > 0. \]

Letting \( R \to \infty \), we get

\[ a_n = 0 \text{ if } n > k. \]

Apropos Schwarz’s Lemma we give here a formula and an inequality which are useful.

□

Theorem 4. Let \( f \) be regular in \( |z - z_0| \leq r \)

\[ f = P(r, \theta) + Q(r, \theta). \]

Then

\[ f'(z_0) = \frac{1}{\pi r} \int_0^{2\pi} P(r, \theta)e^{-i\theta}d\theta. \]

Proof. By Cauchy’s formula,

\[ f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)dz}{(z-z_0)^2} = \frac{1}{2\pi r} \int_0^{2\pi} (P + iQ)e^{-i\theta}d\theta. \]

By Cauchy’s theorem,

\[ 0 = \frac{1}{r^2} \cdot \frac{1}{2\pi i} \int_{|z-z_0|=r} f(z)dz = \frac{1}{2\pi r} \int_0^{2\pi} (P + iQ)e^{i\theta}d\theta. \]

We may change \( i \) to \(-i\) in this relation, and add (i) and (ii).

Then

\[ f'(z_0) = \frac{1}{\pi r} \int_0^{2\pi} P(r, \theta)e^{-i\theta}d\theta. \]

□
Corollary. If $f$ is regular, and $|\text{Re } f| \leq M$ in $|z - z_0| \leq r$, then
\[ |f'(z_0)| \leq \frac{2M}{r}. \]

Aliter: We have obtained a series of results each of which depended on the preceding. We can reverse this procedure, and state one general result from which the rest follow as consequences.

Theorem 5. [11, p.50] Let $f(z) \equiv \sum_{n=0}^{\infty} c_n(z-z_0)^n$ be regular in $|z-z_0| < R$, and $\text{Re } f < \cup$. Then
\[ |e_n| \leq \frac{2(\cup - \text{Re } c_0)}{R^n}, \quad n = 1, 2, 3, \ldots \quad (5.1) \]
and in $|z - z_0| \leq \rho < R$, we have
\[ |f(z) - f(z_0)| \leq \frac{2\rho}{R - \rho} |\cup - \text{Re } f(z_0)| \quad (5.2) \]
\[ \left| \frac{f^{(n)}(z)}{n!} \right| \leq \frac{2R}{(R - \rho)^{n+1}} |\cup - \text{Re } f(z_0)| n = 1, 2, 3, \ldots \quad (5.3) \]

Proof. We may suppose $z_o = 0$.

Set $\phi(z) = \cup - f(z) = \cup - c_0 - \sum_{n=1}^{\infty} c_n z^n$
\[ \equiv \sum_{n=0}^{\infty} b_n z^n, \quad |z| < R. \]
Let $y$ denote the circle $|z| = \rho < R$. Then
\[ b_n = \frac{1}{2\pi i} \int_{\gamma} \frac{\phi(z)dz}{z^{n+1}} = \frac{1}{2\pi \rho^n} \int_{-\pi}^{\pi} (P + iQ)e^{-in\theta} d\theta, n \geq 0, \quad (5.4) \]
where $\phi(r, \theta) = P(r, \theta) + iQ(r, \theta)$. Now, if $n \geq 1$, then $\phi(z)z^{n-1}$ is regular is $\gamma$, so that
\[ 0 = \frac{\rho^n}{2r} \int_{-\pi}^{\pi} (P + iQ)e^{in\theta} d\theta, n \geq 1 \quad (5.5) \]
Changing $i$ to $-i$ in (5.5) and adding this to (5.4) we get
\[
\left( b_n \rho \right)^n = \frac{1}{\pi} \int_{-\pi}^{\pi} Pe^{-n\theta} d\theta, \quad n \geq 1.
\]
But $P = \mathbb{U} - \text{Re } f \geq 0$ in $|z| < R$ and so in $\gamma$. Hence, if $n \geq 1$,
\[
|b_n| \rho^n \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |Pe^{-n\theta}| d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} P d\theta = 2 \text{ Re } b_0, \quad (5.6)
\]
on using (5.4) with $n = 0$. Now letting $\rho \to R$, we get
\[
|b_n| R^n \leq 2 \beta_0 \equiv 2 \text{ Re } b_0.
\]
Since $b_0 = \mathbb{U} - c_0$, and $b_n = -c_n, n \geq 1$, we at once obtain (5.1). We then deduce, for $|z| \leq \rho < R$
\[
|\phi(z) - \phi(0)| = \left| \sum_{n=1}^{\infty} b_n z^n \right| \leq \sum_{n=1}^{\infty} 2 \beta_0 \left( \frac{\rho}{R} \right)^n = \frac{2 \beta_0 \rho}{R - \rho} \quad (5.7)
\]
and if $n \geq 1$,
\[
\left| \phi^{(n)}(z) \right| \leq \sum_{r=n}^{\infty} r(r-1) \ldots (r-n+1) \frac{2 \beta_0 \rho^{r-n}}{R^r} = \left( \begin{array}{c} d \cr d \rho \end{array} \right)^n \sum_{n=0}^{\infty} 2 \beta_0 \left( \frac{\rho}{R} \right)^n = \left( \begin{array}{c} d \cr d \rho \end{array} \right)^n \frac{2 \beta_0 R}{R - \rho} = 2 \beta_0 R \cdot n! (R - \rho)^{n+1} \quad (5.8)
\]
(5.7) and (5.8) yield the required results on substituting $\phi = \mathbb{U} - f$ and $\beta_0 = \mathbb{U} - \text{Re } f(0)$. \hfill \Box
Lecture 4

Entire Functions

An entire function is an analytic function (in the complex plane) which has no singularities except at $\infty$. A polynomial is a simple example. A polynomial $f(z)$ which has zeros at $z_1, \ldots, z_n$ can be factorized as

$$f(z) = f(0) \left(1 - \frac{z}{z_1}\right) \ldots \left(1 - \frac{z}{z_n}\right)$$

The analogy holds for entire functions in general. Before we prove this, we wish to observe that if $G(z)$ is an entire function with no zeros it can be written in the form $e^{g(z)}$ where $g$ is entire. For consider $\frac{G'(z)}{G(z)}$, every (finite) point is an ‘ordinary point’ for this function, and so it is entire and equals $g_1(z)$, say. Then we get

$$\log \left\{ \frac{G(z)}{G(z_0)} \right\} = \int_{z_0}^{z_1} g_1(\zeta) d\zeta = g(z) - g(z_0), \text{ say},$$

so that

$$G(z) = G(z_0) e^{g(z) - g(z_0)} = e^{g(z) - g(z_0) + \log G(z_0)}$$

As a corollary we see that if $G(z)$ is an entire function with $n$ zeros, distinct or not, then

$$G(z) = (z - z_1) \ldots (z - z_n) e^{g(z)}$$
where $g$ is entire. We wish to uphold this in the case when $G$ has an infinity of zeros.

**Theorem 1 (Weierstrass).** [15, p.246] Given a sequence of complex numbers $a_n$ such that

$$0 < |a_1| \leq |a_2| \leq \ldots \leq |a_n| \leq \ldots$$

whose sole limit point is $\infty$, there exists an entire function with zeros at these points and these points only.

**Proof.** Consider the function

$$\left(1 - \frac{z}{a_n}\right)e^{Q_\nu(z)},$$

where $Q_\nu(z)$ is a polynomial of degree $q$. This is an entire function which does not vanish except for $z = a_n$. Rewrite it as

$$\left(1 - \frac{z}{a_n}\right)e^{Q_\nu(z)} = e^{Q_\nu(z) + \log \left(1 - \frac{z}{a_n}\right)} = e^{-\frac{z}{a_n} + \frac{z}{2a_n^2} + \ldots + Q_\nu(z)},$$

and choose

$$Q_\nu(z) = \frac{z}{a_n} + \ldots + \frac{z^\nu}{yq^\nu}$$

so that

$$\left(1 - \frac{z}{a_n}\right)e^{Q_\nu(z)} = e^{-\left(\frac{z}{yq}\right)^{\nu+1}} \ldots \equiv 1 + \mathcal{O}(z), \text{ say.}$$

We wish to determine $\nu$ in such a way that

$$\prod_{1}^{\infty} \left(1 - \frac{z}{a_n}\right)e^{Q_\nu(z)}$$

is absolutely and uniformly convergent for $|z| < R$, however large $R$ may
Choose an $R > 1$, and an $\alpha$ such that $0 < \alpha < 1$; then there exists a $q$ (+ ve integer) such that

$$|a_q| \leq \frac{R}{\alpha}, \quad |a_{q+1}| > \frac{R}{\alpha}.$$ 

Then the partial product

$$\prod_{1}^{q} \left(1 - \frac{z}{a_n}\right)e^{Q(z)}$$

is trivially an entire function of $z$. Consider the remainder

$$\prod_{q+1}^{\infty} \left(1 - \frac{z}{a_n}\right)e^{Q(z)}$$

for $|z| \leq R$.

We have, for $n > q$, $|a_n| > \frac{R}{\alpha}$ or $\left|\frac{z}{a_n}\right| < \alpha < 1$. Using this fact we shall estimate each of the factors $\{1 + U_n(z)\}$ in the product, for $n > q$.

$$|U_n(z)| = \left|e^{\frac{1}{\alpha}(\frac{z}{a_n})^{\nu+1} - \ldots - 1}\right| \leq e^{\frac{1}{\alpha}(\frac{z}{a_n})^{\nu+1} - 1}$$

$\Box$

Since $|e^m - 1| \leq e^{|m|} - 1$, for $e^m - 1 = m + \frac{m^2}{2!} + \ldots$ or $|e^m - 1| \leq 29$

$$|m| + \frac{|m^2|}{2!} + \ldots = e^{|m|} - 1.$$ 

$$|U_n(z)| \leq e^{\frac{1}{\alpha}(\frac{z}{a_n})^{\nu+1}(1+\alpha+\alpha^2+\ldots) - 1},$$

$$\leq e^{\frac{1}{\alpha}(\frac{z}{a_n})^{\nu+1}(1+\alpha^2+\ldots) - 1}$$

$$= e^{\frac{1}{\alpha}(\frac{z}{a_n})^{\nu+1} - 1}$$
$\leq \frac{1}{1 - \alpha} \left| \frac{z}{a_n} \right|^{\nu + 1} \cdot e^{\frac{1}{1 - \alpha} \left| \frac{z}{a_n} \right|^{\nu + 1}}$, since if $x$ is
real, $e^x - 1 \leq xe^x$, for $e^x - 1 = x \left( 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots \right) \leq x(1 + x + \frac{x^2}{2!} + \cdots) = xe^x$

Hence

$|U_n(z)| \leq e^{\frac{1}{1 - \alpha} \left| \frac{z}{a_n} \right|^{\nu + 1}}$

Now arise two cases: (i) either there exists an integer $p > 0$ such that $\sum_{n=1}^{\infty} \frac{1}{|a_n|^p} < \infty$ or (ii) there does not. In case (i) take $\nu = p - 1$, so that

$|U_n(z)| \leq \frac{R^p}{|a_n|^p} \cdot e^{\frac{1}{1 - \alpha}}, \text{ since } |z| \leq R$;

and hence $\sum |U_n(z)| < \infty$ for $|z| \leq R$, i.e. $\prod_{q+1}^{\infty} (1 + U_n(z))$ is absolutely and uniformly convergent for $|z| \leq R$.

In case (ii) take $\nu = n - 1$, so that

$|U_n(z)| < e^{\frac{1}{1 - \alpha} \left| \frac{z}{a_n} \right|^{\nu + 1}}, \text{ if } n > q$

$\left| \frac{z}{a_n} \right| < 1$

$|z| \leq R$

$|a_n| \to \infty$

Then by the ‘root-test’ $\sum |U_n(z)| < \infty$, and the same result follows as before. Hence the product

$\prod_{n=1}^{\infty} \left( 1 - \frac{z}{a_n} \right) e^{Q(z)}$

is analytic in $|z| \leq R$; since $R$ is arbitrary and $\nu$ does not depend on $R$ we see that the above product represents an entire function.
4. Entire Functions

Remark. If in addition \( z = 0 \) is also a zero of \( G(z) \) then \( \frac{G(z)}{z^m G(z)} \) has no zeros and equals \( e^{g(z)} \), say, so that

\[
G(z) = e^{g(z)} \cdot z^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{Q_n(z)}
\]

In the above expression for \( G(z) \), the function \( g(z) \) is an arbitrary entire function. If \( G \) is subject to further restrictions it should be possible to say more about \( g(z) \); this we shall now proceed to do. The class of entire functions which we shall consider will be called “entire functions of finite order”.

Definition. An entire function \( f(z) \) is of finite order if there exists a constant \( \lambda \) such that \( |f(z)| < e^{\lambda r} \) for \( |z| = r > r_0 \). For a non-constant \( f \), of finite order, we have \( \lambda > 0 \). If the above inequality is true for a certain \( \lambda \), it is also true for \( \lambda' > \lambda \). Thus there are an infinity of \( \lambda \)'s satisfying this. The lower bound of such \( \lambda \)'s is called the “order of \( f \)”. Let us denote it by \( \rho \). Then, given \( \varepsilon > 0 \), there exists an \( r_0 \) such that

\[
|f(z)| < e^{\rho + \varepsilon}
\]

for \( |z| = r > r_0 \).

This would imply that

\[
M(r) = \max_{|z| = r} |f(z)| < e^{\rho + \varepsilon}, r > r_0,
\]

while

\[
M(r) > e^{\rho - \varepsilon}
\]

to \( +\infty \)

Taking logs. twice, we get

\[
\frac{\log \log M(r)}{\log r} < \rho + \varepsilon
\]

and

\[
\frac{\log \log M(r)}{\log r} > \rho - \varepsilon
\]

for an infinity of values of \( r \to \infty \).
Hence

\[ \rho = \lim_{r \to \infty} \frac{\log \log M(r)}{\log R} \]

**Theorem 2.** If \( f(z) \) is an entire function of order \( \rho < \infty \), and has an infinity of zeros, \( f(0) \neq 0 \), then given \( \varepsilon > 0 \), there exists an \( R_0 \) such that for \( R \geq R_0 \), we have

\[ n\left(\frac{R}{3}\right) \leq \frac{1}{\log 2} \cdot \log \frac{e^{\rho \varepsilon}}{|f(0)|} \]

*Here \( n(R) \) denotes the number of zeros of \( f(z) \) in \( |z| \leq R \).*

**Proof.** We first observe that if \( f(z) \) is analytic in \( |z| \leq R \), and \( a_1 \ldots a_n \) are the zeros of \( f \) inside \( |z| < R/3 \), then for

\[ g(z) = \frac{f(z)}{(1 - \frac{z}{a_1}) \cdots (1 - \frac{z}{a_n})} \]

we get the inequality

\[ |g(z)| \leq \frac{M}{2^n} \]

where \( M \) is defined by: \( |f(z)| \leq M \) for \( |z| = R \). For, if \( |z| = R \), then since \( |a_p| \leq \frac{R}{3} \), \( p = 1 \ldots n \), we get \( \left| \frac{z}{a_p} \right| \geq 3 \) or \( \left| 1 - \frac{z}{a_p} \right| \geq 2 \).

By the maximum modules theorem,

\[ |g(0)| \leq \frac{M}{2^n} \], i.e. \( |f(0)| \leq \frac{M}{2^n} \)

or

\[ n \equiv n\left(\frac{R}{3}\right) \leq \frac{1}{\log 2} \cdot \log \left(\frac{M}{|f(0)|}\right) \]

If, further, \( f \) is of order \( \rho \), then for \( r > r_0 \), we have \( M < e^{\rho \varepsilon} \) which gives the required results.

N.B. The result is trivial if the number of zeros is finite. \( \square \)
Corollary.  
\[ n(R) = O(R^{\rho + \epsilon}). \]

For  
\[ n \left( \frac{R}{3} \right) \leq \frac{1}{\log 2} \cdot [R^{\rho + \epsilon} - \log |f(0)|], \]

and  
\[ -\log |f(0)| < R^{\rho + \epsilon}, \text{ say.} \]

Then for \( R \geq R' \)
\[ n \left( \frac{R}{3} \right) < \frac{2}{\log 2} R^{\rho + \epsilon} \]

and hence the result.

**Theorem 3.** If \( f(z) \) is of order \( \rho < \infty \), has an infinity of zeros \( a_1, a_2, \ldots, f(0) \neq 0, \sigma > \rho, \) then  
\[ \sum \frac{1}{|a_n|^\sigma} < \infty \]

Proof.  Arrange the zeros in a sequence:  
\[ |a_1| \leq |a_2| \leq \ldots \]

Let  
\[ \alpha_p = |a_p| \]

Then in the circle \( |z| \leq r = \alpha_n \), there are exactly \( n \) zeros. Hence  
\[ n < c \alpha_n^{\rho + \epsilon}, \text{ for } n \geq N \]

or  
\[ \frac{1}{n} > \frac{1}{c} \cdot \frac{1}{\alpha_n^{\rho + \epsilon}} \]

Let \( \sigma > \rho + \epsilon \) (i.e. choose \( 0 < \epsilon < \sigma - \rho \)). Then  
\[ \frac{1}{n^{\sigma/\rho + \epsilon}} > \frac{1}{c^{\sigma/\rho + \epsilon}} \cdot \frac{1}{\alpha_n^{\rho + \epsilon}} \]

or  
\[ \frac{1}{\alpha_n^{\sigma}} < c^{\sigma/\rho + \epsilon} \cdot \frac{1}{n^{\sigma/\rho + \epsilon}} \text{ for } n \geq N \]

Hence  \[ \sum \frac{1}{\alpha_n^{\sigma}} < \infty. \]  \( \square \)
Remark.  (i) There cannot be too dense a distribution of zeros, since 
\[ n < c < \alpha^\rho \varepsilon \]. Nor can their moduli increase too slowly, since, for 
instance, \( \sum \frac{1}{(\log n)^\rho} \) does not converge.

(ii) The result is of course trivial if there are only a finite number of 
zeros.

Definition. The lower bound of the numbers \( \sigma \) for which 
\( \sum \frac{1}{|a_n|^\sigma} < \infty \), is called the exponent of convergence of \( \{a_n\} \). We shall denote it by \( \rho_1 \).

Then 
\[ \sum \frac{1}{|a_n|^\rho + \epsilon} < \infty, \quad \sum \frac{1}{|a_n|^\rho - \epsilon} = \infty, \quad \epsilon > 0 \]

By Theorem 1 we have \( \rho_1 \leq \rho \)

N.B. If the \( a'_n \)s are finite in number or nil, then \( \rho_1 = 0 \). Thus \( \rho_1 > 0 \)
implies that \( f \) has an infinity of zeros. Let \( f(z) \) be entire, of order \( \rho < \infty \); 
\( f(a) \neq 0 \), and \( f(z_n) = 0 \), \( n = 1, 2, 3, \ldots \). Then there exists an integer 
\( (p + 1) \) such that \( \sum \frac{1}{|z_n|^{p+1}} < \infty \)

(By Theorem 1 any integer \( > \rho \) will serve for \( \rho + 1 \)). Thus, by 
Weierstrass’s theorem, we have

\[ f(z) = e^{Q(z)} \prod_{1}^{\infty} \left( 1 - \frac{z}{z_n} \right) e^{\frac{z}{z_n} + \frac{z^2}{2z_n^2} + \cdots + \frac{z^\nu}{\nu!z_n^\nu}}, \]

where \( \nu = p \) (cf. proof of Weierstrass’s theorem).

Definition. The smallest integer \( p \) for which 
\( \sum \frac{1}{|z_n|^p} < \infty \) is the ‘genus’ 
of the ‘canonical product’ 
\[ \Pi \left( 1 - \frac{z}{z_n} \right) e^{\frac{z}{z_n} + \frac{z^2}{2z_n^2} + \cdots + \frac{z^\nu}{\nu!z_n^\nu}}, \]

with zeros \( z_n \). If \( a'_n \)s are finite in number, we (including nil) define \( p = 0 \), and we define the 
product as 
\[ \Pi \left( 1 - \frac{z}{z_n} \right) \]

Examples. (i) \( z_n = n, \ p = 1 \)  (ii) \( z_n = e^n, \ p = 0 \)
(iii) \( z_1 = \frac{1}{2} \log 2, \ z_n = \log n, n \geq 2, \) no finite \( p \).
Remark. If $\sigma > \rho_1$, then $\sum \frac{1}{|z_n|^\sigma} < \infty$

Also,

$$\sum \frac{1}{|z_n|^{p+1}} < \infty$$

But

$$\sum \frac{1}{|z_n|^p} = \infty$$

Thus, if $\rho_1$ is not an integer, $p = [\rho_1]$, and if $\rho_1$ is an integer, then either

$$\sum \frac{1}{|z_n|^{\rho_1}} < \infty \text{ or } \sum \frac{1}{|z_n|^{\rho_1}} = \infty; \text{ in the first case, } p + 1 = \rho_1, \text{ which in the second case } p = \rho_1. \text{ Hence,}$$

$$\phi \leq \rho_1$$

But $\rho_1 \leq \rho$. Thus

$$p \leq \rho_1 \leq \rho$$

Also

$$p \leq \rho_1 \leq p + 1, \text{ since } \sum \frac{1}{|z_n|^{p+1}} < \infty.$$
Lecture 5

Entire Functions (Contd.)

**Theorem 1** (Hadamard). Let $f$ be an entire function with zeros $|z_1| \leq |z_2| \leq \ldots \to \infty \{z_n\}$ such that $f(o) \neq 0$. Let $f$ be of order $\rho < \infty$. Let $G(z)$ be the canonical product with the given zeros.

Then $f(z) = e^{Q(z)}G(z)$, where $Q$ is a polynomial of degree $\leq \rho$.

We shall give a proof without using Weierstrass’s theorem. The proof will depend on integrating the function \( \frac{f'(z)}{f(z)} \cdot \frac{1}{z^{\mu}(z-x)} \) over a sequence of expending circles.

Given $R > 0$, let $N$ be defined by the property:

\[ |z_N| \leq R, \quad |z_{N+1}| > R. \]

We need the following

**Lemma.** Let $0 < \varepsilon < 1$. Then

\[ \left| \frac{f'(z)}{f(z)} - \sum_{n+1}^{N} \frac{1}{z - z_n} \right| < cR^{\rho-1+\varepsilon} \]

for $|z| = \frac{R}{2} > R_0$, and $|z| = \frac{R}{2}$ is free from any $z_n$. 

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For we can choose an $R_0$ such that for $|z| = R > R_0$,

(i) $|f(z)| < e^{R\rho + \epsilon}$,

(ii) no $z_n$ lies on $|z| = \frac{R}{2}$, and

(iii) $\log \frac{1}{|f(0)|} < (2R)^{\rho + \epsilon}$

For such an $R > R_0$, define

$$g_R(z) = \frac{f(z)}{f(0)} \prod_{n=1}^{N} \left(1 - \frac{z}{z_n}\right)^{-1}$$

Then for $|z| = 2R$, we have

$$|g_R(z)| < \frac{1}{|f(0)|} e^{(2R)^{\rho + \epsilon}}$$

since $|f| < e^{(2R)^{\rho + \epsilon}}$, and $|1 - \frac{z}{z_n}| \geq 1$ for $1 \leq n \leq N$.

Hence, by the maximum modulus principle,

$$|g_R(z)| < \frac{1}{f(0)} e^{(2R)^{\rho + \epsilon}} \text{ for } |z| = R.$$ 

That is

$$\log |g_R(z)| < 2^{\rho+2} R^{\rho+\epsilon}. \quad (0 < \epsilon < 1)$$

If $h_R(z) \equiv \log g_R(z)$, and the log vanishes for $z = 0$, then, since $g_R(z)$ is analytic and $g_R(z) \neq 0$ for $|z| \leq R$, we see that $h_R(z)$ is analytic for $|z| \leq R$, and $h_R(0) = 0$. Further, for $|z| = R$,

$$\Re h_R(z) = \log |g_R(z)| < 2^{\rho+2} R^{\rho+\epsilon}$$

Hence, by the Borel-Caratheodory inequality, we have for $|z| = R/2$,

$$|h_R'(z)| < cR^{\rho-1+\epsilon},$$

which gives the required lemma.
Proof of theorem. Let \( \mu = [\rho] \) Then consider

\[
I \equiv \int \frac{f'(z)}{f(z)} \cdot \frac{1}{z^\mu (z-x)} \, dz
\]

taken along \( |z| = R/2 \), where \( |x| < \frac{R}{4} \). The only poles of the integrand are: 0, \( x \) and those \( z'_n \)s which lie inside \( |z| = R/2 \). Calculating the residues, we get

\[
\frac{1}{2\pi i} \int_{|z|=R/2} f'(z) \frac{1}{f(z) z^\mu (z-x)} \, dz = \sum_{|z_n|<R/2} \frac{1}{z_n^\mu (z_n-x)} + \frac{f'(x)}{f(x)} \cdot \frac{1}{x^\mu} + \frac{1}{(\mu-1)!} (D^{\mu-1})_{z=0} \left( \frac{f'(z)}{f(z)} - \frac{1}{z-x} \right)
\]

We shall prove that the l.h.s tends to \( o \) as \( R \to \infty \) in such a way that \( |z| = R/2 \) is free from any \( z_n \). Rewriting \( I \) as

\[
I = \int \left( \frac{f'}{f} - \sum_{n=1}^N \frac{1}{z-z_n} \right) \frac{dz}{z^\mu (z-x)} + \sum_{n=1}^N \int \frac{1}{z-z_n} \cdot \frac{dz}{z^\mu (z-x)}
\]

\[
\equiv I_1 + I_2,
\]

we get for \( |z| = R/2 \) (by the Lemma),

\[
|I_1| = O(R^{\mu-1+\epsilon}) = O(1).
\]

Let \( k \) denote the number of \( z'_n \)s lying inside \( |z| = R/2 \)

Then

\[
I_2 = \sum_{n=1}^k \int_{|z|=R/2} \ldots + \sum_{n=k+1}^N \int \ldots 
\]

\[
\equiv I_{2,1} + I_{2,2}, \text{ say.}
\]
The value of \( I_{2,1} \) remains unaltered when we integrate along \(|z| = 3R/4\), and along the new path, \(|z - z_n| \geq R/4\) since \( n \leq k \). Since \( N = O(R^{\rho + \epsilon}) \) we get
\[
I_{2,1} = O(R^{\rho - \mu - 1 + \epsilon}) = O(1)
\]
Similarly integrating \( I_{2,2} \) over \(|z| = R/4\), we get
\[
I_{2,2} = O(1)
\]
Thus \( I = o(1) \). Hence
\[
\frac{f''(x)}{f(x)} \frac{1}{x^\mu} + \sum_{n=1}^{\infty} \frac{1}{z_n^\mu(z-x)} \left[ D_0^{\mu-1} \frac{f''}{f} \frac{1}{z-x} \right] = 0
\]
i.e.
\[
\frac{f''(x)}{f(x)} + \sum_{n=0}^{\mu-1} b_n x^n \sum_{n=1}^{\infty} \left( \frac{x^{-1}}{z_n^\mu(z-x)} \right) = 0, \quad b_n = a_n, a_n = -\frac{\left( \frac{\mu}{1-x} \right)^{(\mu-n)}}{(\mu-n)!}
\]
i.e.
\[
\frac{f''(x)}{f(x)} = \sum_{n=0}^{\mu-1} c_n x^n + \sum_{n=1}^{\infty} \left( \frac{x^{\mu-1}}{c_n^\mu} + \frac{x^{\mu-2}}{c_n^{\mu-1}} + \cdots + \frac{1}{x-z_n} \right)
\]
Integrating and raising to the power of \( e \) we get the result:
\[
f(x) = e^{\sum_{n=0}^{\mu} d_n x^n} \prod_{n=1}^{\infty} \left( 1 - \frac{x}{z_n} \right)^{e^{x/z_n^\mu} + x^\mu/c_n^\mu}, \quad d_n = \frac{c_n - 1}{n}
\]
If \( f \) is an entire functions of order \( \rho \), which is not a constant, and such that \( f(o) \neq 0 \), and \( f(z_n) = 0 \), \( n = 1, 2, 3, \ldots |z_n| \leq |z_{n+1}| \), then
\[
f(z) = e^{Q_1(z)} G_1(z),
\]
where \( Q_1(z) \) is a polynomial of degree \( q \leq \rho \) and
\[
G_1(z) = \prod_{n=1}^{\infty} \left( 1 - \frac{z}{z_n} \right)^{e^{z/z_n^\mu} + z/(z_n)^\mu},
\]
where \( p + 1 \) is the smallest integer for which \( \sum_{|z_n|^{p+1}} < \infty \), this integer existing, since \( f \) is of finite order.
Remark. This follows at once from Weierstrass’s theorem but can also be deduced from the infinite-product representation derived in the proof of Hadamard’s theorem (without the use of Weierstrass’s theorem). One has only to notice that
\[ \prod e^{-\frac{1}{p+1}(\frac{z}{p})} \]
is convergent, where \( p \) is the genus of the entire function in question, so that \( f(z) = e^{Q(z)}G(z) \) may be multiplied by this product yielding \( f(z) = e^{Q(z)}G_1(z) \), where \( Q_1 \) is again of degree \( \leq \mu \).

We recall the convention that if the \( z_n \)s are finite in number (or if there are no \( z_n \)s), then \( p = 0 \) and \( \rho_1 = 0 \).

We have also seen that
\[ q \leq \rho, \rho_1 \leq \rho \text{ i.e. } \max(q, \rho_1) \leq \rho \]

We shall now prove the following

**Theorem 2.** \( \rho \leq \max(q, \rho_1) \)

Let
\[ E(u) \equiv (1 - u)e^{\frac{u^2}{2} + \frac{u^p}{p} + \cdots} \]

If \(|u| \leq \frac{1}{2} \), then
\[ |E(u)| = \left| e^{\log(1-u) + u + \cdots + \frac{u^p}{p}} \right| = \left| e^{\frac{u^2+1}{2} - \frac{u^{p+2}+\cdots}{p+2}} \right| \]
\[ \leq e^{2|u|+1} \leq e^{12|u|+1} \leq e^{12|u|^\tau}, \text{ if } \tau \leq p + 1 \]

If \(|u| > \frac{1}{2} \), then
\[ |E(u)| \leq (1 + |u|)e^{12|u|+\cdots+\frac{|u|^p}{p}} \]
\[ \leq (1 + |u|)e^{12|u|(|u|^{p+\cdots+1})} \]
\[ \leq (1 + |u|)e^{12|u|(2^{p+\cdots+1})} \]
\[< e^{2|u|^{\tau} \log(1 + |u|)} \]
\[\leq e^{2|u|^{\tau} + \log(1 + |u|)}, \text{ if } p \leq \tau, \text{ since } |2u| > 1.\]

Hence, for all \(u\), we have
\[|E(u)| \leq e^{2|u|^{\tau} + \log(1 + |u|)},\]
if \(p \leq \tau \leq p + 1\).

It is possible to choose \(\tau\) in such a way that

(i) \(p \leq \tau \leq p + 1\),

(ii) \(\tau > 0\),

(iii) \(\sum \frac{1}{|z_n|^\tau} < \infty\), (if there are an infinity of \(z_n\)'s) and

(iv) \(p_1 \leq \tau \leq p_1 + \varepsilon\)

For, if \(\sum \frac{1}{|z_n|^{p_1}} < \infty\) (if the series is infinite this would imply that \(p_1 > 0\)), we have only to take \(\tau = p_1 > 0\). And in all other cases we must have \(p \leq p_1 < p + 1\). [For, if \(p_1 = p + 1\), then \(\sum \frac{1}{|z_n|^{p_1}} < \infty\) and so we choose \(\tau\) such that \(p_1 < \tau < p + 1\); this implies again that \(\sum \frac{1}{|z_n|^\tau} < \infty\) and \(\tau > 0\). For such a \(\tau\) we can write the above inequality as]
\[|E(u)| \leq e^{c_1 |u|^{\tau}}, \quad c_1 \text{ is a constant.}\]

Hence
\[|f(z)| \leq e^{Q(z) + c_1} \cdot \sum_{n=1}^{\infty} \frac{|z|^\tau}{|z_n|^\tau},\]
\[\text{i.e. } M(r) < e^{c_2 r^\delta + c_3 r^\tau}, \mid z \mid = r > 1, \text{ and } c_1 \sum |1/z_n|^\tau = c_3.\]

Hence
\[\log M(r) = O(r^\delta) + O(r^\tau), \text{ as } r \to \infty,\]
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\[ = O(r^{\max(q, \tau)}), \text{ as } r \to \infty \quad (2.1) \]

so that

\[ \rho \leq \max(q, \tau) \]

Since \( \tau \) is arbitrarily near \( \rho_1 \), we get

\[ \rho \leq \max(q, \rho_1), \]

and this proves the theorem.

**Theorem 3.** If the order of the canonical product \( G(z) \) is \( \rho' \), then \( \rho' = \rho_1 \)

**Proof.** As in the proof of Theorem 2, we have

\[ |E(u)| \leq e^{c_1|u|^p} \left\{ \begin{array}{l} p \leq \tau \leq p + 1; \\
\tau > 0 \\
\sum_{|z_n|^p} \frac{1}{|z_n|^p} < \infty \\
\rho_1 \leq \tau \leq \rho + \varepsilon, \quad \varepsilon > 0 \end{array} \right. \]

Hence

\[ c_1 \sum_{|z_n|^p} \left| \frac{z}{z_n} \right|^p \]

\[ |G(z)| \leq e \leq e^{C_3 r}, \quad |z| = r. \]

If \( M(r) = \max_{|z| = r} |G(z)| \), then

\[ M(r) < e^{C_3 r} \]

Hence

\[ \rho' \leq \tau \]

which implies \( \rho' \leq \rho_1 \); but \( \rho_1 \leq \rho' \). Hence \( \rho' = \rho_1 \)

\[ \square \]

**Theorem 4.** If, either
(i) \( \rho_1 < q \) or

(ii) \( \sum \frac{1}{|z_n|^\rho_1} < \infty \) (the series being infinite),

then

\[
\log M(r) = O(r^\beta), \text{ for } \beta = \max(\rho_1, q).
\]

Proof. (i) If \( \rho_1 < q \), then we take \( \tau < q \) and from (2.1) we get

\[
\log M(r) = O(r^\tau) = O(r^{\max(\rho_1, q)}).
\]

(ii) If \( \sum_{n=1}^{\infty} \frac{1}{|z_n|^\rho_1} < \infty \), then we may take \( \tau = \rho_1 > 0 \) and from (2.1) we get

\[
\log M(r) = O(r^\tau) + O(r^{\rho_1}) = O(r^\rho)
\]

\[\square\]

**Theorem 5.** (i) We have

\[
\rho = \max(\rho_1, q)
\]

(The existence of either side implies the other)

(ii) If \( \rho > 0 \), and if \( \log M(r) = O(r^\rho) \) does not hold, then \( \rho_1 = \rho \), \( f(z) \) has an infinity of zeros \( z_n \), and \( \sum |z_n|^{-\tau} \) is divergent.

Proof. The first part is merely an affirmation of Hadamard’s theorem, \( \rho_1 \leq \rho \) and Theorem\[2\]

For the second part, we must have \( \rho_1 = \rho \); for if \( \rho_1 < \rho \), then \( \rho = q \) by the first part, so that \( \rho_1 < q \), which implies, by Theorem\[3\] that \( \log M(r) = O(r^\rho) = O(r^\rho) \) in contradiction with our hypothesis. Since \( \rho > 0 \), and \( \rho_1 = \rho \) we have therefore \( \rho_1 > 0 \), which implies that \( f(z) \) has an infinity of zeros. Finally \( \sum_{n=1}^{\infty} |z_n|^{\rho_1} \) is divergent, for if \( \sum |z_n|^{\rho_1} < \infty \), then by Theorem\[4\] we would have \( \log M(r) = O(r^\rho) \) which contradicts the hypothesis. \[\square\]
Remarks. (1) $\rho_1$ is called the ‘real order’ of $f(z)$, and $\rho$ the ‘apparent order’ $\max(q, p)$ is called the ‘genus’ of $f(z)$.

(2) If $\rho$ is not an integer, then $\rho = \rho_r$, for $q$ is an integer and $\rho = \max(\rho_1, q)$. In particular, if $\rho$ is non-integral, then $f$ must have an infinity of zeros.
Lecture 6

The Gamma Function

1 Elementary Properties [15, p.148]

Define the function $\Gamma$ by the relation.

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \ x \text{ real.}$$

(i) This integral converges at the ‘upper limit’ $\infty$, for all $x$, since $t^{x-1}e^{-t} = t^{-2}t^{x+1}e^{-t} = O(t^{-2})$ as $t \to \infty$; it converges at the lower limit 0 for $x > 0$.

(ii) The integral converges uniformly for $0 < a \leq x \leq b$. For

$$\int_0^\infty t^{x-1}e^{-t}dt = \int_0^1 + \int_1^\infty = O\left(\int_0^1 t^{x-1}dt\right) + O\left(\int_1^\infty t^{b-1}e^{-t}dt\right)$$

$$= O(1), \ \text{independently of } x.$$

Hence the integral represents a continuous function for $x > 0$.

(iii) If $z$ is complex, $\int_0^\infty t^{z-1}e^{-t}dt$ is again uniformly convergent over any finite region in which $\text{Re } z \geq a > 0$. For if $z = x + iy$, then

$$|t^{z-1}| = t^{x-1},$$

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and we use (ii). Hence \( \Gamma(z) \) is an *analytic function for* \( \Re z > 0 \)

(iv) If \( x > 1 \), we integrate by parts and get

\[
\Gamma(x) = \left[ -t^{x-1}e^{-t} \right]_0^{\infty} + (x - 1) \int_0^{\infty} t^{x-2}e^{-t} dt
\]

\[
= (x - 1)\Gamma(x - 1).
\]

However,

\[
\Gamma(1) = \int_0^{\infty} e^{-t} dt = 1; \text{ hence }
\]

\[
\Gamma(n) = (n - 1)!,
\]

if \( n \) is a +ve integer. Thus \( \Gamma(x) \) is a generalization of \( n! \)

(v) We have

\[
\log \Gamma(n) = (n - \frac{1}{2}) \log n - n + c + O(1),
\]

where \( c \) is a constant.

\[
\log \Gamma(n) = \log(n - 1)! = \sum_{r=1}^{n-1} \log r.
\]

We estimate \( \log r \) by an integral: we have

\[
\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \int_{-\frac{1}{2}}^{1/2} \log t \ dt \ ds = \int_{1/2}^{1} \log(s + r) ds = \int_{0}^{1} + \int_{-1/2}^{0} \frac{1}{s + r} \ dt
\]

\[
= \int_0^{1/2} \log(t + r) + \log(r - t) \ dt
\]
1. Elementary Properties

\[
\log r + c_r, \quad \text{where } c_r = O \left( \frac{1}{r^2} \right).
\]

Hence

\[
\log \Gamma(n) = \sum_{r=1}^{n-1} \left( \int_{r^{-1/2}}^{r^{1/2}} \log t \, dt - c_r \right)
\]

\[
= \int_{1/2}^{n-1/2} \log t \, dt - \sum_{r=1}^{n-1} c_r
\]

\[
= \left( \left( n - \frac{1}{2} \right) \log \left( n - \frac{1}{2} \right) - \frac{1}{2} \log \frac{1}{2} \right) - (n - \frac{1}{2}) + \frac{1}{2}
\]

\[
- \sum_{r=1}^{\infty} c_r + \sum_{n} C_r
\]

\[
= (n - \frac{1}{2}) \log n - n + c + o(1), \quad \text{where } c \text{ is a constant.}
\]

Hence, also

\[
(n!) = a \cdot e^{-n} n^{n+1/2} e^{o(1)},
\]

where ‘a’ is a constant such that \( c = \log a \)

(vi) We have

\[
\frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} = \int_{0}^{\infty} \frac{t^{x-1}}{(1 + t)^{x+y}} \, dt, \quad x > 0, \ y > 0.
\]

For

\[
\Gamma(x)\Gamma(y) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt \cdot \int_{0}^{\infty} s^{y-1} e^{-s} \, ds, \quad x > a, \ y > 0
\]
Put \( s = tv \); then

\[
\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1} e^{-t} \, dt \int_0^\infty v^{y-1} e^{-tv} \, dv
\]

\[
= \int_0^\infty v^{y-1} \, dv \int_0^\infty t^{x+y-1} e^{-t(1+v)} \, dt
\]

\[
= \int_0^\infty v^{y-1} \, dv \int_0^\infty u^{x+y-1} e^{-u} \frac{1}{(1+v)^{x+y}} \, du
\]

\[
= \Gamma(x+y) \int_0^\infty v^{y-1} (1+v)^{-x-y} \, dv.
\]

The inversion of the repeated integral is justified by the use of the fact that the individual integrals converge uniformly for \( x \geq \epsilon > 0 \), \( y \geq \epsilon > 0 \).

(vii) Putting \( x = y = \frac{1}{2} \) in (vii), we get [using the substitution \( t = \tan^2 \theta \)]

\[
(\Gamma(\frac{1}{2}))^2 = 2\Gamma(1) \int_0^{\pi/2} d\theta = \pi.
\]

Since \( \Gamma(\frac{1}{2}) > 0 \), we get \( \Gamma(\frac{1}{2}) = \sqrt{\pi} \).

Putting \( y = 1-x \), on the other hand, we get the important relation

\[
\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{u^{x-1}}{1+u} \, du = \frac{\pi}{\sin(1-x)\pi}, \quad 0 < x < 1,
\]

since

\[
\int_0^\infty \frac{x^{a-1}}{1+x} \, dx = \frac{\pi}{\sin a\pi} \quad \text{for} \quad 0 < a < 1, \quad \text{by contour}
\]
integration. Hence
\[ \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x\pi}, \quad 0 < x < 1. \]

2 Analytic continuation of \( \Gamma(z) \) [15, p. 148]

We have seen that the function
\[ \Gamma(z) = \int_0^\infty e^{-t}t^{z-1} \, dt \]
is a regular function for \( \text{Re} \, z > 0 \). We now seek to extend it analytically into the rest of the complex \( z \)-plane. Consider
\[ I(z) \equiv \int_C e^{-\zeta}(-\zeta)^{z-1}d\zeta, \]
where \( C \) consists of the real axis from \( \infty \) to \( \delta > 0 \), the circle \( |\zeta| = \delta \) in the ‘positive’ direction \( \varpi \) and again the real axis from \( \delta \) to \( \infty \).

We define
\[ (-\zeta)^{z-1} = e^{(z-1)\log(-\zeta)} \]
by choosing \( \log(-\zeta) \) to be real when \( \zeta = -\delta \).

The integral \( I(z) \) is uniformly convergent in any finite region of the \( Z \)-plane, for the only possible difficulty is at \( \zeta = \infty \), but this was covered by case (iii) in \( \S7 \). Thus \( I(z) \) is regular for all finite values of \( z \).

We shall evaluate \( I(z) \). For this, set
\[ \zeta = \rho e^{i\varphi} \]
so that
\[ \log(-\zeta) = \log \rho + i(\varphi - \pi) \]
on the contour [so as to conform with the requirement that \( \log(-\zeta) \) is real when \( \zeta = -\delta \)].

Now the integrals on the portion of \( C \) corresponding to \((\infty, \delta)\) and \((\delta, \infty)\) give [since \( \varphi = 0 \) in the first case, and \( \varphi = 2\pi \) in the second]:
\[ \int_0^\delta e^{-\rho(\zeta-1)(\log \rho + i\pi)} \, d\rho + \int_\delta^\infty e^{-\rho(\zeta-1)(\log \rho + i\pi)} \, d\rho \]
\[
= \int_\delta^\infty \left[ -e^{-\rho+(z-1)(\log \rho-i\pi)} + e^{-\rho+(z-1)(\log \rho+i\pi)} \right] d\rho \\
= \int_\delta^\infty e^{-\rho+(z-1)\log \rho} \left[ e^{(z-1)i\pi} - e^{-(z-1)i\pi} \right] d\rho \\
= -2i \sin \pi \int_\delta^\infty e^{-\rho} \cdot \rho^{z-1} d\rho
\]

On the circle \(|\zeta| = \delta\), we have
\[
|(-\zeta)^{-z-1}| = |e^{(z-1)\log(-\zeta)}| = |e^{(z-1)(\log \delta + i(\phi-\pi))}| \\
= e^{(z-1)\log \delta + i\phi - \pi} \\
= O(\delta^{x-1}).
\]

The integral round the circle \(|\zeta| = \delta\) therefore gives
\[
O(\delta^{x}) = O(1), \text{ as } \delta \to 0, \text{ if } x > 0.
\]

Hence, letting \(\delta \to 0\), we get
\[
I(z) = -2i \sin \pi z \int_0^\infty e^{-\rho} \rho^{z-1} d\rho, \text{ Re } z > 0 \\
= -2i \sin \pi z \Gamma(z), \text{ Re } z > 0.
\]

We have already noted that \(I(z)\) is regular for all finite \(z\); so
\[
\frac{1}{2} iI(z) \cosec \pi z
\]
is regular for all finite \(z\), except (possibly) for the poles of \(\cosec \pi z\) namely, \(z = 0, \pm 1, \pm 2, \ldots\); and it equals \(\Gamma(z)\) for \(\text{Re } z > 0\).

**Hence** - \(\frac{1}{2} iI(z) \cosec \pi z\) is the analytic continuation of \(\Gamma(z)\) all over the \(z\)-plane.
We know, however, by (iii) of §1, that $\Gamma(z)$ is regular for $z = 1, 2, 3, \ldots$. Hence the only possible poles of $\frac{1}{2}iI(z)\csc \pi x$ are $z = 0, -1, -2, \ldots$. These are actually poles of $\Gamma(z)$, for if $z$ is one of these numbers, say $-n$, then $(-e^{-\zeta})^{-1}$ is single-valued in 0 and $I(z)$ can be evaluated directly by Cauchy’s theorem. Actually
\[
(-1)^{n+1} \int_C \frac{e^{-\zeta}}{\zeta^{n+1}} d\zeta = \frac{2\pi i}{n!} (-1)^{n+1} = \frac{-2\pi i}{n!}
\]
i.e. $I(-n) = -\frac{2\pi i}{n!}$.

So the poles of $\csc \pi z$, at $z = 0, -n$, are actually poles of $\Gamma(z)$. The residue at $z = -n$ is therefore
\[
\lim_{z \to -n} \left( -\frac{2\pi i}{n!} \right) \left( \frac{z + n}{-2i \sin \pi z} \right) = \frac{(-1)^n}{n!}
\]
The formula in (viii) of §1 can therefore be upheld for complex values. Thus
\[
\Gamma(z) \cdot \Gamma(1 - z) = \pi \csc \pi z
\]
for all non-integral values of $z$. Hence, also,
\[
\frac{1}{\Gamma(z)}
\]
is an entire function.

(Since the poles of $\Gamma(1 - z)$ are cancelled by the zeros of $\sin \pi z$)

3 The Product Formula

We shall prove that $\frac{1}{\Gamma(z)}$ is an entire function of order 1.

Since $I(z) = -2i \sin \pi z \Gamma(z)$, and
\[
\Gamma(z) \cdot \Gamma(1 - z) = \frac{\pi}{\sin \pi z},
\]
we get
\[
I(1 - z) = -2i \sin(\pi - z\pi)\Gamma(1 - z)
\]
6. The Gamma Function

\[ = -2i \sin \pi \frac{\pi}{\sin \pi z} \cdot \frac{1}{\Gamma(z)} \quad = \frac{2i\pi}{\Gamma(z)} \]

or

\[ \frac{1}{\Gamma(z)} = \frac{I(1-z)}{-2i\pi}. \]

Now

\[ I(z) = O \left\lfloor e^{\pi |z|} \left( \int_{0}^{1} e^{-t|z|} dt + \int_{1}^{\infty} e^{-t|z|} dt \right) \right\rfloor \]

\[ = O \left\lfloor e^{\pi |z|} \left( 1 + \int_{1}^{\infty} e^{-t|z|} dt \right) \right\rfloor \]

\[ = O \left\lfloor e^{\pi |z|} \left( 1 + \int_{1}^{1+|z|} e^{-t|z|} dt \right) \right\rfloor \]

\[ = O \left\lfloor e^{\pi |z|} (1 + |z| + 1) \right\rfloor \]

\[ = O \left\lfloor e^{\pi |z| + \epsilon} \right\rfloor, \quad \epsilon > 0. \]

Hence \( \frac{1}{\Gamma(z)} \) is of order \( \leq 1. \) However, the exponent of convergence of its zeros equals 1. Hence the order is 1, and the genus of the canonical product is also equal to 1.

Thus

\[ \frac{1}{\Gamma(z)} = z, e^{az+b} \prod_{n=1}^{\infty} (1 + \frac{z}{n}) e^\frac{-\pi}{n} \]

by Hadamard’s theorem. However, we know that \( \lim_{z \to 0^+} z \Gamma(z) = 1 \) and \( \Gamma(1) = 1. \) These substitutions give \( b = 0, \) and

\[ 1 = e^a \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) e^{-1/n} \]

or,

\[ 0 = a + \sum_{1}^{\infty} \left[ \log \left( 1 + \frac{1}{n} \right) - \frac{1}{n} \right] \]
3. The Product Formula

\[
\begin{align*}
\frac{1}{\Gamma(z)} &= e^{\gamma z} \prod_{n=1}^{\infty} \left\{ 1 + \frac{z}{n} \right\} e^{-z/n} \\
\Gamma(z) &= \lim_{n \to \infty} n^z \cdot \frac{1}{z} \cdot \frac{1}{z+1} \cdot \frac{2}{z+2} \cdot \ldots \cdot \frac{n}{z+n} \\
\Gamma(z) &= \lim_{n \to \infty} \frac{n^z \cdot n!}{z(z+1) \ldots (z+n)} \\
\Gamma(z) &= \lim_{n \to \infty} \Gamma_n(z).
\end{align*}
\]

Hence

\[
\Gamma(z) = \lim_{n \to \infty} \prod_{n=1}^{\infty} \left\{ 1 + \frac{z}{n} \right\} e^{-z/n}
\]

As a consequence of this we can derive Gauss's expression for \( \Gamma(z) \).

Further, we get by logarithmic differentiation,

\[
\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \sum_{n=1}^{\infty} \left( \frac{1}{z+n} - \frac{1}{n} \right).
\]
Thus
\[
\frac{d^2}{dz^2} \left[ \log \Gamma(z) \right] = \sum_{n=0}^{\infty} \frac{1}{(n + z)^2};
\]

Hence
\[
\frac{d^2}{dz^2} \left[ \log \Gamma(z) \right] > 0 \text{ for real, positive } z.
4 The Bohr-Mollerup-Artin Theorem

[7, Bd.I, p.276]

We shall prove that the functional equation of $\Gamma(x)$, namely $\Gamma(x+1) = x\Gamma(x)$, together with the fact that $\frac{d^2}{dx^2} \log \Gamma(x) > 0$ for $x > 0$, determines $\Gamma(x)$ ‘essentially’ uniquely-essentially, that is, except for a constant of proportionality. If we add the normalization condition $\Gamma(1) = 1$, then these three properties determine $\Gamma$ uniquely. N.B. The functional equation alone does not define $\Gamma$ uniquely since $g(x) = p(x)\Gamma(x)$, where $p$ is any analytic function of period 1 also satisfies the same functional equation.

We shall briefly recall the definition of ‘Convex functions’. A real-valued function $f(x)$, defined for $x > 0$, is convex, if the corresponding function $\phi(y) = \frac{f(x+y) - f(x)}{y}$, defined for all $y > -x$, $y \neq 0$, is monotone increasing throughout the range of its definition.

If $0 < x_1 < x < x_2$ are given, then by choosing $y_1 = x_1 - x$ and...
\[ y_2 = x_2 - x, \] we can express the condition of convexity as:
\[
f(x) \leq \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)
\]

Letting \( x \to x_1 \), we get \( f(x_1 + 0) \leq f(x_1) \); and letting \( x_2 \to x \), we get \( f(x) \leq f(x + 0) \) and hence \( f(x_1 + 0) = f(x_1) \) for all \( x_1 \). Similarly \( f(x_1 - 0) = f(x_1) \) for all \( x_1 \).

Thus a convex function is continuous.

Next, if \( f(x) \) is twice (continuously) differentiable, we have
\[
\varphi'(y) = \frac{y f'(x + y) - f(x + y) + f(x)}{y^2}
\]
and writing \( x + y = u \), we get
\[
f(x) = f(u - y) = f(u) - y f'(u) + \frac{y^2}{2} f''[u - (1 - \theta)y], 0 \leq \theta \leq 1
\]
which we substitute in the formula for \( \varphi'(y) \) so as to obtain
\[
\varphi'(y) = \frac{1}{2} f''[x + \theta y].
\]
Thus if \( f'''(x) \geq 0 \) for all \( x > 0 \), then \( \varphi'(y) \) is monotone increasing and \( f \) is convex. [The converse is also true].

Thus log \( \Gamma(x) \) is a convex function; by the last formula of the previous section we may say that \( \Gamma(x) \) is ‘logarithmically convex’ for \( x > 0 \). Further \( \Gamma(x) > 0 \) for \( x > 0 \).

**Theorem.** \( \Gamma(x) \) is the positive function uniquely defined for \( x > 0 \) by the conditions:

1. \( \Gamma(x + 1) = x\Gamma(x) \)
2. \( \Gamma(x) \) is logarithmically convex
3. \( \Gamma(1) = 1. \)
Proof. Let \( f(x) > 0 \) be any function satisfying the above three conditions satisfied by \( \Gamma(x) \).

Choose an integer \( n > 2 \), and \( 0 < x \leq 1 \).

Let 
\[
\begin{align*}
    n - 1 < n < n + x \leq n + 1.
\end{align*}
\]

By logarithmic convexity, we get
\[
\frac{\log f(n - 1) - \log f(n)}{(n - 1) - n} < \frac{\log f(n + x) - \log f(n)}{(n + x) - n} \leq \frac{\log f(n + 1) - \log f(n)}{(n + 1) - n}
\]

Because of conditions 1 and 3, we get
\[
\begin{align*}
    f(n - 1) &= (n - 2)!, \quad f(n) = (n - 1)!, \quad f(n + 1) = n! \\
    f(n + x) &= x(x + 1) \ldots (x + n - 1)f(x).
\end{align*}
\]

Substituting these in the above inequalities we get
\[
\begin{align*}
    \log(n - 1)x &< \log \frac{f(n + x)}{(n - 1)!} \leq \log n^x,
\end{align*}
\]

which implies, since \( \log \) is monotone,
\[
\begin{align*}
    (n - 1)x &< \frac{f(n + x)}{(n - 1)!} \leq n^x \\
    \frac{(n - 1)!n^x}{x(x + 1) \ldots (x + n - 1)} &< f(x) \leq \frac{(n - 1)!n^x}{x(x + 1) \ldots (x + n - 1)}
\end{align*}
\]

Replacing \( (n - 1) \) by \( n \) in the first inequality, we get
\[
\begin{align*}
    \Gamma_n(x) &< f(x) \leq \Gamma_n(x) \cdot \frac{x + n}{n}, \quad n = 2, 3, \ldots
\end{align*}
\]

where \( \Gamma_n \) is defined as in Gauss’s expression for \( \Gamma \).

Letting \( n \to \infty \), we get
\[
\begin{align*}
    f(x) &= \Gamma(x), \quad 0 < x \leq 1.
\end{align*}
\]

For values of \( x > 1 \), we uphold this relation by the functional equation.  \( \square \)
5 Gauss’s Multiplication Formula. [7, Bd.I, p.281]

Let \( p \) be a positive integer, and

\[
f(x) = p^p \Gamma \left( \frac{x}{p} \right) \Gamma \left( \frac{x + 1}{p} \right) \ldots \Gamma \left( \frac{x + p - 1}{p} \right)
\]

[If \( p = 1 \), then \( f(x) = \Gamma(x) \)]. For \( x > 0 \), we then have

\[
\frac{d^x}{dx^2} \log f(x) > 0.
\]

Further

\[
f(x + 1) = x f(x).
\]

Hence by the Bohr-Artin theorem, we get

\[
f(x) = a_p \Gamma(x).
\]

Put \( x = 1 \). Then we get

\[
a_p = p \Gamma \left( \frac{1}{p} \right) \cdot \Gamma \left( \frac{2}{p} \right) \ldots \Gamma \left( \frac{p}{p} \right)
\]  \hspace{1cm} (5.1)

\([a_1 = 1, \text{by definition}]. \) Put \( k = 1, 2, \ldots, p \), in the relation

\[
\Gamma_n \left( \frac{k}{p} \right) = \frac{n^{k/p} n! p^{n+1}}{k(k + p) \ldots (k + np)},
\]

and multiply out and take the limit as \( n \to \infty \). We then get from (5.1),

\[
a_p = p \lim_{n \to \infty} \frac{n^{p+1} (n!)^p p^{np+p}}{(np + p)!}
\]

However

\[
(np + p)! = (np)! (np)^p \left[ (1 + \frac{1}{np}) (1 + \frac{2}{np}) \ldots (1 + \frac{p}{np}) \right]
\]

Thus, for fixed \( p \), as \( n \to \infty \), we get

\[
a_p = p \lim_{n \to \infty} \frac{(n!)^p p^{np}}{(np)! n^{(p-1)/2}}
\]
Now by the asymptotic formula obtained in (v) of §1 [see p.49]

\[(n!)^p = a_p p^{np+p/2} e^{-np} e^{o(1)}\]

\[(np)! = an^{p+1} p^{np+1/2} e^{-np} e^{o(1)}\]

Hence

\[a_p = \sqrt{p} \cdot a^{p-1}.\]

Putting \(p = 2\) and using (5.1) we get

\[a = \frac{1}{\sqrt{2}}, a_2 = \frac{1}{\sqrt{2}} 2 \Gamma \left(\frac{1}{2}\right) \Gamma(1) = \sqrt{2\pi}\]

Hence

\[a_p = \sqrt{p(2\pi)^{(p-1)/2}}.\]

Thus

\[p^x \Gamma \left(\frac{x}{p}\right) \ldots \Gamma \left(\frac{x + p - 1}{p}\right) = \sqrt{p(2\pi)^{(p-1)/2}} \Gamma(x)\]

For \(p = 2\) and \(p = 3\) we get:

\[\Gamma \left(\frac{x}{2}\right) \Gamma \left(\frac{x + 1}{2}\right) = \sqrt{\pi} 2^{x-1} \Gamma(x); \quad \Gamma \left(\frac{x}{3}\right) \Gamma \left(\frac{x + 1}{3}\right) \Gamma \left(\frac{x + 2}{3}\right) = \frac{2\pi}{3^{x-1}} \Gamma(x).\]

6 Stirling’s Formula [15, p.150]

From (v) of §1 p.49 and (5.2) we get

\[(n!) = \sqrt{2\pi} \cdot e^{-n} \cdot n^{p+1/2} e^{o(1)},\]

which is the same thing as

\[\log(n-1)! = \left(n - \frac{1}{2}\right) \log n - n + \log \sqrt{2\pi} + o(1)\]

Further

\[1 + \frac{1}{2} + \cdots + \frac{1}{N - 1} - \log N = \gamma + o(1), \text{ as } N \to \infty\]
and
\[
\log(N + z) = \log N + \log \left(1 + \frac{z}{N}\right)
\]
\[
= \log N + \frac{z}{N} + O\left(\frac{1}{N^2}\right), \quad \text{as } N \to \infty. \quad (6.3)
\]

We need to use (6.1)-(6.3) for obtaining the asymptotic formula for \(\Gamma(z)\) where \(z\) is complex.

From the product-formula we get
\[
\log \Gamma(z) = \sum_{n=1}^{\infty} \left\{ \frac{z}{n} - \log \left(1 + \frac{z}{n}\right) \right\} - \gamma z - \log z, \quad (6.4)
\]
each logarithm having its principal value.

Now
\[
\int_{0}^{N} \frac{[u] - u + \frac{1}{2}}{u + z} du = \sum_{n=0}^{N-1} \int_{n}^{n+1} \left(\frac{n + \frac{1}{2} + z}{u + z} - 1\right) du
\]
\[
= \sum_{n=0}^{N-1} \left(\frac{n + \frac{1}{2} + z}{u + z} - \log(n + 1 + z) - \log(n - z)\right) - N
\]
\[
= \sum_{n=0}^{N-1} n[\log(n + 1 + z) - \log(n + z)] + \left(z + \frac{1}{2}\right) \sum_{n=0}^{N-1} \left[\log(n + 1 + z) - \log(n + z)\right] - N
\]
\[
= (N - 1) \log(N + z) - \sum_{n=1}^{N-1} \log(n + z) + \left(z + \frac{1}{2}\right) \log(N + z)
\]
\[
- \left(z + \frac{1}{2}\right) \log z - N
\]
\[
= \left(\frac{N}{2} + z\right) \log(N + z) - \left(z + \frac{1}{2}\right) \log z
\]
\[
- N - \sum_{n=1}^{N-1} \left[\log n + \log \left(1 + \frac{z}{n}\right)\right]
\]
6. Stirling’s Formula

\[ = \left( N - \frac{1}{2} + z \right) \log(N + z) - \left( z + \frac{1}{2} \right) \log z - N - \log(N - 1)! \]

\[ + \sum_{n=1}^{N-1} \left( \frac{z}{n} - \log \left( 1 + \frac{z}{n} \right) \right) - z \sum_{n=1}^{N-1} \frac{1}{n} \]

Now substituting for \( \log(N + z) \) etc., from (6.1)-(6.3), we get

\[ \int_{0}^{N} \frac{[u] - u + \frac{1}{2}}{u + z} du = (N - \frac{1}{2} + z) \left\{ \log N + \frac{z}{N} + O \left( \frac{1}{N^2} \right) \right\} - \left( z + \frac{1}{2} \right) \log z - N - \left( N - \frac{1}{2} \right) \log N + N - c + O(1) + \]

\[ \sum_{n=1}^{N-1} \left( \frac{z}{n} - \log \left( 1 + \frac{z}{n} \right) \right) - z \sum_{n=1}^{N-1} \frac{1}{n} \]

\[ = z \left( \log N - \sum_{n=1}^{N-1} \frac{1}{n} \right) + z + O(1) - \log \sqrt{2\pi} \]

\[ - \left( z - \frac{1}{2} \right) \log z - \log z + \sum_{n=1}^{N-1} \left( \frac{z}{n} - \log \left( 1 + \frac{z}{n} \right) \right) \]

Now letting \( N \to \infty \), and using (6.4), and (6.2), we get

\[ \int_{0}^{\infty} \frac{[u] - u + \frac{1}{2}}{u + z} du + \frac{1}{2} \log 2\pi - z + \left( z - \frac{1}{2} \right) \log z = \log \Gamma(z) \quad (6.5) \]

If \( \varphi(u) = \int_{0}^{u} \left( [v] - v + \frac{1}{2} \right) dv \), then

\[ \varphi(u) = O(1), \quad \text{since} \quad \varphi(n + 1) = \varphi(n), \]

if \( n \) is an integer.

Hence

\[ \int_{0}^{\infty} \frac{[u] - u + \frac{1}{2}}{u + z} du = \int_{0}^{\infty} \frac{d\varphi(u)}{u + z} = \int_{0}^{\infty} \frac{\varphi(u)}{(u + z)^2} \]
If we write \( z = r e^{i\theta} \), and \( u = ru_1 \), then

\[
\int_0^\infty \frac{[u] - u + \frac{1}{2}}{u + z} du = O \left( \frac{1}{r \int_0^\infty \frac{du_1}{|u_1 + \theta i|^2}} \right)
\]

The last integral is finite if \( \theta \neq \pi \). Hence

\[
\int_0^\infty \frac{[u] - u + \frac{1}{2}}{u + z} du = O \left( \frac{1}{r} \right), \text{ uniformly for } |\text{arg } z| \leq \pi - \varepsilon < \pi
\]

Thus we get from (6.5):

\[
\log \Gamma(z) = \left( z - \frac{1}{2} \right) \log z - z + \frac{1}{2} \log 2\pi + O \left( \frac{1}{|z|} \right)
\]

for \( |\text{arg } z| \leq \pi - \varepsilon < \pi \)

**Corollaries.**

1. \( \log \Gamma(z + \alpha) = (z + \alpha - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + O \left( \frac{1}{|z|} \right) \)
   as \( |z| \to \infty \) uniformly for \( |\text{arg } z| \leq \pi - \varepsilon < \pi \). \( \alpha \) bounded.

2. For any fixed \( x \), as \( y \to \pm \infty \)
   \[
   |\Gamma(x + iy)| \sim e^{-\frac{1}{2} |y|} |y|^{\frac{1}{2} - \frac{1}{2}} \sqrt{2\pi}
   \]

3. \( \frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + O \left( \frac{1}{|z|^2} \right) \) \( |\text{arg } z| \leq \pi - \varepsilon < \pi \) \[11, p.57\]
   This may be deduced from (1) by using
   \[
   f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta,
   \]
   where \( f(z) = \log \Gamma(z) - \left( z - \frac{1}{2} \right) \log z + z - \frac{1}{2} \log 2\pi \) and \( C \) is a circle with centre at \( \zeta = z \) and radius \( |z| \sin \varepsilon/2 \).
Lecture 8

The Zeta Function of Riemann

1 Elementary Properties of $\zeta(s)$ [16, pp.1-9]

We define $\zeta(s)$, for $s$ complex, by the relation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it, \quad \sigma > 1. \quad (1.1)$$

We define $x^s$, for $x > 0$, as $e^{s \log x}$, where $\log x$ has its real determination.

Then $|n^s| = n^\sigma$, and the series converges for $\sigma > 1$, and uniformly in any finite region in which $\sigma \geq 1 + \delta > 1$. Hence $\zeta(s)$ is regular for $\sigma \geq 1 + \delta > 1$. Its derivatives may be calculated by termwise differentiation of the series.

We could express $\zeta(s)$ also as an infinite product called the Euler Product:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1, \quad (1.2)$$

where $p$ runs through all the primes (known to be infinite in number!). It is the Euler product which connects the properties of $\zeta$ with the properties of primes. The infinite product is absolutely convergent for $\sigma > 1,$
since the corresponding series
\[ \sum_{p} \frac{1}{p^{s}} = \sum_{p} \frac{1}{p^{\sigma}} < \infty, \quad \sigma > 1 \]

Expanding each factor of the product, we can write it as
\[ \prod_{p} \left(1 + \frac{1}{p^{s}} + \frac{1}{p^{2s}} + \cdots\right) \]

Multiplying out formally we get the expression (1.1) by the unique factorization theorem. This expansion can be justified as follows:
\[ \prod_{p \leq P} \left(1 + \frac{1}{p^{s}} + \frac{1}{p^{2s}} + \cdots\right) = 1 + \frac{1}{n_{1}^{s}} + \frac{1}{n_{2}^{s}} + \cdots, \]
where \( n_{1}, n_{2}, \ldots \) are those integers none of whose prime factors exceed \( P \). Since all integers \( \leq P \) are of this form, we get
\[ \sum_{n=1}^{\infty} \frac{1}{n^{s}} - \prod_{p \leq P} \left(1 - \frac{1}{p^{2s}}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^{s}} - 1 - \frac{1}{n_{1}^{s}} - \frac{1}{n_{2}^{s}} - \cdots \leq \frac{1}{(P+1)^{s}} + \frac{1}{(P+2)^{s}} + \cdots \]

\( \to 0, \) as \( P \to \infty, \) if \( \sigma > 1. \)

Hence (1.2) has been established for \( \sigma > 1, \) on the basis of (1.1).

As an immediate consequence of (1.2) we observe that \( \zeta(s) \) has no zeros for \( \sigma > 1, \) since a convergent infinite product of non-zero factors is non-zero.

We shall now obtain a few formulae involving \( \zeta(s) \) which will be of use in the sequel. We first have
\[ \log \zeta(s) = - \sum_{p} \log \left(1 - \frac{1}{p^{\sigma}}\right), \quad \sigma > 1. \quad \text{(1.3)} \]
If we write \( \pi(x) = \sum_{p \leq x} 1 \), then

\[
\log \zeta(s) = - \sum_{n=2}^{\infty} (\pi(n) - \pi(n - 1)) \log \left( 1 - \frac{1}{n^s} \right)
\]

\[
= - \sum_{n=2}^{\infty} \pi(n) \left[ \log \left( 1 - \frac{1}{n^s} \right) - \log \left( 1 - \frac{1}{(n+1)^s} \right) \right]
\]

\[
= \sum_{n=2}^{\infty} \pi(n) \int_{n}^{n+1} \frac{s}{x(x^s - 1)} \, dx
\]

Hence

\[
\log \zeta(s) = s \int_{\frac{1}{2}}^{\infty} \frac{\pi(x)}{x(x^s - 1)} \, dx, \quad \sigma > 1. \tag{1.4}
\]

It should be noted that the rearrangement of the series preceding the above formula is permitted because

\[
\pi(n) \leq n \quad \text{and} \quad \log \left( 1 - \frac{1}{n^s} \right) = O(n^{-\sigma}).
\]

A slightly different version of (1.3) would be

\[
\log \zeta(s) = \sum_p \sum_m \frac{1}{mp^{ms}}, \quad \sigma > 1 \tag{1.3}'
\]

where \( p \) runs through all primes, and \( m \) through all positive integers. Differentiating (1.3), we get

\[
- \frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{p^{-s} \log p}{1 - p^{-s}} = \sum_{p,m} \frac{\log p}{p^{ms}},
\]

or

\[
\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}, \quad \sigma > 1, \tag{1.5}
\]
where $\wedge(n) = \begin{cases} \log p, & \text{if } n \text{ is a positive power of a prime } p \\ 0, & \text{otherwise.} \end{cases}$

Again

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \sigma > 1 \quad (1.6)$$

where $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $n$ is the product of $k$ different primes, $\mu(n) = 0$ if $n$ contains a factor to a power higher than the first.

We also have

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \quad \sigma > 1$$

where $d(n)$ denotes the number of divisors of $n$, including 1 and $n$. For

$$\zeta^2(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{\mu=1}^{\infty} \frac{1}{\mu^s} \sum_{mn=\mu} 1$$

More generally

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \sigma > 1 \quad (1.7)$$

where $k = 2, 3, 4, \ldots$, and $d_k(n)$ is the number of ways of expressing $n$ as a product of $k$ factors.

Further

$$\zeta(s) \cdot \zeta(s - a) = \sum_{m=1}^{\infty} \frac{1}{m^s} \sum_{n=1}^{\infty} \frac{n^a}{n^s} = \sum_{\mu=1}^{\infty} \frac{1}{\mu^s} \sum_{mn=\mu} n^a,$$

so that

$$\zeta(s) \cdot \zeta(s - a) = \sum_{\mu=1}^{\infty} \frac{\sigma_2(\mu)}{\mu^s} \quad (1.8)$$
where \( \sigma_a(\mu) \) denotes the sum of the \( a \)-th powers of the divisors of \( \mu \).

If \( a \neq 0 \), we get, from expanding the respective terms of the Euler products,
\[
\zeta(a) \zeta(s - a) = \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \right) \left( 1 + \frac{p^a}{p^s} + \frac{p^{2a}}{p^{2s}} + \cdots \right)
\]
\[
= \prod_p \left( 1 + \frac{1}{p^s} + \frac{1 + p^a + p^{2a}}{p^{2s}} + \cdots \right)
\]
\[
= \prod_p \left( 1 + \frac{1 - p^{2a}}{1 - p^a} \cdot \frac{1}{p^s} + \cdots \right)
\]

Using (1.8) we thus get
\[
\sigma_2(n) = \frac{1 - p_1^{(m_1+1)a}}{1 - p_1^a} \cdots \frac{1 - p_r^{(m_r+1)a}}{1 - p_r^a} \tag{1.9}
\]

if \( n = p_1^{m_1} \cdot p_2^{m_2} \cdots p_r^{m_r} \) by comparison of the coefficients of \( \frac{1}{n^s} \).

More generally, we have
\[
\frac{\zeta(s) \cdot \zeta(s - a) \cdot \zeta(s - b) \cdot \zeta(s - a - b)}{\zeta(2s - a - b)} = \prod_p \frac{1 - p^{-2s+a+b}}{(1 - p^{-s})(1 - p^{-s+a})(1 - p^{-s+b})(1 - p^{-s+a+b})}
\]

for \( \sigma > \max \{1, \Re a + 1, \Re b + 1, \Re(a + b) + 1\} \). Putting \( p^{-s} = z \), we get the general term in the right hand side equal to
\[
1 - p^a z^2 \frac{1}{(1 - z) \cdot (1 - p^a z) \cdot (1 - p^b z) (1 - p^{a+b} z)}
\]
\[
= \frac{1}{(1 - p^a)(1 - p^b)} \left\{ \frac{1}{1 - z} - \frac{p^a}{1 - p^a z} - \frac{p^b}{1 - p^b z} - \frac{p^{a+b}}{1 - p^{a+b} z} \right\}
\]
\[
= \frac{1}{(1 - p^a)(1 - p^b)} \sum_{m=0}^{\infty} \left\{ 1 - p^{(m+1)a} - p^{(m+1)b} + p^{(m+1)(a+b)} \right\} z^m
\]
\[
= \frac{1}{(1 - p^a)(1 - p^b)} \sum_{m=0}^{\infty} \left\{ 1 - p^{(m+1)a} \right\} \left\{ 1 - p^{(m+1)b} \right\} z^m
\]
Hence
\[
\frac{\zeta(s) \cdot \zeta(s-a) \cdot \zeta(s-b) \cdot \zeta(s-a-h)}{\zeta(2s-a-b)} = \prod_p \sum_{m=0}^{\infty} \frac{1 - p^{(m+1)a}}{1 - p^a} \cdot \frac{1 - p^{(m+1)b}}{1 - p^b} \cdot \frac{1}{p^{ms}}
\]

Now using (1.9) we get
\[
\frac{\zeta(s) \cdot \zeta(s-a) \cdot \zeta(s-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} \quad (1.10)
\]
\[
\sigma > \max\{1, \text{Re } a + 1, \text{Re } b + 1, \text{Re}(a + b) + 1\}
\]

If \( a = b = 0 \), then
\[
\frac{[\zeta(s)]^4}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{(d(n))^2}{n^s}, \quad \sigma > 1. \quad (1.11)
\]

If \( \alpha \) is real, and \( \alpha \neq 0 \), we write \( \alpha i \) for \( a \) and \( -\alpha i \) for \( b \) in (1.10), and get
\[
\frac{\zeta^2(s)\zeta(s-\alpha i)\zeta(s+\alpha i)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\sigma_{\alpha i}(n)|^2}{n^s}, \quad \sigma > 1, \quad (1.12)
\]
where \( \sigma_{\alpha i}(n) = \sum_{d|n} \alpha^{i}d^{\frac{i}{2}}. \)
The Zeta Function of Riemann (Contd.)

2 Elementary theory of Dirichlet Series

The Zeta function of Riemann is the sum-function associated with the Dirichlet series \( \sum \frac{1}{n^s} \). We shall now study, very briefly, some of the elementary properties of general Dirichlet series of the form

\[ \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}, \quad \alpha \leq \lambda_1 < \lambda_2 < \ldots \to \infty \]

Special cases are: \( \lambda_n = \log n \); \( \lambda_n = n \), \( e^{-x} = x \). We shall first prove a lemma applicable to the summation of Dirichlet series.

**Lemma.** Let \( A(x) = \sum_{\lambda_n \leq x} a_n \), where \( a_n \) may be real or complex. Then, if \( x \geq \lambda_1 \), and \( \phi(x) \) has a continuous derivative in \((0, \infty)\), we have

\[
\sum_{\lambda_n \leq \omega} a_n \phi(\lambda_n) = - \int_{\lambda_1}^{\omega} A(x) \phi'(x) dx + A(\omega) \phi(\omega).
\]
If \( A(\omega)\phi(\omega) \rightarrow 0 \) as \( \omega \rightarrow \infty \), then
\[
\sum_{n=1}^{\infty} a_n \phi(\lambda_n) = -\int_{\lambda_1}^{\infty} A(x)\phi'(x)dx,
\]
provided that either side is convergent.

**Proof.**
\[
A(\omega)\phi(\omega) - \sum_{\lambda_n \leq \omega} a_n \phi(\lambda_n)
\]
\[
= \sum_{\lambda_n \leq \omega} a_n [\phi(\omega) - \phi(\lambda_n)]
\]
\[
= \sum_{\lambda_n \leq \omega} \int_{\lambda_n}^{\omega} a_n \phi'(x)dx
\]
\[
= \int_{\lambda_1}^{\infty} A(x)\phi'(x)dx
\]

\[\square\]

**Theorem 1.** If \( \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \) converges for \( s = s_0 \equiv \sigma_0 + it_0 \), then it converges uniformly in the angular region defined by
\[
|am(s - s_0)| = \theta < \pi/2, \quad \text{for} \quad 0 < \theta < \pi/2
\]

**Proof.** We may suppose that \( s_0 = 0 \). For
\[
\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s_0} e^{-\lambda_n(t-s_0)} \equiv \sum_{n=1}^{\infty} b_n e^{-\lambda_n s'}, \quad \text{say},
\]
and the new series converges at \( s' = 0 \). By the above lemma, we get
\[
\sum_{\mu+1}^{v} a_n e^{-\lambda_n s} = \left( \sum_{1}^{v} - \sum_{1}^{\mu} \right) a_n e^{-\lambda_n s'}
\]
2. Elementary theory of Dirichlet Series

\[
\begin{align*}
= & \int_{\lambda_\nu}^{\lambda_\nu} A(x) \cdot e^{-xs} \cdot sdx + A(\lambda_\nu)e^{-\lambda_\nu s} - A(\lambda_\mu)e^{-\lambda_\mu s} \\
= & s \int_{\lambda_\mu}^{\lambda_\mu} [A(x) - A(\lambda_\mu)]e^{-xs}dx + [A(\lambda_\nu) - A(\lambda_\mu)]e^{-\lambda_\mu s}
\end{align*}
\]

We have assumed that \(\sum a_n\) converges, therefore, given \(\epsilon > 0\), there exists an \(n_0\) such that for \(x > \lambda_\mu \geq n_0\), we have

\[|A(x) - A(\lambda_\mu)| < \epsilon\]

Hence, for such \(\mu\), we have

\[
\left| \sum_{\mu+1}^{\nu} a_n e^{-\lambda_\mu s} \right| \leq \epsilon|s| \int_{\lambda_\nu}^{\lambda_\nu} e^{-x\sigma}dx + \epsilon e^{-\lambda_\nu \sigma}
\]

\[\leq 2\epsilon \frac{|s|}{\sigma} + \epsilon\]

\[< 2\epsilon (\sec \theta + 1),\]

if \(\sigma \neq 0\), since \(\frac{|s|}{\sigma} < \sec \theta\), and this proves the theorem. As a consequence of Theorem II, we deduce

\[\square\]

**Theorem 2.** If \(\sum a_n e^{-\lambda s}\) converges for \(s = s_0\), then it converges for \(Re s > \sigma_0\), and uniformly in any bounded closed domain contained in the half-plane \(\sigma > \sigma_0\). We also have

**Theorem 3.** If \(\sum a_n e^{-\lambda s}\) converges for \(s = s_0\) to the sum \(f(s_0)\), then \(f(s) \to f(s_0)\) as \(s \to s_0\) along any path in the region \(|\text{arg}(s - s_0)| \leq \theta < \pi/2\).

A Dirichlet series may converge for all values of \(s\), or some values of \(s\), or no values of \(s\).

**Ex.1.** \(\sum a_n n^{-s}\), \(a_n = \frac{1}{n!}\) converges for all values of \(s\).
Ex.2. \( \sum a_n n^{-s} \), \( a_n = n! \) converges for no values of \( s \).

Ex.3. \( \sum n^{-s} \) converges for \( \Re s > 1 \), and diverges for \( \Re s \leq 1 \).

If a Dirichlet series converges for some, but not all, values of \( s \), and if \( s_1 \) is a point of convergence while \( s_2 \) is a point of divergence, then, on account of the above theorems, we have \( \sigma_1 > \sigma_2 \). All points on the real axis can then be divided into two classes \( L \) and \( U \); \( \sigma \in U \) if the series converges at \( \sigma \), otherwise \( \sigma \in L \). Then any point of \( U \) lies to the right of any point of \( L \), and this classification defines a number \( \sigma_\sigma \) such that the series converges for every \( \sigma > \sigma_\sigma \) and diverges for \( \sigma < \sigma_\sigma \), the cases \( \sigma = \sigma_\sigma \) begin undecided. Thus the region of convergence is a half-plane. The line \( \sigma = \sigma_\sigma \) is called the line of convergence, and the number \( \sigma_\sigma \) is called the abscissa of convergence. We have seen that \( \sigma_\sigma \) may be \( \pm \infty \). On the basis of Theorem 4 we can establish

**Theorem 4.** A Dirichlet series represents in its half-plane of convergence a regular function of \( s \) whose successive derivatives are obtained by termwise differentiation.

We shall now prove a theorem which implies the ‘uniqueness’ of Dirichlet series.

**Theorem 5.** If \( \sum a_n e^{-\lambda s} \) converges for \( s = 0 \), and its sum \( f(s) = 0 \) for an infinity of values of \( s \) lying in the region:

\[ \sigma \geq \epsilon > 0, \quad |\text{am } s| \leq \theta < \pi/2, \]

then \( a_n = 0 \) for all \( n \).

**Proof.** \( f(s) \) cannot have an infinite number of zeros in the neighbourhood of any finite point of the giver region, since it is regular there. Hence there exists an infinity of values \( s_n = \sigma_n + i\eta_n \), say, with \( \sigma_{n+1} > \sigma_n \), \( \lim \sigma_n = \infty \) such that \( f(s_n) = 0 \).

However,

\[ g(s) \equiv e^{i\lambda s} f(s) = a_1 + \sum_{2}^{\infty} a_n e^{-(\lambda \eta - \lambda_0)s}, \]
2. Elementary theory of Dirichlet Series

(here we are assuming \( \lambda_1 > 0 \)) converges for \( s = 0 \), and is therefore uniformly convergent in the region given; each term of the series on the right \( \rightarrow 0 \), as \( s \rightarrow \infty \) and hence the right hand side, as a whole, tends to \( a_1 \) as \( s \rightarrow \infty \). Thus \( g(s) \rightarrow a_1 \) as \( s \rightarrow \infty \) along any path in the given region. This would contradict the fact that \( g(s_n) = 0 \) for an infinity of \( s_n \rightarrow \infty \), unless \( a_1 = 0 \). Similarly \( a_2 = 0 \), and so on. □

Remark. In the hypothesis it is essential that \( \varepsilon > 0 \), for if \( \varepsilon = 0 \), the origin itself may be a limit point of the zeros of \( f \), and a contradiction does not result in the manner in which it is derived in the above proof.

The above arguments can be applied to \( \sum a_n e^{-\lambda_n s} \) so as to yield the existence of an abscissa of absolute convergence \( \tilde{\sigma} \), etc. In general, \( \tilde{\sigma} \geq \sigma_0 \). The strip that separates \( \tilde{\sigma} \) from \( \sigma_0 \) may comprise the whole plane, or may be vacuous or may be a half-plane.

Ex. 1. \( \sigma_0 = 0, \tilde{\sigma} = 1 \)

\[
(1 - 2^{1-s})\zeta(s) = \left( \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \right) - 2 \left( \frac{1}{2^s} + \frac{1}{4^s} + \cdots \right) \\
= \left( \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots \right)
\]

Ex. 2. \( \sum \frac{(-1)^n}{\sqrt{n}} (\log n)^{-s} \) converges for all \( s \) but never absolutely. In the simple case \( \lambda_n = \log n \), we have

Theorem 6. \( \tilde{\sigma} - \sigma_0 \leq 1 \).

For if \( \sum a_n n^{-s} < \infty \), then \( |a_n| \cdot n^{-\sigma} = O(1) \), so that \( \sum \frac{|a_n|}{n^{s+1+\varepsilon}} < \infty \) for \( \varepsilon > 0 \).

While we have observed that the sum-function of a Dirichlet series is regular in the half-plane of convergence, there is no reason to assume that the line of convergence contains at least one singularity of the function (see Ex 1. above!). In the special case where the coefficients are positive, however we can assert the following

Theorem 7. If \( a_n \geq 0 \), then the point \( s = \sigma_0 \) is a singularity of the function \( f(s) \).
Proof. Since \( a_n \geq 0 \), we have \( \sigma_0 = \bar{\sigma} \), and we may assume, without loss of generality, that \( \sigma_0 = 0 \). Then, if \( s = 0 \) is a regular point, the Taylor series for \( f(s) \) at \( s = 1 \) will have a radius of convergence > 1. (since the circle of convergence of a power series must have at least one singularity). Hence we can find a negative value of \( s \) for which

\[
f(s) = \sum_{\nu=0}^{\infty} \frac{(s - 1)^\nu}{\nu!} f^{(\nu)}(1) = \sum_{\nu=0}^{\infty} \frac{(1 - s)^\nu}{\nu!} \sum_{n=1}^{\infty} a_n \lambda_n^\nu e^{-\lambda_n}
\]

Here every term is positive, so the summations can be interchanged and we get

\[
f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n} \sum_{\nu=0}^{\infty} \frac{(1 - s)^\nu \lambda_n^\nu}{\nu!} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^s}
\]

Hence the series converges for a negative value of \( s \) which is a contradiction. \( \square \)
Lecture 10

The Zeta Function of Riemann (Contd)

2 (Contd). Elementary theory of Dirichlet series

We wish to prove a formula for the partial sum of a Dirichlet series. We shall give two proofs, one based on an estimate of the order of the sum-function \[10\], and another \[14, Lec. 3\] which is independent of such an estimate. We first need the following

**Lemma 1.** If \(\sigma > 0\), then

\[
\frac{1}{2\pi i} \int_{\sigma-i0}^{\sigma+i0} \frac{e^{us}}{s} ds = \begin{cases} 
1, & \text{if } u > 0 \\
0, & \text{if } u < 0 \\
\frac{1}{2}, & \text{if } u = 0
\end{cases}
\]

where \(\int_{\sigma-10}^{\sigma+i0} = \lim_{T \to \infty} \int_{\sigma-iT}^{\sigma+iT}\).

**Proof.** The case \(u = 0\) is obvious. We shall therefore study only the cases \(u \geq 0\). Now

\[
\int_{\sigma-iT}^{\sigma+iT} \frac{e^{us}}{s} ds = \int_{\sigma-iT}^{\sigma+iT} \frac{e^{us}}{us} ds + \int_{\sigma-iT}^{\sigma+iT} \frac{e^{us}}{us^2} ds
\]
However \[ |e^{iu(\sigma+iT)}| = e^{u\sigma} \]
and \(|s| > T\); hence, letting \(T \to \infty\), we get
\[
\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{us}}{s} ds = \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{us}}{us^2} ds \equiv I, \text{ say.}
\]

We shall evaluate \(I\) on the contour suggested by the diagram. If \(u < 0\), we use the contour \(L + C_R\); and if \(u > 0\), we use \(L + C_L\).

If \(u < 0\), we have, on \(C_R\),
\[
\left| \frac{e^{us}}{s^2} \right| \leq \frac{e^{u\sigma}}{r^2}, \text{ since } \Re s > \sigma.
\]

If \(u > 0\), we have, on \(C_L\),
\[
\left| \frac{e^{us}}{s^2} \right| \leq \frac{e^{u\sigma}}{r^2}, \text{ since } \Re s < \sigma.
\]

Hence
\[
\left| \int_{C_R \cup C_L} \frac{e^{us}}{us^2} ds \right| \leq 2\pi r \cdot \frac{e^{u\sigma}}{|u| r^2} \to 0, \text{ as } r \to \infty.
\]

By Cauchy’s theorem, however, we have
\[
\int_L = - \int_{C_R}.
\]
hence

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{us} ds = 0, \quad \text{if } u < 0.$$  

If, however, $u > 0$, the contour $L + C_L$ encloses a pole of the second order at the origin, and we get

$$\int_L = - \int_{C_L} + 1,$$

so that

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{as} ds = 1, \quad \text{if } u > 0.$$  

\[\square\]

**Remarks.** We can rewrite the Lemma as: if $a > 0$,

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{(\lambda - \lambda_n)s}}{s} ds = \begin{cases} 0, & \text{if } \lambda < \lambda_n \\ 1, & \text{if } \lambda > \lambda_n \\ \frac{1}{2}, & \text{if } \lambda = \lambda_n \end{cases}$$

Formally, therefore, we should have

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} f(s) \frac{e^{as}}{s} ds = \frac{1}{2\pi i} \sum_{1}^{\infty} a_n \int_{a-i\infty}^{a+i\infty} \frac{e^{(\lambda - \lambda_n)s}}{s} ds = \sum_{\lambda_n < \lambda} a_n$$

the dash denoting that if $\lambda = \lambda_n$ the last term should be halved. We wish now to establish this on a rigorous basis.

We proceed to prove another lemma first.
Lemma 2. Let \( \sum a_n e^{-\lambda_n s} \) converge for \( s = \beta \) real. Then

\[
\sum_1^n a_n e^{-\lambda_n s} = o(|t|),
\]

the estimate holding uniformly both in \( \sigma \geq \beta + \epsilon > \beta \) and in \( n \). In particular, for \( n = \infty \), \( f(s) = o(|t|) \) uniformly for \( \sigma \geq \beta + \epsilon > \beta \).

Proof. Assume that \( \beta = 0 \), without loss of generality. Then \( a_\nu = O(1) \) and \( A(\lambda_\nu) - A(\lambda_\mu) = O(1) \) for all \( \nu \) and \( \mu \).

If \( 1 < N < n \), then

\[
\sum_1^n a_n e^{-\lambda_n s} = \sum_1^N a_n e^{-\lambda_n s} + \sum_{N+1}^n a_n e^{-\lambda_n s}
\]

\[
= \sum_1^N a_n e^{-\lambda_n s} + s \int_{\lambda_N}^{\lambda_n} [A(x) - A(\lambda_n)] e^{-xs} dx
\]

\[
= O(N) + O(1) + O\left( |s| \int_{\lambda_N}^{\lambda_n} e^{-xs} dx \right)
\]

\[
= O(N) + O\left( |s| e^{-\lambda_N \sigma} \right)
\]

\[
= O(N) + O\left( |t| e^{-\lambda_N \epsilon} \right)
\]

since \( |s|^2 = t^2 + \sigma^2 \) and \( \sigma \geq \epsilon \).

Hence

\[
\sum_1^n a_n e^{-\lambda_n s} = O(N) + O(|t| e^{-\lambda_N \epsilon}), \text{ if } 1 < N < n
\]

If \( N \geq n \), then, trivially,

\[
\sum_1^n a_n e^{-\lambda_n s} = O(N).
\]
2. (Contd). Elementary theory of Dirichlet series

If we choose $N$ as a function of $|t|$, tending to $\infty$ more slowly than $|t|$, we get

$$\sum_{n=1}^{N} a_n e^{-\lambda_n s} = O(|t|)$$

in either case. □

**Theorem (Perron’s Formula)** Let $\sigma_0$ be the abscissa of convergence of $\sum a_n e^{-\lambda_n s}$ whose sum-function is $f(s)$. Then for $\sigma > \sigma_0$ and $\sigma > 0$, we have

$$\sum_{\lambda_n \leq \sigma} a_n = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{f(s)}{s} e^{\omega s} ds,$$

where the dash denotes the fact that if $\omega = \lambda_n$ the last term is to be halved.

**Proof.** Let $\lambda_m \leq \omega < \lambda_{m+1}$, and

$$g(s) = e^{i\omega s} \left\{ f(s) - \sum_{n=1}^{m} a_n e^{-\lambda_n s} \right\}$$

(This has a meaning since Re $s \equiv \sigma > \sigma_0$). Now

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{g(s)}{s} ds$$

$$= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{f(s) e^{i\omega s}}{s} ds - \sum_{\lambda_n \leq \sigma} a_n$$

by the foregoing lemma. We have to show that the last expression is zero.

Applying Cauchy’s theorem to the rectangle

$$\sigma - iT_1, \sigma + iT_2, \Omega + iT_2, \Omega - iT_1,$$
where $T_1, T_2 > 0$, and $\Omega > \sigma$, we get

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{g(s)}{s} ds = 0,$$

for

$$\frac{1}{2\pi i} \int_{\sigma-iT_1}^{\sigma+iT_2} \frac{g(s)}{s} ds = \frac{1}{2\pi i} \left[ \int_{\sigma-iT_1}^{\sigma-iT_2} + \int_{\sigma+iT_1}^{\sigma+iT_2} \right] = I_1 + I_2 + I_3,$$

Now, if we fix $T_1$ and $T_2$ and let $\Omega \to \infty$, then we see that $I_2 \to 0$, for $g(s)$ is the sum-function of $\sum_{n=1}^{\infty} b_n^{-\mu_n s}$, where $b_n = a_{m+n}, \mu_n = \lambda_{m+n} - \omega$, with $\mu_1 > 0$, and hence $g(s) = o(1)$ as $s \to \infty$ in the angle $|\text{ams}| \leq \frac{\pi}{2} - \delta$.

Thus

$$\frac{1}{2\pi i} \int_{\sigma-iT_1}^{\sigma+iT_2} \frac{g(s)}{s} ds = \frac{1}{2\pi i} \left[ \int_{\sigma-iT_1}^{\sigma+iT_2} \right]$$

if the two integrals on the right converge.

Now if

$$h(s) = g(s)e^{(\lambda_{m+1} - \omega)s} = a_{m+1} + a_{m+2}e^{-(\lambda_{m+2} - \lambda_{m+1})s} + \ldots$$

then by lemma we have

$$|h(s)| < \epsilon T_2$$

for $s = \sigma + iT_2, \sigma \geq \sigma_0 + \epsilon, T_2 \geq T_0$. Therefore for the second integral on the right we have

$$\left| \int_{\sigma+iT_2}^{\sigma+iT_2} \frac{g(s)}{s} ds \right| < \frac{\epsilon T_2}{\sqrt{\sigma^2 + T_2^2}} \int_{\sigma}^{\infty} e^{-\mu_1 \xi} d\xi < \epsilon/\mu_1$$

$\mu_1 > 0$. This proves not only that the integral converges for a finite $T_2$, but also that as $T_2 \to \infty$, the second integral tends to zero. Similarly for the first integral on the right. \qed
N.B. An estimate for \( g(s) \) alone (instead of \( h(s) \)) will not work!

**Remarks.** More generally we have for \( \sigma > \sigma_0 \) and \( \sigma > \sigma^* \)

\[
\sum_{\lambda_n \leq \phi} \lambda_n e^{-\lambda_n s^*} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{f(s)}{s} e^{\omega(s-s^*)} ds,
\]

for \( \sum \lambda_n e^{-\lambda_n s^*} = \sum a_n e^{-\lambda_n s} = f(s) \),

and writing \( s' = s - s^* \) and \( b_n = a_n e^{-\lambda_n s'} \), we have \( f(s) \equiv f(s') = \sum b_n e^{-\lambda_n s'} \).

We shall now give an alternative proof of Person’s formula without using the order of \( f(s) \) as \( s \to \infty \) \[14, Lec. 4\].

**Aliter.** If \( f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \) has \( \sigma_0 \neq \infty \) as its abscissa of convergence, and if \( a > 0, a > \sigma_0 \), then for \( \omega \neq \lambda_n \), we have

\[
\sum_{\lambda_n < \omega} \lambda_n e^{-\lambda_n s} = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(s) \frac{e^{\omega s}}{s} ds
\]

**Proof.** Define

\[
f_m(s) = \sum_{n=1}^{m} a_n e^{-\lambda_n s}
\]

and

\[
r_m(s) = \sum_{n=m+1}^{\infty} a_n e^{-\lambda_n s} = f(s) - f_m(s)
\]

Then

\[
\text{(A)} \quad \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(s) \frac{e^{\omega s}}{s} ds = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f_m(s) \frac{e^{\omega s}}{s} ds + \frac{1}{2\pi i} \int_{a-iT}^{a+iT} r_m(s) \frac{e^{\omega s}}{s} ds
\]

Here the integral on the left exists since \( f(s) \) is regular on \( \sigma = a > \sigma_0 \). By Lemma\[\text{[4]}\] as applied to the first integral on the right hand side, we
get

\[
(B) \lim_{T \to \infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(s) \frac{e^{i\omega s}}{s} ds - \sum_{\lambda_n < \omega} \lambda_n = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} r_m(s) \frac{e^{i\omega s}}{s} ds
\]

the limits existing.

Since \(\lambda_n \to \infty\), we can choose \(m_0\) such that for all \(m \geq m_0\) we have \(\lambda_m > \omega\). For all such \(m\) the last relation holds.

Now write \(b_n = a_n e^{-\lambda_n \sigma'}\), \(\phi(x) = e^{-x(s-\sigma')}\), so that \(\sum a_n e^{-\lambda_n s} = \sum b_n\phi(\lambda)\). Write \(B(x) = \sum_{n \leq x} b_n\).

Then applying the Lemma of the 9th lecture, we get

\[
\sum_{m+1}^{N} a_n e^{-\lambda_m s} = \sum_{1}^{N} - \sum_{1}^{m} \int_{\lambda_m}^{\lambda_N} B(x)e^{-x(s-\sigma')} dx
\]

\[
= (s - \sigma') \int_{\lambda_m}^{\lambda_N} B(x)e^{-x(s-\sigma')} dx + B(\lambda_N)e^{-\lambda_N(s-\sigma') - B(m)e^{-\lambda_m(s-\sigma')}}
\]

\[
= (B(\lambda_N) - B(\lambda_m))e^{-\lambda_N(s-\sigma')} + (s - \sigma') \int_{\lambda_m}^{\lambda_N} [B(x) - B(\lambda_m)]e^{-x(s-\sigma')} dx.
\]

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Now choose \(\sigma' > 0\) and \(a > \sigma' > \sigma_0\). [If \(\sigma_0 \geq 0\) the second condition implies the first; if \(\sigma_0 < 0\), then since \(a > 0\), choose \(0 < \sigma' < a\).

Then \(B(x)\) tends to a limit as \(x \to \infty\), so that \(B(x) = o(1)\) for all \(x\); while \(|e^{-x(s-\sigma')}| = e^{-x(\sigma-\sigma')} \to 0\) as \(x \to \infty\) for \(\sigma > \sigma'\). Hence, letting \(N \to \infty\), we get

\[
r_m(s) = (s - \sigma') \int_{\lambda_m}^{\infty} [B(x) - B(\lambda_m)]e^{-x(s-\sigma')} dx,
\]
or

\[ |r_m(s)| \leq e \frac{|s - \sigma'|}{\sigma - \sigma'} e^{-\lambda_m (\sigma - \sigma')} \]

Now consider

\[ \frac{1}{2\pi i} \int_{L + C_R} r_m(s) \frac{e^{\omega s}}{s} ds \]

where \( L + C_R \) is the contour indicated in the diagram.

\( r_m(s) \) is regular in this contour, and the integral is therefore zero. Hence

\[ \int_L = - \int_{C_R} \]

On \( C_R \), however, \( s = \sigma' + re^{i\theta} \) and \( \sigma \geq a \), so that

\[ |r_m(s)| \leq C \cdot \frac{r}{\alpha - \sigma'} \cdot e^{-\lambda_m r \cos \theta} \]

\[ |e^{\omega s}| = e^{\omega(\sigma' + r \cos \theta)}, |s| \geq r. \]

Hence

\[ \left| \frac{1}{2\pi i} \int_{C_R} r_m(s) \frac{e^{\omega s}}{s} ds \right| \leq \frac{cr}{2\pi (\alpha - \sigma')} e^{\omega \sigma'} \int_{-\pi/2}^{\pi/2} e^{(\omega - \lambda_m) r \cos \theta} d\theta \]

\[ = \frac{cr}{\pi (\alpha - \sigma')} e^{\omega \sigma'} \int_{0}^{\pi/2} e^{(\omega - \lambda_m) r \sin \theta} d\theta \]
However, $\sin \theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \pi/2$; hence
\[
\left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} r_m(s) e^{\omega s} \frac{e^{\omega s}}{s} ds \right| \leq \frac{c e^{\omega \sigma'}}{\pi(a - \sigma')} \int_0^{\pi/2} e^{\frac{1}{2}(\omega - \lambda_m)\theta} d\theta
\]
\[
\leq \frac{c e^{\omega \sigma'}}{2(a - \sigma')(\lambda_m - \omega)}
\]
for all $T$. Hence
\[
(C) \quad \lim_{T \to \infty} \left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} r_m(s) e^{\omega s} \frac{e^{\omega s}}{s} ds \right| \leq \frac{c_1}{(\lambda_m - \omega)}
\]
for all $m \geq m_0$.

Now reverting to relation (A), we get
\[
(D) \quad \lim_{T \to \infty} \left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(s) e^{\omega s} \frac{e^{\omega s}}{s} ds - \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f_m(s) e^{\omega s} \frac{e^{\omega s}}{s} ds \right|
\]
\[
= \lim_{T \to \infty} \left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} r_m(s) e^{\omega s} \frac{e^{\omega s}}{s} ds \right|
\]
for every $m$. However,
\[
\lim_{T \to \infty} \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f_m(s) e^{\omega s} \frac{e^{\omega s}}{s} ds = \sum_{\lambda_m < \omega} a_n
\]
independently of $m$. Hence\(^1\)

\[^1\]We use the fact that if
\[
\lim_{T \to \infty} |f(T) - g(T)| = \alpha
\]
and $\lim_{T \to \infty} g(T) = \beta$, then
\[
\lim_{T \to \infty} |f(T) - \beta| \leq \alpha
2. (Contd). Elementary theory of Dirichlet series

\[
\lim_{T \to \infty} \left| \frac{1}{2\pi i} \int_{a-iT}^{a+iT} f(s) e^{as} \frac{ds}{s} - \sum_{\lambda_n < \omega} a_n \right| \leq \text{left hand side in (D)} \leq \frac{c_1}{\lambda_m - \omega} \text{ for } m > m_0.
\]

Letting \( m \to \infty \) we get the result. \( \square \)

Remarks. (i) If \( \omega = \lambda_n \) the last term in the sum \( \sum_{\lambda_n < \omega} a_n \) has to be multiplied by \( \frac{1}{2} \).

(ii) If \( a < 0, (a > \sigma_0) \) then in the contour-integration the residue at the origin has to be taken, and this will contribute a term \( f(0) \).

In the proof, the relation between \( |s| \) and \( r \) has to be modified.
Lecture 11

The Zeta Function of Riemann (Contd)

3 Analytic continuation of $\zeta(s)$. First method [16, p.18]

We have, for $\sigma > 0$,

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \, dx.$$  

Writing $nx$ for $x$, and summing over $n$, we get

$$\sum_{n=1}^\infty \frac{\Gamma(s)}{n^s} = \sum_{n=0}^\infty x^{s-1} e^{-nx} dx = \int_0^\infty \left( \sum_{n=1}^\infty e^{-nx} \right) x^{s-1} \, dx,$$

if $\sigma > 1$.

since

$$\sum_{n=1}^\infty \left| \int_0^\infty x^{s-1} e^{-nx} \, dx \right| \leq \sum_{n=1}^\infty \int_0^\infty x^{s-1} e^{-nx} \, dx = \sum_{n=1}^\infty \frac{\Gamma(s)}{n^\sigma} < \infty,$$
if \( \sigma > 1 \). Hence

\[
\sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} = \int_0^\infty \frac{e^{-x}}{1 - e^{-x}} x^{s-1} \, dx = \int_0^\infty \frac{x^{s-1} \, dx}{e^x - 1}
\]

or

\[
\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1} \, dx}{e^x - 1}, \quad \sigma > 1
\]

In order to continue \( \zeta(s) \) analytically all over the \( s \)-plane, consider the complex integral

\[
I(s) = \int_C \frac{z^{s-1}}{e^z - 1} \, dz
\]

where \( C \) is a contour consisting of the real axis from \( +\infty \) to \( \rho \), \( 0 < \rho < 2\pi \), the circle \( |z| = \rho \); and the real axis from \( \rho \) to \( \infty \). \( I(s) \), if convergent, is independent of \( \rho \), by Cauchy's theorem.

Now, on the circle \( |z| = \rho \), we have

\[
|z^{s-1}| = |e^{(s-1)\log \rho}| = |e^{[(s-1)+i\theta] \log \rho}| = e^{(s-1)\log \rho - i \arg z} = |\rho|^{s-1} e^{2\pi i \theta},
\]

while

\[
|e^z - 1| > A|z|
\]

Hence, for fixed \( s \),

\[
\left| \int_{|z|=\rho} \frac{2\pi \rho \cdot \rho^{s-1}}{A\rho} \cdot e^{2\pi i \theta} \right| \to 0 \text{ as } \rho \to 0, \text{ if } \sigma > 1.
\]

Thus, on letting \( \rho \to 0 \), we get, if \( \sigma > 1 \),

\[
I(s) = -\int_0^\infty \frac{x^{s-1} \, dx}{e^x - 1} + \int_0^\infty \frac{(xe^{2\pi i})^{s-1} \, dx}{e^x - 1}
\]
3. Analytic continuation of \( \zeta(s) \). First method

\[ \zeta(s) = -\Gamma(s)\zeta(s) + e^{2\pi is}\Gamma(s)\zeta(s) = \Gamma(s)\zeta(s)(e^{2\pi s} - 1). \]

Using the result

\[ \Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \]

we get

\[ I(s) = \frac{\zeta(s)}{\Gamma(1-s)} \cdot 2\pi i \cdot e^{2\pi is} - 1 \]

\[ = \frac{\zeta(s)}{\Gamma(1-s)} \cdot 2\pi i \cdot e^{\pi is}, \]

or

\[
\zeta(s) = \frac{e^{-\pi s} \Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1} dz}{e^z - 1}, \quad \sigma > 1.
\]

The integral on the right is uniformly convergent in any finite region of the \( s \)-plane (by obvious majorization of the integrand), and so defines an entire function. Hence the above formula, proved first for \( \sigma > 1 \), defines \( \zeta(s) \), as a meromorphic function, all over the \( s \)-plane. This only possible poles are the poles of \( \Gamma(1-s) \), namely \( s = 1, 2, 3, \ldots \). We know that \( \zeta(s) \) is regular for \( s = 2, 3, \ldots \) (As a matter of fact, \( I(s) \) vanishes at these points). Hence the only possible pole is at \( s = 1 \).

Hence

\[ I(1) = \int_C \frac{dz}{e^z - 1} = 2\pi i, \]

while

\[ \Gamma(1-s) = -\frac{1}{s-1} + \cdots \]

Hence the residue of \( \zeta(s) \) at \( s = 1 \) is 1.

We see in passing, since

\[ \frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + B_1 \frac{z}{2!} - B_2 \frac{z^3}{4!} + \cdots \]

that

\[ \zeta(0) = -\frac{1}{2}, \zeta(-2m) = 0, \zeta(1-2m) = \frac{(-1)^m B_m}{2m}, m = 1, 2, 3, \ldots \]
4 Functional Equation (First method) [16, p.18]

Consider the integral
\[ \int \frac{z^{s-1}dz}{e^z - 1} \]
taken along \( C_n \) as in the diagram.

Between \( C \) and \( C_n \), the integrand has poles at
\[ \pm 2i\pi, \ldots, \pm 2n i\pi. \]

The residue at
\[ 2m\pi i \]
is
\[ (2m\pi e^{\pi i})^{s-1} \]
while the residue at \(-2m\pi i\) is \((2m\pi e^{3\pi i})^{s-1}\); taken together they amount to
\[ \frac{(2m\pi e^{\pi i(s-1)})}{2} \left[ e^{\pi i(s-1)} + e^{-\pi i(s-1)} \right] \]
\[ = (2m\pi)^{s-1} e^{\pi i(s-1)} 2 \cos \frac{\pi}{2} (s - 1) \]
\[ = -2(2m\pi)^{s-1} e^{\pi i s} \sin \frac{\pi}{2} s \]
Hence

\[ I(s) = \int_{C_n} \frac{z^{s-1}dz}{e^z - 1} + 4\pi i \frac{\sin \pi s}{2} e^{\pi i s} \sum_{m=1}^{\infty} (2m\pi)^{s-1} \]

by the theorem of residues.

Now let \( \sigma < 0 \), and \( n \to \infty \). Then, on \( C_n \),

\[ |z^{s-1}| = O(|z|^\sigma - 1) = O(n^{\sigma - 1}), \]

and

\[ \frac{1}{e^z - 1} = O(1), \]

for

\[ |e^z - 1|^2 = |e^{x+iy} - 1|^2 = |e^y(\cos y + i \sin y) - 1|^2 \\
= e^{2y} - 2e^y \cos y + 1, \]

which, on the vertical lines, is \( \geq (e^x - 1)^2 \) and, on the horizontal lines, \( = (e^x + 1)^2 \). (since \( \cos y = -1 \) there).

Also the length of the square-path is \( O(n) \). Hence the integral round the square \( \to 0 \) as \( n \to \infty \).

Hence

\[ I(s) = 4\pi i e^{\pi i s} \frac{\sin \pi s}{2} \sum_{m=1}^{\infty} (2m\pi)^{s-1} \]

\[ = 4\pi i e^{\pi i s} \frac{\pi s}{2} (2\pi)^{s-1} \zeta(1 - s), \text{ if } \sigma < 0. \]

or

\[ \zeta(s)\Gamma(s)(e^{2\pi i s} - 1) = (4\pi i)(2\pi)^{s-1} e^{\pi i s} \frac{\pi s}{2} \zeta(1 - s) \]

\[ = 2\pi i e^{\pi i s} \frac{\zeta(s)}{\Gamma(1 - s)} \]

Thus

\[ \zeta(s) = \Gamma(1 - s)\zeta(1 - s)2^s \pi^{s-1} \sin \frac{\pi s}{2}. \]
for \( \sigma < 0 \), and hence, by analytic continuation, for all values of \( s \) (each side is regular except for poles!).

This is the functional equation.

Since
\[
\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \frac{\sqrt{\pi}}{2^{x-1}}\Gamma(x),
\]
we get, on writing \( x = 1 - s \),
\[
2^{-s}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \sqrt{\pi}\Gamma(1-s);
\]
also
\[
\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\sin \frac{\pi s}{2}}
\]
Hence
\[
\Gamma(1-s) = 2^{-s}\Gamma\left(\frac{1-s}{2}\right)\left(\Gamma\left(\frac{s}{2}\right)\right)^{-1} \sqrt{\pi}\sin \left(\frac{\pi s}{2}\right)^{-1}
\]
Thus
\[
\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)
\]

If
\[
\xi(s) \equiv \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)
\]
\[
\equiv \frac{1}{2}s(s-1)\eta(s),
\]
then \( \eta(s) = \eta(1-s) \) and \( \xi(s) = \xi(1-s) \). If \( \equiv (z) = \xi\left(\frac{1}{2} + iz\right) \), then \( \equiv (z) \equiv (-z) \).

5 Functional Equation (Second Method) [16, p.13]

Consider the lemma given in Lecture 9 and write \( \lambda_n = n, a_n = 1, \phi(x) = x^{-s} \) in it. We then get
\[
\sum_{n \leq x} n^{-s} = s \int_1^x \left\lfloor \frac{x}{n} \right\rfloor \frac{dx}{x^{s+1}} + \frac{[X]}{X^s}, \text{ if } X \geq 1.
\]
5. Functional Equation (Second Method)

$$\zeta(s) = \frac{s}{s-1} - \frac{s}{(s-1)X^{s-1}} - \frac{1}{X^{s-1}} + \frac{1}{X^s} - X - [X] \frac{1}{X^s}$$

Since

$$\frac{1}{X^{s-1}} = 1/X^{s-1}, \text{ and } \frac{X - [X]}{X^s} \leq 1/X^s,$$

we deduce, on making $X \to \infty$,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{x - [x]}{x^{s+1}} dx, \text{ if } \sigma > 1$$

or

$$\zeta(s) = s \int_1^\infty \frac{[x] - x + \frac{1}{2}}{x^{s+1}} dx + \frac{1}{s-1} + \frac{1}{2}, \text{ if } \sigma > 1$$

5.1 Since $[x] - x + \frac{1}{2}$ is bounded, the integral on the right hand side converges for $\sigma > 0$, and uniformly in any finite region to the right of $\sigma = 0$. Hence it represents an analytic function of $s$ regular for $\sigma > 0$, and so provides the continuation of $\zeta(s)$ up to $\sigma = 0$, and $s = 1$ is clearly a simple pole with residue 1.

For $0 < \sigma < 1$, we have, however,

$$\int_0^1 \frac{[x]}{x^{s+1}} dx = -\int_0^1 x^{-s} dx = \frac{1}{s-1},$$

and

$$\frac{s}{2} = \int_1^\infty \frac{dx}{x^{s+1}} = \frac{1}{2}$$

Hence
5.2

\[ \zeta(s) = s \int_0^\infty \frac{[x] - x}{x^{s+1}} \, dx, \quad 0 < \sigma < 1 \]

We have seen that (5.1) gives the analytic continuation up to \( \sigma = 0 \).

By refining the argument dealing with the integral on the right-hand side of (5.1) we can get the continuation all over the \( s \)-plane. For, if

\[ f(x) \equiv [x] - x + \frac{1}{2}, \quad f_1(x) \equiv \int_1^x f(y) \, dy, \]

then \( f_1(x) \) is also bounded, since \[ \frac{n+1}{n} \int_1^n f(y) \, dy = 0 \] for any integer \( n \).

Hence

\[ \int_{x_1}^{x_2} \frac{f(x)}{x^{s+1}} \, dx = \left[ \frac{f_1(x)}{x^{s+1}} \right]_{x_1}^{x_2} + (s + 1) \int_{x_1}^{x_2} \frac{f_1(x)}{x^{s+2}} \, dx \]

\[ \to 0, \quad \text{as} \ x_1 \to \infty, x_2 \to \infty, \]

[if \( \sigma > -1 \).]

Hence the integral in (5.1) converges for \( \sigma > -1 \).

Further

\[ s \int_0^1 \frac{[x] - x + \frac{1}{2}}{x^{s+1}} \, dx = \frac{1}{s-1} + \frac{1}{2}, \quad \text{for} \ \sigma < 0; \]

\[ \zeta(s) = s \int_0^\infty \frac{[x] - x + \frac{1}{2}}{x^{s+1}} \, dx, \quad -1 < \sigma < 0 \quad (5.3) \]

Now the function \( [x] - x + \frac{1}{2} \) has the Fourier expansion

\[ \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi} \]
if $x$ is not an integer. The series is boundedly convergent. If we substitute it in (5.3), we get

$$
\zeta(s) = \frac{s}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{\infty} \frac{\sin 2n\pi x}{x^{s+1}} dx
$$

$$
= \frac{s}{\pi} \sum_{n=1}^{\infty} \frac{(2n\pi)^s}{n} \int_{0}^{\infty} \frac{\sin y}{y^{s+1}} dy
$$

$$
\zeta(s) = \frac{s}{\pi} (2\pi)^s \{-\Gamma(1-s)\} \sin \frac{s\pi}{2} \zeta(1 - s).
$$

5.4 If termwise integration is permitted, for $-1 < \sigma < 0$. The right hand side is, however, analytic for all values of $s$ such that $\sigma < 0$. Hence (5.4) provides the analytic continuation (not only for $-1 < \sigma < 0$) all over the $s$-plane.

The term-wise integration preceding (5.4) is certainly justified over any finite range since the concerned series is boundedly convergent. We have therefore only to prove that

$$
\lim_{X \to \infty} \sum_{n=1}^{\infty} \frac{1}{n} \int_{X}^{\infty} \frac{\sin 2n\pi x}{x^{s+1}} dx = 0, -1 < \sigma < 0
$$

Now

$$
\int_{X}^{\infty} \frac{\sin 2n\pi x}{x^{s+1}} dx = \left[ -\frac{\cos 2n\pi x}{2n\pi x^{s+1}} \right]_{X}^{\infty} - \frac{s + 1}{2n\pi} \int_{X}^{\infty} \frac{2n\pi x}{x^{s+2}} dx
$$

$$
= O\left( \frac{1}{nX^{\sigma+1}} \right) + O\left( \frac{1}{n} \int_{X}^{\infty} \frac{dx}{x^{\sigma+2}} \right)
$$

$$
= O\left( \frac{1}{nX^{\sigma+1}} \right)
$$
and hence
\[
\lim_{X \to \infty} \sum_{n=1}^{\infty} \frac{1}{n} \int_{X}^{\infty} \frac{\sin 2n\pi x}{x^{s+1}} \, dx = 0, \quad \text{if} \quad -1 < \sigma < 0
\]

This completes the analytic continuation and the proof of the Functional equation by the second method.

As a consequence of (5.1), we get
\[
\lim_{s \to 1} \left\{ \zeta(s) - \frac{1}{s-1} \right\} = \int_{1}^{\infty} \left\lfloor x \right\rfloor - x + \frac{1}{2} \frac{dx}{x^2} + \frac{1}{2}
\]
\[
= \lim_{n \to \infty} \int_{1}^{n} \left\lfloor x \right\rfloor - x + \frac{1}{2} \frac{dx}{x^2} + 1
\]
\[
= \lim_{n \to \infty} \left\{ \sum_{m=1}^{n-1} \frac{m+1}{m} + 1 - \log n \right\}
\]
\[
= \gamma
\]

Hence, near \( s = 1 \), we have
\[
\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)
\]
Lecture 12

The Zeta Function of Riemann (Contd)

6 Some estimates for $\zeta(s)$ [11, p.27]

In any fixed half-plane $\sigma \geq 1 + \epsilon > 1$, we know that $\zeta(s)$ is bounded, since
\[ |\zeta(s)| \leq \zeta(\sigma) \leq \zeta(1 + \epsilon). \]

We shall now use the formulae of §5 to estimate $|\zeta(s)|$ for large $t$, where $s = \sigma + it$.

**Theorem.**

\[ |\zeta(s)| < A \log t, \quad \sigma \geq 1, \quad t \geq 2. \]
\[ |\zeta(s)| < A(\delta)^{1-\delta}, \quad \sigma \geq \delta, \quad t \geq 1, \text{ if } 0 < \delta < 1. \]

**Proof.** From (5.1) we get
\[ \zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{x-[x]}{x^{s+1}} \, dx, \quad \sigma > 0 \]
Also from §5 of the 11th Lecture,

\[ \sum_{n \leq X} \frac{1}{n^s} = s \int_1^X \frac{\lfloor x \rfloor}{x^{s+1}} \, dx + \frac{\lfloor X \rfloor}{X^s}, \text{ if } X \geq 1, \, s \neq 1 \]

If \( \sigma > 0, \, t \geq 1, \, X \geq 1 \), we get

\[ \zeta(s) - \sum_{n \leq X} \frac{1}{n^s} = -s \int_X^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}} \, dx + \frac{1}{(s - 1)X^{s-1}} + \frac{X - \lfloor X \rfloor}{X^s} \]

Hence

\[ |\zeta(s)| < \sum_{n \leq X} \frac{1}{n^\sigma} + \frac{1}{tX^{\sigma-1}} + \frac{1}{X^\sigma} + |s| \int_X^\infty \frac{dx}{x^{\sigma+1}} \]

\[ < \sum_{n \leq X} \frac{1}{n^\sigma} + \frac{1}{tx^{\sigma-1}} + \frac{1}{X^\sigma} + \left(1 + \frac{t}{\sigma}\right) \frac{1}{X^\sigma}, \text{ since } |s| < \sigma + t. \]

If \( \sigma \geq 1 \),

\[ |\zeta(s)| < \sum_{n \leq X} \frac{1}{n^\sigma} + \frac{1}{\sigma} + \frac{1}{X} + \frac{1}{X} \]

\[ \leq (\log X + 1) + 3 + \frac{t}{X}, \text{ since } t \geq 1, \quad X \geq 1. \]

Taking \( X = t \), we get the first result.

If \( \sigma \geq \eta \), where \( 0 < \eta < 1 \),

\[ |\zeta(s)| < \sum_{n \leq X} \frac{1}{n^\eta} + \frac{1}{tX^{\eta-1}} + \left(2 + \frac{1}{\eta}\right) \frac{1}{X^\eta} \]

\[ < \int_0^{\lfloor X \rfloor} \frac{dx}{x^\eta} + \frac{X^{1-\eta}}{t} + \frac{3t}{\eta X^\eta} \]

\[ \leq \frac{X^{1-\eta}}{1-\eta} + X^{1-\eta} + \frac{3t}{\eta X^\eta} \]
Taking $X = t$, we deduce

$$|\zeta(s)| < t^{1-\eta} \left( \frac{1}{1-\eta} + 1 + \frac{3}{\eta} \right), \quad \sigma \geq \eta, \quad t \geq 1.$$  

\[\square\]

7 Functional Equation (Third Method) [16, p.21]

The third method is based on the ‘theta-relation’

$$\vartheta(x) = \frac{1}{\sqrt{x}} \vartheta \left( \frac{1}{x} \right),$$

where

$$\vartheta(x) = \sum_{n=\infty}^{\infty} e^{-\pi n^2 x}, \text{Re } x > 0$$

This relation can be proved directly, or obtained as a special case of ‘Poisson’s formula’. We shall here give a proof of that formula [2, p.37]

Let $f(x)$ be continuous in $(-\infty, \infty)$. Let

$$\phi(\alpha) = \int_{-\infty}^{\infty} f(x) e^{2\pi i \alpha x} \, dx \equiv \lim_{T \to \infty} \int_{-T}^{T} f(x) e^{2\pi i \alpha x} \, dx,$$

if the integral exists. Then, for $\alpha$ and pro-positive integers, we have

$$\int_{-p^{1/2}}^{p^{1/2}} f(x) e^{2\pi i \alpha x} \, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{ \sum_{k=-p}^{p} f(x+k) \right\} e^{2\pi i \alpha x} \, dx.$$

If we assume that

$$\lim_{p \to \infty} \sum_{k=-p}^{p} f(x+k) = g(x) \quad (7.1)$$
uniformly in $-\frac{1}{2} \leq x \leq \frac{1}{2}$, then $g(x)$ is continuous, and on letting $p \to \infty$ in (7.1), we get

$$
\phi(\alpha) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x)e^{2\pi i \alpha x} dx.
$$

Thus $\phi(\alpha)$ is the Fourier coefficient of the continuous periodic function $g(x)$. Hence, by Fejer’s theorem,

$$g(0) = \sum_{(\varepsilon,1)} \phi(\alpha)$$

If we assume, however, that $\sum_{-\infty}^{\infty} \phi(\alpha) < \infty$, then $\sum_{(\varepsilon,1)} \phi(\alpha) = \sum_{-\infty}^{\infty} \phi(\alpha)$

Hence

$$\lim_{p \to \infty} \sum_{-p}^{p} f(k) = \lim_{p \to \infty} \sum_{-p}^{p} \phi(\alpha)$$

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$$\sum_{-\infty}^{\infty} f(k) = \sum_{-\infty}^{\infty} \phi(\alpha) \quad (7.2)$$

This is Poisson’s formula. We may summarize our result as follows. **Poisson’s Formula.** If

(i) $f(x)$ is continuous in $(-\infty, \infty)$,

(ii) $\sum_{-\infty}^{\infty} f(x + k)$ converges uniformly in $-\frac{1}{2} \leq x < \frac{1}{2}$ and

(iii) $\sum_{-\infty}^{\infty} \phi(\alpha)$ converges,

then

$$\sum_{-\infty}^{\infty} f(k) = \sum_{-\infty}^{\infty} \phi(\alpha)$$

**Remarks.** This result can be improved by a relaxation of the hypotheses.
The Theta Relation. If we take \( f(x) = e^{-\pi x^2}, t > 0, \) in the above discussion, we obtain the required relation. However, we have to show that for this function the conditions of Poisson’s formula are satisfied. This is easily done. \( f(x) \) is obviously continuous in \((−\infty, \infty)\). Secondly

\[
e^{-\pi(x+k)^2}t \leq e^{\|k\|t} \text{ for } |k| \geq k_0(t)
\]

This proves the uniform convergence of

\[
\sum_{k=−\infty}^{\infty} e^{-\pi(x+k)^2}t, \quad t > 0, \quad \frac{1}{2} \leq x < \frac{1}{2}
\]

Thirdly we show that, for \( t > 0 \)

\[
\phi(\alpha) \equiv \int_{−\infty}^{\infty} e^{-\pi x^2} e^{2\pi\alpha y} dy = \frac{1}{2\sqrt{t}} e^{-\pi\alpha^2/t}
\]

For

\[
\phi(\alpha) = e^{-\pi\alpha^2/t} \int_{−\infty}^{\infty} e^{-\pi t \left(y - \frac{\alpha}{t}\right)^2} dy, \text{ and}
\]

If \( \alpha \geq 0, \) we consider the contour integral

\[
\int_{C} e^{-\pi t s^2} ds, \quad s = \sigma + i\tau
\]

taken over \( C \) as indicated. Then on the vertical lines, we have
\[ e^{-n \text{Re}(\pm \sigma + it)^2} = e^{-n(\sigma^2 - t^2)} \leq e^{-n(\sigma^2 - \frac{t^2}{r})} \]

Letting \( \sigma \to \infty \), we thus see that the integrals along the vertical lines vanish. Hence we have, for all real \( \alpha \),

\[
\int_{-\infty}^{\infty} e^{-n \alpha^2} d\sigma = \int_{-\infty}^{\infty} e^{-n(\sigma - \frac{1}{r} t)^2} d\sigma
\]

Thus

\[
\phi(\alpha) = e^{-n\alpha^2/it} \int_{-\infty}^{\infty} e^{-n\alpha^2} d\sigma
\]

\[
= e^{-n\alpha^2/it} \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-nu^2} du
\]

\[
= e^{-n\alpha^2/it} \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-v^2} dv = e^{-n\alpha^2/it} \frac{1}{\sqrt{t}} \int_{0}^{\infty} e^{-z^2/2} dz
\]

\[
= \frac{e^{-n\alpha^2/it}}{\sqrt{t}} \frac{1}{\sqrt{\pi}} \Gamma(\frac{1}{2}) = \frac{e^{-n\alpha^2/it}}{\sqrt{t}}
\]

Hence we have the desired relation:

\[
\sum_{-\infty}^{\infty} e^{-nk^2/t} = \frac{1}{\sqrt{t}} \sum_{-\infty}^{\infty} e^{-nk^2/t}, \text{ for } t > 0, \quad (7.3)
\]

or, writing

\[
\psi(t) = \sum_{1}^{\infty} e^{-nk^2/t}, \text{ we get}
\]

\[
2\psi(t) + 1 = \frac{1}{\sqrt{t}} \left\{ 2\psi \left( \frac{1}{t} \right) + 1 \right\} \quad (7.4)
\]

Remark. (7.3) holds for \( t \) complex, with \( \text{Re}(t) > 0 \).
7. Functional Equation (Third Method)

We have, for \( \sigma > 0 \),

\[
\Gamma\left( \frac{s}{2} \right) = \int_{0}^{\infty} e^{-x} x^{s/2-1} \, dx.
\]

If \( \sigma > 0 \), \( n > 0 \), we then have on writing \( \pi n^2 x \) for \( x \),

\[
\Gamma\left( \frac{s}{2} \right) = \int_{0}^{\infty} e^{-\pi n^2 x} (\pi n^2 x)^{s/2-1} \pi n^2 \, dx
\]

or

\[
\frac{1}{n^s} = \frac{\pi^{s/2}}{\Gamma(s/2)} \int_{0}^{\infty} e^{-\pi n^2 x} x^{s/2-1} \, dx.
\]

Now if \( \sigma > 1 \), we have

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{\pi^{s/2}}{\Gamma(s/2)} \sum_{n=1}^{\infty} \int_{0}^{\infty} e^{-\pi n^2 x} x^{s/2-1} \, dx
\]

the inversion being justified by absolute convergence. Hence, for \( \sigma > 1 \), we have

\[
\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \int_{0}^{\infty} \psi(x) x^{s/2-1} \, dx.
\]

Using (7.4) we rewrite this as

\[
\pi^{-s/2} \Gamma(s/2) \zeta(s) = \int_{0}^{1} x^{s/2-1} \psi(x) \, dx + \int_{1}^{\infty} \psi(x) x^{s/2-1} \, dx
\]
\[ \int_0^1 x^{s/2 - 1} \left\{ \frac{1}{\sqrt{x}} \psi \left( \frac{1}{x} \right) + \frac{1}{2 \sqrt{x}} - \frac{1}{2} \right\} \, dx + \int_1^\infty \psi(x) x^{s/2 - 1} \, dx \]
\[ = \frac{1}{s - 1} - \frac{1}{s} + \int_0^1 x^{s/2 - 3/2} \psi \left( \frac{1}{x} \right) \, dx + \int_1^\infty \psi(x) x^{s/2 - 1} \, dx \]
\[ = \frac{1}{s(s - 1)} + \int_1^\infty \left( x^{-s/2 - 1/2} + x^{s/2 - 1} \right) \psi(x) \, dx, \tag{7.5} \]

for \( \sigma > 1 \).

Now the integral on the right converges uniformly in \( a \leq s \leq b \), for if \( x \geq 1 \), we have
\[ \left| x^{-s/2 - 1/2} + x^{s/2 - 1} \right| \leq x^{b/2 - 1} + x^{-a/2 - 1/2} \]
while
\[ \psi(x) < \sum_1^\infty e^{-\pi n x} = \frac{1}{e^{\pi x} - 1}, \]
and hence the integral on the right hand side of (7.5) is an entire function. Hence (7.5) provides the analytic continuation to the left of \( \sigma = 1 \).

It also yields the functional equation directly. We have also deduced that
\[ \pi^{-s/2} \Gamma(s/2) \zeta(s) - \frac{1}{s(s - 1)} \]
is an entire function; so is \( \pi^{s/2} \left[ \Gamma(s/2) \right]^{-1} \). Hence
\[ \zeta(s) - \frac{1}{s(s - 1)} \cdot \frac{\pi^{s/2}}{\Gamma(s/2)} = \zeta(s) - \frac{1}{s - 1} \cdot \frac{\pi^{s/2}}{2\Gamma(s/2 + 1)} \]
is an entire function. But \( \frac{\pi^{1/2}}{2\Gamma(1/2 + 1)} = 1 \). Hence \( \zeta(s) - \frac{1}{s - 1} \) is an entire function.
Lecture 13

The Zeta Function of Riemann (Contd)

8 The zeros of $\zeta(s)$

In the $11^{th}$ Lecture, (p.98) we defined

$$\xi(s) \equiv \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \equiv \frac{1}{2} s(s-1)\eta(s).$$

The following theorem about $\xi(s)$ is a consequence of the results which we have already proved.


(i) $\xi(s)$ is an entire function, and

$$\xi(s) = \xi(1 - s);$$

(ii) $\xi(s)$ is real on $t = 0$ and $\sigma = \frac{1}{2};$

(iii) $\xi(0) = \xi(1) = \frac{1}{2}.$

Proof. That $\xi(s) = \xi(1 - s)$ is immediate from the functional equation of $\zeta(s)$ (cf. p.97, Lecture 11). Clearly $\xi(s)$ is regular for $\sigma > 0,$ and since $\xi(s) = \xi(1 - s)$ it is also regular for $\sigma < 1,$ and hence it is entire.
This proves (i)
\(\xi(s)\) is obviously real for real \(s\). Hence by a general theorem it takes conjugate values at conjugate points. Thus \(\xi(\frac{1}{2} + ti)\) and \(\xi(\frac{1}{2} - ti)\) are conjugate; they are equal by the functional equation. So they are real. This proves (ii).

To prove (iii) we observe that
\[\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2} + 1\right) \cdot (s - 1)\xi(s),\]
so that
\[\xi(1) = \pi^{-1/2} \Gamma(3/2) \cdot 1 = \frac{1}{2},\]
and by the functional equation, \(\xi(0) = \frac{1}{4}\). □

We shall next examine the location of the zeros of \(\xi(s)\) and of \(\zeta(s)\).

**Theorem 2.** (p.48) (i) The zeros of \(\xi(s)\) (if any!) are all situated in the strip \(0 \leq \sigma \leq 1\), and lie symmetrical about the lines \(t = 0\) and \(\sigma = \frac{1}{2}\).

(ii) The zeros of \(\zeta(s)\) are identical (in position and order of multiplicity) with those of \(\xi(s)\), except that \(\zeta(s)\) has a simple zero at each of the points \(s = -2, -4, -6, \ldots\).

(iii) \(\xi(s)\) has no zeros on the real axis.

**Proof.** (i) \(\xi(s) = (s - 1)\pi^{-s/2} \Gamma\left(\frac{s}{2} + 1\right)\zeta(s)\)
\[\equiv h(s)\zeta(s), \text{ say.}\]

Now \(\zeta(s) \neq 0\); for \(\sigma > 1\); (Euler Product!) also \(h(s) \neq 0\), for \(\sigma > 1\); hence \(\xi(s) \neq 0\), for \(\sigma > 1\); hence \(\xi(s) \neq 0\), for \(\sigma < 0\); since \(\xi(s) = \xi(1 - s)\).
The zeros, if any, are symmetrical about the real axis since \( \xi(\sigma \pm it) \) are conjugates, and about the point \( s = \frac{1}{2} \), since \( \xi(s) = \xi(1 - s) \); they are therefore also symmetrical about the line \( \sigma = \frac{1}{2} \).

(ii) The zeros of \( \zeta(s) \) differ from those of \( \xi(s) \) only in so far as \( h(s) \) has zeros or poles. The only zero of \( h(s) \) is at \( s = 1 \). But this is not a zero of \( \xi(s) \) since \( \xi(1) = \frac{1}{2} \), nor of \( \zeta(s) \) for it is a pole of the latter.

The poles of \( h(s) \) are simple ones at \( s = -2, -4, -6, \ldots \). Since these are points at which \( \xi(s) \) is regular and not zero, they must be simple zeros of \( \zeta(s) \). We have already seen this by a different argument on p.95.

(iii) Since \( \xi(s) \neq 0 \) for \( \sigma < 0 \) and \( \sigma > 1 \), it is enough, in view of (ii), to show that

\[ \zeta(\sigma) \neq 0, \quad \text{for } 0 < \sigma < 1. \]

Now

\[ (1 - 2^{1-s})\zeta(s) = (1 - 2^{-s}) + (3^{-s} - 4^{-s}) + \ldots, \quad \text{for } \sigma > 0. \]

To prove this we first notice that the relation is obvious for \( \sigma > 1 \), and secondly each side is regular for \( \sigma > 0 \); the left side is obviously regular for \( \sigma > 0 \) (in fact it is entire); and, as for the right side, we have

\[
\left| \frac{1}{(2n-1)^s} - \frac{1}{(2n)^s} \right| = \left| \frac{2n}{s} \int_{2n-1}^{2n} \frac{dx}{x^{s+1}} \right|
\]
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\[ \leq \frac{|s|}{(2n - 1)^{\sigma + 1}} < \frac{\Delta}{(2n - 1)^{\delta + 1}} \]

if \( \sigma > \delta \) and \( |s| < \Delta \) for any fixed \( \delta \) and \( \Delta \)

But, if \( 0 < \sigma < 1 \), the above relation gives

\[(1 - 2^{1-\sigma})\zeta(\sigma) > 0, \quad \text{or} \quad \zeta(\sigma) < 0.\]

This establishes (iii). \( \square \)

**Remarks.** The strip \( 0 \leq \sigma \leq 1 \) is called the ‘critical strip’; the line \( \sigma = \frac{1}{2} \) the ‘critical line’; the zeros at \(-2, -4, -6, \ldots\) are the ‘trivial zeros’ of \( \zeta(s) \). We still have to show that \( \xi(s) \) actually has zeros, i.e. \( \zeta(s) \) has ‘non-trivial’ zeros. This is done by an appeal to the theory of entire functions gives in the first few lectures.

**Theorem 3.** [11, p.56] If \( M(r) \equiv \max_{|s|=r} |\xi(s)| \), then

\[ \log M(r) \sim \frac{1}{2} r \log r, \quad \text{as} \quad r \to \infty \]

**Proof.** For \( \sigma \geq 2 \), we have

\[ |\zeta(s)| \leq \zeta(2), \]

and for \( \sigma \geq \frac{1}{2}, |t| \geq 1 \), we have

\[ |\zeta(s)| < c_1 |t|^{1/2}, \]

so that

\[ |\zeta(s)| < c_2 |s|^{1/2}, \quad \text{for} \quad \sigma \geq \frac{1}{2}, |s| > 3. \]

Applying Stirlingl’s formula to \( \Gamma(s/2) \), we get, for

\[ \sigma \geq \frac{1}{2}, \quad |s| = r > 3, \]

\[ |\xi(s)| < e^{\frac{1}{2} s \log \frac{1}{2} s + c_3 s} < e^{\frac{1}{2} r \log r + c_4 r}, \]
8. The zeros of $\zeta(s)$

since $\log \frac{s}{2} = \log |s| - \log 2 + i \arg s, |\arg s| < \frac{\pi}{2}$. From the equation $\xi(s) = \xi(1 - s)$, we infer that

$$|\xi(s)| < e^{\frac{1}{2}|1-s|\log |1-s| + c_1|1-s|}$$
$$< e^{\frac{1}{2}r \log r + c_5 r}$$

for $\sigma \leq \frac{1}{2}, |s| = r > 4$. Combining the results we get

$$M(r) < e^{\frac{1}{2}r \log r + c_5 r}, r > 4.$$ 

On the other hand, if $r > 2$,

$$M(r) \geq \xi(r) > 1.\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{2}r\right) e^{\frac{1}{2}r \log r - c_7 r}$$

Thus we have, for $r > 4$,

$$\frac{1}{2}r \log r - C_7 r < \log M(r) < \frac{1}{2}r \log r + c_6 r,$$

and hence the result. $\Box$

Theorem 4. [11, p.57]

(i) $\xi(s)$ has an infinity of zeros;

(ii) If they are denoted by ‘$\rho$’, then $\sum |\rho|^{-\alpha}$ converges for $\alpha > 1$, diverges for $\alpha \leq 1$;

(iii) $\xi(s) = e^{b_0 + b_1 s} \pi^{\frac{1}{2} \int (1 - s \rho)} e^{i \rho}$,

where $b_0, b_1$ are constants;

(iv) $\frac{\xi'(s)}{\xi(s)} = b_1 + \sum \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right)$

(v) $\frac{\xi'(s)}{\xi(s)} = b - \frac{1}{s - 1} - \frac{i}{2} \Gamma \left(\frac{s}{2} + 1\right) \sum \left(\frac{1}{s - \rho} + \frac{1}{\rho}\right)$
where \( b = b_1 + \frac{1}{2} \log 2\pi \).

**Proof.** \( \xi(0) = \frac{1}{2} \neq 0 \), and we apply the theory of entire functions as developed in the earlier lectures. By Theorem 3 \( \xi(s) \) is of order 1, and the relation \( \log M(r) = O(r) \) does not hold. Hence \( \rho_1 = 1 \), \( \xi(s) \) has an infinity of zeros and \( \sum \frac{1}{|\rho|} \) diverges (by a previous theorem). This proves (i), (ii) and (iii). (iv) follows by logarithmic differentiation. (v) is obtained from \( \xi(s) = (s-1)\pi^{-s/2}\Gamma(\frac{s}{2}) \xi(s) \).

We shall now show that \( \zeta(s) \) has no zeros on the line \( \sigma = 1 \).

**Theorem 5.**\(^1\) \( \zeta(1 + it) \neq 0 \).

**First Proof.** \( 2(1 + \cos \theta)^2 = 3 \geq 4 \cos \theta + \cos 2\theta \geq 0 \) for real \( \theta \). Since

\[
\log \zeta(s) = \sum_m \sum_p \frac{1}{mp^m}, \text{ for } \sigma > 1,
\]

we have

\[
\log |\zeta(s)| = \Re \sum_{n=2}^\infty c_n n^{-\sigma - ti} = \sum_{n=2}^\infty c_n n^{-\sigma} \cos(t \log n),
\]

where \( c_n = \begin{cases} 1/m & \text{if } n \text{ is the } m^{th} \text{ power of a prime} \\ 0 & \text{otherwise}. \end{cases} \)

Hence

\[
\log |\zeta^3(\sigma) \cdot \zeta(\sigma + ti) \zeta(\sigma + 2ti)| = \sum c_n n^{-\sigma} \left( 3 + 4 \cos(t \log n) + \cos(2t \log n) \right) \geq 0,
\]

because \( c_n \geq 0 \) and we have the trigonometric identity above.

Thus

\[
((\sigma - 1)\zeta(\sigma))^3 \left| \frac{\zeta(\sigma + ti)}{\sigma - 1} \right|^4 \left| \zeta(\sigma + 2ti) \right| \geq \frac{1}{\sigma - 1}.
\]
8. The zeros of \( \zeta(s) \)

if \( \sigma > 1 \).

Now suppose \( 1 + ti \) (\( t \neq 0 \)) were a zero of \( \zeta(s) \); then, on letting \( \sigma \to 1 + 0 \), we see that the left hand side in the above inequality would tend to a finite limit, viz. \( |\zeta'(1 + ti)|^2|\zeta(1 + 2ti)| \), while the right hand side tends to \( \infty \). Hence \( \zeta(1 + it) \neq 0 \).

**Second Proof.** [11, p.89] We know from the 8th Lecture (1.12) that if \( \alpha \) is real, and \( \alpha \neq 0 \), then

\[
f(s) \equiv \frac{\zeta^2(s)\zeta(s - \alpha i)\zeta(s + \alpha i)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\sigma_{11}(n)|^2}{n^s}, \sigma > 1 \quad (8.1)
\]

Let \( \sigma_0 \) be the abscissa of convergence of the series on the right hand side. Then \( \sigma_0 \leq 1 \). The sum-function of the series is then regular in the half-plane \( \sigma > \sigma_0 \), and hence, by analytic continuation, (8.1) can be upheld for \( \sigma > \sigma_0 \). And since the coefficients of the series are positive, the point \( s = \sigma_0 \) is a singularity of \( f(s) \).

Now, if \( 1 + \alpha i \) is a zero of \( \zeta(s) \), then so is \( 1 - \alpha i \), and these two zeros cancel the double pole of \( \xi^2(s) \) at \( s = 1 \). Hence \( f(s) \) is regular on the real axis, as far to the left as \( s = -1 \), where \( \zeta(2s) = 0 \). Hence \( \sigma_0 = -1 \). This is, however, impossible since (8.1) gives

\[
f\left(\frac{1}{2}\right) \geq |\sigma_{11}(1)|^2 = 1,
\]

while \( f\left(\frac{1}{2}\right) = 0 \). Thus \( \zeta(1 + ai) \neq 0 \).

**Remarks.** (i) We shall show later on that this fact (\( \zeta(1 + it) \neq 0 \)) is equivalent to the Prime Number Theorem.

(ii) If \( \rho = \beta + \gamma i \) is a zero of \( \xi(s) \), then we have seen that \( 0 \leq \beta \leq 1 \), and since \( \xi(1 + it) \neq 0 \), we actually have \( 0 \leq \beta < 1 \), and by symmetry about the line \( \sigma = \frac{1}{2} \), we have \( 0 < \beta < 1 \).
Lecture 14

The Zeta Function of Riemann (Contd)

9 Riemann-Von Magoldt Formula [11, p.68]

Let \( N(T) \) denote the number of zeros (necessarily finite) of \( \zeta(s) \) in \( 0 \leq \sigma \leq 1 \), \( 0 \leq t \leq T \). Then, by what we have proved, \( N(T) \) is the number of zeros \( \rho = \beta + iy \) of \( \zeta(s) \), or of \( \xi(s) \), for which \( 0 < \gamma \leq T \).

**Theorem 1.** \( N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \) as \( T \to \infty \).

**Proof.** Suppose first that \( T \) is not equal to any \( \gamma \). Also let \( T > 3 \). Consider the rectangle \( R \) as indicated. \( \xi(s) \) has \( 2N(T) \) zeros inside it, and none on the boundary. Hence, by the principle of the argument,

\[
2N(T) = \frac{1}{2\pi} [\arg \xi(s)]_R,
\]

where \( [\arg \xi(s)]_R \) denotes the increase in \( \arg \xi(s) \), as \( s \) describes the boundary of \( R \).
14. The Zeta Function of Riemann (Contd)

\[ \arg \xi(s)_R = \arg \frac{1}{2} s(s - 1)_R + \arg \eta(s)_R, \]

where \( \eta(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) \). Here

\[ \arg \frac{1}{2} s(s - 1)_R = 4\pi, \]

and \( \eta(s) \) has the properties: \( \eta(s) = \eta(1 - s) \), and \( \eta(\sigma \pm it) \) are conjugates. Hence

\[ \arg \eta(s)_R = 4\arg \eta(s)_{ABC} \]

Hence

\[ \pi N(T) = \pi + \arg [\pi^{-s/2}] + \arg \Gamma(s/2)_{ABC} + \arg \xi(s)_{ABC} \quad (9.1) \]

Now

\[ \arg \pi^{-s/2}_{ABC} = \left[ -\frac{T}{2} \log \pi \right]_{ABC} = -\frac{T}{2} \log \pi. \quad (9.2) \]

Next, since \( \Gamma(z + \alpha) = (z + \alpha - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + O(|z|^{-1}) \) as \( |z| \to \infty \), in any angle \( |\arg z| \leq \pi - \varepsilon < \pi \), we have
\[
\left[ \arg \Gamma \left( \frac{1}{2} \right) \right]_{\text{ABC}} = [\im \log \Gamma(s/2)]_{\text{ABC}} \\
= \im \log \Gamma \left( \frac{1}{4} + \frac{1}{2} iT \right) - \im \log \Gamma(1) \\
= \im \left\{ \left( -\frac{1}{4} + \frac{1}{2} iT \right) \log \left( \frac{1}{2} iT \right) - \frac{1}{2} iT + \frac{1}{2} \log 2\pi + O(T^{-1}) \right\} \\
= \frac{1}{2} T \log \left( \frac{1}{2} T \right) - \frac{1}{8} \pi - T^{-1} + O(T^{-1}), \quad \text{as } T \to \infty \quad (9.3)
\]

Using (9.2) and (9.3) in (9.1), we get
\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + \frac{1}{\pi} [\arg \zeta(s)]_{\text{ABC}} + \frac{1}{T} \quad (9.4)
\]

Consider the broken line ABC exclusive of the end-points A and C. If \( m \) is the number of distinct points \( s' \) on ABC at which \( \text{Re}\zeta(s) = 0 \), then
\[
[\arg \zeta(s)]_{\text{ABC}} \leq (m + 1)\pi, \quad (9.5)
\]

since \( \arg \zeta(s) \) cannot vary by more than \( \pi \) (since \( \text{Re}\zeta \) does not change sign here) on any one of the \( m + 1 \) segments into which ABC is divided by the point \( s' \).

Now no point \( s' \) can be on AB, for
\[
\text{Re}\zeta(2 + iT) \geq 1 - \sum_{2}^{\infty} \frac{1}{n^2} > 1 - \frac{1}{2^2} - \int_{\frac{1}{2}}^{\infty} \frac{du}{u^2} = \frac{1}{4} \quad (9.6)
\]

Thus \( m \) is the number of distinct points \( \sigma \) of \( \frac{1}{2} < \sigma < 2 \) at which \( \text{Re}\zeta(\sigma + iT) = 0 \), i.e. the number of distinct zeros of
\[
g(s) = \frac{1}{2} \{ \zeta(s + iT) + \zeta(s - iT) \}
\]
on the real axis for which \( \frac{1}{2} < \sigma < 2 \), because \( g(\sigma) = \text{Re}\zeta(\sigma + iT) \) for real \( \sigma \), since \( \zeta(\sigma \pm iT) \) are conjugate.
Now since $g(s)$ is regular except for $s = 1 \pm iT$, $m$ must be finite, and we can get an upper bound for $m$ by using a theorem proved earlier. Consider the two circles: $|s - 2| \leq \frac{7}{4}$, $|s - 2| \leq \frac{3}{2}$. Since $T > 3$, $g(s)$ is regular in the larger circle. At all points $s$ of this circle, we have

$\sigma \geq \frac{1}{4}, \quad 1 < |t \pm T| < 2 + T.$

Hence, by an appeal to the other of $\zeta(s)$ obtained in the $12^{th}$ Lecture (with $\delta = \frac{1}{4}$), we get

$$|g(s)| < \frac{1}{2} c_1 (|t + T|^{3/4} + |t - T|^{3/4}) < c_1 (T + 2)^{3/4},$$

on the larger circle. At the centre, we have

$$g(2) = \text{Re } \zeta(2 + iT) > \frac{1}{4} \text{ by (9.6)}$$

Hence, by using the theorem (given in the Appendix), we get

$$\left(\frac{7}{6}\right)^m < \frac{c_1 (T + 2)^{3/4}}{\frac{1}{4}} < T, \text{ for } T > T_0 \geq 3.$$ 

Thus

$$m < c_2 \log T, \text{ for } T > T_0.$$ 

Substituting this into (9.5) and (9.4) we get

$$|N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi} + \frac{T}{2\pi}| \leq c_3 \log T, \quad T > T_0,$$

provided that $T \neq \gamma$ for any ‘$\gamma$’. The last restriction may be removed by first taking $T'$ larger than $T$ and distinct from the $\gamma$’s and then letting $T' \to T + 0$. □

**Remarks.** The above theorem was stated by Riemann but proved by Von Mangoldt in 1894. The proof given here is due to Backlund (1914).
Corollary 1. If \( h > 0 \) (fixed), then

\[
N(T + h) - N(T) = O(\log T), \text{ as } T \to \infty
\]

Proof. If we write

\[
f(t) = \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi},
\]

we get

\[
f(t + h) - f(t) = hf'(t + \theta h), 0 < \theta < 1,
\]

where \( f'(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} \). Now use the Theorem. \( \square \)

Corollary 2.

\[
S \equiv \sum_{0 \leq \gamma \leq T} \frac{1}{\gamma} = O(\log^2 T); S' \sum_{\gamma > T} \frac{1}{\gamma^2} = O\left(\frac{\log T}{T}\right)
\]

(summed over all zeros \( \rho \) whose imaginary parts \( \gamma \) satisfy the given conditions).

Let

\[
s_m = \sum_{m < \gamma \leq m + 1} \frac{1}{\gamma}, \quad m \geq 1, \text{ and}
\]

\[
s'_m = \sum_{m < \gamma \leq m + 1} \frac{1}{\gamma^2}.
\]

Then

\[
S \leq \sum_{m=0}^{[T]} s_m, \quad S' \leq \sum_{m=[T]}^{\infty} s'_m
\]

If \( m \geq 1 \), the number of terms in \( s_m \) (or \( s'_m \)) is

\[
N(m + 1) - N(m) = \nu_m, \text{ say.}
\]

By Corollary 1, \( \nu_m = O(\log m) \) as \( m \to \infty \); and

\[
s_m \leq \frac{\nu_m}{m}, \quad s'_m \leq \frac{\nu_m}{m^2}
\]
Hence, as $T \to \infty$,

$$S = O(1) + O \left( \sum_{2}^{17} \frac{\log m}{m} \right) = O(\log^2 T)$$

$$S' = O \left( \sum_{17}^{\infty} \frac{\log m}{m^2} \right) = O \left( \frac{\log T}{T} \right)$$

**Corollary 3.** If the zeros ‘ρ’ for which $\gamma > 0$ are arranged as a sequence $\rho_n = \beta_n + i\gamma_n$, so that $\gamma_{n+1} \geq \gamma_n$, then

$$|\rho_n| \sim \gamma_n \sim \frac{2\pi n}{\log n}, \text{ as } n \to \infty$$

**Proof.** Since $N(\gamma_n - 1) < n \leq N(\gamma_n + 1)$, and

$$2\pi N(\gamma_n \pm 1) \sim (\gamma_n \pm 1) \log(\gamma_n \pm 1) \sim \gamma_n \log \gamma_n,$$

we have first

$$2\pi n \sim \gamma_n \log \gamma_n$$

whence

$$\log n \sim \log \gamma_n,$$

and

$$\gamma_n \sim \frac{2\pi n}{\log \gamma_n} \sim \frac{2\pi n}{\log n}$$

And

$$\gamma_n \leq |\rho_n| < \gamma_n + 1. \quad \Box$$

**Remarks.** (i) The main theorem proves incidentally the existence of an infinity of complex zeros of $\zeta(s)$, without the use of the theory of entire functions.

(ii) We see from corollary [3] that

$$\sum \frac{1}{|\rho|(|\rho|)^\alpha}$$

converges for $\alpha > 2$ and diverges for $\alpha \leq 2$. 
(iii) Corollary [119] may be obtained directly by applying the theorem
(on the number of zeros) to \( \zeta(s) \) and to two circles with centre at
\( c + iT \) and passing through \( \frac{1}{2} + (T + 2h)i, \frac{1}{2} + (T + h)i \) respectively,
where \( c = c(h) \) is sufficiently large, and using the symmetry about
\( \sigma = 1/2 \)

(iv) If \( N_0(T) \) denotes the number of complex zeros of \( \zeta(s) \) with real
part \( \frac{1}{2} \) and imaginary part between 0 and \( T \), then Hardy and Littlewood proved that
\( N_0(T) > cT \) in comparison with \( N(T) \sim \frac{1}{2\pi} T \log A \). Selberg has shown, however, that
\( N_0(T) > cT \log T \).

Appendix

**Theorem.** Let \( f(z) \) be regular in \( |z - z_0| \leq R \) and have \( n \) zeros (at least)
in \( |z - z_0| \leq r < R \). Then, if \( f(z_0) \neq 0 \),
\[
\left( \frac{R}{r} \right)^n \leq \frac{M}{|f(z_0)|},
\]
where \( M = \max_{|z-z_0|=R} |f(z)| \).

**Proof.** Multiple zeros are counted according to multiplicity. Suppose
\( z_0 = 0 \) (for otherwise put \( z = z_0 + z' \)). Let \( a_1, \ldots, a_n \) be the zeros of \( f(z) \)
in \( |z| \leq r \).

Then
\[
f(z) = \varphi(z) \prod_{\nu=1}^{n} \frac{R(z - a_{\nu})}{R^2 - \lambda_{\nu}z},
\]
where \( a_{\nu}, \overline{a_{\nu}} \) are conjugates and \( \varphi \) is regular in \( |z| \leq R \). On \( |z| = R \), each
factor has modulus 1; hence
\[
|\varphi(z)| = |f(z)| \leq M \text{ on } |z| = R
\]

Since \( \varphi \) is regular in \( |z| \leq R \), we have
\[
|\varphi(0)| \leq M
\]
Hence
\[
|f(0)| = |\varphi(0)| \prod_{\nu=1}^{n} \frac{|a_{\nu}|}{R} \leq M \left( \frac{1}{R} \right)^n \text{ and } f(0) \neq 0, \text{ hence the result.}
\]
\[\square\]
Theorem. If $f$ is analytic inside and on $C$ and $N$ is the number of zero inside $C$, then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz = N$$

$$= \Delta_c \log|f(z)|$$

so that

$$N = \frac{1}{2\pi} \Delta_c \arg f(z).$$
Lecture 15

The Zeta Function of Riemann (Contd)

10 Hardy’s Theorem [16, p.214]

We have proved that $\zeta(s)$ has an infinity of complex zeros in $0 < \sigma < 1$.  
Riemann conjectured that all these zeros lie on the line $\sigma = \frac{1}{2}$. Though this conjecture is yet to be proved, Hardy showed in 1914 that an infinity of these zeros do lie on the ‘critical’ line. We shall prove this result.

In obtaining the functional equation by the third method, we have used the formula:

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \psi(x) \left( x^{s-1} + x^{-s/2-1/2} \right) dx,$$

where $\psi(x) = \sum_{n=1}^\infty e^{-n^2 \pi x}, x > 0$. Multiplying by $\frac{1}{2}s(s-1)$, and writing $x = \frac{1}{2} + it$, we get

$$\xi\left( \frac{1}{2} + it \right) = \Xi(t) = \frac{1}{2} - \left( \pi^2 + \frac{1}{4} \right) \int_1^\infty \psi(x)x^{-3/4} \cos\left( \frac{1}{2}t \log x \right) dx$$

It is a question of showing that $\Xi(t)$, which is an even, entire function of $t$, real for real $t$, has an infinity of real zeros.
Writing \( x = e^{4u} \), we get

\[ \equiv (t) = \frac{1}{2} - 4 \left( t^2 + \frac{1}{4} \right) \int_0^\infty \psi(t^{4u}) e^{4u} \cos(2ut) du. \]

If we write \( \psi(u) = \psi(e^{4u}) e^u \), we obtain

\[ \equiv \left( \frac{t}{2} \right) = \frac{1}{2} - (t^2 + 1) \int_0^\infty \phi(u) \cos(ut) du \]

We wish now to get rid of the factor \((t^2 + 1)\) and the additive constant \(\frac{1}{2}\)
so as to obtain \(\equiv (t/2)\) as a cosine transform of \(\phi\). This we can achieve
by two integrations (by parts). However, to get rid of the extra term
arising out of partial integration we need to know \(\phi'(0)\). We shall show
that \(\phi'(0) = -\frac{1}{2}\). For, by the functional equation of the theta-function,

\[ 1 + 2\psi(x) = \frac{1}{\sqrt{x}} \left[ 1 + 2\psi \left( \frac{1}{x} \right) \right] \]

or

\[ 1 + 2\psi(e^{4u}) = e^{-2u} \left[ 1 + 2\psi(e^{-4u}) \right] \]

or

\[ e^u + 2e^u\psi(e^{4u}) = e^{-u} + 2e^{-u}\psi(e^{-4u}) \]

or

\[ e^u + 2\phi(u) = e^{-u} + 2\phi(-u) \]

so that

\[ \frac{d}{du} [e^u + 2\phi(u)]_{u=0} = 0 \]

i.e. \( \phi'(0) = -\frac{1}{2} \).

We also know that \(\psi(x) = O(e^{-\pi x})\) as \(x \to \infty\). Hence

\[ t^2 \int_0^\infty \phi(u) \cos(ut) du = \frac{1}{2} - \int_0^\infty \phi''(u) \cos(ut) du \]
Hence
\[ \equiv (t/2) = \int_{0}^{\infty} [\phi''(u) - \phi(u)] \cos ut \, du \] (10.1)

By actual computation it may be verified that
\[ \phi''(t) - \phi(t) = 4[6e^{5t}\psi'(e^{4t}) + 4e^{9t}\psi''(e^{4t})] \]

Again if we write
\[ \phi(u) = e^{u}\psi(e^{4u}) + \frac{1}{2}e^{u} = \phi(u) + \frac{1}{2}e^{u} \]
\[ = \frac{1}{2}e^{u}\theta(e^{4u}), \]
then by the theta relation, we have
\[ \Phi(u) = \Phi(-u), \text{ and } \Phi''(u) - \Phi(u) = \phi''(u) - \phi(u) \]

Let us now write (10.1) in the form
\[ \equiv (t) = 2 \int_{0}^{\infty} f(u) \cos ut \, du \]

We wish to deduce that
\[ f(u) = \frac{1}{\pi} \int_{0}^{\infty} \equiv (t) \cos ut \, dt; \] (10.2)

by Fourier’s integral theorem. We need to establish this result.

**Fourier Inversion Formula.**[5] p.10] If \( f(x) \) is absolutely integrable in \((-\infty, \infty)\), then
\[ \phi(u) \equiv \int_{-\infty}^{\infty} f(x)e^{ixu} \, dx \]
exists for every real \( \alpha, -\infty < \alpha < \infty \), is continuous and bounded in \((-\infty, \infty)\). We wish to find out when

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(\alpha) e^{-i\alpha x} d\alpha
\]

for a given \( x \). The conditions will bear on the behaviour of \( f \) in a neighbourhood of \( x \).

If

\[
S_R(x) = \frac{1}{2\pi i} \int_{-R}^{R} e^{-i\alpha x} \phi(\alpha) d\alpha
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \sin R t \frac{f(x + t) + f(x - t)}{2} dt,
\]

and

\[
g_s(t) = \frac{1}{2} [f(x + t) + f(x - t)] - f(x),
\]

then

\[
S_R(x) - f(x) = \frac{2}{\pi} \int_{0}^{\infty} \sin R t g_s(t) \, dt
\]

\[
= \frac{2}{\pi} \left[ \int_{0}^{\epsilon} + \int_{\epsilon}^{\infty} \right] = I_1 + I_2, \text{ say},
\]

for a fixed \( \epsilon > 0 \).

\( I_2 \to 0 \) as \( R \to \infty \), by the Riemann-Lebesgue Lemma. So, if \( \frac{g_s(t)}{t} \)
is absolutely integrable in \((0, \epsilon)\), then \( I_1 \to 0 \) as \( \epsilon \to 0 \). Hence we have the
Theorem. [5, p.10] If \( g_x(t) \) is absolutely integrable in \((0, \varepsilon)\), then
\[
\lim_{R \to \infty} S_R(x) = f(x).
\]

If, in particular, \( f \) is absolutely integrable in \((-\infty, \infty)\) and differentiable in \((-\infty, \infty)\) then
\[
\lim_{R \to \infty} S_R(x) = f(x) \text{ for every } x \in (-\infty, \infty).
\]

It should be noted that sufficient conditions for the validity of the theorem are: (i) \( f(x) \) is of bounded variation in a neighbourhood of \( x \), and (ii) \( f(x) \) is absolutely integrable in \((-\infty, \infty)\). Then
\[
\lim_{R \to \infty} S_R(x) = \frac{1}{2} [f(x + 0) + f(x - 0)].
\]

We now return to (10.2) which is an immediate deduction from the formula for \( \equiv (t) \), on using Fourier’s inversion formula.

Now we know that \( \equiv (t) = \xi (-\frac{1}{2} + it) = O(t^{\frac{1}{4}} e^{-\pi t}) \); and \( \equiv (t) \) is an entire function of \( t \). Hence we are justified in obtaining, by differentiation of (10.2),
\[
f^{(2n)}(u) = \frac{(-1)^n}{\pi} \int_0^\infty \equiv (t) t^{2n} \cos ut \, dt.
\]

Since \( \psi(x) \) is regular for \( \text{Re} \, x > 0 \), \( f(u) \) is regular for \(-\pi/4 < \text{Im} \, u < \pi/4\) (Note that \( x = e^{2n} \)). Let
\[
f(iu) = c_0 + c_1 u^2 + c_2 u^4 + \cdots
\]
Then
\[
c_n = \frac{\equiv (-1)^n f^{(2n)}(0)}{(2n)!} = \frac{1}{(2n)! \pi} \int_0^\infty \equiv (t) t^{2n} \, dt
\]

If \( \equiv (t) \) had only a finite number of zeros, then \( \equiv (t) \) would be of constant sign for \( t > T \). Let \( \equiv (t) > 0 \) for \( t > T \). Then \( c_n > 0 \) for \( n > 2n_0 \), since
\[
\int_0^\infty \equiv (t) t^{2n} \, dt > \int_T^{T+2} \equiv (t) t^{2n} \, dt - \int_0^T \equiv (t) t^{2n} \, dt
\]
\[
> (T + 1)^{2n} \int_T^{T+2} \equiv (t) dt - T^{2n} \int_0^T \equiv (t) dt
\]
> 0 for \( n > 2n_0 \), say.

Hence \( f^{(n)}(iu) \) increases steadily with \( u \) for \( n > 2n_0 \). However we shall see that \( f(u) \), and all its derivatives, tend to zero as \( u \to \frac{\pi i}{4} \) along the imaginary axis.

We know that

\[
\psi(x) = x^{-1/2} \psi\left(\frac{1}{x}\right) + \frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{2}
\]

Further

\[
\psi(i + \delta) = \sum_{n=1}^{\infty} e^{-n^2(i+\delta)\pi} = \sum_{n=1}^{\infty} (-1)^n e^{-n^2\pi \delta}
\]

\[
= 2 \sum_{n=1}^{\infty} e^{-(2n^2\pi \delta)} - \sum_{n=1}^{\infty} e^{-n^2\pi \delta}
\]

\[
= 2\psi(4\delta) - \psi(\delta)
\]

\[
= \frac{1}{\sqrt{\delta}} \psi\left(\frac{1}{4\delta}\right) - \frac{1}{\delta} \psi\left(\frac{1}{\delta}\right) - \frac{1}{2}
\]

i.e.

\[
\frac{1}{2} + \psi(i + \delta) = \frac{1}{\sqrt{\delta}} \psi\left(\frac{1}{4\delta}\right) - \frac{1}{\sqrt{\delta}} \psi\left(\frac{1}{\delta}\right)
\]

Thus \( \frac{1}{2} + \psi(x) \to 0 \) as \( \delta \to 0 \), and so do its derivatives (by a similar argument). Hence \( \frac{1}{2} + \psi(u) \to 0 \) as \( u \to \frac{\pi i}{4} \) along the imaginary axis, and so do its derivatives. Thus \( f(u) \) and all its derivatives \( \to 0 \) as \( u \to \frac{\pi i}{4} \).
Lecture 16

The Zeta Function of Riemann (Contd)

**Hamburger’s Theorem.**[13]

We have seen that the functional equation of the Gamma function, together with its logarithmic convexity, determine the Gamma function ‘essentially’ uniquely (i.e. up to a constant). A corresponding result can be proved for a the zeta function.

**Theorem 1.** Let \( f(s) = \frac{G(s)}{P(s)} \), where \( G(s) \) is an entire function of finite order and \( P(s) \) is a polynomial, and let

\[
f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},
\]

(1.1)

the series on the right converging absolutely for \( \sigma > 1 \).

Let

\[
f(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2} = g(1-s)\Gamma\left(\frac{1}{2} - \frac{s}{2}\right)\pi^{\frac{(1-s)}{2}}
\]

(1.2)

where

\[
g(1-s) = \sum_{n=1}^{\infty} \frac{b_n}{n^{1-s}},
\]

(1.3)
The series on the right converging absolutely for \( \sigma < -\alpha < 0 \). Then 
\[ f(s) = a_1 \zeta(s) = g(s). \]

For the proof we need to evaluate two integrals:

\[ e^{-y} = \frac{1}{2\pi i} \int_{1-\infty i}^{1+\infty i} y^{-s} \Gamma(s) ds, \quad y > 0 \quad (1) \]

\[ \int_0^\infty e^{-a^2 x - \frac{\beta^2}{x}} \frac{dx}{\sqrt{x}} = \frac{\sqrt{\pi}}{a} e^{-2ab}, \quad a > 0, \quad b \geq 0. \quad (2) \]

The first formula is a classic example of Mellin’s inversion formula which can be obtained as an instance of Fourier’s inversion formula discussed in the last lecture. We have seen that if \( f(x) \in L_1(-\infty, \infty) \), then its Fourier transform \( \phi(\alpha) \) is defined in \(-\infty < \alpha < \infty\), and if in a neighbourhood of \( x_0 \), \( f(x) \) is differentiable, then

\[ f(x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\alpha} d\alpha \int_{-\infty}^{\infty} f(y)e^{iy\alpha} dy. \]

If we define

\[ \mathcal{F}(s) = \int_0^\infty f(x)x^{s-1} dx, \quad (3) \]

for \( s = c + it, \ c > 0 \), where \( f(x)x^{c-1} \in L_1(0, \infty) \), so that the integral on the right exists absolutely, we can look upon this as a Fourier transform by a change of variable: \( x \to e^y \), for

\[ \mathcal{F}(c + it) = \int_{-\infty}^{\infty} f(e^y)e^{ycz} e^{ity} dy \]

Hence, provided that \( f(e^y)e^{ycz} \) satisfies a suitable ‘local’ condition, such as differentiability in the neighbourhood of a point, we can invert this
relation and obtain
\[ f(e^y) \cdot e^{yc} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(c + it)e^{-yt} dt \]
or
\[ f(e^y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(c + it)e^{-(c+it)y} dt \]
\[ = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{F}(s)e^{-ys} ds \]
or
\[ f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{F}(s)x^{-s} ds, \quad c > 0. \tag{4} \]

Relations (3) and (4) give the Mellin inversion formulae. Since
\[ \Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx, \quad \text{Re} \, s > 0, \]
we get (1) as in immediate application.

We know that, for \( k > 0 \)
\[ \int_0^A e^{-(k-ia)x} dx = \frac{1}{k - ia} \left(1 - e^{-kA} e^{iaA}\right) \]
so that
\[ 2 \int_0^\infty e^{-kx} \cos \alpha x dx = \frac{2k}{k^2 + \alpha^2} \]
and by Fourier’s inversion formula,
\[ \int_0^\infty \frac{\cos \alpha x}{k^2 + \alpha^2} d\alpha = \frac{\pi e^{-kx}}{2k}, \quad k > 0 \tag{5} \]
On the other hand, we have for \( k \geq 0 \)
\[ \frac{\Gamma(1)}{x^2 + k^2} = \int_0^\infty e^{-(x^2+k^2)y} dy, \]
so that
\[
\int_0^\infty \cos \alpha x \frac{dx}{x^2 + k^2} = \sqrt{\frac{\pi}{2}} \int_0^\infty y^{-1} e^{-\left(\frac{k^2 y^2}{\pi} + \frac{\alpha^2}{4y^2}\right)} dy
\]
\[= \frac{\pi}{2k} e^{-|\alpha|}, \quad \text{from (5)}.
\]
Setting \(k^2 = a^2, \alpha^2 = 4b^2\), we get
\[
\int_0^\infty e^{-a^2 x - b^2 x} \frac{dx}{\sqrt{x}} = \sqrt{\frac{\pi}{a}} e^{-2ab}, \quad a > 0, b \geq 0
\]
which proves formula (2).

**Proof of Theorem 1.** For any \(x > 0\), we have by (1.1) and (1.2), (Here \(f(s) = O(1)\))

\[
S_1 = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} f(s) \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} x^{-s/2} ds
\]
\[= \sum_{n=1}^{\infty} \frac{a_n}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s/2) (\pi n^2 x)^{-s/2} ds
\]
\[= 2 \sum_{n=1}^{\infty} a_n e^{-\pi n^2 x}
\]
We have, however, by (1.2), \(S_1 = S_2\), where
\[
S_2 = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} g(1-s) \Gamma\left(\frac{1-s}{2}\right) \pi^{-\frac{s-1}{2}} x^{-s/2} ds.
\]

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\]

Now we wish to move the line of integration here from \(\sigma = 2\) to \(\sigma = -1 - \alpha\).

By the hypothesis on the ‘order’ of \(f(s)\) and (1.2), it follows that there exist two numbers \(T > 0, \gamma > 0\), such that for \(|t| \geq T\) and \(-1 - \alpha \leq\)
\( \sigma \leq 2 \), the function \( g(1 - s) \) is regular and \( O(e^{\vert t \vert}) \). By (1.3), we see that \( g(1 - s) = O(1) \) on \( \sigma = -1 - \alpha \); and, since \( f(s) = O(1) \) on \( \sigma = 2 \), while

\[
\frac{\Gamma(s/2)}{\Gamma(\frac{1-s}{2})} = O(|t|^{3/2}), \text{ on } \sigma = 2,
\]

it follows that \( g(1 - s) = O(|t|^{3/2}) \) on \( \sigma = 2 \). Hence, by the principle of Phragmen-Lindelöf, we observe that

\[
g(1 - s) = O(|t|^{3/2})
\]

for \( |t| \geq T, -\alpha - 1 \leq \sigma \leq 2 \). Taking a suitable rectangle, and applying Cauchy's theorem, we get

\[
S_2 = \frac{1}{2\pi i} \int_{-\alpha - 1 - i\infty}^{\alpha - 1 + i\infty} g(1 - s)\Gamma\left(1 - \frac{s}{2}\right)\pi^{\frac{1-s}{2}}\chi^{-s/2} ds + \sum_{v=1}^{m} R_v
\]

where \( R_v \) is the residue of the integrand at the pole \( s_v \) lying in the region \(-\alpha - 1 < \sigma < 2\). The integrand, however, equals
The Zeta Function of Riemann (Contd)

\[ f(s) \Gamma \left( \frac{s}{2} \right) \pi^{-s/2} x^{-s/2} \]

The residue of this at a pole \( s_\nu \) of order \( q_\nu \) is

\[ x^{\frac{s_\nu}{2}} \left( A_{q_\nu-1}^{(\nu)} \log^{q_\nu-1} x + \cdots + A_1^{(\nu)} \log x + A_0^{(\nu)} \right) \]

Hence

\[ \sum_{\nu=1}^m R_\nu = \sum_{\nu=1}^m x^{-s_\nu/2} Q_\nu(\log x) = Q(x) \text{, say, where} \]

\( Q_\nu \) is a polynomial

Here \( \text{Re } s_\nu \leq 2 - \theta \), \( \theta > 0 \). (i.e. we are using the absolute convergence of \( \sum a_n n^{-\sigma} \) only for \( \sigma > 2 - \theta \)).

Hence

\[ S_2 = \frac{1}{\sqrt{x}} \sum_{n=1}^\infty \frac{b_n}{2 \pi i} \int_{\alpha-1+i\infty}^{\alpha-1-i\infty} \frac{\Gamma(1-s)}{2} \left( \pi n^2 \right)^{-(1-s)} ds + Q(x) \]

\[ = \frac{2}{\sqrt{x}} \sum_{n=1}^\infty \frac{b_n}{2 \pi i} \int_{\frac{1}{2}+i\infty}^{\frac{1}{2}+1-i\infty} \Gamma(s) \left( \frac{\pi n^2}{x} \right)^{-s} ds + Q(x) \]

\[ = \frac{2}{\sqrt{x}} \sum_{n=1}^\infty b_n e^{-\pi n^2/x} + Q(x) \]

Hence (since \( S_1 = S_2 \)) we have

\[ 2 \sum_{n=1}^\infty a_n e^{-\pi n^2/x} = \frac{2}{\sqrt{x}} \sum_{n=1}^\infty b_n e^{-\pi n^2/x} + Q(x) \]

Multiplying this by \( e^{-\pi t^2 x}, t > 0, \) and integrating over \((0, \infty)\) w.r.t. \( x \), we get

\[ 2 \sum_{n=1}^\infty \frac{a_n}{\pi(t^2 + n^2)} = 2 \sum_{n=1}^\infty \frac{b_n}{t} e^{-2\pi nt} + \int_0^\infty Q(x) e^{-\pi t^2 x} dx \]
The last integral is convergent. Since \( \text{Re } s_{\nu} \leq 2 - \theta \), \( \nu = 1, 2, \ldots, m \), each term of \( Q(x) \) is \( O(x^{-1+\theta}) \) as \( x \to 0 \).

Now

\[
\int_0^\infty Q(x)e^{-\pi x^2}dx = \frac{1}{\pi^2} \int_0^\infty Q\left(\frac{x}{\pi}\right)e^{-\pi x}dx = \sum_{\nu=1}^{m} t^{\nu-2}H_{\nu}(\log t),
\]

where \( H_{\nu} \) is a polynomial, \( = H(t) \), say.

Hence

\[
\sum_{n=1}^{\infty} a_n \left( \frac{1}{t + ni} + \frac{1}{t - ni} \right) - \pi t H(t) = 2\pi \sum_{n=1}^{\infty} b_n e^{-2\pi nt}
\]

The series on the left hand side is uniformly convergent in any finite region of the \( t \)-plane which excludes \( t = \pm ki \), \( k = 1, 2, \ldots \); It is a meromorphic function with poles of the first order at \( t = \pm ki \).

\( H(t) \) is regular and single-valued in the \( t \)-plane with the negative real axis deleted and \( t \neq 0 \).

The right hand side is a periodic function of \( t \) with period \( i \) for \( \text{Re } t > 0 \). Hence, by analytical continuation, so is the function on the left. Hence the residues at \( ki, (k+1)i \) are equal. So \( a_k = a_{k+1} \)
Lecture 17

The Zeta Function of Riemann (Contd)

12 The Prime Number Theorem [11, pp.1-18]

The number of primes is infinite. For consider any finite set of primes; let $P$ denote their product, and let $Q = P + 1$. Then $(P, Q) = 1$, since any common prime factor would divide $Q - P$, which would be impossible if $(P, Q) \neq 1$. But $Q > 1$, and so divisible by some prime. Hence there exists at least one prime distinct from those occurring in $P$. If there were only a finite number of primes, by taking $P$ to be their product we would arrive at a contradiction.

Actually we get a little more by this argument: if $p_n$ is the $n^{th}$ prime, the integer $Q_n = p_1 \ldots p_n + 1$ is divisible by some $p_m$ with $m > n$, so that

$$p_{n+1} \leq p_m \leq Q_n$$

from which we get, by induction, $p_n < 2^{2^n}$. For if this inequality was true for $n = 1, 2, \ldots N$, then

$$p_{N+1} \leq p_1 \ldots p_N + 1 < 2^{2+4+\cdots+2^N} + 1 < 2^{2^{N+1}}$$

and the inequality is known to be true for $n = 1$.  

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The fact that the number of primes is infinite is equivalent to the statement that \( \pi(x) = \sum_{p \leq x} 1 \to \infty \) as \( x \to \infty \). Actually far more is true; we shall show that \( \pi(x) \sim \frac{x}{\log x} \). Again the relation \( p_n < 2^{2^n} \) is far weaker than the asymptotic formula: \( p_n \sim n \log n \), which is know to be equivalent to the formula: \( \pi(x) \sim \frac{x}{\log x} \) (The Prime Number Theorem!).

We shall prove the Prime Number Theorem, and show that it is equivalent to the non-vanishing of the zeta function on the line \( \sigma = 1 \). We shall state a few preliminary results on primes.

**Theorem 1.** The series \( \sum \frac{1}{p} \) and the product \( \prod \left(1 - \frac{1}{p}\right)^{-1} \) are divergent.

**Proof.** Let \( S(x) = \sum_{p \leq x} \frac{1}{p} \), \( P(x) = \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1}, \ x > 2. \)

Then, for \( 0 < u < 1 \),

\[
\frac{1}{1-u} > \frac{1-u^{m+1}}{1-u} = 1 + u + \cdots + u^m
\]

Hence

\[
P(x) \geq \prod_{p \leq x} (1 + p^{-1} + \cdots + p^{-m}),
\]

where \( m \) is any positive integer. The product on the right is \( \sum \frac{1}{n} \) summed over a certain set of positive integers \( n \), and if \( m \) is so chosen that \( 2^{m+1} > x \), this set will certainly include all integers from 1 to \( [x] \). Hence

\[
P(x) > \sum_{i=1}^{[x]} \frac{1}{n} > \int_{1}^{[x]+1} \frac{du}{u} > \log x. \quad (1)
\]

Since

\[- \log(1-u) - u < \frac{1}{2} u^2, \text{ for } 0 < u < 1, \]

we have

\[
\log P(x) - S(x) < \sum_{p \leq x} \frac{p^{-2}}{2(1 - p^{-1})}
\]
12. The Prime Number Theorem

\[ < \sum_{n=2}^{\infty} \frac{1}{2n(n-1)} = \frac{1}{2} \]

Hence, by (1),

\[ S(x) > \log \log x - \frac{1}{2}. \]  \hspace{1cm} (2)

(1) and (2) prove the theorem.

The Chebyshev Functions. Let

\[ \vartheta(x) = \sum_{p \leq x} \log p, \quad \psi(x) = \sum_{p^m \leq x} \log p, \quad x > 0. \]

Grouping together terms of \( \psi(x) \) for which \( m \) has the same value, we get

\[ \psi(x) = \vartheta(x) + \vartheta(x^{1/2}) + \vartheta(x^{1/3}) + \cdots, \]  \hspace{1cm} (3)

this series having only a finite number of non-zero terms, since \( \vartheta(x) = 0 \) if \( x < 2 \).

Grouping together terms for which \( p \) has the same value \( (\leq x) \), we obtain

\[ \psi(x) = \sum_{p \leq x} \left[ \frac{\log x}{\log p} \right] \log p, \]  \hspace{1cm} (4)

since the number of values of \( m \) associated with a given \( p \) is equal to the number of positive integers \( m \) satisfying \( m \log p \leq x \), and this is \[ \left\lfloor \frac{\log x}{\log p} \right\rfloor \]

\[ \square \]

Theorem 2. The functions

\[ \frac{\pi(x)}{x / \log x}, \quad \frac{\vartheta(x)}{x}, \quad \frac{\psi(x)}{x} \]

have the same limits of indetermination as \( x \to \infty \).

Proof. Let the upper limits (may be \( \infty \)) be \( L_1, L_2, L_3 \) and the lower limits be \( l_1, l_2, l_3 \), respectively. Then by (3) and (4),

\[ \vartheta(x) \leq \psi(x) \leq \sum_{p \leq x} \frac{\log x}{\log p} \log p = \pi(x) \log x. \]
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Hence
\[ L_2 \leq L_3 \leq L_1. \]  (5)

On the other hand, if \( 0 < \alpha < 1, \ x > 1, \) then
\[ \vartheta(x) \geq \sum_{x^\alpha < p \leq x} \log p \geq \{ \pi(x) - \pi(x^\alpha) \} \log x^\alpha \]
and \( \pi(x^\alpha) < x^\alpha, \) so that
\[ \frac{\vartheta(x)}{x} \geq \alpha \left( \frac{\pi(x) \log x}{x} - \frac{\log x}{x^{1-\alpha}} \right) \]

Keep \( \alpha \) fixed, and let \( x \to \infty; \) then \( \frac{\log x}{x^{1-\alpha}} \to 0 \)
Hence
\[ L_2 \geq \alpha L_1, \]
i.e.
\[ L_2 \geq L_1, \] as \( \alpha \to 1 - 0. \)

This combined with (5) gives \( L_2 = L_3 = L_1, \) and similarly for the \( l \)’s. \( \square \)

**Corollary.** If \( \frac{\pi(x)}{x/\log x} \) or \( \frac{\vartheta(x)}{x} \) or \( \frac{\psi(x)}{x} \) tends to a limit, then so do all, and the limits are equal.

**Remarks.** If \( \land(n) = \begin{cases} \log p, & \text{if } n \text{ is a } +\text{ve power of a prime } p \\ 0, & \text{otherwise,} \end{cases} \)
then
\[ \psi(x) = \sum_{p^n \leq x} \log p = \sum_{n \leq x} \land(n), \] and
\[ -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\land(n)}{n^s}, \quad \sigma > 1 \ [\text{p.69}] \]

Also
\[ -\frac{\zeta'(s)}{\zeta(s)} = s \int_{1}^{\infty} \frac{\psi(x)}{x^{s+1}} \, dx, \quad \sigma > 1. \]
The Wiener-Ikehara theorem. [3, 4] Let $A(u)$ be a non-negative non-decreasing function defined for $0 \leq u < \infty$. Let

$$f(s) \equiv \int_{0}^{\infty} A(u)e^{-us}du, \quad s = \sigma + i\tau,$$

converge for $\sigma > 1$, and be analytic for $\sigma \geq 1$ except at $s = 1$, where it has a simple pole with residue 1. Then $e^{-u}A(u) \to 1$ as $u \to \infty$.

Proof. If $\sigma > 1$,

$$\frac{1}{s-1} = \int_{0}^{\infty} e^{-(s-1)u}du$$

Therefore

$$g(s) \equiv f(s) - \frac{1}{s-1} = \int_{0}^{\infty} [A(u) - e^u]e^{-us}du$$

$$\equiv \int_{0}^{\infty} [B(u) - 1]e^{-(s-1)u}du,$$

where

$$B(u) \equiv A(u)e^{-u}$$

Take

$$s = 1 + \varepsilon + i\tau.$$ 

Then

$$g(1 + \varepsilon + i\tau) \equiv g_\varepsilon(\tau) = \int_{0}^{\infty} [B(u) - 1]e^{-\varepsilon u} \cdot e^{-i\tau u}du$$

Now $g(s)$ is analytic for $\sigma \geq 1$; therefore

$$g_\varepsilon(t) \to g(1 + it) as \varepsilon \to 0.$$ 

uniformly in any finite interval $-2\lambda \leq t \leq 2\lambda$. 

We should like to form the Fourier transform of \( g_\varepsilon(\tau) \), but since we do not know that it is bounded on the whole line, we shall introduce a smoothing kernel. Thus

\[
\frac{1}{2} \int_{-2\lambda}^{2\lambda} g_\varepsilon(\tau) \left( 1 - \frac{|\tau|}{2\lambda} \right) e^{iy\tau} d\tau
\]

\[
= \frac{1}{2} \int_{-2\lambda}^{2\lambda} e^{iy\tau} \left( 1 - \frac{|\tau|}{2\lambda} \right) d\tau \int_{0}^{\infty} \{ B(u) - 1 \} e^{-\varepsilon u - i\tau u} du.
\]

For a fixed \( y \), a finite \( \tau \)-interval, and an infinite \( u \)-interval we wish to change the order of integration. This is permitted if

\[
\int_{0}^{\infty} \{ B(u) - 1 \} e^{-\varepsilon u} - i\tau u du
\]

converges uniformly in \(-2\lambda \leq \tau \leq 2\lambda\). This is so, because it is equal to

\[
\int_{0}^{\infty} B(u) e^{-\varepsilon u} - i\tau u du - \int_{0}^{\infty} e^{-\varepsilon u} e^{-\varepsilon u} du.
\]

For a fixed \( \varepsilon > 0 \), the second converges absolutely and uniformly in \( \tau \). In the first we have

\[
B(u) \cdot e^{(-\varepsilon/2)u} = A(u) \cdot e^{-(1+\varepsilon/2)u} \to 0 \text{ as } u \to \infty
\]

Hence \( B(u) = O(e^{(\varepsilon/2)u}) \), which implies the first integral converges uniformly and absolutely.

Thus

\[
\frac{1}{2} \int_{-2\lambda}^{2\lambda} e^{iy\tau} \left( 1 - \frac{|\tau|}{2\lambda} \right) g_\varepsilon(\tau) d\tau
\]

\[
= \left[ \int_{0}^{\infty} \{ B(u) - 1 \} e^{-\varepsilon u} du \int_{-2\lambda}^{2\lambda} \frac{1}{2} e^{i(y-u)\tau} \left( 1 - \frac{|\tau|}{2\lambda} \right) d\tau \right]
\]
12. The Prime Number Theorem

\[
\frac{1}{\lambda(y-u)^2} \int_0^\infty [B(u) - 1] e^{-\varepsilon u} \sin^2 \lambda (y-u) \, du
\]

Now consider the limit as \( \varepsilon \to 0 \). The left hand side tends to

\[
\frac{1}{2} \int_{-2\lambda}^{2\lambda} e^{i\lambda t} \left( 1 - \frac{|t|}{2\lambda} \right) g(1 + it) \, dt
\]

since \( g \) is analytic, or rather, since \( g(\tau) \to g(1 + i\tau) \) uniformly. On the right side, we have

\[
\int_0^\infty e^{-\varepsilon u} \frac{\sin^2 \lambda (y-u)}{\lambda(y-u)^2} \, du \to \int_0^\infty \frac{\sin^2 \lambda (y-u)}{\lambda(y-u)^2} \, du.
\]

Hence

\[
\lim_{\varepsilon \to 0} \int_0^\infty e^{-\varepsilon u} B(u) \frac{\sin^2 \lambda (y-u)}{\lambda(y-u)^2} \, du =
\]

\[
\int_0^\infty \frac{\sin^2 \lambda (y-u)}{\lambda(y-u)^2} \, du + \int_{-2\lambda}^{2\lambda} e^{i\lambda t} \left( 1 - \frac{|t|}{2\lambda} \right) g(1 + it) \, dt
\]

The integrand of the left hand side increases monotonically as \( \varepsilon \to 0 \); it is positive. Hence by the monotone-convergence theorem,

\[
\int_0^\infty B(u) \frac{\sin^2 \lambda (y-u)}{\lambda(y-u)^2} \, du = \int_0^\infty \frac{\sin^2 \lambda (y-u)}{\lambda(y-u)^2} \, du
\]

\[
+ \int_{-2\lambda}^{2\lambda} e^{i\lambda t} \left( 1 - \frac{|t|}{2\lambda} \right) g(1 + it) \, dt
\]

Now let \( y \to \infty \); then the second term on the right-hand side tends to zero by the Riemann-Lebesgue lemma, while the first term is

\[
\lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2} \, dv = \int_{-\infty}^{\infty} \frac{\sin^2 v}{v^2} \, dv = \pi
\]
Hence
\[
\lim_{y \to \infty} \int_{-\infty}^{\lambda y} B\left(y - \frac{v}{\lambda}\right) \sin^2 \frac{v}{\lambda^2} dv = \pi \text{ for every } \lambda.
\]

To prove that \(\lim_{u \to \infty} B(u) = 1\). \hfill \Box

Second Part.

(i) \(\lim B(u) \leq 1\).

For a fixed ‘a’ such that \(0 < a < \lambda y\), we have
\[
\lim_{y \to \infty} \int_{-a}^{a} B\left(y - \frac{v}{\lambda}\right) \sin^2 \frac{v}{\lambda^2} dv \leq \pi
\]

By definition \(B(u) = A(u)e^{-u}\), where \(A(u) \uparrow\), so that
\[
e^{(y-a)/\lambda} B\left(y - \frac{v}{\lambda}\right) = e^{(y-a)/\lambda} B\left(y - \frac{v}{\lambda}\right)
\]
or
\[
B\left(y - \frac{v}{\lambda}\right) \geq e^{(y-a)/\lambda} B\left(y - \frac{a}{\lambda}\right).
\]

Hence
\[
\lim_{y \to \infty} \int_{-a}^{a} B\left(y - \frac{a}{\lambda}\right) e^{(y-a)/\lambda} \sin^2 \frac{v}{\lambda^2} dv \leq \pi
\]
or
\[
\int_{-a}^{a} e^{\frac{-2a}{\lambda}} \frac{\sin^2 \frac{v}{\lambda^2}}{\lambda} \lim_{y \to \infty} B(y) \cdot dv \leq \pi
\]
or
\[
e^{-2a/\lambda} \int_{-a}^{a} \frac{\sin^2 \frac{v}{\lambda^2}}{\lambda} \lim_{y \to \infty} B(y) \cdot dv \leq \pi
\]

Now let \(a \to \infty, \lambda \to \infty\) in such a way that \(\frac{a}{\lambda} \to 0\).

Then
\[
\pi \lim_{y \to \infty} B(y) \leq \pi
\]
12. The Prime Number Theorem

(ii) \( \lim_{u \to \infty} B(u) \geq 1. \)

\[
\int_{-\infty}^{\lambda y} B\left(y - \frac{\nu}{\lambda}\right) \frac{\sin^2 \nu \nu^2}{v^2} \, dv = \int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{\lambda y} \tag{7}
\]

We know that \( |B(u)| \leq c \), so that

\[
\int_{-\infty}^{\lambda y} \leq c \left[ \int_{-\infty}^{-a} \frac{\sin^2 \nu \nu^2}{v^2} \, dv + \int_{-a}^{a} \frac{\sin^2 \nu \nu^2}{v^2} \, dv \right] + \int_{a}^{\lambda y} \]

As before we get

\[
e^{\nu + a/\lambda} B\left(y + \frac{a}{\lambda}\right) \geq e^{\nu - a/\lambda} B\left(y - \frac{\nu}{\lambda}\right)
\]

or

\[
B\left(y - \frac{\nu}{\lambda}\right) \leq B\left(y + \frac{a}{\lambda}\right) e^{a/\lambda} \leq B\left(y + \frac{a}{\lambda}\right) e^{2a/\lambda}
\]

Therefore

\[
\int_{-a}^{a} B\left(y - \frac{\nu}{\lambda}\right) \frac{\sin^2 \nu \nu^2}{v^2} \, dv \leq \int_{-a}^{a} B\left(y + \frac{a}{\lambda}\right) e^{2a/\lambda} \frac{\sin^2 \nu \nu^2}{v^2} \, dv
\]

\[
\leq e^{2a/\lambda} B\left(y + \frac{a}{\lambda}\right) \int_{-a}^{a} \frac{\sin^2 \nu \nu^2}{v^2} \, dv
\]

Hence taking the \( \lim \) in (7) we get, since

\[
\lim_{y \to \infty} \int_{-\infty}^{\lambda y} = \lim_{y \to \infty} = \pi
\]
and
\[ \lim (c + \psi(y)) \leq c + \lim \psi(y), \]
that
\[
\pi \leq c \left( \int_{-\infty}^{-a} - \int_{-a}^{\infty} \right) \frac{\sin^2 v}{v^2} dv + \lim \int_{-a}^{a} \frac{\sin^2 v}{v^2} dv
\]
\[
\leq \cdot \cdot \cdot + \lim B(y + \frac{a}{\lambda}) e^{2a/\lambda} \int_{-a}^{a} \frac{\sin^2 v}{v^2} dv
\]
\[
\leq \cdot \cdot \cdot + e^{2a/\lambda} \int_{-a}^{a} \lim B(y) \frac{\sin^2 v}{v^2} dv
\]

Let \( a \to \infty, \lambda \to \infty \) in such a way that \( \frac{a}{\lambda} \to 0 \).

Then
\[
\pi \leq \pi \lim B(y)
\]
Thus
\[
\lim_{u \to \infty} B(u) = \lim_{u \to \infty} A(u)e^{-u} = 1.
\]

**Proof of the Prime Number Theorem.**

We have seen [p.143] that
\[
-\zeta'(s) \quad \frac{s}{\zeta(s)} = \int_{1}^{\infty} \frac{\psi(x)}{e^{s-1}} dx, \quad \sigma > 1
\]
\[
= \int_{0}^{\infty} e^{-st} \psi(e^t) dt, \quad \sigma > 1.
\]

Since \( \zeta(s) \) is analytic for \( \sigma \geq 1 \) except for \( s = 1 \), where it has a simple pole with residue 1, and has no zeros for \( \sigma \geq 1 \), and \( \psi(e^t) \geq 0 \) and non-decreasing, we can appeal to the Wiener-Ikehara theorem with \( f(s) = -\frac{\zeta'(s)}{\zeta(s)} \), and obtain
\[
\psi(e^t) \sim e^t \quad \text{as} \quad t \to \infty
\]
or \( \psi(x) \sim x \).

13 Prime Number Theorem and the zeros of \( \zeta(s) \)

We have seen that the Prime Number Theorem follows from the Wiener-Ikehara Theorem if we assume that \( \zeta(1+it) \neq 0 \). On the other hand, if we assume the Prime Number Theorem, it is easy to deduce that \( \zeta(1+it) \neq 0 \). If

\[
\psi(x) = \sum_{n \leq x} \Lambda(n),
\]

then, for \( \sigma > 1 \), we have

\[
\int_1^\infty \frac{\psi(x) - x}{x^{\sigma+1}} dx = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{1-s} \equiv \phi(s), \text{ say.}
\]

Now \( \phi(s) \) is regular for \( \sigma > 0 \) except (possibly) for simple poles at the zeros of \( \zeta(s) \). Now, if \( \psi(x) = x + O(x) \), [which is a consequence of the p.n. theorem], then, given \( \varepsilon > 0 \),

\[
|\psi(x) - x| < \varepsilon x, \text{ for } x \geq x_0(\varepsilon) > 1.
\]

Hence, for \( \sigma > 1 \),

\[
|\phi(s)| < \int_1^{x_0} \frac{|\psi(x) - x|}{x^2} dx + \int_{x_0}^\infty \frac{\varepsilon}{x^\sigma} dx < K + \frac{\varepsilon}{\sigma - 1}
\]

where \( K = K(x_0) = K(\varepsilon) \). Thus

\[
|(\sigma - 1)\phi(\sigma + it)| < K(\sigma - 1) + \varepsilon < 2\varepsilon,
\]

for \( 1 < \sigma < \sigma_0(\varepsilon, K) = \sigma_0(\varepsilon) \). Hence, for any fixed \( t \),

\[
(\sigma - 1)\phi(\sigma + it) \to 0 \text{ as } \sigma \to 1 + 0.
\]

This shows that \( 1 + it \) cannot be a zero of \( \zeta(s) \), for in that case

\[
(\sigma - 1)\phi(\sigma + it)
\]

would tend to a limit different from zero, namely the residue of \( \phi(s) \) at the simple pole \( 1 + it \).
14 Prime Number Theorem and the magnitude of $p_n$

It is easy to see that the p.n. theorem is equivalent to the result: $p_n \sim n \log n$. For if
\[
\frac{\tau(x) \log x}{x} \rightarrow 1
\]
then
\[
\log \pi(x) + \log \log x - \log x \rightarrow 0,
\]
hence
\[
\frac{\log \pi(x)}{\log x} \rightarrow 1
\]
Now (1) and (2) give
\[
\pi(x) \cdot \log \pi(x) \rightarrow 1
\]
Taking $x = p_n$, we get $p_n \sim n \log n$, since $\pi(p_n) = n$.

Conversely, if $n$ is defined by: $p_n \leq x < p_{n+1}$, then $p_n \sim n \log n$ implies
\[
p_{n+1} \sim (n + 1) \log(n + 1) \sim n \log n,
\]
as $x \rightarrow \infty$. Hence
\[
x \sim n \log n
\]
or
\[
x \sim y \log y, \quad y = \pi(x) = n.
\]
Hence
\[
\log x \sim \log y, \quad \text{as above, so that}
\]
\[
y \sim x/\log x
\]
Bibliography


