SEMINAR

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1. Automorphisms of Surfaces

1.1. **Background.** The classification of homeomorphisms of surfaces is related to the topology and geometry of 3-manifolds in the following way. In his paper in the Bull. A.M.S. 1982, Thurston conjectured that the interior of every compact 3-manifold has a canonical decoposition into pieces which have geometric structures. He established the following result: let S be a surface and $\phi : S \to S$ a diffeomorphism; let M_ϕ be the 3-manifold defined by $M_\phi = S \times [0,1]/\sim_\phi$, where the relation is given by $(x,0) \sim_\phi (\phi(x),1)$; then M_ϕ has a hyperbolic structure if and only if ϕ is pseudo-Anosiov. This last term denotes one of the three types of automorphisms of surfaces, according to Thurston's classification; the other two types are homeomoprhisms of finite order and homeomorphisms that leave invariant a finite, pairwise disjoint, set of curves. The classification is up to homotopy, as explained in the next paragraph.

Given an oriented, smooth surface S we denote by $Hom(S)$, $Hom^+(S)$ and $Hom_0(S)$ the groups of homeomorphisms of S, orientation-preserving homeomorphisms and homeomorphisms homotopic to the identity, respectively. The extended mapping class group is the quotient group $Mod^*(S) = Hom(S)/Hom_0(S)$; the mapping class group is the index 2 subgroup given by $Mod(S) = Hom^+(S)/Hom_0(S)$.

Two remarks: first, a homeomorphism on S is homotopic to a diffeomorphism, so one can work in the smooth category. Secondly, by a result of Baer, Mangler and Epstein, two elements in $Hom^+(S)$ are homotopic if and only if they are isotopic, if S is of finite topological type (a compact surface minus a finite number of points and discs). References: [1] D.B.A. Epstein, "Curves on 2-manifolds and isotopies", Acta Math. 115 (1966), 83-107 and [2] W. Mangler, "Die Klassen von topologischen Abbildungen einer geschlossene Fläche auf sich", Math. Z. 44 (1939), 541-554.

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The Dehn-Nielsen-Baer theorem states that $Mod^*(S)$ is isomorphic to $Out(\pi_1(S)) = Aut(\pi_1(S))/Inn(\pi_1(S))$. Here $Inn(\pi_1(S))$ denotes the group of inner automorphisms of the fundamental group of S, that is, automorphisms given by conjugation by elements of $\pi_1(S)$. Strictly speaking we should write the base point in the above notation, but since change of base point is given by a conjugation, and we are taking quotients by inner automorphisms, the base point does not matter (as long as the same base point is used in the definition, of course). References: [1] B. Farb and D. Margalit, "A primer on mapping class groups" and [2] N. Ivanov, "Mapping Class Groups".

1.2. **Automorphisms of the torus.** Let T denote a torus, that is, a compact surface of genus 1. We identify T with the quotient $\mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts by translations, for example, $(x, y) \mapsto (x + m, y + n)$, with m and n integers. Since we are interested on a topological object, the mapping class group, the particular action of $\mathbb{Z}^2$ on the plane is not important; however, when we want to give some examples we might change this action to suit our needs.

Since $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ is abelian we have that the inner automorphisms group is trivial. Thus $Mod^*(T)$ is isomorphic to $GL(2, \mathbb{Z})$, the group of rank 2 matrices with integer coefficients and determinant ± 1. The subgroup $Mod(T)$ is isomorphic to $SL(2, \mathbb{Z})$, that is, those matrices with determinant equal to 1.

If h is a mapping class then we have a unique element α of $SL(2, \mathbb{Z})$ associated to it by the above isomorphism. If $\alpha = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ then its characteristic polynomial is $t^2 - (tr\alpha)\, t + 1$, where tr denotes the trace. Let λ and $\lambda^{-1} = 1/\lambda$ be the eigenvalues of α. There are three different possibilities:

(1) λ and λ^{-1} are complex but not real; in this case they must have absolute value 1 and an easy computation shows that $tr(\alpha)$ can take only the values 0, 1 and -1. We have that h has finite order equal to 2, 3, 4 or 6. To construct one example take $\tau = \exp(\pi i/3)$, and let us consider T as the quotient of the complex plane by the group of transformations $z \mapsto z + m + n\tau$. Then T has an automorphism given by $h(z) = \tau\, z$, with associated matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. It is easy to check that h has order 6.

(2) $\lambda = \lambda^{-1} = \pm 1$, then $tr(\alpha) = \pm 2$. We have that α has an integral eigenvector that projects to an invariant closed curve on the torus. One possible way of looking at this is, for example, considering α as a Möbius transformation acting on the complex plane

by $z \mapsto \frac{pz+q}{rz+s}$. Then if $\binom{x}{y}$ is an eigenvector for α if and only if the real number x/y is a fixed point of the corresponding Möbius transformation. In the particular case of $\lambda = \lambda^{-1} = \pm 1$ we have that the fixed point in the (extended) real line is the number $\frac{p-s}{2r}$, which is rational. So the eigenvector induces a closed curve on the torus. Conversely, if p/q is a rational number the transformation $\begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}$ has eigenvector $\binom{p}{q}$. The maps of this type are Dehn twists (or power of such maps) as the one given in the figure.

(3) $|\lambda| > 1 > |\lambda^{-1}|$; in this case, by considerations of Kleinian groups the associated Möbius transformation cannot have a rational fixed point (see Maskit: II.C.6), so the invariant curved induced by the eigenvalue λ is not closed. One has that the mapping h "stretches" in the direction of the eigenvector corresponding to λ and "shrinks" in the direction of the other eigenvector, corresponding to $1/\lambda$. These transformations are called Anosov.

1.3. **Automorphisms of the torus and Riemann surfaces.** The above discussion suggest that one looks at $SL(2,\mathbb{Z})$ as a group acting on the space of tori as follows: from the complex point of view, different actions of $\mathbb{Z}^2$ on the complex plane give different tori. From the theory of elliptic functions we have that any torus can be representedas $\mathbb{C}/\mathcal{L}_\tau = T_\tau$, where $\mathcal{L}_\tau$ is the group of translations of the form $z \mapsto z + m + n\tau$, with m and n integers. One has that τ is a complex number with positive imaginary part, so we have a parameter space for torus given by the upper half plane $\tau \in \mathbb{H} = \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$. Elements of the modular group $SL(2,\mathbb{Z})$ act as Möbius transformations on $\mathbb{H}$; they are of three types: elliptic, parabolic and hyperbolic. Elliptic transformations have a fixed point in $\mathbb{H}$ and correspond to automorphisms of finite order in a torus. Parabolic transformations fix a rational point on the real line (or the point of infinity) and correspond to the transformations of type (2) above, that have a fixed closed curve. Finally hyperbolic mappings correspond to Anosov homeomorphisms. Observe that the following two transformations, which are Dehn twists, generate $SL(2,\mathbb{Z})$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

1.4. **Generalization.** Thurston's classification of automorphisms of surfaces generalizes to higher genus in a way similar to what we explained in the last paragraph. There is a a parameter space of a surface S, of genus at least 2, called **Teichmüller space** of S, $T(S)$. Any element of the mapping class group $Mod(S)$ acts naturally on $T(S)$ in a continuous way. The space $T(S)$ is an open ball in $\mathbb{R}^{6g-6}$; the first step in Thurston's classification is to compactify $T(S)$ by means of some classes of foliations, so $\overline{T(S)}$ is homeomorphic to the closed ball in $\mathbb{R}^{6g-6}$. The action of $Mod(S)$ on $T(S)$ extends to a continuous action on the compactified space. Thus, by Brouwer's fixed point, any element ϕ of $Mod(S)$ has a fixed point in $\overline{T(S)}$. There are three possibilities: (1) ϕ fixes a point in the interior: $T(S)$; it is easy to see that ϕ acts as a holomorphic (invertible) mapping on some surface S' and, by Hurwitz's theorem, ϕ must have finite order. (2) If ϕ has only one fixed point in the boundary of $T(S)$ then that point corresponds to a collection of closed, pairwise disjoint curves, and ϕ leaves those curves invariants; in this case we talk of **reducible** mappings. (3) Finally, ϕ can have two fixed points on the boundary of $T(S)$, these points corresponds to foliations on S, and ϕ stretches in one direction by a factor of λ and shrinks in a "complementary" direction by a factor of λ; the mapping (or its class) is called **pseudo-Anosov**.

2. Hyperbolic Geometry

2.1. **Basic Hyperbolic Geometry.** Let $\mathbb{D}$ denote the unit disc on the complex plane, that is $\mathbb{D} = \{z \in \mathbb{C};\ |z|, 1\}$. For an open subset of $\mathbb{C}$ we denote by $Aut(G)$ the group of biholomorphic mappings of G (bijective holomorphisms from G to G).

An easy consequence of Schwarz's lemma is that the group $Aut(\mathbb{D})$ consists of the Möbius transformations of the form $g(z) = \frac{az+\bar{c}}{cz+\bar{a}}$; where a and c are complex number satisfying $|a|^2 - |c|^2 = 1$. An easy computation shows that, for such mapping g, one has

$$\frac{|g'(z)|}{1-|g(z)|^2} = \frac{1}{1-|z|^2},$$

for all z in $\mathbb{D}$. This means that the metric $\lambda(z)|dz| = \frac{2}{1-|z|^2}\,|dz|$ is invariant under $Aut(\mathbb{D})$. More precisely, if $\gamma : [a,b] \to \mathbb{D}$, is a smooth curve, we define the length of γ by $l(\gamma) = \int_\gamma \lambda(z)|dz| = \int_a^b \lambda(\gamma(t))\,|\gamma'(t)|\,dt$, then $l(\gamma) = l(g\circ\gamma)$. We define the **hyperbolic distance** on $\mathbb{D}$ by $d(z,w) = \inf l(\gamma)$, where the infimum is taken over all (smooth) curves joining z and w.

Theorem 2.1. *The hyperbolic distance d above defined is a distance; it is invariant under $Aut(\mathbb{D})$ (equivalently, the elements of $Aut(\mathbb{D})$ are isometries with respect to d) and given any two points in $\mathbb{D}$ there is a (smooth) curve joining them whose length is the distance between the points ($(\mathbb{D}, d)$ is a geodesic path space).*

The standard proof of this theorem is starts from the last statement as follows: by an automorphism of the disc we can assume that $z = 0$; composing with a rotation then we can further assume w is a point in the open interval $(0, 1)$, say t_0. Let $\gamma : [0, 1] \to \mathbb{D}$ be a (smooth) curve with $\gamma(0) = 0$ and $\gamma(1) = t_0$; write $\gamma(t) = x(t) + iy(t)$. Then

$$l(\gamma) = \int_0^1 \frac{2\sqrt{x'(t)^2 + y'(t)^2}}{1 - (x(t)^2 + y(t)^2)}\, dt \geq \int_0^1 \frac{2\sqrt{x'(t)^2}}{1 - x(t)^2}\, dt \geq \int_0^1 \frac{2x'(t)}{1 - x(t)^2}\, dt = \log \frac{1 + t_0}{1 - t_0}.$$

Equality occurs only if $x'(t) > 0$ and $y(t) = 0$ for all t, that is, γ is the line segment from 0 to t_0 without *backtracking*.

Thus we have $d(0, z) = \log \frac{1+|z|}{1-|z|}$; applying a Möbius transformation on $Aut(\mathbb{D})$ we get

$$d(z, w) = \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}.$$

From the above construction and the fact that Möbius transformations take lines and circles to lines and circles we have that the geodesics in the hyperbolic metric are the lines and circles orthogonal to S^1. Moreover we see that geodesics are global in the sense that they realized the distance between any two points on their images.

From the above formula of the hyperbolic distance we get that the circle of center the origin and (hyperbolic) radius r, C_r, is just the Euclidean circle of center the origin and radius R where $r = \log \frac{1+R}{1-R}$. Thus hyperbolic circles (and discs) are Euclidean circles (and discs), with different radius and, generally, different center. Hence we have that the topology induced by the hyperbolic metric on the unit disc coincides with the topology induced by the Euclidean metric.

2.2. **Length, Area and the Gauss-Bonnet Theorem.** It is easy to compute by direct integration that $l(C_r) = 2\pi \sinh r$. Here $\sinh r = (e^r - e^{-r})/2$; thus $l(C_r) \sim e^r$ as $r \to \infty$ (in the hyperbolic metric, or in Euclidean terms $r \to 1$), as opposed to the Euclidean metric where the length of a circle grows linearly with the radius.

The (hyperbolic) area of a subset $E \in \mathbb{D}$ is given by $Area(E) = \int_E \lambda^2(z)\, dx\, dy$; in particular, for a disc of radius r, say D_r, we have $Area(D_r) = 4\pi \sinh^2(r/2) = 2\pi(\cosh r -$

1). So $l(C_r) \sim Area(D_r)$ as $r \to \infty$, another difference with respect to the Euclidean metric.

A triangle is the region enclosed by three geodesics. We allow the vertices of a triangle to be in the unit circle. The angles in the hyperbolic plane are measured as in the Euclidean case (that is, the angles between the tangents for the curves); if two geodesics meet on a point on S^1 we declare the angle to be zero.

The Gauss-Bonnet Theorem for triangle states that a triangle T with (interior) angles α, β and γ has area equal to $\pi - (\alpha+\beta+\gamma)$. The proof for the case of a triangle with two zero angles is just by a simple integration; the rest of the cases (one angle equal to zero and no zero angles) can be deduced from this case by expressing a triangle as the differences of two triangles in the previous cases. See the figures at the end of the section (where the triangles are in the upper half plane, rather than in the unit disc). For a polygon with n sides and angles $\alpha_1, \dots, \alpha_n$ we get that the area is equal to $(n-2)\pi - (\alpha_1 + \cdots + \alpha_n)$.

2.3. **The Upper Half Plane Model.** Let $\mathbb{H}$ denote the upper half plane in the complex plane, $\mathbb{H} = \{z \in \mathbb{C};\ Im(z) > 0\}$; the Cayley transform $T(z) = \frac{z-i}{z+i} : \mathbb{H} \to \mathbb{D}$ is a biholomorphic mapping between the upper half plane and the unit disc, so we can put a hyperbolic metric on $\mathbb{H}$ using T. The expression for this metric is $\frac{1}{Im(z)}\,|dz|$. The properties of the metric on $\mathbb{D}$ are satisfied for this metric on $\mathbb{H}$. For example, the figures below show the proof (up to an integration) of the Gauss-Bonnet theorem on $\mathbb{H}$.

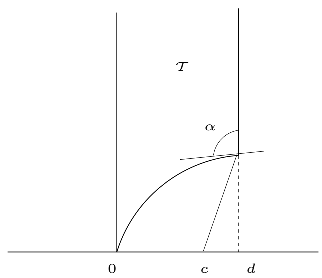

FIGURE 1. Triangle with two zero angles.

2.4. **Gromov's hyperbolicity.** A **geodesic** on a metric space X is a curve $g : [a, b] \to X$ such that $d(g(t), g(s)) = |t - s|$, for all t, s in $[a, b]$. A geodesic triangle is the region of X enclosed by three geodesics. A (geodesic) triangle is called δ-thin (for some $\delta > 0$) if any side is contained in the δ-neighborhood of the other two sides. A metric space is called

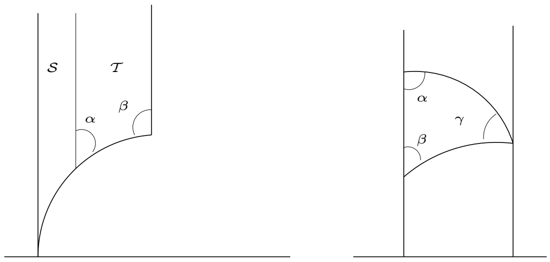

FIGURE 2. Gauss-Bonet Theorem.

δ-hyperbolic if all geodesic triangles are δ-thin for some fixed δ. In that case X is alse said to be **Gromov-hyperbolic**.

Hyperbolic space (either the unit disc model or the upper half plane one) is δ-hyperbolic. One can easily see that as follows: if a point p in a side of a triangle T is not in the *delta*-neighborhood of the other two sides then one can put a half disc of radius δ inside T; see figure below. From the area formulæof the disc and the triangle we get

$$\pi(\cosh\delta - 1) \leq \pi - (\alpha+\beta+\gamma) \leq \pi \Rightarrow \cosh\delta - 1 \leq 1 \Rightarrow \delta \leq \log(2+\sqrt{3}).$$

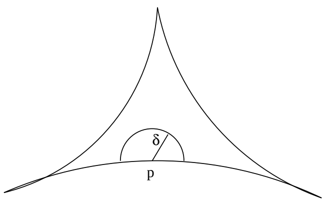

FIGURE 3. Gromov's hperbolicity.

2.5. **Isometries of Hyperbolic Space.** Every isometry of $\mathbb{D}$ (with the hyperbolic metric) is of the form $z \mapsto g(z)$ or $z \mapsto g(\bar{z})$, where $g \in Aut(\mathbb{D})$. To see this, let h be an isometry; composing with an element of $Aut(\mathbb{D})$ we can assume that $h(0) = 0$; by composing with a rotation we can further assume that $h(1/2) = 1/2$. But then h must fix every point in $(-1, 1)$ (by a simple distance calculation) and from that it is easy to see that for every $z \in \mathbb{D}$ we have $h(z)$ is either z or $\bar{z}$. A simple continuity argument shows that $h(z) = z$ or $h(z) = \bar{z}$ for **all** z in the unit disc.

Thus we have that the orientation-preserving isometris of $\mathbb{D}$ or $\mathbb{H}$ are Möbius transformations; moreover, this subgroup coincides with the group of biholomorphic mappings of

$\mathbb{D}$ or $\mathbb{H}$, making thus the hyperbolic metric natural from the Complex Analysis point of view.

As we mentioned ealier, an element of $Aut(\mathbb{D})$ is of the form $g(z) = \frac{az+\bar{c}}{cz+\bar{a}}$; the elements of $Aut(\mathbb{H})$ are of the form $g(z) = \frac{az+b}{cz+d}$, where a, b, c and d are real numbers satisfying $ad - bc = 1$. In any case we can assign to an element of $Aut(\mathbb{D})$ or $Aut(\mathbb{H})$ an element of $\mathrm{PSL}(2, \mathbb{C})$ (or $\mathrm{PSL}(2, \mathbb{R})$ in the case of the upper half plane), say $\left[\begin{smallmatrix} a & \bar{c} \\ c & \bar{a} \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ respectively.

From Complex Analysis we have the following classification of $Aut(\mathbb{H})$, up to conjugacy, that is, any element of $Aut(\mathbb{H})$ is conjugate to one of the following:

1. Identity.
2. Elliptic transformations: conjugate to rotations, with corresponding matrices of the form $\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right]$, for $0 < \theta < \pi$;
3. Parabolic transformations. These are conjugate to the translations $z \mapsto z + 1$ or $z \mapsto z - 1$;
4. Hyperbolic transformations, conjugate to $z \mapsto k^2 z$, for k real and $k > 1$.

In terms of their fixed points, elliptic elements have one fixed point in the hyperbolic plane, parabolics have one fixed point on the boundary (S^1 or $\mathbb{R} \cup \{\infty\}$) and hyperbolic transformations have two fixed points on the boundary.

Given an isometry g of the hyperbolic plane we define the **translation length**, $\tau(g)$ by

$$\tau(g) = \inf\{d(z, g(z));\ z \in \mathbb{D}\,(\mathbb{H})\}.$$

The identity has $\tau(g) = 0$ but it is not of interest. Elliptic elements have $\tau(g) = 0$ and that value is realized at the fixed point of the transformation. Parabolics have $\tau(g) = 0$ (consider the translation $z \mapsto z + 1$ and make $Im(z) \to +\infty$) but this value is not realized for any point of the hyperbolic plane. Hyperbolic isometris satisfy $\tau(g) > 0$, with the value realized on a geodesic, called the axis of the transformation A_g. Moreover, we have

$$\sinh(d(z, g(z))/2) = \sinh(\tau(g)/2)\, \cosh(d(z, A_g));$$

so the further z is from A_g the more it is translated by g.

3. Fuchsian Groups

3.1. **Riemann surfaces and Fuchsian groups.** A **Riemann surface** S is a (connected) surface where the transition functions are holomorphic functions. It is easy to see that if $\pi : \tilde{S} \to S$ is the universal covering of S then it is possible to put on $\tilde{S}$ a (unique) Riemann surface structure so that π becomes a holomorphic function. Thus a natural question is to find what simply connected Riemann surface exists. The Riemann Mapping Theorem gives us a first result on the complex plane: any simply connected subset of $\mathbb{C}$ is either the complex plane of the unit disc, up to biholomorphisms.

Theorem 3.1 (Uniformization Theorem)**.** *Any simply connected Riemann surface is biholomorphic to one (and only one) of the following surfaces: the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the complex plane $\mathbb{C}$ or the upper half plane $\mathbb{H}$.*

The unit disc $\mathbb{D}$ is conformally equivalent to the upper half plane; we can use for example the Cayley transform as done ealier in these notes.

We have then that any Riemann surface can be written as $\widetilde{S}/\Gamma$, where $\widetilde{S}$ is a subset of the Riemann sphere and Γ is the group of covering transformations. It is easy to see that the elements of Γ are biholomorphic mappings of $\widetilde{S}$. The groups of automorphisms (biholomorphic self-mappings) of the three simply connected surfaces $\widehat{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{H}$ are groups of Möbius transformations, so we conclude that any Riemann surface can be expressed as the quotient of an open subset of the Riemann sphere by a group of Möbius transformations.

The group of Möbius transformations can be identified with PSL$(2\mathbb{C})$, the quotient of SL$(2, \mathbb{C})$ by the complex numbers of absolute value 1. Thus we can identify a Möbius transformation with a matrix of complex coefficients and determinant 1, up to multiplication by a number in the unit circle. The classification of Möbius transformations, up to conjugacy, is well known; we have three different types of transformations:

1. the identity;
2. elliptic elements, conjugate to rotations $z \mapsto \lambda z$, with $|\lambda| = 1$; the trace of such elements satisfies $0 \leq tr^2 <$;
3. parabolic transformations, conjugate to $z \mapsto z + 1$ with $tr^2 = 4$;

4. loxodromic elements, conjugate to $z \mapsto \lambda^2 z$, where $|\lambda| > 1$. Among the loxodromic elements one has the hyperbolic transformations, where λ is real (and greater than 1) and satisfy $tr^2 > 4$.

Lemma 3.2. *A non-identity Möbius transformation has at least one fixed point and at most two in the Riemann sphere* $\widetilde{\mathbb{C}}$.

The group $Aut(\widehat{\mathbb{C}})$ is the group of Möbius transformations. Since covering transformations act fixed-points free we have that if $\widetilde{S} = \widehat{\mathbb{C}}$ then Γ must be trivial and so S should be the Riemann sphere too.

The group $Aut(\mathbb{C})$ consists of the Möbius transformations of the form $z \mapsto az + b$, where a and b are complex numbers and $a \neq 0$; such a transformation has a fixed point if and only $a \neq 1$, so if $S = \mathbb{C}/\Gamma$ then Γ consists of translations, $z \mapsto z + b$. It is easy to see that Γ can be only of three different types:

1. Γ is trivial and $S = \mathbb{C}$;
2. Γ is cyclic, thus $\Gamma \cong \mathbb{Z}$ generated by a single translation, say $z \mapsto z + 1$, and $S \cong \mathbb{C} \backslash \{0\}$ (use, for example, the exponential mapping as the covering map);
3. $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by two translations, $z \mapsto z + 1$ and $z \mapsto z + \tau$, where τ is a complex number with positive imaginary part; the surface S is a torus.

For all other Riemann surfaces we have that S is biholomorphic to $\mathbb{H}/\Gamma$, where Γ is a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ called a Fuchsian group.

3.2. **Basic properties of Fuchsian Groups.** We will now restric to the study of groups Γ for surfaces of the form $S = \mathbb{H}/\Gamma$ (or, equivalently, we can assume Γ is a subgroup of $Aut(\mathbb{D})$).

The group $\mathrm{PSL}(2, \mathbb{R})$ (or $\mathrm{PSL}(2, \mathbb{C})$) has a natural topology given by "coefficients convergence"; a sequence of Möbius transformations A_n converges to A is we can find representatives $A_n(z) = \frac{a_n z + b_n}{c_n z + d_n}$ and $A(z) = \frac{az+b}{cz+d}$ such that $a_n \to a$, …, $d_n \to d$. A subgroup $G \leq \mathrm{PSL}(2, \mathbb{R})$ is **discrete** if there is no sequence of distinct elements $\{g_n\} \subset G$ such that $g_n \to A$, for some Möbius transformation A.

A subgroup Γ of $\mathrm{PSL}(2, \mathbb{C})$ is said to act **properly discontinuously** at a point $z_0 \in \widehat{\mathbb{C}}$ if there exists a neighbourhood U of z_0 such that the number of elements $g \in G$ with

$g(U) \cap U \neq \emptyset$ is finite. Clearly, since Γ is a covering group we have that it acts properly discontinuously on the upper half plane.

Lemma 3.3. *If $G \leq PSL(2, \mathbb{C})$ acts properly discontinuously on some non-empty set then G is discrete.*

The proof is simple: if G is not discrete then the images of any point z under the sequence $g_{n+1}^{-1} \circ g_n$ converge to z.
The converse of the above statement is false; the Picard group, consisting of Möbius transformations of the form $z \mapsto \frac{az+b}{cz+d}$, with $ad - bc = 1$ and a, b, c , d in $\mathbb{Z}[i]$ is clearly discrete (the entries are almost integers) but it does not act properly discontinuously anywhere on the Riemann sphere. However, for subgroups of $\mathrm{PSL}(2, \mathbb{R})$ the two definitions are equivalent.

Proposition 3.4. *If $\Gamma \leq PSL(2, \mathbb{R})$ then the following are equivalent:*

(1) Γ *acts properly discontinuously on* $\mathbb{H}$*;*
(2) *there is a point* $z_0 \in \mathbb{H}$ *such that* Γ *acts properly discontinuously at* z_0*;*
(3) Γ *is discrete.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3). To prove the remaining implication we will work with $\Gamma \subset Aut(\mathbb{D})$. It is easy to see that if Γ does not act properly discontinuously on $\mathbb{H}$ then we can find $z_0 \in \mathbb{H}$ and sequences $\{z_n\} \subset \mathbb{H}$ of distinct points, all Γ-equivalent to z_0 and such that $z_n \to z_0$. Thus we have a sequence of elements $\{\gamma_n\} \subset \Gamma$ such that $\gamma_n(z_n) = z_0$; since the points z_n are distinct the transformations γ_n are also distinct.

Consider $\phi_n(z) = \frac{z - z_n}{1 - \bar{z_n} z}$ and set $C_n = \phi_{n+1} \circ \gamma_{n+1}^{-1} \circ \gamma_n \circ \phi_n^{-1}$. Then we have C_n are automorphisms of the unit disc with $C_n(0) = 0$, so $C_n(z) = \lambda_n z$, with $|\lambda_n| = 1$. Therefore there is a subsequence of λ_n that converges to λ_0 (let C_0 be the corresponding rotation). Since $\phi_n \to \phi_0$ where $\phi_0(z) = \frac{z - z_0}{1 - \bar{z_0} z}$ we have that $\gamma_{n+1}^{-1} \circ \gamma_n = \phi_{n+1}^{-1} \circ C_n \circ \phi_n \to \phi_0^{-1} \circ C_0 \circ \phi_0$ and thus Γ is not discrete. $\square$

A **Fuchsian group** is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ or a group conjugate to one such subgroup. The above result is the motivation for the Thurston definition of **Kleinian group** as a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$. The reasons is that the action of Möbius transformations on $\widehat{\mathbb{C}}$ extends naturally to hyperbolic 3-space, $\mathbb{H}^3$, and one has that if G is a discrte group of Möbius transformations then it acts properly discontinuously on $\mathbb{H}^3$ and $\mathbb{H}^3/G$ is a manifold (see Maskit's book).

3.3. **Uniformization of compact surfaces.** The goal of this subsection is to show that if $S = \mathbb{H}/\Gamma$ is a compact surface then all non-trivial elements of Γ are hyperbolic. We will prove this statement with a series of lemmas.

Lemma 3.5. *If a Möbius transformation fixes a disc or a half space if and only if* $tr^2 \geq 0$.

One implication is simple (an elliptic, parabolic or hyperbolic transformation fixes a disc or half plane); for the other see Maskit's book.

Thus we have that loxodromic elements of a Fuchsian group are actually hyperbolic.

Lemma 3.6. *If A and B are Möbius transformations sharing exactly one fixed point and A has two fixed points then $ABA^{-1}B^{-1}$ is parabolic.*

Proof. Writing the transformation in matrix form we can assume that $A = \left[\begin{smallmatrix} k & 0 \\ 0 & k^{-1} \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right]$. Then $ABA^{-1}B^{-1} = \left[\begin{smallmatrix} 1 & ab(k^2-1) \\ 0 & 1 \end{smallmatrix}\right]$. Since $a \neq 0$, $b \neq 0$ and $k^2 \neq 1$ we have the result. □

Lemma 3.7. *If A is a loxodromic transformation and B is a Möbius transformation such that A and B share exactly one fixed point then the group generated by A and B is not discrete.*

Proof. By the previous lemma one can assume that A is parabolic. By conjugation we can write $A(z) = z + b$ and $B(z) = k^2 z$, with $|k| > 1$. An easy computation shows that $B^{-m}AB^m$ converges to the identity. □

Proposition 3.8 (Shimizu-Leutbecher Lemma)**.** *If G is a discrete subgroup of $PSL(2, \mathbb{C})$ containing $f(z) = z + 1$. If $g == \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is in G, with $ad - bc = 1$ then either $c = 0$ or $|c| \geq 1$.*

Corollary 3.9. *If G is a subgroup of $PSL(2, \mathbb{R})$ and $f(z) = z + 1$, and $\mathcal{H} = \{z \in \mathbb{H};\ Im(z) > 1\}$ then for any $g \in G$ either $g(\mathcal{H}) = \mathcal{H}$ or $g(\mathcal{H}) \cup \mathcal{H} = \emptyset$.*

Proof. If g fixes ∞ then it cannot be hyperbolic or elliptic. So g must be parabolic, of the form $g(z) = z + b$ ($b \in \mathbb{R}$), and thus $g(\mathcal{H}) = \mathcal{H}$. In case of g not fixing ∞ (that is, $c \neq 0$ in the above notation) we have that $g(\mathcal{H})$ is a disc bounded by a circle tangent to the ral axis at $g(\infty) = a/c$ and diameter $1/c^2$, achieved at the point $g(-d/c + i) = a/c + i/c^2$. Since $|c| \geq 1$ we get $g(\mathcal{H}) \cap \mathcal{H} = \emptyset$. $\square$

Corollary 3.10. *If S is a compact surface written as $S = \mathbb{H}/\Gamma$ where Γ is a discrete subgroup of $PSL(2,\mathbb{R})$ then all non-trivial elements of Γ are hyperbolic transformations.*

3.4. **Dirichlet region.** A (convex) polygon D in $\mathbb{H}$ is defined as the intersection of countably many open half-planes (with respect to the hyperbolic metric), where only finitely many of the defining geodesics meet any compact subset of $\mathbb{H}$.

If Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ and D is a polygon in the upper half plane we say that D is a **fundamental polygon** for Γ if the following conditions are satisfied:

(1) for all $\gamma \in \Gamma \backslash \{id\}$, $\gamma(D) \cap D = \emptyset$;
(2) for all $x \in \mathbb{H}$ there exists an $\gamma \in \Gamma$ such that $\gamma(x) \in \overline{D}$;
(3) the sides of D are paired by elements of Γ (that is, for each side s there is another side s' and an element γ_s such that $\gamma_s(s) = s'$; also $\gamma_{s'} = \gamma_s^{-1}$ and $(s')' = s$);
(4) any compact set of $\mathbb{H}$ meets only finitely many Γ-translates of D.

Construction on fundamental polygons (or more generally, fundamental regions) for Fuchsian groups is a key point in the theory, since it is eeasy to understand the action of a group on $\mathbb{H}$ if we know a fundamental polygon (where the action in the interior is trivial and on the boundary is given by the sides identification/pairing). We give here an example of one such construction. We begin with some tehcnical, simple result.

Lemma 3.11. *In the above conditions there is a point $z_0 \in \mathbb{H}$ such that $\gamma(z_0) = z_0$ if and only if γ is the identity.*

Choose one such point z_0 and for every non-trivial $\gamma \in \Gamma$ set $D_\gamma = \{z \in \mathbb{H};\ d(z, z_0) < d(z, \gamma(z_0))\}$. We have D_γ is a half-plane in $\mathbb{H}$. The **Dirichlet region** for Γ centered at z_0 is defined by $D = \cap D_\gamma$, where the intersection runs over all $\gamma \in \Gamma \backslash \{id\}$.

Theorem 3.12. *D is a fundamental polygon for Γ.*

Proof. Since there are only finitely many Γ translates of z_0 in any compact set we have that D is a convex polygon. Also, if K is compact, we can assume that K is the ball of radius ρ centered at z_0; only finitely many translates of z_0 meet the ball of radius 2ρ so if $d(\gamma^{-1}(z_0), z_0) > 2\rho$ we have that $\gamma(D) \cap K = \emptyset$, which proves property (4) above.

Since $\gamma(z_0) = z_0$ only for the identity eah side of D corresponds to a unique element of Γ.

To prove (1) consider γ non-trivial and $x \in D$. Then

$$d(\gamma(x), \gamma(z_0)) = d(x, z_0) < d(x, \gamma^{-1}(z_0)) = d(\gamma(x), z_0),$$

so $\gamma(x) \notin D$.

To show (2), let $x \in \mathbb{H}$; then there exists $\gamma \in \Gamma$ sich that $d(x, \gamma(z_0)) \leq d(x, h(z_0)$, for all $h \in \Gamma$, and thus

$$d(\gamma^{-1}(x), z_0) = d(x, \gamma(z_0)) \leq (x, \gamma \circ h(z_0)) = d(\gamma^{-1}(x), h(z_0));$$

since $\gamma \circ h$ covers Γ as h varies over Γ we get that $\gamma^{-1}(x) \in \overline{D_h}$ for all $h \in \Gamma$ and so $\gamma^{-1}(x) \in \overline{D}$.

Finally, for (3) suppose that x is in the (relative) interior of a side s; that means that there exists a unique $\gamma \in \Gamma$ such that $x \in \overline{D_\gamma}$. In other words,

$$\begin{cases} d(x, z_0) < d(x, h(z_0)), \ \forall \ h \neq \gamma, \\ d(x, z_0) = d(x, \gamma(z_0)). \end{cases}$$

These two statements imply

$$\begin{cases} d(\gamma^{-1}(x), h(z_0)) = d(x, \gamma \circ h(z_0)) > d(x, z_0) = d(\gamma^{-1}(x), z_0), \ \forall \ h \neq \gamma^{-1} \\ d(\gamma^{-1}(x), z_0) = d(x, z_0) = d(\gamma^{-1}(x), \gamma^{-1}(z_0)). \end{cases}$$

Thus $\gamma^{-1}(x)$ belongs to a side s' of D with $\gamma^{-1}(s) = s'$. □

The following two results (we do not include their proofs here) show useful applications of the Dirichlet region. We need a little notation first: if Γ is a Fuchsian group by Λ_Γ we mean the set of points in $\mathbb{R} \cup \{\infty\}$ where Γ does not act properly discontinuously. The statements are not as simple as they could be, but we do not want to introduce more concepts; if the reader knows about Kleinian groups then the statements can be simplied.

Lemma 3.13. *Let D be a Dirichlet region for the Fuchsian group Γ.*

1. $\mathbb{H}/\Gamma$ is compact if and only if the Euclidean closure of D is a compact subset of $\mathbb{H}$.

2. If $\overline{D}$ is compact then D has only finitely many sides.

Theorem 3.14. *Let D be a Dirichlet region for the Fuchsian group Γ and assume that Λ_Γ has more than two points. The the following are equivalent:*

(1) *D has finitely many sides;*
(2) *the quotient $(\widehat{\mathbb{C}}\backslash\Lambda_\Gamma)/\Gamma$ is a union of compact Riemann surface with finitely many (possibly none) points removed;*
(3) *Γ is finitely generated (by the side-pairing transformations);*
(4) *the hyperbolic area of (convex hull(Λ_Γ))$/\Gamma$ is finite.*

4. Kleinian Groups

4.1. **Basic results on Kleinian groups.** Let G denotes a subgroup of $\mathrm{PSL}(2,\mathbb{C})$. We say that G acts **properly discontinuously** at $z \in \widehat{\mathbb{C}}$ if there is a neighbourhood U of z such that the number of elements $g \in G$ with $g(U) \cap U \neq \emptyset$ if finite. If only the identity satisfies $g(U) \cap U \neq \emptyset$ then we say that G acts freely discontinuously at z. The set of points where G acts properly discontinuously (resp. freely discontinuously) is called the **regular set** (resp. **free regular set**) and it is denoted by $\Omega(G)$ or just Ω (resp. ${}^o\Omega(G)$ or ${}^o\Omega$). We say that G is **Kleinian** (or some times, **Kleinian of the second kind**) if ${}^o\Omega \neq \emptyset$.

A point $x \in \widehat{\mathbb{C}}$ is called a **limit point** if there exists $z \in {}^o\Omega$ and a sequence $\{g_m\}$ of distinct elements of G such that $g_m(z) \to x$. The set of limit points, called the **limit set** is denoted by $\Lambda(G)$ or simply Λ.

Proposition 4.1. *Λ is a closed, G-invariant, nowhere dense subset of $\widehat{\mathbb{C}}$. If it contains more than 2 points then it is a perfect set (and therefore, uncountable).*

Proposition 4.2. *$\widehat{\mathbb{C}}$ is the disjoint union of Ω and Λ.*

4.2. **Kleinian groups and Riemann surfaces.**

Theorem 4.3. *If G is a Kleinian group then Ω/G is a union of Riemann surfaces.*

If $\Delta \subset \Omega$ is a component of the regular set of G we denote by G_Δ its stabilizer in G, that is, $G_\Delta = \{g \in G;\ g(\Delta) = \Delta\}$. Two components Δ and Δ' are conjugate if there is an element $g \in G$ such that $g(\Delta) = \Delta'$. In that case we have that the stabilizers of the two components are conjugate subgroups of G, and therefore the quotient of the components by the stabilizers are the same Riemann surface. Thus we see that if Δ_1, Δ_2, ..., is a list of non-conjugate components of Ω, then $\Omega/G \cong \Delta_1/G_{\Delta_1} \cup \Delta_2/G_{\Delta_2} \cup \cdots$ (in particular, Ω/G consists of at most countably many Riemann surfaces).

Let $z_0 \in \Omega$ and suppose that it has non-trivial stabilizer, i.e. the subgroup $G_{z_0} = \{g \in G;\ g(z_0) = z_0\}$ is not the identity. It is not difficult to see that G_{z_0} must be a cyclic group of finite order, say n. The projection $\pi : \Omega \to \Omega/G$ is then like $z \mapsto z^n$ near z_0; we say that z_0 is a **ramification point** and π is ramifed over $p_0 = \pi(z_0)$.

The Kleinian group G is said to be of **finite type** over a component Δ if Δ/G_Δ is a compact surface of genus g, with $n \geq 0$ points removed (punctures), and the projection $\pi : \Delta \to \Delta/G_\Delta$ is ramified over finitely many points, say $m \geq 0$, with orders $\nu_1, \ldots, \nu_m$. Set $s = n + m$. Then (g, p) is called the type of Δ/G_Δ and $(g, p; \nu_1, \ldots, \nu_m, \infty, \infty)$ the **signature** (where the symbolc ∞ appears n times). We say that G is of **finite type** or **analytically finite** if it has only finitely many non-conjugate components (of Ω) and it is of finite type over each them.

Theorem 4.4 (Ahlfors Finiteness Theorem)**.** *A finitely generated Kleinian group is analytically finite.*

Remark: the definition of analytically finite requires to consider the whole regular set, otherwise the above result might not hold. For example, if G is the group of translations generated by $z \mapsto z + 1$, then G acts properly discontinuously on the upper half plane $\mathbb{H}$ with quotient the puncture disc $\mathbb{D}\backslash\{0\}$, which is not of finite type. However, the regular set of G is the whole complex plane and $\mathbb{C}/G$ is the punctured plane $\mathbb{C}\backslash\{0\} = \widehat{\mathbb{C}}\backslash\{0, \infty\}$, which is a surface with signature $(0, 2; \infty, \infty)$.

If Λ has more than 2 points then the group G is called **non-elementary**. In that case we have that each component of Ω has a hyperbolic metric (since the universal covering of each component is the upper half plane, by the Uniformization theorem done earlier).

Theorem 4.5 (Bers' First Area Theorem). *If G is a finitely generated non-elementrary Kleinian group with n generators then* $area(\Omega/G) \leq 4\pi(n-1)$.

Theorem 4.6 (Bers' Second Area Theorem). *If G is a finitely generated non-elementrary Kleinian group and Δ_0 is an invariant component (i.e., $g(\Delta_0) = \Delta_0$ for all g in G), then* $area(\Omega/G) \leq 2\, area(\Delta/G)$.

Conjucture 1 (Ahlfors zero measure conjecture). *If G is a finitely generated discrete subgroup of $PSL(2, \mathbb{C})$ then either $\Lambda = \widehat{\mathbb{C}}$ or Λ has zero measure in S^2.*

This conjecture has been recently settled in the affirmative (2004) by Agol and Calegari-Gabai (independently) as a consequence of Marden's Tameness Conjecture: every hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold.

4.3. **Uniformization and Kleinian groups.** We have already seen that most Riemann surfaces can be expressed as the quotient of $\mathbb{H}$ by a Fuchsian group, as a consequence of the Uniformization Theorem. In particular if S is a compact surface of genus $g \geq 2$ then its universal covering is $\mathbb{H}$.

Let C_1, C_1', …, C_n, C_n' be $2n$ disjoint circles ($n > geq2$) in the Riemann sphere which bound a domain D. Let $g_1, \ldots, g_n$ be Möbius transformations such that $g_j(C_j) = C_j'$ and g_j sends the exterior of C_j to the interior of C_j'. The group generated by these transformation is called a **Schottky group** on $g_1, \ldots, g_n$. Such group is also called a classical Schottky group; if one takes Jordan curves instead of circles then one gets a (general) Schottky group.

Proposition 4.7. *Let G be a Schottky group on the generators $g_1, \ldots, g_n$. Then:*

1. *G is a function group (a Kleinian group with an invarian component Δ such that Δ/G is of finite type) with no non-invariant components.*
2. *G is free on $g_1, \ldots, g_n$.*
3. *G is purely loxodromic.*
4. *D is a fundamental doamin for G.*

Once we have that D is a fundamental domain for G we have Ω/G is a compact surface of genus g.

Theorem 4.8 (Schottky-Klein-Koebe Retrosection Theorem)**.** *Any compact surface of genus $g \geq 2$ can be represented by a Schottky group.*

Theorem 4.9. *Let G be a free, locally analytically finite (i.e., G is of finite type over every component of Ω), purely loxodromic Kleinian group. Then G is a Schottky group; in particular G is finitely generated.*

A **b-group** is a function group where the invariant component is simply connected. A **quasi-fuchsian** group is a b-group with exactly two components, both invariant.

Theorem 4.10 (Simultaneous Uniformization)**.** *Given two Riemann surfaces X and Y of genus $g \geq 2$ there exists a quasi-fuchsian group Γ, with invariant components Δ_1 and Δ_2 such that $\Delta_1/\Gamma \cong X$ and $\Delta_2/\Gamma \cong Y$.*

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