Teichmüller Spaces

Contents

These notes were written for some expository talks on Teichmüller theory, given at "Conference on Low-dimensional manifolds and Groups", held in the Indian Statistical Institute, Bangalore, July 12-24, 2004. I have tried to make §§3 and 4 a little independent from §§1 and 2 (if one has a little background in the theory) and of each other. So some definitions might appear in §3 and also in §4.

The notes are far from complete, proofs are not given; they are meant as a guide for lecturing, rather than a formal introduction to the theory of Teichmüller spaces.

1. Some preliminary results

1.1. **Automorphisms of certain domains in the Riemann sphere.** Let $\mathbb{C}$ the set of complex numbers (complex plane) and $\widehat{\mathbb{C}}$ the **Riemann sphere**, the compactification of $\mathbb{C}$ by adding the point ∞. We will use the following notation for certain subsets of the complex plane:

$$\text{upper half plane,} \quad \mathbb{H} = \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$$

$$\text{lower half plane,} \quad \mathbb{L} = \{z \in \mathbb{C};\ \mathrm{Im}(z) < 0\}$$

$$\text{unit disc,} \quad \mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$$

A holomorphic mapping with a (necessarily holomorphic) inverse is called biholomorphic. By an automorphism of an open set we understand a biholomorphic mapping of the set onto itself.

The following results, describing the group of automorphisms of some domains in the Riemann sphere are classical. But first we remark that when we talk of mappings on $\widehat{\mathbb{C}}$, or with values in the Riemann sphere, we do not make any difference regarding the point of ∞, and thus we talk of holomorphic functions to include meromorphic functions as well (think of holomorphic functions on Riemann surfaces).

Prop 1.1. *The group of automorphisms of the Riemann sphere consists of the group of Möbius transformations; that is, mappings of the form $z \mapsto \frac{az+b}{cz+d}$, where a, b, c and d are complex numbers satisfying $ad - bc \neq 0$.*

Clearly we can multiply all coefficients of a Möbius transformation by a non-zero number and the mapping does not change, so we can assume, if needed, that $ad - bc = 1$. Then the group of Möbius transformations is then identified with the quotient of $\mathrm{GL}(2, \mathbb{C})$ by the multiples of the identity, or with the quotient of $\mathrm{SL}(2, \mathbb{C})$ by $\pm Id$. We denote these quotients by $\mathrm{PGL}(2, \mathbb{C})$ and $\mathrm{PSL}(2, \mathbb{C})$, respectively. We say that a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ represents a Möbius transformation M if M is given by $M(z) = (az + b)/(cz + d)$.

Prop 1.2. *The group of holomorphic automorphisms of the complex plane consists of the Möbius transformations of the form $z \mapsto az + b$, where $a \neq 0$.*

Prop 1.3. *The group of holomorphic automorphisms of the upper half plane consists of the Möbius transformations of the form $z \mapsto \frac{az+b}{cz+d}$, with real coefficients satisfying $ad - bc > 0$, or equivalently $ad - bc = 1$.*

Prop 1.4. *The group of holomorphic automorphisms of the unit disc consists of the Möbius transformations of the form* $z \mapsto \lambda \frac{z-w}{1-\overline{w}z}$*, where* $|\lambda| = 1$ *and* $|w| < 1$ *are complex numbers.*

A more general statement is the following result of B. Maskit [33], which we will not use but quote here for general reference.

Theorem 1.5 (Maskit)**.** *Let* D *be a domain (connected, open set) of the plane and* G *the group of automorphisms of* D*. Then there exists a one-to-one holomorphic mapping* $f : D \to D'$ *onto a domain* D' *so that every element of* $f\,G\,f^{-1}$ *is a Möbius transformation.*

1.2. **Möbius transformations.** The group of Möbius transformations, $\mathrm{PGL}(2, \mathbb{C})$, gets a natural topology from the group $\mathrm{GL}(2, \mathbb{C})$ (which is an open subset of $\mathbb{C}^4$). More precisely, a sequence of transformations $\{A_n\}$ converges to a transformation A_0 if and only if there exists matrices, $\left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right)$, representing A_n and A_0 respectively, such that $a_n \to a_0$, $b_n \to b_0$, $c_n \to c_0$ and $d_n \to d_0$.

Prop 1.6. *A non-identity Möbius transformation has at least one and at most two fixed points on the Riemann sphere.*

Möbius transformations can be classified in four different, mutually exclusive types as follows:

(i) the identity;

(ii) **parabolic** if it has only one fixed point. In that case the transformation is conjugate to $z \mapsto z + 1$;

(iii) **elliptic** if it has two fixed points and it is conjugate to a transformation of the form $z \mapsto \lambda z$, where $|\lambda| = 1$;

(iv) **loxodromic** if it is conjugate to a transformation of the form $z \mapsto \lambda z$, where $|\lambda| \neq 0, 1$. In the particular case of λ real, positive and not equal to 0 or 1, the transformation is called **hyperbolic** (we include hyperbolic transformations in the loxodromic case).

This classification can also be made with the square of the trace the matrices representing Möbius transformations. Let M be a Möbius transformation and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ a matrix representing it; denote by $tr^2(M)$ the number $(a + d)^2$. We have then the following way of classifying M:

(i) $tr^2(M) = 4$ if and only if M is the identity or parabolic;

(i) $tr^2(M)$ is real and in the interval $[0, 4)$ if and only if M is elliptic;

(iii) $tr^2(M)$ is real and greater than 4 if and only if M is hyperbolic;

(iv) $tr^2(M)$ if not in the real interval $[0, \infty)$ if and only if M is loxodromic but not hyperbolic.

Prop 1.7. *A Möbius transformation* M *keeps a disc invariant if and only if* $tr^2(M) \geq 0$*.*

Prop 1.8. *Let $\mathfrak{C}$ be the collection of lines and circles in the complex plane. Let C be an element of $\mathfrak{C}$ and M a Möbius transformation. Then $M(C)$ is also an element of $\mathfrak{C}$.*

1.3. **Kleinian and Fuchsian groups.** Let G be a group of Möbius transformations. We say that G acts **properly discontinuously** at a point z of $\widehat{\mathbb{C}}$ if there exists an open neighbourhood U of z such that the subset of G defined by

$$\{g \in G/\ g(U) \cap U \neq \emptyset\}$$

is finite. Clearly if G acts properly discontinuously at z then G will act properly discontinuously at all points of U, so we have that the set of points where G acts properly discontinuously is an open set of the Riemann sphere. We call this set the **set of discontinuity** of G and denote it by $\Omega(G)$ (or simply Ω). Its complement on the Riemann sphere is called the **limit set** of G, $\Lambda(G)$ (or just Λ).

Definition 1.9. *A group of Möbius transformations is called* **Kleinian** *if $\Omega(G) \neq \emptyset$.*

With the topology of Möbius transformations explained above we have the following result.

Prop 1.10. *A Kleinian group is discrete subgroup of* $\mathrm{PLS}(2, \mathbb{C})$.

Remark: frequently the term Kleinian is used for a discrete groups of Möbius transformations, even if $\Omega(G) = \emptyset$ (Thurston's definition). This is motivated by the fact that a discrete group of Möbius transformations acts properly discontinuously on the 3-dimension hyperbolic space. See proposition 1.17 below for a similar result for Fuchsian groups.

Prop 1.11. *If G is a Kleinian group then Λ is closed, G-invariant, nowhere dense subset of $\widehat{\mathbb{C}}$.*

An interesting property of Kleinian groups is that their limit sets have at most two points or are uncountable (a similar property holds for iteration of rational functions on the Riemann sphere).

Prop 1.12. *If G is a Kleinian group then Λ consists of zero, one, two or uncountably many points. In the latter case, Λ is perfect.*

A Kleinian group whose limit set consists of at most two points is called an **elementary** group.

Prop 1.13. *For a Kleinian group G we have that the Riemann sphere is the disjoint union of Ω and Λ.*

Elementary groups appear as groups of deck transformations of branched coverings of certain Riemann surfaces. We can list them all, but before that we need to set up some notation. Let $\tilde{X}$ and

X be two Riemann surfaces and $\pi : \tilde{X} \to X$ a mapping. We say that π is a **branched covering** if at every point p of X we have that π is either a covering in the usual sense (called regular covering by some authors), or π is locally like $z \mapsto z^k$ for some positive integer $k > 1$. In the latter case we say that p is a ramification point and k is called the ramification value of π. We also require that the set of points with ramification is a discrete subset of X. If X is a compact surface of genus g, $\tilde{X}$ is simply connected, and π is ramified over $p_1, \ldots, p_n$, with ramification values $\nu_1, \ldots, \nu_n$, we say that X has **signature** $(g, n; \nu_1, \ldots, \nu_n)$. For notation purposes we allow the symbol ∞ in a signature, to include punctures. A **puncture** is just a "missing" point of X (or, more formally, a domain on X conformally equivalent to the unit disc minus the origin). We include the punctures of a surface in its signature with the symbol ∞; for example, the complex plane is equal to the Riemann sphere with one puncture, so its signature is $(0, 1; \infty)$, and the complex plane minus the origin has signature $(0, 2; \infty, \infty)$. With this notation we can describe all elementary groups.

Suppose G is an elementary, non trivial group with no limit points. Then G is the deck transformations of a covering $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, branched over two or three points. The possible signatures of such covering are:

$$(0, 2; \nu, \nu),\ \ \nu > 1; \quad (0, 3; 2, 2, \nu),\ \ \nu > 1; \quad (0, 3; 2, 3, \nu),\ \ \nu = 3, 4, 5.$$

If G has one limit point then G (after conjugation) acts properly on the complex plane; that is, it is the deck transformation group of a (possibly branched) covering of the form $\mathbb{C} \to X$. We have the following possible signatures:

$$(1, 0);\ \ (0, 2; \infty, \infty);\ \ (0, 3; \infty, 2, 2);\ \ (0, 3; 2, 3, 6); (0, 3; 3, 3, 3);\ \ (0, 3; 2, 4, 4);\ \ (0, 3; 2, 2, 2, 2).$$

The first signature corresponds to a torus and the second one to the complex plane minus one point; the other signatures correspond to different branched coverings of the complex plane or the Riemann sphere.

If G has two limit points we have that it acts (after conjugation) on $\mathbb{C} \backslash \{0\}$; the corresponding coverings are those of a torus (signature $(1, 0)$) and a sphere with four ramification points of order 2 (signature $(0, 4; 2, 2, 2, 2)$).

A Kleinian group is called **Fuchsian** if it leaves a disc U invariant. Here by a disc (in the Riemann sphere) we understand some circular disc or a half-plane. The boundary of U is called the circle at infinity of G. We say that the group is of the **first kind** if every point of the circle at infinity is a limit point ($\Lambda = U$); otherwise the group is called of the **second kind** ($U \subset \Lambda$). By conjugation in $\mathrm{PSL}(2,\mathbb{C})$ we can assume, when necessary, that a Fuchsian group leaves the upper half plane invariant; in that case G can be considered as a subgroup of $\mathrm{PSL}(2,\mathbb{R})$.

Prop 1.14. *Let G be a Fuchsian group of the second kind. Then Λ is nowhere dense in the circle at infinity.*

We set the convention $\frac{1}{\infty} = 0$ and for a signature $(g, s; n_1, \dots, n_s)$ we put $\chi = 2g - 2 + n - \sum_{j=1}^{s} \frac{1}{n_j}$ (the negative of the Euler characteristic of the covering). Then we have that for an elementary group with no limit points, $\chi < 0$, while for an elementary group with one or two limit points we get $\chi = 0$. The next result says that if a signature satisfies $\chi > 0$ then it is the signature of a (possibly branched) covering defined on the upper half plane.

Prop 1.15. *Let σ be a signature satisfying $\chi > 0$. Then there exists a Fuchsian group of the first kind G, acting on the upper half plane, and a Riemann surface X, such that $\mathbb{H} \to X$ is a branched covering with the given signature and covering transformation group G.*

Theorem 1.16. *Let G be a non-elementary group in which $tr^2(g) \geq 0$ for all $g \in G$. Then G is Fuchsian.*

Prop 1.17. *Let G be a subgroup of* $\mathrm{PSL}(2,\mathbb{R})$*. Then the following are equivalent:*

(i) *G is acts properly discontinuously on the upper half;*

(ii) *G is discrete.*

This result is the motivation for the definition of a Kleinian group as a discrete group of Möbius transformations: one such group will act properly discontinuously on the 3-dimensional hyperbolic sphere (whose boundary is the Riemann sphere). The conclusion of the proposition is not true if we remove the condition of the group being Fuchsian. For example the group consisting of all Möbius transformations with entries in the Gaussian integers, the Picard group, is discrete, but it does not act properly discontinuously at any point of the Riemann sphere.

1.4. **Hyperbolic metrics.** Let $\lambda_{\mathbb{H}}$ denote the positive function $\lambda_{\mathbb{H}}(z) = \frac{1}{Im(z)}$ defined on the upper half plane. One can define a metric on $\mathbb{H}$ using this function as follows: for a piecewise smooth curve $\gamma : [a, b] \to \mathbb{H}$, we define the length of γ by the integral

$$|\gamma| = \int_{\gamma} \lambda_{\mathbb{H}} = \int_a^b \frac{1}{\mathrm{Im}(\gamma(t)} \, |\gamma'(t)| \, dt.$$

The distance between two points in $\mathbb{H}$, called the **Poincaré distance**, is defined as the infimum of the lengths of (piecewise smooth) curves joining them. We called $\lambda_{\mathbb{H}}|dz|$ the Poincaré metric on $\mathbb{H}$.

Theorem 1.18. *The group* $\mathrm{PSL}(2,\mathbb{R})$ *acts by isometries on* $\mathbb{H}$ *with respect to the Poincaré metric.*

By the Riemann Mapping Theorem, if D is a proper, simply connected subset of $\mathbb{C}$, then there exists a biholomorphic function $f : \mathbb{H} \to D$. We can then use the Poincaré metric on the upper half plane to get a metric on D. In particular, in the unit disc $\mathbb{D}$ we have that the expression of the metric is given by the function $\lambda_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}$.

It is easy to show that the Poincaré metric has constant negative curvature equal to -1; or some other negative constant, if the metric is a positive multiple of the expression given above, as some textbooks do.

1.5. **Quasiconformal mappings.** In this subsection all mappings are assumed to orientation-preserving, defined between domains (open, connected sets) of the complex plane. We will use $w : D \to D'$ for one such mapping. We can define the following differential operators where w has partial derivatives:

$$w_z = \frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) \quad \text{and} \quad w_{\overline{z}} = \frac{\partial w}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right).$$

We define the **complex dilatation** of w at a point z_0, where w is differentiable with $Jac(w)(z_0) = |w_z(z_0)|^2 - |w_{\overline{z}}(z_0)|^2 \neq 0$, by $\mu(z_0) = \mu_w(z_0) = w_{\overline{z}}(z_0)/w_z(z_0)$. Observe that $|\mu(z_0)| < 1$ (since w is orientation-preserving) and $w_z(z_0) \neq 0$.

For a mapping w and a point z in D we can set $L(z,r) = \max_\zeta\{|w(\zeta) - w(z)|;\ |\zeta - z| = r\}$ and $l(z,r) = \min \zeta\{|w(\zeta) - w(z)|;\ |\zeta - z| = r\}$ (these expressions make sense for small values of r). If w is C^1 we have that

$$\lim_{r\to 0} \frac{L(z,r)}{l(z,r)} = \frac{1+|\mu(z)|}{1-|\mu(z)|}.$$

This equation motivates our definition of quasiconformal mapping.

Definition 1.19. *An orientation-preserving homeomorphism* $w : D \to D'$ *between domains of the complex plane is called* **quasiconformal (qc)** *if the circular dilatation at* z*,*

$$H(z) = \limsup_{r\to 0} \frac{L(z,r)}{l(z,r)}$$

is bounded on D*. If* $H(z) \leq K$ *a.e. on* D *then we say that* w *is* K*-quasiconformal.*

Another equivalent definition is as follows.

Definition 1.20. *An orientation-preserving homeomorphism* $w : D \to D'$ *between domains of the complex plane is called* **quasiconformal (qc)** *if* w *has distributional derivatives in* L^p_{loc}*, for some*

$p \geq 1$, (locally in L^p), and $|w_{\overline{z}}(z)| \leq k|w_z(z)|$ almost everywhere on D, for some k satisfying $0 \leq k < 1$. The map w is called K-qc for any K with $K \geq (1+k)/(1-k)$.

One can easily compute that a holomorphic mapping is 1-qc, or equivalently that its complex dilatation is identically 0 (this is simply the Cauchy-Riemann equations, $f_{\overline{z}} = 0$).

Prop 1.21. *A 1-qc homeomorphism is a biholomorphism.*

Prop 1.22. *The inverse of a K-qc mapping is also K-qc. The composition of a K_1-qc mapping and a K_2-qc mapping is a K_1K_2-qc mapping.*

The existence of qc mappings on the Riemann sphere with a given dilatation (solutions of the Beltrami equation $w_{\overline{z}} = \mu w_z$) is an important result extensively used in the theory of Teichmüller spaces.

Theorem 1.23. *Given any $\mu \in L^\infty(\mathbb{C})$ with $||\mu||_\infty < 1$, there exists a unique quasiconformal homeomorphism of $\widehat{\mathbb{C}}$, with dilatation μ, and fixing ∞, 0 and 1 (pointwise). We will denote this mapping by w^μ.*

The following stronger result, that the solutions given above depend holomorphically on the dilatation μ, is known as the *measurable Riemann mapping theorem.*

Theorem 1.24 (Ahlfors-Bers [3])**.** *For every fixed $z \in \mathbb{C}$, the mapping $\mu \mapsto w^\mu(z)$ is holomorphic: if μ depends holomorphically on variables $(t_1, \ldots, t_k) \in \mathbb{C}^k$, then $w^\mu(z)$ is a holomorphic function of the variables $(t_1, \ldots, t_k)$.*

1.6. **Beltrami differentials for groups.** An orientation-preserving homeomorphism between two Riemann surfaces if K-qc if and only if it is K-qc when expressed on local coordinates. This definition makes sense since changes of coordinates are holomorphic functions, and post-composition of qc mappings with holomorphic functions does not change the type of a qc mapping.

Let $L^\infty_{(-1,1)}(X)$ denote the set of tensors of type $(-1,1)$, essentially bounded, on X; that is, the set of L^∞ sections of the bundle $\kappa^{-1} \otimes \overline{\kappa}$, where κ is the cotangent bundle of X. Each such tensor will be given by certain functions on local coordinates of X satisfying a compatibility condition, namely, if z and w are two local coordinates on X, and μ and ν are the corresponding functions of one such tensor, then

$$\mu(z(w)) \left(\frac{dz}{dw}\right)^{-1} \left(\overline{\frac{dz}{dw}}\right) = \nu(w).$$

Suppose $f : X \to Y$ is a qc mapping between two surfaces; we can express f in terms of local coordinates z and w (on X and Y, respectively), by $f(z) = w$. Then the expression $\mu_f(z) = f_{\overline{z}}/f_z$ defines an element of $L^\infty_{(-1,1)}(X)$.

Conversely, if $\mu \in L^\infty_{(-1,1)}(X)$, with $||\mu||_\infty < 1$, on each coordinate patch (U_α, z_α) of X we can find a qc mapping w_α with Beltrami coefficient $\mu(z_\alpha)$. These $(U_\alpha, \{w_\alpha\})$ provide a complex analytic atalas on X; the new local charts are qc with respect to the original structure of X. We denote the surface X with this atlas by X_μ.

Prop 1.25. *The identity mapping $1_X : X \to X_\mu$ is a qc homeomorphism of X with complex dilatation given by the $(-1, 1)$ tensor μ.*

We can do a similar analysis regarding groups rather than surfaces in the following way. Let G be a Fuchsian group and w a qc homeomorphism of $\widehat{\mathbb{C}}$ with complex dilatation μ. We say that w is *compatible* with G is wgw^{-1} is a Möbius transformation for all $g \in$G.

Lemma 1.26. *The mapping w is compatible with G if and only if its complex dilatation $\mu_w = \mu$ satisfies*

$$(\mu \circ g)\, \overline{g'}/g' = \mu,$$

a.e. on $\widehat{\mathbb{C}}$, for all g on G.

We denote by $L^\infty(\mathbb{H}, G)$ the set of L^∞ functions defined on $\mathbb{H}$ which are compatible with G. The elements of this space are called **Beltrami coefficients for *G***. Given one element μ in the unit ball of $L^\infty(\mathbb{H}, G)$, we can extend it to the lower half plane in two ways: by 0 or by reflection $(\mu(z) = \overline{\mu(\overline{z})}$ for z in the lower half plane). The corresponding solutions of the Beltrami equations fixing ∞, 0 and 1 will be denoted by w^μ and w_μ respectively. Since μ is compatible with G we have that $G^\mu = w^\mu G(w^\mu)^{-1}$ and $G_\mu = w_\mu G w_\mu^{-1}$ are Kleinian groups. We have that G_μ is actually Fuchsian (it leaves the upper half plane invariant). The group G^μ is called **quasi-Fuchsian**; the image of the real axis under w^μ is a Jordan curve on the Riemann sphere, known as a quasi-circle, and G^μ keeps each component of the complement of that curve invariant.

Prop 1.27. *The element $g^\mu = w^\mu \circ g \circ (w^\mu)^{-1}$ depends complex analytically on μ for any fixed $g \in G$. Similarly, $g_\mu = w_\mu \circ g \circ w_\mu^{-1}$ depends real analytically on μ.*

Prop 1.28. *The homomorphism $E_\mu : G \to G_\mu$ given by $E_\mu(g) = g_\mu$ preserves the type (1.2) of each element g. The groups G and G_μ are both of the first kind or both of the second kind. If G is finitely* *generated of the first kind then G_μ is finitely generated Fuchsian group and $X_\mu = \mathbb{H}/G_\mu$ has the same signature as $X = \mathbb{H}/G$.*

1.7. **Automorphic forms and quadratic differentials on surfaces.** Let G be a Fuchsian group acting on $\mathbb{H}$. Given a positive integer $q \geq 2$, a ***q*-form** for a G on $\mathbb{H}$ is a function σ defined on $\mathbb{H}$, satisfying $(\sigma \circ g)\,(g')^q = \sigma$ for all $g \in G$. One can talk of measurable, holomorphic, etc, forms, depending on the character of the function σ (in the case of measurable functions the equality should be satisfied a.e. on $\mathbb{H}$).

We defined the supremum norm of an automorphic form by weighting it by the Poincaré metric:

$$||\sigma||_{q,\infty} = ||\lambda_{\mathbb{H}}(z)^{-q}\,\sigma(z)||_\infty,$$

and denote by $L^\infty_q(\mathbb{H}, G)$ the (Banach) space of q-forms with finite supremum norm. We denote by $B_q(\mathbb{H}, G)$ the subspace of $L^\infty_q(\mathbb{H}, G)$ consisting of holomorphic forms.

Similarly one can consider the L^1-norm, defined by

$$||\sigma||_q = \int_{\mathbb{H}/G} \lambda_{\mathbb{H}}(z)^{2-q}\,|\sigma(z)|\,dx\,dy.$$

The integration is done over a fundamental domain (for the action of G on the upper half plane). Since the expression in the integral is G-invariant, the choice of fundamental domain does not change the value of the above integral. We denote by $L^1_q(\mathbb{H}, G)$ the space of q-forms with finite L^1-norm, and by $A_q(\mathbb{H}, G)$ the subspace of holomorphic forms (one can define L^p -norms in a similar way, with exponent $2-pq$ in the Poincaré metric term of the above integral; however the case we are interested is when $p = 1$).

One has $B_q(\mathbb{H}, G) \subset A_q(\mathbb{H}, G)$, if $\mathbb{H}/G$ has finite Poincaré area. If $\mathbb{H}/G$ is a compact surface with a finite number of punctures we actually have $B_q(\mathbb{H}, G) = A_q(\mathbb{H}, G)$.

There is a paring, called the Petersson pairing, between forms defined by

$$L^1_q(\mathbb{H}, G) \times L^\infty_q(\mathbb{H}, G) \to \mathbb{C}$$

$$(\alpha, \beta) \mapsto \int_{\mathbb{H}/G} \lambda_{\mathbb{H}}(z)^{2-2q}\,\alpha(z)\,\overline{\beta(z)}\,dx\,dy.$$

Theorem 1.29. *The restriction of the Petersson pairing to holomorphic forms, $A_q(\mathbb{H}, G)\times B_q(\mathbb{H}, G)$, is non-singular (and B_q becomes the dual of A_q).*

Forms invariant with respect to groups are related to sections of certain bundles on the quotient surfaces. More precisely, if X is a compact Riemann surface, possibly with punctures, with universal covering space given by the upper half plane, we can write $X = \mathbb{H}/G$ (assuming that $2g-2+n > 0$, where g and n are the genus and number of punctures of X, respectively). Then $A_2(\mathbb{H}, G)$ corresponds to integrable holomorphic sections of κ^2, (recall that κ denotes the cotangent bundle of X); that is, integrable holomorphic quadratic differentials on X. To make sense of the integrable part we have the following result.

Prop 1.30. *Let X be a surface and p a puncture on X. A holomorphic quadratic differential φ on X is integrable on a neighbourhood of p if and only if φ has either a removable singularity or a pole of order 1 at p.*

If we denote by $A_2(X)$ the space of holomorphic quadratic differentials with

$$\int_X |\varphi| < \infty,$$

then $A_2(X)$ can be identified with $A_2(\mathbb{H}, G)$. The Riemann-Roch theorem gives the dimension of this space.

Prop 1.31. *If X is a compact surface of genus g with n punctures, then $A_2(X)$ is a complex vector space of dimension $3g - 3 + n$.*

Prop 1.32. *Let G be a Fuchsian group acting on the upper half plane. Then $A_2(\mathbb{H}, G)$ is finite dimensional if and only if G is finitely generated of the first kind. In this case, $B_2(\mathbb{H}, G) = A_2(\mathbb{H}, G)$.*

1.8. **Nielsen's theorem.** Let X be a surface. Let $Hom(X)$ denote the group of (self) homeomorphisms of X, and $Hom_0(X)$ the subgroup of homeomorphisms homotopic to the identity. Choose a point x of X and let $Aut(\pi_1(X, x))$ be the group of automorphisms of the fundamental group of X (based at x). Let $Inn(\pi_1(X, x))$ be the subgroup of inner automorphisms (automorphisms given by conjugation). If $f \in Hom(X)$ satisfies $f(x) = x$, then it induces a homomorphism on the fundamental group of X, $f_* : \pi_1(X, x) \to \pi_1(X, x)$. There is a natural homomorphism

$$Hom(X)/Hom_0(X) \to Out(\pi_1(X, x)) = Aut(\pi_1(X, x))/Inn(\pi_1(X, x)).$$

Theorem 1.33 (Nielsen)**.** *If X is a compact Riemann surface of positive genus then the above homomorphism is actually an isomorphism.*

For reference we quote the following interesting result.

Theorem 1.34 (Baer-Mangler-Epstein)**.** *Let X be an orientable compact surface, with finitely many (possibly none) punctures. Two orientation-preserving homeomorphism of X are homotopic if and only if they are isotopic.*

2. TeichmüllerTheory

While studying two Riemann surfaces it is natural to consider them equivalent if there exists a holomorphic homeomorphism between them. The set of such equivalence classes is the moduli or Riemann space. It was Teichmüller who realized that introducing a new relation, based on quasiconformal mappings (and some homotopy conditions) one gets a space, known as the Teichmüller space, which is simpler to study than the Riemann space. This new space is the universal covering space of Riemann space; the mapping class group (see Nielsen's theorem 1.33) becomes the covering group. Thus one reduces the study of Riemann space to the study of Teichmüller space and the mapping class group.

In the first two subsections we study some examples of classification of certain Riemann surfaces where it is possible to do computations explicitly. The rest of the subsections are devoted to the definition and properties of Teichmüller spaces of Riemann surfaces and Fuchsian groups. The basic material of this section is taken from S. Nag's book [41]; other good references are the books by Abikoff [1] (for the real analytic theory), Gardiner [20] and Y. Imayoshi and M. Taniguchi [26].

2.1. **The Riemann Mapping Theorem and the Uniformization Theorem.** One can ask when two domains (connected, open subsets of the complex plane) are **conformally equivalent**; that is, when there exists a holomorphic homeomorphism between them. The case of simply connected domains is answered by Riemann Mapping Theorem.

Theorem 2.1. *If Ω is a simply connected subset of the complex plane, not equal to $\mathbb{C}$, then there exists a holomorphic homeomorphism from the unit disc onto Ω.*

Thus we have that there are two classes of simply connected domains in the complex plane, namely the complex plane and the unit disc. In particular the upper half plane is conformally equivalent to the unit disc; we will use this fact to make most of our statements on $\mathbb{H}$ rather than on $\mathbb{D}$.

We now consider the problem of classifying circular annuli, that is regions on the complex plane determined by two concentric circles. Set $A(z_0; r_1, r_2) = \{z \in \mathbb{C}; r_1 < |z - z_0| < r_2\}$, for $z_0 \in \mathbb{C}$ and $0 < r_1 < r_2$. We have that $A(z_0, r_1, r_2)$ is equivalent to $A(0; 1, r_2/r_1)$. So we need to study only annuli of this form, which we denote by A_R ($R = r_2/r_1 > 1$). An analytic approach to this problem is given by the following result [44].

Theorem 2.2. *The annuli $A_1 = A_{R_1}$ and $A_2 = A_{R_2}$ are conformally equivalent if and only if $R_1 = R_2$.*

Proof. Let $f : A_1 \to A_2$ be a biholomorphic function. Let C be the circle of centre the origin and radius $\sqrt{R_2}$. Since $f^{-1}(C)$ is compact we have that there exists an $\epsilon > 0$, such that $A(1, 1+\epsilon)$ and $f^{-1}(C)$ have empty intersection. The set $V = f(A(1, 1+\epsilon))$ is connected and does not intersect C, so $V \subset A(1, \sqrt{R_2})$ or $V \subset A(\sqrt{R_2}, R_2)$. In the second case we can consider the function R_2/f instead of f, so we can assume $V \subset A(1, r)$.

If $\{z_n\}$ is a sequence of complex numbers with $1 < |z_n| < 1+\epsilon$ and $|z_n| \to 1$, then $\{f(z_n)\}$ is a sequence in V without limit point in A_2, so $|f(z_n)| \to 1$. Similarly $|f(z_n)| \to R_2$ when $z_n \to R_1$.

Let $\alpha = \log R_2 / \log R_1$. Set $u(z) = 2\log|f(z)| - 2\alpha \log|z|$ on A_1. Since $\partial(2\log|f|) = \partial(\log f\overline{f}) = f'/f$ we have $\partial u = f'/f - \alpha/z$, so u is harmonic on A_1, and by the above remarks, u extends to a continuous function on $\overline{A_1}$; then we get $u = 0$ and therefore $f'(z)/f(z) = \alpha/z$.

Let $\gamma(t) = \sqrt{R_1}e^{it}$, for $-\pi \le t \le \pi$ and $\Gamma = f \circ \gamma$. Then we have

$$\alpha = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = Ind_\Gamma(0);$$

where $Ind_\Gamma(0)$ denotes the index of 0 with respect to Γ. So α is a positive integer, and $\partial(z^{-\alpha}f(z)) = 0$ on A_1, which implies that $f(z) = cz^\alpha$ for some constant c. Since f is one-to-one we get $\alpha = 1$ and so $R_1 = R_2$. ☐

The classification of simply connected Riemann surfaces is given by the Uniformization theorem.

Theorem 2.3 (Poincaré-Koebe Uniformization Theorem)**.** *Any simply connected Riemann sphere is biholomorphically equivalent to the Riemann sphere, the complex plane or the unit disc. These three surfaces are not conformally equivalent.*

Using the Uniformization theorem one can approach the question of classification of annuli from a different point of view. First of all, it is easy to see that the universal covering of an annulus A_R is given by $\pi_R : \mathbb{H} \to A_R$, where $\pi_R(z) = \exp(-2\pi i \log(z)/\log(\lambda))$; here λ is a real number greater than 1 satisfying $\exp(2\pi/\log(\lambda)) = R$. The covering group, G_λ, is generated by the transformation $g_\lambda(z) = \lambda z$. We will have that the annuli G_{R_1} and G_{R_2} are conformally equivalent if the corresponding groups, say G_λ and G_μ, are conjugate. This implies that there exists a Möbius transformation A satisfying $g_\lambda = A\, g_\mu\, A^{-1}$. It is easy to see that then $g_\lambda = g_\mu$; that is, $R_1 = R_2$, which is the result proved above.

2.2. **Abel's theorem and the uniformization of tori.** Abel's theorem implies that any tori is a quotient of the form $S_\tau = \mathbb{C}/L(1,\tau)$, where τ is a complex number with positive imaginary part. Here $L(1,\tau)$ denotes the lattice of all transformations of the form $z \mapsto z + m + n\tau$, n and m integers.

Theorem 2.4. *Two tori S_τ and S_σ are conformally equivalent if and only if there exists a transformation $M \in SL(2,\mathbb{Z})$ such that $M(\sigma) = \tau$.*

Proof. If S_τ and S_σ are conformally equivalent by, say $\varphi : S_\tau \to S_\sigma$, then we have that φ lifts to an automorphism of the complex plane, $\widetilde{\varphi}(z) = az + b$, where $a \neq 0$. So we have, for any integers m and n, $\widetilde{\varphi}(z + m + n\tau) = \widetilde{\varphi}(z) \text{mod } L(1,\sigma)$. We then get $a\tau = \mu_1\sigma + \nu_1$ and $a = \mu_2\sigma + \nu_2$. In other words, $\tau = M\sigma$, where M is the element of $GL(2,\mathbb{Z})$ given by $M = \left(\begin{smallmatrix} \mu_1 & \nu_1 \\ \mu_2 & \nu_2 \end{smallmatrix}\right)$. Since M is invertible (apply the above reasoning to φ^{-1}) we have that $\det M = \pm 1$. But τ and σ both have positive imaginary parts, so we must have $\det M = 1$. □

Choose a reference torus, say S_i, with generators of its fundamental group given by the projections of the straight segments $[0,1]$ and $[0,i]$ (call them A and B respectively). A *marked torus* is a triple (S_i, f_τ, S_τ), where S_τ is a torus, and $f_\tau : S_i \to S_\tau$ a homeomorphism; we have in S_τ two generators of its fundamental group given by $A_\tau = f_\tau(A)$ and $B_\tau = f_\tau(B)$. Suppose (S_i, f_τ, S_τ) and $(S_i, f_\sigma, S_\sigma)$ are two marked tori; we consider them equivalent if there is a biholomorphic mapping $\varphi : S_\tau \to S_\sigma$ with $f_\sigma^{-1} \circ \varphi \circ f_\tau$ homotopic to the identity; that is, if φ_* is the induced mapping between homotopy groups, then $\varphi_*(A_\tau) = A_\sigma$ and $\varphi_*(B_\tau) = B_\sigma$. As before, φ lifts to $\widetilde{\varphi}(z) = (\mu_2\sigma + \nu_2)z$ with M as in the above proof. Then φ_* is given by

$$\begin{pmatrix} B_\tau \\ A_\tau \end{pmatrix} \mapsto M, \begin{pmatrix} B_\sigma \\ A_\sigma \end{pmatrix}.$$

So $\varphi_* = id$ if and only if $M = \pm Id$ which implies $\sigma = \tau$.
This shows that the space of marked tori (Teichmüller space) of the torus is given by the upper half plane $\mathbb{H}$, and the moduli (Riemann) space is the quotient of $\mathbb{H}$ by the action of the (modular) group $SL(2,\mathbb{Z})$.

2.3. **Teichmüller spaces of Riemann surfaces, moduli spaces and modular groups.** For the rest of this notes, unless otherwise state, X denotes a compact Riemann surface of genus g with n punctures. We put the condition that $2g - 2 + n > 0$, so the universal covering space of X is the upper half plane.

Fix one such surface X. A **marked Riemann surface** is a triple (X, f, X_1), where X_1 is a Riemann surface and $f : X \to X_1$ is a quasiconformal homeomorphism. Let Marked(X) denote the space of marked Riemann surfaces (with X fixed). We define two equivalence relations in Marked(X) as follows:

- $(X, f, X_1) \sim (X, g, x_2)$ are Teichmüller equivalent, if there exists a biholomorphic mapping

$\varphi : X_1 \to X_2$ such that $g^{-1} \circ \varphi \circ f$ is homotopic to the identity mapping of X;

• $(X, f, X_1) \sim_R (X, g, X_2)$ are Riemann equivalent, if there is a biholomorphic mapping $\varphi : X_1 \to X_2$.

The quotient spaces $T(X) = \text{Marked}(X)/\sim$ and $M(X) = \text{Marked}(X)/\sim_R$ are called the **Teichmüller** and **Riemann** (or **moduli**) spaces of X respectively. The notation $T(g, n)$ and $M(g, n)$ for these two spaces, respectively, is also frequently used. It is the moduli space the one we are interested on (it classifies Riemann surfaces up to biholomorphisms); however, it turns out that it is easier to study properties of Teichmüller spaces.

To introduce a distance in $T(X)$ we start by setting a pseudo-metric on Marked(X) by the expression

$$\hat{\tau}((X, f, X_1),\ (X, g, X_2)) = \frac{1}{2} \log K(g \circ f^{-1}),$$

where $K(h) = (1 + ||\mu_h||_\infty)(1 - ||\mu_h||_\infty)$. However $\hat{\tau}$ is not a metric since it vanishes when $g \circ f^{-1}$ is conformal (see proposition 1.21), so we quotient the space of marked Riemann by a new relation: $(X, f, X_1) \approx (X, g, X_2)$ if the mapping $g \circ f^{-1}$ is conformal; let Marked $^*(X)$ denote the resulting quotient space.

Prop 2.5. *The quotient mapping from Marked(X) to $M(X)$ factors via Marked $^*(X)$ and $T(X)$ so the following natural mappings are well defined:*

$$\text{Marked}(X) \to \text{Marked }^*(X) \to T(X) \to M(X).$$

Lemma 2.6. Marked $^*(X)$ *can be canonically identified with the unit ball of the space of Beltrami differentials of X, $L^\infty_{(-1,1)}(X) = L^\infty(\overline{\kappa} \otimes \kappa^{-1})$.*

Proof. Let $B(L^\infty_{(-1,1)}(X))$ denote the unit ball of the space. Consider the mapping Marked $^*(X) \to B(L^\infty_{(-1,1)}(X))$ defined by $[X, f, X_1] \mapsto \mu_f$. One has $\mu_f = \mu_g$ implies that $g \circ f^{-1}$ is conformal.

To construct the inverse mapping let $\mu \in B(L^\infty_{(-1,1)}(X))$. We have $X = \widetilde{X}/G$, where $\widetilde{X}$ is the universal covering space of X and G a group of Möbius transformations. Then μ lifts to a G-invariant form in $L^\infty_{(-1,1)}$. Let w^μ be the corresponding solution, X_μ the quotient $w^\mu(\widetilde{X})/w^\mu, G, (w^\mu)^{-1}$ and $(w^\mu)^* : X \to X_\mu$ the induced map. Then we map μ to the class of $(X, (w^\mu)^*, X_\mu)$. □

The function $\hat{\tau}$ defines a distance on $T(X)$ by

$$d_T([X, f, X_1], [X, g, X_2]) = \inf_{\varphi} \frac{1}{2} \log K(\varphi),$$

where the φ varies over all quasiconformal mappings from X_1 to X_2 homotopic to $g^{-1} \circ f$. We call this function the **Teichmüller distance**.

Prop 2.7. *The space $T(X)$ with the Teichmüller distance is complete.*

Let $Q(X)$ denote the set of quasiconformal homeomorphisms of X, and $Q_0(X)$ those homotopic to the identity. We define the **modular group** or **mapping class group**, $\mathrm{Mod}(X)$ as the quotient $Q(X)/Q_0(X)$. This group acts on $T(X)$ by $\varphi^*([X, f, X_1]) = [X, f \circ \varphi^{-1}, X_1]$.

Lemma 2.8. *$T(X)/\mathrm{Mod}(X)$ can be naturally identified with $R(X)$.*

Proof. If $\varphi^*([X, f, X_1]) = [X, g, X_2]$ then obviously $(X, f, X_1) \sim_R (X, g, X_2)$. Conversely, if $\phi : X_1 \to X_2$ is biholomorphic, set $\varphi = g^{-1}\,\phi\, f$, and then we have

$$\varphi^*([X, f, X_1]) = [X, \phi^{-1}\, g, X_1] = [X, g, X_2].$$

□

The modular group does not act effectively on $T(2, 0)$ since the hyperelliptic involution acts like the identity. The following result gives the general statement about this fact.

Prop 2.9. *The modular group does not act effectively on $T(g, n)$ only if (g, n) is one of the following:* $(2, 0),\ (1, 2),\ (1, 1),\ (0, 4),\ (0, 3)$.

The extended modular group is defined similarly to $Mod(X)$ but including orientation-reversing mapping too. By Nielsen's theorem we have that the extended modular group is isomorphic to $Out(X)$. The mapping class group is isomorphic to $Aut^+(\pi_1(X))/Inn(\pi_1(X))$, where $Aut^+()_1(X)X$ is the subgroup of index 2 of $Aut(\pi_1(X))$ corresponding to the π_1-actions of orientation-preserving homeomorphisms of X.

Prop 2.10. *Teichmüller space is isomorphic to the quotient of the unit ball of $L^\infty_{(-1,1)}(X)$ by $Q_0(X)$.*

Proof. Let $B(L^\infty_{(-1,1)}(X))$ denote the unit ball of the space of Beltrami differential of X. Define a mapping θ as follows:

$$\theta : T(X) \to B(L^\infty_{(-1,1)}(X))/Q_0(X)$$

$$[X, f, X_1] \mapsto \mu_f (\operatorname{mod} Q_0(X))$$

The inverse of θ is defined as follows. Given μ in $L^\infty_{(-1,1)}(X)_1$, let X_μ be the complex structure on X induced by μ (see proposition 1.25); then $\theta^{-1}([\mu]) = [X, 1_X, X_\mu]$. □

2.4. **Teichmüller spaces of Fuchsian groups.** From now onwards, unless stated otherwise, we will assume that G is a finitely generated, torsion free Fuchsian group of the first kind, leaving the upper half plane invariant (the theory we explain below works in greater generality than this setting).

We define the space of bounded Beltrami coefficients for G on the upper half plane by

$$L^\infty(\mathbb{H},G) = \{\mu \in L^\infty(\mathbb{H}) \ ; \ (\mu \circ g)`, \overline{g'}/g = \mu, \text{ a.e. on } \mathbb{H}, \text{ for all } g \in G\},$$

If μ is in the unit ball of $L^\infty(\mathbb{H},G)$ (we will denote this unit ball by $B(L^\infty(\mathbb{H},G))$), let w_μ denote the solution of the Beltrami equation, fixing ∞, 0 and 1 pointwise, obtained by extending μ to the lower half plane by reflection. Let $E_\mu : G \to G_\mu \subset \mathrm{PSL}(2,\mathbb{C})$ be the group homomorphism given by $E_\mu(g) = w_\mu \, g \, (w_\mu)^{-1}$. Assume μ and ν are elements of $B(L^\infty(\mathbb{H},G))$ such that the groups G_μ and G_ν are equal; set $X = \mathbb{H}/G$, $Y = \mathbb{H}/G_\mu = \mathbb{H}/G_\nu$. We have functions f_μ, $f_\nu : X \to Y$ induced by w_μ and w_ν respectively.

Prop 2.11. *The following are equivalent:*

(i) f_μ *is homotopic to* f_ν*;*

(ii) $w_\mu|_\mathbb{R} = w_\nu|_\mathbb{R}$*;*

(iii) $w_\mu \, g \, (w_\mu)^{-1} = w_\nu \, g \, (w_\nu)^{-1}$, *for all* $g \in G$*.*

We define the **Teichmüller** $T(G)$ and **Riemann** $M(G)$ spaces of G as the quotients of the unit ball $B(L^\infty(\mathbb{H},G))$ by the relations $\sim$ and $\sim_R$, defined as follows:

- $\mu \sim \nu$ if $w_\mu|_\mathbb{R} = w_\nu|_\mathbb{R}$;
- $\mu \sim_R \nu$ if G_μ and G_ν are conjugate subgroups of $\mathrm{PSL}(2,\mathbb{C})$.

To define the modular group of G we need to set up some notation:

$$Q(G) = \{w : \mathbb{H} \to \mathbb{H}; w \text{ quasiconformal }, wGw^{-1} \subset \mathrm{PSL}(2,\mathbb{R})\};$$

$$Q_0(G) = \{w \in Q(G); \ w|_\mathbb{R} = \text{identity}\};$$

$$N_{qc}(G) = \{w \in Q(G); \ wGw^{-1} = G\}.$$

We set $mod(G) = N_{qc}(G)/\left(N_{qc}(G)G \cap Q_0(G)\right)$ and the **modular group** of G by $Mod(G) = mod(G)/G$. If $X = \mathbb{H}/G$ then the groups $Mod(X)$ and $Mod(G)$ are isomorphic (to obtain that $Mod(G)$ is isomorphic to $Mod(X)$, one has to quotient $mod(G)$ by G since any homeomorphism of X has G-lifts to $\mathbb{H}$).

The space $T(G)$ has a pseudo-metric τ given by

$$\tau([w_1],[w_2]) = \inf \frac{1}{2} \log K(w),$$

where w is a qc mapping, and w and $w_1 \circ w_2^{-1}$ coincide on the real axis.

Prop 2.12. *The pseudo-metric τ on $T(G)$ is actually a metric. Moreover, if $[\mu]$ and $[\nu]$ in $T(G)$ with $\tau([\mu],[\nu]) = d$, then for any given μ_0 such that $[\mu_0] = [\mu]$, there exists ν_0 such that $[\nu_0] = [\nu]$ and $\frac{1}{2}\log K(w_{\mu_0} \circ w_{\nu_0}^{-1}) = d$. τ is a complete metric on $T(G)$.*

Prop 2.13. *If G is a (non-elementary, torsion free) Fuchsian group and $X = \mathbb{H}/G$, then there exists a canonical isometry between $T(G)$ and $T(X)$.*

Proof. By conjugation we can assume that G is *normalized*; that is, ∞, 0 and 1 are points in the limit set of G. Let μ be an element of $L^\infty(\mathbb{H}, G)$; then we assigned to it (the Teichmüller class of) the marked Riemann surface $(X, f_\mu, X_\mu = \mathbb{H}/G_\mu)$, where f_μ is induced by w_μ as above.

For the inverse mapping, given a marked surface (X, f, X_1), write $X_1 = \mathbb{H}/G_1$; then f lifts to $\tilde{f} : \mathbb{H} \to \mathbb{H}$ such that $\tilde{f}G\tilde{f}^{-1} = G_1$, and $\tilde{f}$ fixes ∞, 0 and 1 pointwise. We assign to the given marked Riemann surface the Beltrami coefficient $\mu_{\tilde{f}}$. □

Theorem 2.14. *The modular group $Mod(G)$ ($Mod(X)$) acts properly discontinuously by isometries on $T(G)$ ($T(X)$ respectively).*

The proof of the theorem for G is based on the next two results. For a surface X one uses the isometries between $T(G)$ and the identification between the groups $Mod(G)$ and $Mod(X)$.

Lemma 2.15. *If G is Fuchsian, non-cyclic, then the normalizer of G in* $\mathrm{PSL}(2,\mathbb{R})$*, $N(G)$, is also Fuchsian. Moreover, if G is of the firs kind then $N(G)/G$ is a finite group, isomorphic to the automorphism group of the surface $\mathbb{H}/G$.*

Lemma 2.16. *If G is Fuchsian of the first kind then the set of traces squares of elements of G is a discrete subset of $\mathbb{R}$.*

2.5. **Fricke coordinates.** We now want to give some coordinates to $T(X)$, for X a compact surface, possible with punctures. We start with a lemma that says that all non-trivial elements of a Fuchsian group uniformizing a compact surface are of the same type, namely hyperbolic transformations (see subsection 1.2).

Lemma 2.17. *Let G be a torsion-free Fuchsian group with $X = \mathbb{H}/G$ compact, $E : G \to$* $\mathrm{PSL}(2,\mathbb{R})$ *an injective homomorphism with discrete image. Then all non-identity elements of $E(G)$ are hyperbolic.*

If X as above is compact of genus g (≥ 2), we can take a set of "standard loops", $A_1, \ldots, A_g,$, $B_1, \ldots, B_g$, which generate the fundamental group of X (which is isomorphic to G) and satisfy the relation $\prod_{j=1}^{g}[A_j, B_j] = Id$ (here $[A, B] = ABA^{-1}B^{-1}$ denotes the commutator of two elements). Denote by the same letters the corresponding set of generators of G. We can normalized these generators so that B_g has attractive fixed point at ∞ and repelling fixed point at 0 while A_g has attractive fixed point at 1. If $E : G \to \mathrm{PSL}(2, \mathbb{R})$ is an injective group homomorphism then we require that $E(B_g)$ and $E(A_g)$ are also normalized in this form. Such homomorphism is called a **normalized Fricke** homomorphism. Consider now the sequence of generators of $E(G)$ given by $E(A_1), E(B_1), \ldots, E(A_g), E(B_g)$. Each of these transformations can be represented by a matrix with real entries, $\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ satisfying $a_i \geq 0$ and $b_i > 0$ if $a_i = 0$. Moreover, if the image of E is discrete then these transformations are hyperbolic.

Prop 2.18. *A Fricke normalized monomorphism with discrete image is determined by the tuple*

$$(a_1, b_1, c_1, \ldots, a_{2g-2}, b_{2g-2}, c_{2g-2}) \in \mathbb{R}^{6g-6}.$$

Theorem 2.19. $F : T(X) \to \mathbb{R}^{6g-6}$ *is a real analytic embedding with real analytic inverse of* $T(X)$ *onto an open domain (where* $T(X)$ *has a complex structure explained later).*

In the case of a surface with punctures we have a similar result, where the image of F lies in $\mathbb{R}^{6g-6+2n}$. The group G is generated by $2g + n$ elements, say $A_1, \ldots, A_g$, $B_1, \ldots, B_g$ and $C_1, \ldots, C_n$, satisfying $(\prod_{j=1}^{g}[A_j, B_j])C_1 \cdots C_n = Id$. The elements A_j and B_j are hyperbolic; we require that for any matrix $\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$ representing any of these elements, we have $a_i \geq 0$ and $b_i > 0$ if $a_i = 0$. The elements C_i are parabolic; our requirements in this case are that the matrix $\left(\begin{smallmatrix} a'_i & b'_i \\ c'_i & d'_i \end{smallmatrix}\right)$ representing C_i satisfies $a'_i + d'_i = 2$. The a Fricke normalized homomorphism is determined uniquely by the following tuple of real numbers:

$$(a_1, b_1, c_1, \ldots, a_{2g-2}, b_{2g-2}, c_{2g-2}, a'_1, c'_1, \ldots, a'_n, c'_n).$$

We also have that the mapping $F : T(X) \to \mathbb{R}^{6g-6+2n}$ is a real analytic embedding onto an open domain. However in this case we have that the elements of the group $E(G)$ (with discrete image) are either hyperbolic or parabolic (the latter corresponding to loops are around the punctures; they are conjugate in $E(G)$ to one of the elements $E(C_1), \ldots, E(C_n)$).

2.6. **Teichmüller's theorem.** Teichmüller's extremal problem can be stated as follows: given a homeomorphism $f : X \to X_1$ between Riemann surfaces, find a qc homeomorphism $f_T : X \to X_1$, homotopic to f, such that $K(f_T)$ is infimum in this class. We can state the problem in terms of groups as well: given $\mu \in B(L^\infty(\mathbb{H}, G))$, find μ_T with $\mu_t \sim \mu$ and $||\mu_T||_\infty$ infimum in this class. Existence is given by normality properties of quasiconformal mapping (proposition 2.12).

As usual, we assume X is of type (g,n) with $2g-2+n>0$. Let $A_2(X)$ be the space of integrable holomorphic quadratic differentials on X.

Lemma 2.20. *For $\varphi \in A_2(X)\backslash\{0\}$ and $k \in [0,1)$, set $\mu_t = \mu(k,\varphi) = k\frac{\overline{\varphi}}{|\varphi|}$. Then μ_T belongs to the unit ball of $L^\infty_{(-1,1)}(X)$ and $|\mu_T|$ is essentially constant on X, with $||\mu_T||_\infty = k$. Moreover, $\mu_T(k,\varphi) = \mu_T(l,\phi)$ if and only if $k=l$ and $\varphi = t\,\phi$, for some $t>0$. μ_T is called a* **Teichmüller-Beltrami differential** *form on X.*

Theorem 2.21 (Teichmüller's Theorem)**.** *Given $[X,f,X_1]$ in $T(g,n)$, there exists a unique extremal mapping f_T solving Teichmüller's extremal problem. The complex dilatation of f_T is a Teichmüller-Beltrami differential for a unique $k \in (0,1)$ and a unique, up to a positive multiple, element $\varphi \in A_2(X)$, if $[X,f,X_1]$ is not the origin of $T(g,n)$ (for the origin, f_T is conformal so $k=0$).*

One can construct explicitly Teichmüller mappings for once-punctured tori as follows. First of all, if S_τ is a torus given by $\mathbb{C}/L(1,\tau)$, the group of translations on $\mathbb{C}$ induces a group of biholomorphic mappings on S_τ that acts transitively. So one can assume that S_τ has a puncture at (the projection of) the origin. A mapping $f_\tau : S_i \to S_\tau$ is given by the affine mapping taking the parallelogram of S_i (with vertices at 0, 1, i and $1+i$) to the parallelogram of S_τ (with vertices at 0, 1, τ and $1+\tau$):

$$f_\tau([x+iy]\,mod\,L(1,i)) = [x+y\tau]\,mod\,L(1,\tau).$$

To show this mapping is extremal, it is enough to show that μ_{f_τ} is a Teichmüller-Beltrami differential. The lift of f_τ to the complex plane is given by $\widetilde{f_\tau}(z) = \frac{1}{2}\left((1-i\tau)z + (1+i\tau)\overline{z}\right)$, so

$$\mu_{f_\tau} = \frac{1+i\tau}{1-i\tau}\frac{d\overline{z}}{dz} = \zeta\frac{\overline{\varphi}}{|\varphi|}, \qquad \zeta = \frac{1+i\tau}{1-i\tau}, \qquad \varphi = dz^2,$$

which shows that μ_{f_τ} is a Teichmüller-Beltrami differential. Teichmüller distance can then be easily computed:

$$d_T([X_i,Id,X_i],[X_i,f_\tau,X_\tau]) = \frac{1}{2}\log K(f_\tau) = \frac{1}{2}\log\frac{1+|\zeta}{1-|\zeta|} = \frac{1}{2}\log\frac{|1-i\tau|+|1+i\tau|}{|1-i\tau|-|1+i\tau|},$$

which is precisely the Poincaré distance between i and τ in the upper half plane.

Theorem 2.22. *Let $B(A_2(X))$ denote the unit ball of the space of quadratic differentials on X; the mapping $H_T : B(A_2(X)) \to T(X)$ given by $H_T(\varphi) = \Phi(\mu_T(||\varphi||,\varphi)$ is a homeomorphism, called* **Teichmüller's homeomorphism***.*

Corollary 2.23. *$T(X)$ (X a compact surface with punctures and universal covering space the upper half plane) and the Teichmüller metric is simply connected (more than that, it is contractible).*

The boundary of $T(X)$ is Teichmüller embedding is a sphere of dimension $6g-7+2n$.

2.7. **Teichmüller's lemma and the complex structure of Teichmüller spaces.** Consider the Fricke homeomorphism $F : T(X) \to \mathbb{R}^{6g-6+2n}$ (given by Fricke coordinates on Teichmüller space) and let $\widetilde{F} : B(L^\infty_{(-1,1)}(X)) \to \mathbb{R}^{6g-6+2n}$ be a lift of F.

Theorem 2.24 (Teichmüller's Lemma)**.** *The set*

$$A_2(X)^\perp = \left\{ \nu \in L^\infty_{(-1,1)}(X)_1;\ \int_X \nu\,\varphi = 0,\ \text{for all } \varphi \in A_2(X) \right\}$$

is the kernel of $d_0\widetilde{F}$*, the differential of* $\widetilde{F}$ *at the origin* $0 \in L^\infty_{(-1,1)}$.

This theorem can be used to give a complex structure to the Teichmüller space $T(X)$. For that we need some general facts on mappings between complex and real manifolds, and when such mappings induce complex structures (on the real manifold). Let $f : X \to Y$ be a C^1 submersion from a complex Banach manifold X onto a real C^1 manifold Y. We say that f *induces a well-defined almost complex structure* on Y if for all $y \in Y$ the space T_yY inherits via the differential d_xf a unique complex structure, independent of the choice of $x \in f^{-1}(y)$.

Theorem 2.25. *Let* $f : X \to Y$ *be a surjective* C^1 *submersion from a complex manifold* X *onto a* $2d$*-dimensional* C^1 *manifold* Y*. Then there is a complex structure on* Y*, compatible with the* C^1 *structure, making* f *holomorphic if and only if* f *induces a well-defined almost complex structure on* Y*. The complex structure, if it exists,is unique.*

We can apply the above result to the mapping $\widetilde{F} = F \circ \Phi$, define from the unit ball of $L^\infty(\mathbb{H}, G)$ to $Im(F) \subset \mathbb{R}^{6g-6+2n}$, where F is the Fricke mapping. To check that $\widetilde{F}$ induces a well-defined almost complex structure one needs to use the right-translation mappings, R_θ, from the unit ball of $L^\infty(\mathbb{H}, G)$ to itself, defined by

$$R_\theta : B(L^\infty(\mathbb{H}, G)) \to B(L^\infty(\mathbb{H}, G))$$

$$\lambda \mapsto \text{complex dilatation of } (w_\lambda \circ w_\theta).$$

Theorem 2.26. *For a Riemann surface* X *of type* (g, n)*, with* $2g-2+n > 0$*, there is a unique complex structure of a* $(3g-3+n)$*-dimensional complex manifold on* $Im(F)$ *such that it is compatible with the real-analytic structure of the open domain* $Im(F) \subset \mathbb{R}^{6g-6+2n}$ *and that makes* $\widetilde{F}$ *a holomorphic submersion.*

The details of the proof of this result can be found in [41, Chapter 3].

The Fricke homeomorphism $F : T(X) \to Im(F)$ gives then *a complex structure* to Teichmüller space $T(X)$. The mapping F becomes biholomorphic and the projection Φ is then a holomorphic submersion.

The discontinuous action of the modular group on Teichmüller space gives us some information about the moduli space of surfaces.

Theorem 2.27. *The moduli space $M(g,n)$ of compact surfaces of type (g,n) ($2g-2+n>0$) is a normal complex space of dimension $3g-3+n$. The space $M(g,0)$ is simply connected.*

2.8. **Bers embedding and the Bers boundary of Teichmüller space.** There is another way of giving a complex structure to $T(G)$ that works for any group, not only those uniformizing surfaces of finite type. Although the above results provides us with a complex structure on $T(X)$, we will explain here this other method, since it is important on its own and it appears in many papers. The complex structure obtained above and the one from the result in this section coincide in the case of X being a surface of type (g,n) (and, of course, $2g-2+n>0$).

Lemma 2.28. *Let G be a Fuchsian group, μ and ν elements of $L^\infty(\mathbb{H},G)_1$; then the following are equivalent:*

(i) *μ is Teichmüller equivalent to ν;*

(ii) *$w^\mu|_\mathbb{R} = w^\nu|_\mathbb{R}$;*

(iii) *$w^\mu|_\mathbb{L} = w^\nu|_\mathbb{L}$, where $\mathbb{L} = \{z \in \mathbb{C}/\ Im(z) < 0\}$ is the lower half plane.*

Let $B_2(\mathbb{L},G)$ the the space of holomorphic 2-forms for G ($(\sigma \circ g)(g')^2 = \sigma$) on the lower half plane, bounded with respect to the norm $||\sigma||_{2,\infty} = ||\lambda_\mathbb{L}^{-2}\,\sigma(z)||_\infty$, where $\lambda_\mathbb{L}$ is the Poincaré metric on $\mathbb{L}$.

If w is a one-to-one holomorphic function, we define the **Schwarzian** derivative of w by

$$S(w) = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2.$$

We define a mapping (we need to prove that its is well-defined) on the unit ball of the space of bounded Beltrami coefficients on the upper half plane for G as follows:

$$\Phi_\beta : B(L^\infty(\mathbb{H},G)) \to B_2(\mathbb{L},G)$$

$$\Phi_\beta(\mu) = S(w^\mu|_\mathbb{L}).$$

This mapping induces a mapping on the Teichmüller space of G,

$$\beta : T(G) \to B_2(\mathbb{L},G)$$

$$[\mu] \mapsto \Phi_\beta(\mu),$$

called the **Bers embedding** (of Teichmüller space). The image of β is an open domain in $B_2(\mathbb{L},G)$, and thus $T(G)$ gets a complex structure.

Theorem 2.29. *If G is a Fuchsian group uniformizing a surface of genus g with n punctures, then the complex structure of $T(G)$ obtained by the Bers embedding and the complex structure obtained earlier are equal.*

The following results show that β is well defined.

Lemma 2.30. *If $\mu \in B(L^\infty(\mathbb{H}, G))$ then $\Phi_\beta(\mu)$ is a quadratic differential for G on the lower half plane.*

Lemma 2.31. *$\Phi_\beta(\mu) = \Phi_\beta(\nu)$ if and only if $[\mu] = [\nu]$ in $T(G)$.*

Theorem 2.32 (Nehari)**.** *$|Sw(z)|\,(2Im(z))^2 \leq 6$.*

Nehari's theorem and the two lemmas above show that if μ is an element of the unit ball of $L^\infty(\mathbb{H}, G)$, then $\Phi_\beta(\mu)$ is a bounded quadratic differential for G, which depends only on the Teichmüller class of μ.

The map $\Phi_\beta : B(L^\infty(\mathbb{H}, G)) \to B_2(\mathbb{L}, G)$ is a holomorphic submersion with open image a domain denoted by $T_\beta(G)$. The steps on the proof of this fact are the following:

1. Φ_β is holomorphic (in the Banach-space setting).
2. If we set $\Phi^\mu : B(L^\infty(\mathbb{H}^\mu, G^\mu) \to B_2(\mathbb{L}^\mu, G^\mu)$ by $\Phi^\mu(\theta) = S(w^\theta|_{\mathbb{L}^\mu})$, then we have that $\Phi_\beta = \Phi^0$. Moreover, $d_\mu\Phi_\beta$ can be expressed in terms of $d_0\Phi^\mu$.
3. One can find an explicit formula for $d_0|\phi^\mu(\nu)$.
4. Let $\mathcal{H}$ denote the space of Bers' Beltrami differentials defined by

$$\mathcal{H} = \left\{\nu \in B(L^\infty(\mathbb{H}^\mu, G^\mu);\ \nu = \lambda_{\mathbb{H}^\mu}^{-2}\overline{\psi} \text{ for some } \psi \in B_2(\mathbb{H}^\mu, G^\mu)\right\}.$$

Then $\mathcal{H}$ is a complex linear isomorphism onto $B_2(\mathbb{H}^\mu, G^\mu)$.
These steps show that $T(G)$ has a complex structure so that the projection Φ becomes a holomorphic submersion. The Teichmüller space $T(G)$ becomes biholomorphically identified with the domain $T_\beta(G)$ (an open domain in $\mathbb{C}^{3g-3+n}$ if $\mathbb{H}/G$ is a compact surface of genus g with n punctures).

Prop 2.33. *The space $T_\beta(G)$ is precisely the component of the intersection of $T_\beta(\{1\})$ with $B_2(\mathbb{L}, G)$ that contains the zero.*

Let $\mathcal{S}$ be the (closed and bounded) subset of $B_2(\mathbb{L}, \{1\})$ consisting of the Schwarzian derivatives of all one-to-one holomorphic functions defined on the lower half plane.

Theorem 2.34 (Gehring, [21], [22])**.** *The closure of $T_\beta(\{1\})$ in $B_2(\mathbb{L}, \{1\})$ is a proper subset of $\mathcal{S}$. The interior of $\mathcal{S}$ is precisely $T_\beta(\{1\})$.*

A **deformation** of a Fuchsian group pG is a pair, (σ, f), where $\sigma : G \to \mathrm{PSL}(2,\mathbb{C})$ is a homomorphism and $f : \mathbb{L} \to \widehat{\mathbb{C}}$ is a locally one-to-one meromorphic function inducing σ in the sense that $f \circ g = \sigma(g) \circ f$, for all $g \in G$. Two deformations (σ, f) and (σ_1, f_1) are equivalent if the groups $\sigma_1(G)$ and $\sigma(G)$ are conjugate in $\mathrm{PSL}(2,\mathbb{C})$.

Prop 2.35. *The Schwarzian derivative (of f) sets up a canonical bijection between equivalence classes of deformations and the space $Q(\mathbb{L}, G)$ of holomorphic quadratic differentials for G in $\mathbb{L}$.*

The elements of $Q(\mathbb{L}, G)$ are holomorphic functions φ, defined on the lower half plane such that $(\varphi \circ g)\ (g')^2 = \varphi$ for all $g \in G$. We have that the space $B_2(\mathbb{L}, G)$ is a subspace of $Q(\mathbb{L}, G)$, and these two spaces are equal if $\mathbb{L}/G$ is a compact surface (*without* punctures).

The relation between deformations and Teichmüller space is given in the following result.

Theorem 2.36. *$T(G)$ is exactly the set of equivalence classes of deformations of G which are bounded Schwarzians φ such that $Sf = \varphi$ has a globally one-to-one solution f, defined on the lower half plane and which extends to a qc homeomorphism w of $\widehat{\mathbb{C}}$ compatible with G (that is, wGw^{-1} is a group of Möbius transformations).*

The relation between quadratic differentials in $Q(\mathbb{L}, G)$ and deformation is done via the theory of ordinary differential equations. The solutions of the differential equation $Sw = \varphi$ are of the form $w = \eta_1/\eta_2$, where η_1 and η_2 are two linearly independent solutions of the equation $2\eta'' + \varphi\eta = 0$. If φ is an element of $Q(\mathbb{L}, G)$, then the above differential equation makes sense in the Riemann surface $\mathbb{L}/G$. The ratio of two linearly independent solutions (η_1, η_2) on a small region of the surface, can be analytically continued along closed paths on the surface. Each such path corresponds to an element g of G; on the other hand, after analytic continuation one reaches a pair of linearly independent solutions, so these new solutions should be linear combinations of η_1 and η_2). Therefore the ratio is changed by a Möbius transformation, $\sigma(g)$.

Define two sets $S(G)$ and $K(G)$ by

$$S(G) = \{\varphi \in B_2(\mathbb{L}, G);\ \text{if } Sf^{\varphi} = \varphi,\ \text{the function } f^{\varphi} \text{ is globally one} - \text{to} - \text{one on } \mathbb{L}\}$$

$$K(G) = \{\varphi \in B_2(\mathbb{L}, G);\ \sigma^{\varphi}(G) \text{ is Kleinian}\}$$

Theorem 2.37. *$T_\beta(G)$ is the interior of $S(G)$. $T_\beta(G)$ is the component of the interior of $K(G)$ containing the origin.*

A **projective atlas** on a surface X is simply an atlas where the changes of coordinates are given by Möbius transformations. Two projective atlases are equivalent if their union is again a projective atlas.

Prop 2.38. *Let G be a torsion-free Fuchsian group and X the surface $\mathbb{L}/G$. There are canonical one-to-one correspondences between:*
(i) *equivalence classes of projective atlases on X;*
(ii) *equivalence classes of deformations of G;*
(iii) *the elements of the vector space $Q(\mathbb{L}, G)$.*

When φ is in $T_\beta(G)$, then we know that the deformation associated to it is given by a qc homeomorphism of the Riemann sphere (and thus the image group is of the same type as G). But we have that any point in $Q(\mathbb{L}, G)$ induces a deformation of G. Bers [4] studied the deformations induced by elements of the boundary of $T_\beta(G)$. Denote this boundary, the **Bers boundary of Teichmüller space**, by $\partial T_\beta(G)$.

Theorem 2.39. *Let G be a Fuchsian group and $\varphi \in \partial T_\beta(G)$. Then f^φ is a globally one-to-one holomorphic function defined on the lower half plane; consequently $\sigma^\varphi(G) = G^\varphi$ is a Kleinian group, which acts properly discontinuously on $f^\varphi(L)$. Moreover, G^φ has an invariant component (of its domain of discontinuity) containing $f^\varphi(L)$.*

Prop 2.40. *If G is of the first kind, then for any point $\varphi \in \partial T_\beta(G)$, the component of the region of discontinuity of G^φ containing $f^\varphi(L)$ is precisely $f^\varphi(L)$.*

The main question studied in the paper cited above is the structure of the group G^φ for elements in the boundary of Teichmüller space. Clearly, $f^\varphi(\mathbb{L})/G^\varphi$ is conformally equivalent to $X^* = \mathbb{L}/G$. On the other hand, G^φ cannot uniformize a surface qc equivalent to $\mathbb{H}/G$, because then φ would be a point in Teichmüller space, not in its boundary.

If $\varphi \in T_\beta(G)$, then the isomorphism $\sigma^\varphi : G \to G^\varphi$ preserves the type of every transformation (2.17). We say that a point $\varphi \in \partial T_\beta(G)$ is a **cusp** if there is a hyperbolic element $g \in G$, such that $\sigma^\varphi(g)$ is parabolic (the loop corresponding to g has been pinched to a point).

A Kleinian group Γ is called **totally degenerate** if it is non-elementary and its region of discontinuity is connected and simply connected.

Theorem 2.41. *If G is a finitely generated Fuchsian group of the first kind, and $T(G)$ has positive dimension, then the subset of totally degenerate points of $\partial T_\beta(G)$ is of second category in $\partial T_\beta(G)$.*

This theorem is a consequence of the following two results.

Prop 2.42. *The subset of cusps in the boundary of Teichmüller space is of first category.*

Prop 2.43. *Every element in $\partial T_\beta(G)$ that is not a cusp must be totally degenerate.*

The following result gives us some information about cusp points.

Prop 2.44. *Let G be finitely generated Fuchsian group of the first kind, such that $T(G)$ has positive dimension. If $\varphi \in \partial T_\beta(G)$ is not totally degenerate, then*
(a) *each component of $\Omega(G^\varphi)$ other than $f^\varphi(\mathbb{L})$ is non-invariant;*
(b) *if P is one such component, then there exists a parabolic element of G^φ_P (the subgroup of G^φ that keeps P invariant) such that $(\sigma^\varphi)^{-1}(\gamma)$ is hyperbolic in G.*

Another interesting result about mapping class groups is given below (compared with theorem 3.10).

Prop 2.45. *Any element of $Mod(G)$ that fixes the origin of $T_\beta(G)$ extends not only to $\partial T_\beta(G)$ but to a linear isometry of the Banach space $B_2(\mathbb{L}, G)$.*

The paper [4] states some open (at the time it was written) conjectures regarding the boundary of Teichmüller space.

Conjecture 2.46. *Every finitely generated Kleinian group with exactly one simply connected invariant component is a boundary group of a finitely generated Fuchsian group of the first kind.*

The following conjecture (see also 4.16) was proved by C. McMullen in [39].

Theorem 2.47. *Cusps are dense in $\partial T_\beta(G)$.*

2.9. **Royden's Theorems.** The Kobayashi pseudometric on a complex manifold M can be defined as the largest pseudometric ρ on M satisfying $\rho(f(z), f(w)) \leq d(z, w)$ for all holomorphic mappings $f : \mathbb{D} \to M$, where d is the Poincaré distance on the unit disc $\mathbb{D}$.

Theorem 2.48 (Royden [43])**.** *The Kobayashi metric on $T(g, 0)$ is equal to the Teichmüller metric.*

The extension to the case of surface with punctures was done by C. Earle and I. Kra in [16].

A couple of results also due to Royden (see theorem 4.4 for a similar result regarding the Weil-Petersson metric) are as follows.

Theorem 2.49 (Royden [43])**.** *The group of isometries of $T(g, 0)$ with respect to the Teichmüller metric is the modular group if $g > 2$ and a $\mathbb{Z}_2$-quotient of $Mod(2, 0)$ for the case of $g = 2$.*

Corollary 2.50. *The group of biholomorphisms of Teichmüller space is the mapping class group.*

2.10. **The Patterson and Bers-Greenberg Isomorphism Theorems.** When can the Teichmüller spaces of surfaces of distinct type be equal?

Theorem 2.51 (Patterson Isomorphism Theorem [42], [16]). *The only isometric bijections between two distinct Teichmüller spaces occur for the following cases:*

$$T(2,0) \cong T(0,6), \quad T(1,2) \cong T(0,5), \quad T(1,1) \cong T(0,4).$$

Let G be a Fuchsian group such that the covering $\mathbb{H} \to \mathbb{H}/G$ has signature $(g, n; \nu_1, \ldots, \nu_n)$ (1.15), with $\chi > 0$. Consider the Riemann surface $X = \mathbb{H}_G/G$, where

$$\mathbb{H}_G = \{z \in \mathbb{H};\ z \text{ is not fixed by any elliptic element of } G\};$$

Theorem 2.52 (Bers-Greenberg Isomorphism Theorem [7]). *The Teichmüller spaces $T(X)$ and $T(G)$ are isometric.*

2.11. **The Weil-Petersson metric.** Good references for this section are A. Tromba's book [47] and S. Wolpert's article [52].

Let X be a compact surface, possibly with punctures, uniformized by a Fuchsian group acting on the upper half plane. The cotangent bundle of $T(X)$ at a point $[X, f, X_1]$ can be identified with the space of holomorphic integrable quadratic differentials on X_1 (see Teichmüller's lemma, 2.24), which we have denoted by $A_2(X_1)$ (if X_1 is uniformized by the Fuchsian group G_1 then we can identify $A_2(X_1)$ with $B_2(\mathbb{H}, G_1)$). The Weil-Petersson product of forms (see 1.7) gives a metric on the cotangent bundle of Teichmüller space, and by duality, a metric on $T(X)$, called the **Weil-Petersson metric** (the name of Weil comes from the fact that it was remarked by André Weil in a letter to Lars Ahlfors that the Petersson product should give a metric on Teichmüller spaces).

Let M be a complex Riemannian manifold with almost complex structure $\hat{\Phi}$ and Hermitian metric G_M. We say that G_M is **Kähler** if the two-form Ω_M defined by $\Omega_M(X_p, Y_p) = G_M(\hat{\Phi}(X_p), Y_p)$ is closed (here X_p and Y_p are tangent vectors to M at a point p).

Theorem 2.53 (Ahlfors [2]). *The Weil-Petersson metric is Kähler. The Ricci curvature and the curvatures of holomorphic sections are negative.*

In fact, the holomorphic sectional curvatures are bounded away from zero by a negative constant. For the Ricci curvature we have a bound that depends only on the topology of X.

Theorem 2.54 (Theorem 5.4.18 in [47]). *If X is a compact surface of genus g then the Ricci curvature of $T(X)$ with respect to the Weil-Petersson metric is bounded above by $-1/(8\pi(g-1)$.*

Prop 2.55 (Tromba [46], Wolpert [50]). *The Weil-Petersson metric has negative sectional curvatures.*

The Weil-Petersson metric, unlike the Teichmüller metric, is not complete.

Theorem 2.56 (Theorem 5.5.1 in [47]). *If a Weil-Petersson geodesic has finite length, then a non-trivial closed geodesic on X is shrinking in length to zero as one approaches the limits of definition of the Weil-Petersson geodesic.*

The mapping class group acts by isometries with respect to the Weil-Petersson metric. Thus one gets a metric on moduli space; that metric can be extended to the Deligne-Mumford compactification (see [34] for details).

Theorem 2.57 (Knudsen-Mumford [30], Wolpert [49]). *The compactification of the moduli space of compact surfaces of genus $g \geq 2$ is a projective algebraic variety.*

Although the Weil-Petersson is not complete, its behaviour is in some aspects similar to that of a complete negatively curved Riemannian metric.

Theorem 2.58 (Wolpert [51]). *For each pair of points in $T(X)$ there is a unique Weil-Petersson metric joining them.*

A consequence of this fact is that Teichmüller space is a domain of holomorphy, a result first obtain by Bers and Ehrenpreis in [6].

Another important result in [51] is the solution of the Nielsen realization problem: given a finite subgroup G of the mapping class group, find a surface X such that G is isomorphic to a subgroup of the group of biholomorphisms of X. This question was first answer in the positive by Kerchkhoff [28] using Thurston's "earthquakes" theory. Another way of solving the problem is by finding a real-valued function on Teichmüller space, convex along Weil-Petersson geodesics. Wolpert uses sums of geodesics lengths for that purpose. One more solution of this problem was given by Tromba [48] in a similar way but using the Dirichlet's energy.

3. Thurston's theory

In this section we give a broad outline of the work of Thurston on Teichmüller spaces and the classification of diffeomorphisms of surfaces. The (mainly expository) items [45] and [12] in the bibliography are short introductions to this topic; a more detailed explanation with proofs can be found in [19].

3.1. **Thurston's approach to Teichmüller spaces and diffeomorphisms of surfaces.** Consider a compact surface S of genus g with $n \geq 0$ punctures, such that $2g-2+n>0$. Let $\mathfrak{S}$ be the homotopy (isotopy) classes of essential, simple closed curves, not homotopic to punctures, on S. Given two elements α and β of $\mathfrak{S}$, denote by $i(\alpha,\beta)$ the minimum of the (geometric) intersection of representatives of α and β. For example, if S is a torus (in this case $2g-2+n=0$, but we include it here for the example), we can choose two generators of its fundamental group, say γ and δ. Then any element of $\mathfrak{S}$ is determined by two integers, a and b, by $\alpha(a,b)=a\gamma+b\delta$. It is easy to check that

$$i(\alpha(a,b),\alpha(c,d)) = \left|\det\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right|$$

Prop 3.1. 1. *For every $\alpha \in \mathfrak{S}$, there exists a $\beta \in \mathfrak{S}$, such that $i(\alpha,\beta)\neq 0$.*

2. *If $\alpha_1,\alpha_2 \in \mathfrak{S}$ are not equal, then there exists a $\beta\in\mathfrak{S}$, such that $i(\alpha_1,\beta)\neq i(\alpha_2,\beta)$.*

The intersection number i gives then a mapping from $\mathfrak{S}\times\mathfrak{S}$ into $\mathbb{R}$, and therefore we obtained an induced mapping $i_*:\mathfrak{S}\to\mathbb{R}^{\mathfrak{S}}$. By the first part of the above proposition, i_* is not equal to the zero mapping; by the second part, i_* is injective. Let $P(\mathbb{R}^{\mathfrak{S}})$ denote the corresponding projective space and $p:\mathbb{R}^{\mathfrak{S}}\backslash\{0\}\to P(\mathbb{R}^{\mathfrak{S}})$ the projection mapping. The closure of $p\circ i_*(\mathfrak{S})$, denoted by $\mathcal{PS}(S)$, is compact in $P(\mathbb{R}^{\mathfrak{S}})$. Elements of $\mathcal{PS}(S)$ are given by sequences $\{(t_n,\alpha_n)\}$, where $t_n>0$, $\alpha_n\in\mathfrak{S}$ and such that for all $\beta\in\mathfrak{S}$, the sequence $t_n i(\alpha_n,\beta)$ converges.

Theorem 3.2. *$\mathcal{PS}(S)$ is homeomorphic to a sphere of dimension $6g-7+2n$.*

Let $\mathfrak{S}'$ denote collections of elements of $\mathfrak{S}$, where we allow a curve to be multiplied by positive scalars: if $n>0$ is an integer, and γ and element of $\mathfrak{S}$, then $n\,\gamma$ is represented by n disjoint copies of γ. Geometric intersection number and the space $\mathcal{PS}'(S)$ are defined just as for $\mathfrak{S}$.

Theorem 3.3. *The space $\mathcal{PS}'(S)$ is equal to $\mathcal{PS}(S)$.*

A **measured foliation** $\mathcal{F}$, is simply a foliation where for every regular point of $\mathcal{F}$ one has a neighbourhood U and a chart $\phi: U\to\mathbb{R}^2$, which send the leaves of $\mathcal{F}$ to horizontal lines, and a "transverse" measure given by $|dy|$. The changes of local charts should be of the form $(x, y) \mapsto (f(x, y), c \pm y)$, which clearly preserves the measure $|dy|$. One allows a measured foliation to have a finite number of singular points, where the foliation looks like a "p-pronged saddle" as in figure 1, with $p \geq 3$.

If $\mathcal{F}$ is a measured foliation and a is a simple closed curve, we define $\int_a \mathcal{F}$ as the total variation of the y-coordinates of a, as measured locally with respect to any chart. Then we can extend the intersection number defining, for $\beta \in \mathfrak{S}$, $i(\mathcal{F}, \beta)$ to be the infimum of $\int_b \mathcal{F}$, where b is a simple closed curve representing β.

Given a quadratic differential on a Riemann surface X, with at most simple poles at the punctures, near any point of X there is a local coordinate $\zeta = \xi + i\eta$, so that the quadratic differential has the expression

$$\left(\frac{n+2}{2}\right) \zeta^n \, d\zeta^2,$$

where n is the order of the differential at the given point. The horizontal lines ($\xi =$ constant) define a foliation on X, with transverse measure given by $|d\eta|$. The relation between measured foliations and quadratic differentials has been studied by Hubbard and Masur [25] (to quote from the introduction: *Given any measured foliation F on S and any complex structure X on M, there exists a unique quadratic differential on the Riemann surface X whose horizontal trajectory structure realizes F*), and Kerckhoff [27]).

Two measured foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are **measure equivalent** if for all $\beta \in \mathfrak{S}$ one has $i(\mathcal{F}_1, \beta) = i(\mathcal{F}_1, \beta)$. It is possible to show that this equivalence is the same as Whitehead equivalence, given in figure 2.

We say that two measured foliations are **projectively equivalent** if there exists a real number λ such that for all $\beta \in \mathfrak{S}$ one has $i(\mathcal{F}_1, \beta) = \lambda i(\mathcal{F}_1, \beta)$.

Prop 3.4. *For every measured foliation $\mathcal{F}$ there exists a $\beta \in \mathfrak{S}$ such that $i(\mathcal{F}, \beta) \neq 0$.*

As with curves, the intersection number i for foliations defines a mapping i_* and we can also define the space of projectively equivalent measured foliations $\mathcal{PF}(S)$.

Theorem 3.5. $\mathcal{PF}(S) = \mathcal{PS}(S)$.

Let $T(S)$ denote the Teichmüller space of S; for a point X of $T(S)$ (here we are making an abuse of notation and forgetting the marking homeomorphism) and an element $\alpha \in \mathfrak{S}$, let $i(X, \alpha)$ denote the infimum of the length of a simple closed curve a representing α (it is a classical result that there is a unique geodesic representing α, with length $i(X, \alpha)$; by Schwarz's lemma, holomorphic mappings do not increase hyperbolic distances and lengths, so a biholomorphic mapping becomes an isometry). We give to $T(S)$ the minimum topology that makes all mappings $i(X,\alpha)$ continuous; in this way we get a continuous mapping from $T(S)$ to $P(\mathbb{R}^{\mathfrak{S}})$.

Theorem 3.6. *The closure of $p \circ i_*(T(S)$ is $(p \circ i_*)(T(S)) \cup \mathcal{PS}(S)$. This gives a natural topology to $T(S) \cup \mathcal{PS}(S)$, which becomes homeomorphic to a closed disc of dimension $6g-6+2n$.*

in the above theorem.Any diffeomorphism of S extends to a diffeomorphism of this closure of Teichmüller space with the topology given in the theorem above. The following result is a consequence of Brouwer fixed point theorem.

Theorem 3.7. *Let $\phi : S \to S$ be a diffeomorphism of S. Then either*
(i) *ϕ fixes a point in the Teichmüller space of S, or*
(ii) *ϕ fixes a projective class of measured foliations.*
In the first case we have that there exists a hyperbolic metric on S for which ϕ is isotopic to an isometry ϕ'; the mapping ϕ' has finite order.

We say that a diffeomorphism $\phi : S \to S$ is **reducible** by a finite collection of disjoint simple closed curves $\Gamma = \{\Gamma_1, \dots, \Gamma_n\}$ $(n \geq 1)$, representing distinct elements of $\mathfrak{S}$ if Γ is invariant by ϕ (though ϕ can permute the curves Γ_i). There is an upper bound, depending on the topology of S, to the number of curves in Γ.

We will denote by $\mathcal{F}_1 = \lambda \mathcal{F}_2$ if $\mathcal{F}_i$ are measured foliations that agree as foliation but for any arc α on S we have

$$\int_\alpha \mathcal{F}_1 = \lambda \int_\alpha \mathcal{F}_2.$$

Two measured foliations are transverse if the are transverse in the usual sense (near regular points).

Theorem 3.8. *Let ϕ be a diffeomorphism of S; then ϕ is homotopic (isotopic) to a diffeomorphism ϕ' such that either*
(i) *ϕ' fixes an element of $T(S)$ and has finite order; or*
(ii) *there is a number $\lambda > 1$ and two transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ (the stable and unstable foliations, respectively), such that $\phi'(\mathcal{F}^s) = (1/\lambda)\mathcal{F}^s$ and $\phi'(\mathcal{F}^u) = \lambda \mathcal{F}^u$; or*
(iii) *ϕ' is reducible by Γ. The collection Γ has a ϕ'-invariant tubular neighbourhood $N(\Gamma)$ such that on each component of $S \backslash N(\Gamma)$ the mapping ϕ' satisfies* (i) *or* (ii).

3.2. **Bers' and Thurston's compactifications of Teichmüller space.** The relationship between the compactifications of $T(S)$ by Bers and Thurston was studied by Kerckhoff and Thurston in [29]. To explain the results of that paper we need to introduce a little notation. Let T_g denote the Teichmüller space of a compact surface of genus $g \geq 2$. Given a pair of points X and Y in T_g, there is a quasi-Fuchsian group G, with two invariant components, Ω and Ω', such that $\Omega/G = X$ and $\Omega'/G = Y$. Conversely, a pair in $T_g \times T_g$ determines a quasi-Fuchsian group with an isomorphism to the fundamental group of S. Fixing a element in the first factor of $T_g \times T_g$ we get an embedding of Teichmüller space into the set $V(\pi_1(S))$ of conjugacy classes of representations of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{C})$. The space $V(\pi_1(S))$ has a topology of convergence on finite system of generators. This embedding is holomorphic, and it is equivalent to the Bers embedding; the image is called a Bers slice, and its compactification is the Bers compactification of Teichmüller space. Changes of base point (the fixed point in the first factor of $T_g \times T_g$) induce biholomorphic mappings between Bers slices.

Theorem 3.9. *There are Bers slices for which the change of base point mappings do not extend to homeomorphisms between their compactifications.*

It is also shown in the same paper that Bers' and Thurston's compactifications of Teichmüller space are not equivalent; this is a consequence of the following theorem.

Theorem 3.10. *For $g = 2$ there is a Bers slice for which the action of the modular group does not extend continuously to the compactification.*

The authors sketch a proof of this result for any $g \geq 2$.

3.3. **Bers' approach to Thurston's classification of diffeomorphisms of surfaces.** Bers gave in [5] a different proof of Thurston's classification of diffeomorphisms on a surfaces by consider Teichmüller's extremal problem where not only the map is allowed to vary but also the conformal structure on the surface. More precisely, he considers the following extremal problem. Let f be a homeomorphism of a compact surface of genus with punctures S onto itself. Among all maps isotopic to f, find a map f_0 and a Riemann surface structure X on S such that $f_0 : X \to X$ has minimum dilatation. Such a solution is called absolutely extremal. Let ϕ be an element of the modular group $Mod(S)$, and set $\alpha(\phi) = inf_{X \in T(S)} d(X, \phi(X))$, where d is the Teichmüller distance.

Theorem 3.11. *A map f of S is isotopic to a periodic map if and only if there is a point $X \in T(S)$ and f_0 homotopic to f, such that $f_0 : X \to X$ is conformal.*

One can classify the elements of $Mod(S)$ in a way similar to the classification of Möbius transformations acting on the upper half plane: an element ϕ is **elliptic** if it has a fixed point in $T(S)$ (and therefore $\alpha(\phi) = 0$), **parabolic** if $\alpha(\phi) = 0$ but ϕ does not have a fixed point in $T(S)$, **hyperbolic** if $\alpha(\phi) > 0$ and there is an α-minimal point, and **pseudohyperbolic** if $\alpha(\phi) > 0$ and there is no α-minimal point. Absolutely extremal maps are then either elliptic of hyperbolic.

Theorem 3.12. *An element of $Mod(S)$ is elliptic if and only if it is periodic. An element of $Mod(S)$ is irreducible if and only if it is elliptic of hyperbolic.*

Theorem 3.13. *If an element ϕ of $Mod(S)$ is of infinite order, then $X \in T(S)$ is minimal with respect to $d(Y, \phi(Y))$ if and only if ϕ leaves a Teichmüller line through X invariant.*

Theorem 3.14. *A map $f : X \to X$ is absolutely extremal if and only if it is either conformal or it is a Teichmüller map satisfying any of the following two equivalent conditions:*
(i) *$f \circ f$ is also Teichmüller and the maximal dilatation of it is the square of the dilatation of f;*
(ii) *the initial and terminal differentials of f coincide.*

See Bers' paper for the definition of initial and terminal differentials of a mapping.

Theorem 3.15. *If a mapping is reducible and not isotopic to a periodic map, then the corresponding mapping class is parabolic of pseudohyperbolic.*

4. Some results on the Weil-Petersson metric of Teichmüller spaces

4.1. **The pants and curve complexes of a surface.** We change a little the notation to make it consistent with the papers explained in this section. We start by giving the definitions of new object that appear in those papers, as well as a reformulation of Teichmüller spaces.

Let S denote a (topological) surface, possibly with boundary, with negative Euler characteristic. We denote by $int(S)$ the interior of S. The Teichmüller space of S, $T(S)$, parametrizes finite area hyperbolic structures on S equipped with markings. Points in $T(S)$ are pairs of the form (f, X), where X is a finite hyperbolic area surface, and $f : int(S) \to X$ a homeomorphism, up to the equivalence $(f, X) \sim (g, Y)$ if there exists an isometry $\phi : X \to Y$, such that $\phi \circ f$ is homotopic to g (by the Uniformization theorem, S carries a Riemann surface structure; the area will be finite if the boundary components of X are just points, that is, X is compact with punctures) For simplicity we will say that X is a point in $T(S)$, when we do not need to emphasize the mapping f.

Let $\mathfrak{S}$ denote the homotopy classes of essential simple closed curves on S. A **pair of pants**, or a **pants decomposition**, on S is a maximal collection of distinct elements of $\mathfrak{S}$ such that no two isotopy (homotopy) classes of curves in P have representative that intersect. A pants decomposition consists of $3g - 3 + n$ curves. The **pants complex** $C_P(S)$ is the graph with one vertex for each pants decomposition of S and an edge joining two vertices if the corresponding pants decompositions differ by an elementary move (given in figure 3 below). The pants complex for the once-punctured torus (or for a four-punctured sphere) is given in figure 4; it can be identified with the Farey graph. Setting the length of each edge equal to 1 we get a metric space, which is denoted by $C_P(X)$. The distance function

$$d_P : C_P^0(S) \times C_P^0(S) \to \mathbb{Z}_{\geq 0}$$

on the vertex set $C_P^0(S)$ of $C_P(X)$ counts the minimal number of elementary moves between two pants decompositions.

The **curve complex** $C(S)$ is a complex constructed as follows:

- the vertices of $C(S)$ are identified with the elements of $\mathfrak{S}$ (isotopy classes of essential curves on S);
- a k-simplex in $C(S)$ consists of $k + 1$ curves, $(\alpha_1, \ldots, \alpha_{k+1})$ in $\mathfrak{S}^{k+1}$ such that $\alpha_i \neq \alpha_j$ and the intersection number of these two curves is 0, if $i \neq j$.

4.2. **The pants complex and Teichmüller space are quasi-isometric.** For a point X in the Teichmüller space $T(S)$, and a closed curve α on X (or in S, and its image on X under f), let $l_X(\alpha)$ denote the hyperbolic length of the closed geodesic in the homotopy class of α, with respect to the hyperbolic metric of X. If l is a positive real number, and P a pants decomposition, we define

$$V_l(P) = \left\{ X \in T(S); \max_{\alpha \in P} l_{X_1}(\alpha) < l \right\}.$$

By a result of Bers, there exists a constant L such that $T(S)$ is the union of $V_L(P)$ as P, varies over all pants decompositions of S. Set $V(P) = V_L(P)$. Let $Q : C_P(X) \to T(X)$ be any mapping such that $Q(P) = X_P$ is in $V(P)$.

A **quasi-isometry** [8, pg. 138] between two metric spaces (M, d_M) and (N, d_N) is a function $f : M \to N$ satisfying the following properties:

- there exist real constants $K \geq 1$ and $C \geq 0$ such that, for all p and q in M, the following inequalities are satisfied:

$$\frac{1}{K}\, d_M(p, q) - C \leq d_N(f(p), f(q)) \leq K\; d_M(p, q) + C.$$

- There exists a constant $D \geq 0$, such that for all $q \in N$ there exists a point $p \in M$ such that $d_N(f(p), q) < D$ (the image of f is D-dense in N).

Quasi-isometries are interesting in the sense that two quasi-isometric spaces have the same behaviour *at large.*

Theorem 4.1 (Brock [10])**.** *The mapping Q is a quasi-isometry of $C_P(S)$ into $T(S)$ equipped with the Weil-Petersson metric.*

4.3. **Linear growth of infinite order mapping class group elements.** Fix a set of generators of the mapping class group of S (the group is finitely generated; there are many proofs of this fact, already known to Dehn [14] or [15] for an English translation; see also [32] and [24]) say $\mathfrak{T}$. We assume that the identity is not in $\mathfrak{T}$ and that if f is in $\mathfrak{T}$ then so is f^{-1}. The length of an element f of $Mod(S)$, $||f||$, is defined as the minimum integer n such that f can be expressed as a product of n elements of $\mathfrak{T}$ (with the length of the identity equal to 0).

Theorem 4.2 (Farb, Lubotzky and Minsky [18])**.** *Every Dehn twist t has linear growth in $Mod(S)$; that is, there exists a positive constant c such that $||t^n|| \geq c\,|n|$, for all $n \in \mathbb{Z}$.*

Combining this result with the work of L. Mosher [40] one gets that every element of infinite order in $Mod(S)$ has linear growth (theorem 1.2 in the cited paper).

If we now consider the Teichmüller metric on $T(S)$, and fix a point $X_0 \in T(S)$, we have an embedding of the modular group on Teichmüller space by $g \mapsto g(X_0)$. It is known that if a group acts with certain properties on a space then this mapping is a quasi-isometry (see [8, proposition 8.19]). However we have that this is not true in the case under consideration.

Theorem 4.3. *The word metric on the modular group and the metric induced by inclusion as an orbit in $T(X)$ with the Teichmüller metric are not Lipschitz equivalent.*

The proof is based on the fact that given Dehn twist t, there exists a totally geodesic copy of the upper half plane (Teichmüller disc), invariant under t, that contains X_0 and on which t acts as a parabolic transformation ($z \mapsto z+1$ on $\mathbb{H}$).

4.4. **Isometries of the Weil-Petersson metric.** By Royden's theorem (theorem 2.49) we know that the group $Mod(S)$ is the full group of isometries of $T(S)$ with respect to the Teichmüller metric (except for a few cases).

Theorem 4.4 (Masur and Wolf [37])**.** *If $3g-3+n>1$ and $(g,n) \neq (1,2)$, every isometry of the Weil-Petersson metric of $T(S)$ is induced by an element of the extended mapping class group (where this group is defined as the mapping class group but including also orientation-reversing homeomorphisms).*

By the work of Daskalopoulos and Wentworth [13] we have more precise information about the elements of the mapping class groups as isometries of Teichmüller spaces. First of all, we recall the classification of mapping classes due to Thurston. An element of $Mod(X)$ is called **periodic** if some power is isotopic to the identity, **reducible** if it is not periodic but there is some collection of non-trivial (in homotopy) curves on X invariant under the automorphism, and **pseudo-Anosov** or **irreducible** if it is not of the previous two types. We can introduce a category, **pseudoperiodic** that includes periodic and reducible mappings that are periodic in the reduced components. In particular we call a mapping **strictly pseudoperiodic** if it is pseudoperiodic but not periodic.

If (X,d) is a metric space, and $f : X \to X$ an isometry, one can define the **translation length** of f, $L(f)$ the infimum of $d(x, f(x))$ as x varies on X [8]. We say that f is **semisimple** if there is a point x_0 such that $d(x_0, f(x_0)) = L(f)$.

According to this elements of the mapping class group are classified as the table below [13].

	semisimple	not semisimple
$L(f)=0$	periodic	strictly pseudoperiodic
$L(f)\neq 0$	pseudo-Anosov	reducible but not pseudoperiodic

4.5. **Gromov-hyperbolicity.** The curve complex $C(S)$ can be made into a metric space by making each simplex isometric to the standard simplex in the corresponding Euclidean space. J.L. Harer [23] proved that $C(S)$ is simply connected and homotopic to a wedge of spheres (so it cannot have a CAT(k) metric, for any $k \leq 0$).

Let $\delta > 0$. A geodesic triangle is calls **δ-slim** if each of its sides is contained in the δ-neighbourhood of the other two sides. A space is called **δ-hyperbolic** if all geodesic triangles are δ-slim, and **Gromov hyperbolic** if it is δ-hyperbolic for some positive δ.

Theorem 4.5 (Masur and Minsky [35])**.** *The space $C(S)$ is Gromov-hyperbolic.*

Theorem 4.6 (Masur and Wolf [36])**.** *$T(S)$ with the Teichmüller metric is not Gromov-hyperbolic.*

If we consider the Weil-Petersson metric on $T(S)$ then we have that for low dimension $T(S)$ is Gromov-hyperbolic, while it is not if the dimension is greater than 2.

Theorem 4.7 (Brock [11])**.** *The Weil-Petersson metric on $T(X)$ is Gromov-hyperbolic if and only the dimension of $T(X)$ is less than or equal to* 2.

The proof of the theorem is based on the previous result of Brock that says that $T(S)$ (with the Weil-Petersson metric) and the complex $C_P(S)$ are quasi-isometric. In the case of $T(S)$ being of dimension 1, one has that $C_P(S)$ is equal to the curve complex $C(S)$, which by Masur-Minsky theorem is Gromov-hyperbolic. The case of dimension 2 is proved by method of relative hyperbolicity [17]. For the higher dimensional cases one uses a result of Gromov that says that a Gromov-hyperbolic space has rank 1. The rank of a space is the maximal dimension of a quasi-flat (quasi-isometric image of an Euclidean space) in the space; in the case of $T(S)$ we have the following theorem (the result is in the same paper):

Theorem 4.8. *The rank of the Weil-Petersson metric on $T(S)$ is at least* $[(\dim(T(S)) + 1)/2]$.

There are another couple of interesting results in the same paper regarding properties of metrics on Teichmüller spaces. They are as follows.

Theorem 4.9. *If the dimension of $T(S)$ is at least* 3*, then $T(S)$ admits no proper, geodesically complete, Gromov-hyperbolic, equivariant under the mapping class group, path metric with finite covolume.*

A proper metric space is a metric space where the closed bounded balls are compact. A geodesically complete metric space has finite volume if for every $\epsilon > 0$ there is no infinite collection of pairwise disjoint balls of radius ϵ embedded into the space.

Theorem 4.10. *If the dimension of $T(S)$ is at least 2, then the moduli space $M(S)$ admits no complete Riemannian metric of pinched negative sectional curvature.*

4.6. **The Weil-Petersson visual sphere.** Given a complete metric space M, two geodesic rays $c, c' : [0, \infty) \to M$, are called asymptotic if $d(c(t), c'(t))$ is bounded. The set ∂M of **boundary points** of M is the set of equivalence classes (asymptotic) of geodesic rays. This set is also known as the **visual boundary** of M. If M is complete (and CAT(0)) then its visual boundary has many interesting properties (see [8, Chapter II.8]). For example, one can give a topology to $\overline{M} = M \cup \partial M$, known as the cone topology, so that $\overline{M}$ is compact if M is proper (in a proper space the closed balls are compact) and isometries of M extend naturally to homeomorphisms of $\overline{M}$.

The Weil-Petersson metric on $T(S)$ is not complete, so the above construction is not possible, However, J. Brock [9] has made some construction that resembles the visual boundary as much as possible, and has some of its properties. Let X be a point in $T(S)$; then one defines a Weil-Petersson virtual sphere, $\mathcal{V}_X(S)$, as the space of geodesic rays emanating from X. Some rays will leave $T(S)$ in finite time, which illustrates the non-completion of the Weil-Petersson metric. By geodesic convexity [51] we have that $\mathcal{V}_X(S)$ compactifies $T(S)$.

Theorem 4.11. *The finite rays in $\mathcal{V}_X(S)$ are in bijection with the points of $\overline{T(S)} \backslash T(S)$, where $\overline{T(S)}$ is the metric completion of Teichmüller space with the Weil-Petersson metric.*

Brock studies some properties of the visual sphere, especially those similar to the relation of the Bers' and Thurton's compactifications of Teichmüller space.

Theorem 4.12. *If the dimension of $T(S)$ is at least 2 then the action of the mapping class group does not extend continuously to $\mathcal{V}_X(S)$.*

Corollary 4.13. *Thurston's compactification of $T(S)$ by projective measured laminations is distinct from the compactification by the Weil-Petersson visual sphere: the identity mapping of $T(S)$ does not extend to a homeomorphism between these two compactifications.*

Given two points X and Y in $T(S)$, by the geodesic convexity of the Weil-Petersson metric there exists a unique geodesic $g(X, Y)$ joining them. We can then define a mapping from $T(S)$ to $WP_X(S)$, the space of Weil-Petersson geodesics joining X to points of $T(S)$ by

$$\mathcal{G}_X : T(S) \to WP_X(S)$$
$$Y \mapsto g(X, Y).$$

Theorem 4.14 (Wolpert [52]). *The mapping $\mathcal{G}_X$ defined above is a homeomorphism from $T(S)$ to $WP_X(S)$ (with the topology of pointwise convergence of parametrizations by the interval proportional to arclength).*

There are natural *change of base-point* mappings, $b_{X,Y} : WP_X(S) \to WP_Y(S)$ defined by $b_{X,Y}(\mathcal{G}_X(Z)) = \mathcal{G}_Y(Z)$, which are homeomorphisms.

Theorem 4.15. *If the dimension of $T(S)$ is at least 2, then the mappings $b_{X,Y}$ do not extend to homeomorphisms of the visual spheres $\mathcal{V}_X(S)$ and $\mathcal{V}_Y(S)$.*

Each finite ray in $\mathcal{V}_X(S)$ gives a simplex in the curve complex of S, namely a collection of curves whose lengths tend to zero on surfaces exiting $T(S)$ along the ray. The "limit" surfaces are called cusps. As we mentioned in section 2.8 they play an important role in the Bers boundary of Teichmüller space.

Theorem 4.16 (Cusps are dense). *The finite rays are dense in $\mathcal{V}_X(S)$.*

4.7. **3-manifolds and the Weil-Petersson distance.** In [10] Brock obtains a couple of interesting inequalities relating the Weil-Petersson metric with the volume of certain 3-manifolds and the eigenvalues of the Laplacian on a surface. To explain these results, let S be a surface of type (g, n) $(2g - 2 + n > 0)$; given $(X, Y) \in T(S) \times T(S)$, by Bers' simultaneous uniformization there exists a quasi-Fuchsian group $G(X, Y)$ uniformizing X and Y in its two components. One has that $Q(X, Y) = \mathbb{H}^3/G(X, Y)$ is a hyperbolic 3-manifold, homeomorphic to $X \times \mathbb{R}$, with X and Y in its boundary at infinity. The convex core of $Q(X, y)$ is the convex hull in $\mathbb{H}^3$ of the limit set of $G(X, Y)$; it is the smallest convex subset of $Q(X, Y)$ carrying its fundamental group (and it is homeomorphic to $X \times I$.

Theorem 4.17. *The volume of the convex core of $Q(X, Y)$ is comparable to the Weil-Petersson distance $d(X, Y)$ between X and Y. More precisely, there are real constants, $K_1 > 1$ and $K_2 > 0$, which depend only on X, such that for any $(X, Y) \in T(S) \times T(S)$ one has*

$$\frac{1}{K_1}\, d(X, Y) - K_2 \;\leq\; \mathrm{vol}(\mathrm{core}(Q(X, Y))) \;\leq\; K_1\, d(X, Y) + K_2.$$

Let $\lambda_0(X, Y)$ denote the lowest eigenvalue of the Laplacian on $Q(X, Y)$ and $D(X, Y)$ the Hausdorff dimension of the limit set of $G(X, Y)$.

Theorem 4.18. *There are constants, $K > 0$, C_1, C_2, C_3 and $C_4 > 1$, such that*

$$\frac{C_1}{d(X, Y)^2} \;\leq\; \lambda_0(X, Y) \;\leq\; \frac{C_2}{d(X, Y)},$$

and

$$2 - \frac{C_3}{d(X,Y)} \ \leq \ D(X,Y) \ \leq \ 2 - \frac{C_4}{d(X,Y)^2}.$$

5. The universal Teichmüller space

(The basic reference for this section is [31].)

5.1. **The universal Teichmüller space and the universal modular group.** A quasiconformal mapping of the upper half plane, $f : \mathbb{H} \to \mathbb{H}$ extends to a homeomorphism of the closure of $\mathbb{H}$ in a unique way, so it makes sense to talk of the values of f on the real axis. Let F denote the set of quasiconformal homeomorphisms of $\mathbb{H}$ that fix the points 0, 1 and ∞. We consider two mappings f and g in F **equivalent** if $f|_{\mathbb{R}} = g|_{\mathbb{R}}$.

Definition 5.1. *The* **universal Teichmüllerspace** $T(1)$ *consists of all equivalence classes defined above.*

Let B denote the open unit ball of $L^\infty(\mathbb{H})$; for each measurable function μ in B there exists a unique quasiconformal mapping $f^\mu : \mathbb{H} \to \mathbb{H}$ with complex dilatation μ which fixes the points 0, 1 and ∞. We say that μ and ν in B are equivalent if f^μ and f^ν are equivalent in the sense given above the previous definition. Then $T(1)$ is the set of equivalence classes of B.

Let $h : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism. One says that f is ***k*-quasisymetric** if for all real numbers x and all positive numbers t, the following inequalities are satisfied:

$$\frac{1}{k} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq k.$$

A function is **quasisymetric** if it is k-quasisymetric for some constant k. Let X denote the set of quasisymetric function which which 0 and 1 (since these functions are increasing one can assume that they fix the point of infinity, when we consider the real axis as a subset of the Riemann sphere).

Prop 5.2. *The mapping:*

$$T(1) \to X$$

$$[f] \mapsto f|_{\mathbb{R}}$$

is a bijection.

Let $\mu \in B$; extend μ to the lower half plane $\mathbb{L}$ by zero and let f_μ be the unique quasiconformal mapping of $\mathbb{C}$ that has dilatation equal to μ and fixes 0, 1 and ∞. Since μ is zero on $\mathbb{L}$ we have that f_μ is actually holomorphic on the lower half plane.

Theorem 5.3. $\mu,\ \nu \in B$ *are equivalent if and only if* f_μ *and* f_ν *coincide on the lower half plane.*

If $g : \mathbb{H} \to \mathbb{H}$ is quasiconformal then the mapping $f \circ g^{-1}$ will be also quasiconformal for $f \in F$ but not necessarily normalized (by fixing the three "special" points 0, 1 and ∞). Composing with an element h of $PSL(2, \mathbb{R})$ (that is, a Möbius transformation with real coefficients, or equivalently, a conformal self-homeomorphism of $\mathbb{H}$) we get $h \circ f \circ g^{-1}$ normalized. So g induces a mapping $\omega_{[g]} : T(1) \to T(1)$ by means of this operation: $\omega_{[g]}([f]) = [h \circ f \circ g^{-1}]$. The set of all such mappings is called the **universal modular group**, denoted by M.

If g belongs to F then we don't need the Möbius transformation h since $f \circ g^{-1}$ will be normalized. The subgroup of M consisting of all such transformations (called **right translations** is denoted by M_t.

Prop 5.4. *M_t acts transitively on $T(1)$.*

5.2. **The Teichmüller metric.** For a quasiconformal mapping f (defined somewhere on the complex plane) let K_f denote maximal dilatation. One can define a distance on $T(1)$ by considering the dilatation of mappings within the respective equivalence classes. More precisely, if p and q are points on $T(1)$ (so they are equivalence classes of quasiconformal homeomorphisms of $\mathbb{H}$) we define the **Teichmüller distance** τ by

$$\tau(p, q) = \frac{1}{2} \inf\{\log K_{g \circ f^{-1}};\ f \in p,\ g \in q\}.$$

One can have a similar definition but fixing a representative of the class p; that is, let $f_0 \in p$ and set

$$\tau_1(p, q) = \frac{1}{2} \inf\{\log K_{g \circ f_0^{-1}};\ g \in q\}.$$

From our equivalent definitions of $T(1)$ we can consider classes represented by quasiconformal mappings of the complex plane (rather than of the upper half plane) which are conformal on the lower half plane. Let f_0 and g_0 be two such mappings, representatives of the classes p and q. Consider the set W of all quasiconformal mappings of $\mathbb{C}$ that agree on $\mathbb{L}$ with $g_0 \circ f_0^{-1}$ and define

$$\tau_2(p, q) = \frac{1}{2} \inf\{\log K_\phi;\ \phi \in W\}.$$

Lemma 5.5. $\tau = \tau_1 = \tau_2$.

Some properties of the Teichmüller distance are given in the following results.

Theorem 5.6. *$(T(1), \tau)$ is a complete metric space.*

Theorem 5.7. *$(T(1), \tau)$ is path connected. Moreover, for any two points in $T(1)$ there is always a geodesic (that is, continuous path that realises the distance between any two points in the image) joining them.*

Theorem 5.8. *$(T(1), \tau)$ is contractible.*

5.3. **Schwarzian derivatives and the Universal Teichmüller space.** Let $f : U \to \mathbb{C}$ be a holomorphic function defined on an open set U of the complex plane. Assume that f is locally univalent; then its derivative is never zero, and therefore one can compute the following expressions at all points of U:

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

S_f is known as the **Schwarzian derivative** of f; it measures how much f deviates from a Möbius transformation in the sense that a function satisfies $S_f = 0$ if and only if it is a Möbius transformation. It also give the "best" Möbius approximation to f in the sense that given a point z_0 of U there exists a unique Möbius transformation h such that

$$\lim_{z\to z_0} \frac{(h \circ f)(z) - z}{(z - z_0)^3}$$

is finite. The limit is actually equal to $S_f(z_0)/6$ as an easy computation expressing f as a power series shows.

The Schwarzian derivative satisfies the rule

$$S_{f\circ g} = (S_f \circ g)\,(g')^2 + S_g;$$

so if f is a Möbius transformation then g and $f \circ g$ have the same Schwarzian derivative.

If ψ is a holomorphic function on a simply connected domain U then there is a meromorphic function f in U such that $S_f = \psi$; f is unique up to coposition with a M̈bius transformation.

Let Q denote the space of holomorphic functions on the lower half plane $\mathbb{L}$; equipped Q with a norm given by $||\psi|| = \sup_{z\in\mathbb{L}} 4y^2|\psi(z)|$, where $z = x + iy$. Remember that for $\mu \in B$ we have that f^μ is the quasiconformal mapping of $\mathbb{C}$ that has dilatation μ on $\mathbb{H}$ and is conformal on $\mathbb{L}$; it makes sense then to compute $S_{f|_{\mathbb{L}}}$.

Theorem 5.9. *The mapping from $T(1)$ to Q that sends the class $[f]$ to $S_{f|_{\mathbb{L}}}$ is a homeomorphism onto its image (with respect to the Teichmüller metric and the above defined norm on $T(1)$ and Q respectively).*

Let U denote the subset of Q consisting of Schwarzian derivatives S_f for functions f that are univalent on $\mathbb{L}$. We have that U is closed on Q and $T(1)$ is contained in U. Therefore the closure of $T(1)$ is also contained in U. The relation between U and universal Teichmüller space is given in the following two theorems.

Theorem 5.10. *$T(1)$ is the interior of U.*

Theorem 5.11. *The closure of $T(1)$ (in Q) is not equal to U.*

5.4. **The universal Teichmüller space and the Teichmüller spaces of surfaces.** Let S be a Riemann surface and denote by $T(S)$ its Teichmüller space. Since $T(S)$ can be understood as a space of quasiconformal mappings of $\mathbb{H}$ be have that $T(S) \subset T(1)$. It is in this sense that $T(1)$ can be called the universal Teichmüller space. If τ_S denotes the Teichmüller metric on $T(S)$ (and τ the Teichmüller metric on the universal space) then $\tau|_{T(S)} \leq \tau_S$. Strebel showed that the inequality could be strict, even for compact surfaces (his exmaples are for a punctured torus and for a surface of genus 2).

Let G be a group of Möbius transformations uniformizing X, that is $X = \tilde{X}/G$, where $\tilde{X}$ is the universal covering space of X (that is, the Riemann sphere, the complex plane or the upper half plane, by the Uniformization Theorem). The Teichmüller space of S can be understood as $T(G)$, the space of quasiconformal mappings f of $\mathbb{H}$ such that $f \circ g \circ f^{-1}$ is a Möbius transformation for all $g \in G$, with the appropriate equivalence relation. Similarly we can define the set $Q(G)$ of quadradic differentials with respect to G; that is the set of holomorphic functions ϕ on $\mathbb{L}$ such that $(\phi \circ g)\,(g')^2 = \phi$ for all $g \in G$. $Q(G)$ has the supremum norm defined above.

Theorem 5.12. $T(G) = Q(G) \cap T(1)$.

Let's now assume that S is a hyperbolic surface, that is, its universal covering space if $\mathbb{H}$, with covering group G. Denote by $L^\infty(G)$ the space of bounded measurable $(-1,1)$ differentials compatible with G, and by $L^1(G)$ the space of integrable quadratic differentials compatible with G. Each of these spaces has a norm, given respectively by $||\mu||_\infty = \sum 4y^2|\mu(z)|$ and $||\varphi||_1 = \int_N |\varphi|$, with the supremum taken over $\mathbb{H}$ and N is a fundamental domain for the action of G on $\mathbb{H}$ (with measurable boundary, to avoid problems).

If $\mu \in L^\infty(G)$ and $\varphi \in L^1(G)$ then we can form the **Petersson scalar product**, $\int_N \mu\varphi$.

Let $A(G)$ denote the subset of $L^1(G)$ consisting of holomorphic functions, and similarly $Q(G)$ the subset of $L^\infty(G)$. One has that if S is compact then $A(G) = Q(G)$. If S has finite area (for example a compact surface with finitely many punctures) then $Q(G) \subset A(G)$, while for the trivial group $G = \{id\}$ one has $A(G) \subset Q(G)$ with strict content.

Theorem 5.13. *The Poincaré theta series $\Theta : A(1) \to A(G)$ given by $\Theta f = \sum_G (f \circ g)\,(g')^2$ is a continuous linear surjection of norm less than or equal to* 1*.*

See [38] for the conditions on G when Θ has norm equal to 1.

The space of **infinitesimally trivial deformations** $N(G)$ is given by

$$\left\{\mu \in L^\infty(G);\ \int_N \mu\varphi = 0\ \forall \varphi \in G\right\}.$$

Theorem 5.14. *μ is infinitesimally trivial if and only if the function*

$$T_\mu(z) = \frac{-1}{\pi}\int_{\mathbb{D}} \frac{\mu(\zeta)}{\zeta - z}\, d\xi d\eta$$

vanishes on $\{z \in \mathbb{C};\ |z| > 1\}$. Here $\mathbb{D}$ denotes the unit disc and $\zeta = \xi + i\eta$.

5.5. **Teichmüller theorems.** A mapping is called a **Teichmüller mapping** if its Beltrami coefficient is of the form $\mu = k\frac{\bar{\varphi}}{|\varphi|}$, for some number k (clearly $k = ||\mu||_\infty$) and $\varphi \in Q(G)$.

Theorem 5.15 (Teichmüller's Existence Theorem)**.** *Extremal mappings (that is, those with smallest maximal dilatation in a homotopy class) on compact surfaces are Teichmüller mappings.*

Theorem 5.16 (Teichmüller's Uniqueness Theorem)**.** *On every homotopy class of orientation-preserving mappings on a compact surface there is a unique Teichmüller map.*

Theorem 5.17 (Teichmüller Embedding)**.** *The mapping $T(S) \to Q(S)$ given by $[k\frac{\bar{\varphi}}{|\phi|}] \mapsto k\varphi$, $0 \leq k < 1$, is a homeomorphism of $T(S)$ with the open unit ball of $Q(S)$.*

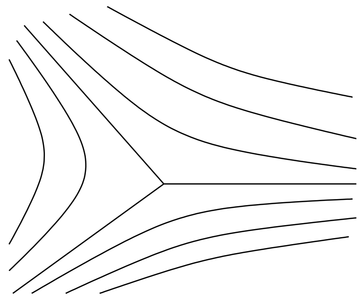
Corollary 5.18. *$T(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$.*

Corollary 5.19. *$T(S)$ is contractible.*

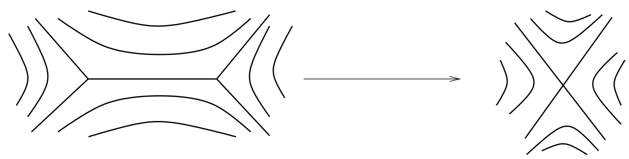
Let φ be a non-zero element of $Q(S)$; the **Teichmüller disc** defined by φ is the set

$$\Delta(\varphi) = \left\{[z\frac{\bar{\varphi}}{|\varphi|}];\ |z| < 1\right\}.$$

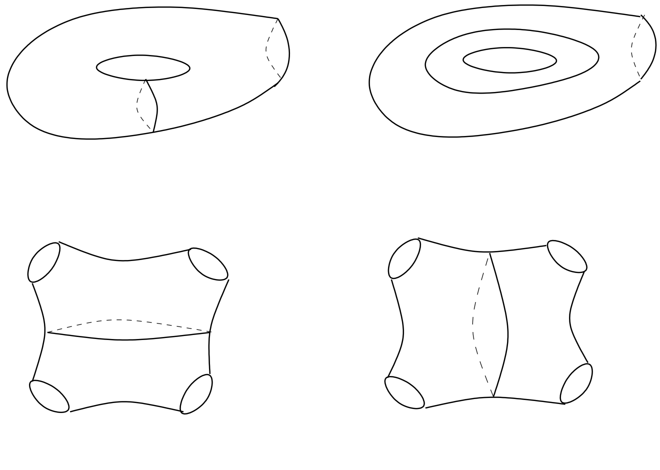
Theorem 5.20. *The mapping from the unit disc $\mathbb{D}$ to $\Delta(\varphi)$ given by $z \mapsto [z\frac{\bar{\varphi}}{|\varphi|}]$ is an isometry when $\mathbb{D}$ has the hyperbolic metric and $\Delta(\varphi)$ the metric induced.*



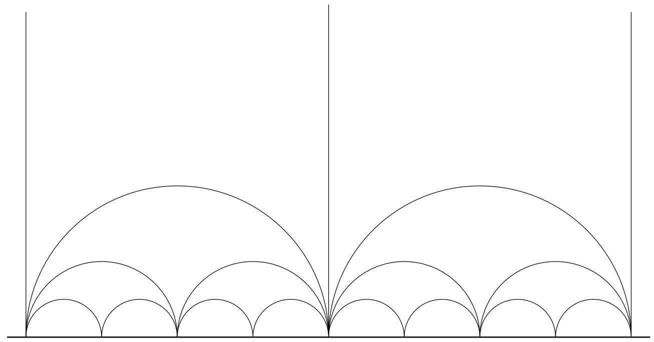
3-pronged saddle



Whitehead equivalence



Elementary moves on pants decompositions



Farey graph in the upper half plane

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