Analytic Aspects of Teichmüller Spaces

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The Riemann Sphere

The complex plane $\mathbb{C}$ can be extended to $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by the one-point compactification. A "basic" neighbourhood U of ∞ consists of the complement of a compact subset K of $\mathbb{C}$:

$$U = \{\infty\} \cup (\mathbb{C} \backslash K) .$$

$\widehat{\mathbb{C}}$ is homeomorphic to the unit sphere in $\mathbb{R}^3$; it is called **the Riemann sphere**.

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Two important subsets of $\widehat{\mathbb{C}}$ are the **upper half-plane** $\mathbb{H} = \{\mathrm{Im}(z) > 0\}$ and the **unit disc** $\mathbb{D} = \{|z| < 1\}$.

Möbius transformations

The automorphisms (bijective, holomorphic, self-maps) of $\widehat{\mathbb{C}}$ are the **Möbius transformations**, $T(z) = \frac{az+b}{cz+d}$, with $ad - bc \neq 0$; we will denote this group by Möb($\mathbb{C}$). One can assume that $ad - bc = 1$ and identify Möb($\mathbb{C}$) with PSL$(2, \mathbb{C})$, the group of equivalences classes of matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

if there is a complex non-zero number t such that $a' = ta, \ldots, d' = td$.

A simple computation shows that

$$Aut(\mathbb{C}) = \{z \mapsto az + b;\ a \neq 0\}.$$

By Schwarz's lemma

$$Aut(\mathbb{D}) = \{z \mapsto \lambda \frac{z - \omega}{1 - \overline{\omega} z};\ |\lambda| = 1,\ |\omega| < 1\}.$$

Using the biholomorphic mapping (Cayley transform) $T(z) = \frac{z-i}{z+i} : \mathbb{H} \rightarrow \mathbb{D}$ we have that $Aut(\mathbb{H})$ is the group of Möbius transformations with real coefficients and positive determinant $(ad - bc > 0)$. $Aut(\mathbb{H})$ can be identified with $\mathrm{PSL}(2, \mathbb{R})$.

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A non-identity Möbius transformation has at least one fixed point in $\widehat{\mathbb{C}}$ and at most two.

Möbius transformations can be classified by their conjugacy classes:

- identity;
- **parabolic**, $z \mapsto z + 1$, $tr^2 = 4$;
- **elliptic**, $z \mapsto e^{i\theta}$, $0 < \theta < 2\pi$, $0 \leq tr^2 < 4$;
- **loxodromic**, $z \mapsto \lambda z$, $|\lambda| \neq 0, 1$. Among the loxodromic one has **hyperbolic** transformations, where λ is real and positive; in this case $tr^2 > 4$.

The above trace statements require the normalization $ad - bc = 1$.

Examples of actions

- The parabolic transformation $T(z) = z + 1$ has the property that all elements converge to ∞ under iteration. The quotient $\mathbb{C}/ <T>$ is a cylinder.
- For the elliptic element $R(z) = e^{2p\pi i/q}z$ all elements have finite orbits and the quotient is a manifold.
- For a hyperbolic transformation $D(z) = kz$, $k > 1$, we have two accumulation points, the origin and the point of infinity. The quotient of $\mathbb{C}\setminus\{0\}$ by the action of D is a torus.

Discrete groups

The group Möb($\mathbb{C}$) has a natural topology coming from the space of matrices: we say that a sequence $\{A_n\}$ converges to A if there exist representatives $A_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $a_n \to a, \dots, d_n \to d$.

A subgroup G of Möb($\mathbb{C}$) is **discrete** if there is no sequence of distinct elements $\{g_n\}$ of G such that $g_n \to g$, where $g \in$ Möb($\mathbb{C}$). A trivial example of a discrete subgroup is the cyclic group generated by a parabolic transformation. A discrete subgroup is (sometimes) called a **Kleinian group** (**of the first kind**).

The action of a Möbius transformation on the Riemann sphere can be naturally extended to an action on hyperbolic 3-space.

Proposition

If G is a discrete subgroup of Möb($\mathbb{C}$) then $\mathbb{H}^3/G$ is a 3-manifold.

Discontinuous actions

A subgroup G of Möb($\mathbb{C}$) **acts freely discontinuously** at $z \in \widehat{\mathbb{C}}$ if there is a neighbourhood U of z such that $g(U) \cap U = \emptyset$ for all non-identity elements g of G. We say that G **acts discontinuously** if $g(U) \cap U \neq \emptyset$ for only finitely many $g \in G$. The set of points where G acts freely discontinuously (resp. discontinuously) is denoted by Ω^* (resp. Ω). The set Ω is called the **regular set** of G.

We say G is a **Kleinian group** (**of the second kind**) if $\Omega^* \neq \emptyset$. (We will use the term Kleinian in this sense.)

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If G is Kleinian and $z \in \widehat{\mathbb{C}}$ we say that z is a **limit point** if there is a point $w \in \Omega^*$ and a sequence of distinct elements $\{g_n\} \subset G$ such that $g_n(w) \to z$. The set of limit points, known as **limit set**, will be denoted by Λ.

Properties of the regular and limit set

We assume that G is Kleinian in the sense that $\Omega^* \neq \emptyset$.

- Λ is a closed, G-invariant, nowhere dense subset of $\widehat{\mathbb{C}}$.
- If Λ contains more than two points then it is a perfect set. If Λ has at most two points then G is called **elementary**.
- If G is non-elementary then Λ is the closure of the fixed points of the loxodromic elements of G.
- If G is non-elementary and E is a non-empty, G-invariant closed set of $\widehat{\mathbb{C}}$ then $\Lambda \subset E$.
- $\widehat{\mathbb{C}}$ is the disjoint union of Ω and Λ.
- $\Omega \backslash \Omega^*$ is a discrete subset of Ω.
- Ω^* is dense in Ω.

Proposition

If G is Kleinian then Ω^*/G is a Hausdorff space. Moreover, Ω/G is a union of Riemann surfaces.

Ahlfors' Theorem

Let Δ be a component of (the regular set of) G; the **stabilizer** of Δ in G is the subgroup

$$Stab_G(\Delta) = \{g \in G;\ g(\Delta) = \Delta\}.$$

One says that G is of **finite type** over Δ if $\Delta / Stab_G(\Delta)$ is obtained from a compact surface by removing finitely many points and the covering $\Delta \to \Delta / Stab_G(\Delta)$ is ramified over finitely many points.

Theorem (Ahlfors' Finiteness Theorem)

If G is a finitely generated Kleinian group then Ω has only finitely many non-conjugate components and G is of finite type over all of them.

Fuchsian groups

A Kleinian group is called **Fuchsian** if it keeps invariant some disc.

A Möbius transformation A preserves a disc if and only if $tr^2(A) \geq 0$.

Let G be a non-elementary Kleinian group G such that $tr^2(g) \geq 0$ for all $g \in G$. Then G is Fuchsian.

Theorem

Let G be a subgroup of $Aut(\mathbb{D})$. Then the following are equivalent:

- *G acts discontinuously on $\mathbb{D}$.*
- *G acts discontinuously at some point of $\mathbb{D}$.*
- *G is discrete.*

Uniformization

The first basic result on "uniformization" is the following well-known theorem.

Theorem (Riemannn Mapping Theorem)

Let U be a simply connected subset of the complex plane, $U \neq \mathbb{C}$. Then there exists a biholomorphic surjective function, $\phi : \mathbb{D} \to U$.

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A more complete result is the famous Uniformization Theorem.

Theorem (Poincaré-Koebe-Klein Uniformization Theorem)

If S is a simply connected Riemann surface then S is biholomorphic to one and only one of the following surfaces: the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$.

Examples of Uniformization

(Since $\mathbb{H}$ and $\mathbb{D}$ are biholomorphic we will use the former as expressions are easier.)

Let $T(z) = z + 1$. Then $<T>$ acts discontinuously on $\mathbb{H}$ and the quotient is a punctured disc, $\mathbb{D}^* = \{0 < |z| < 1\}$; the covering map is given by the exponential function: $z \mapsto exp(2\pi iz)$.

If $D_k(z) = k^2 z$, $k > 1$, then the quotient of $\mathbb{H}$ by the action of $<D_k>$ is an annulus $\mathbb{A}_r = \{r < |z| < 1\}$, for some $0 < r < 1$. The covering is given again by an exponential function: $z \mapsto exp(\pi i \log(z)/\log(k))$.

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Since a parabolic and a hyperbolic transformation are not conjugate we obtain the following result.

Proposition

$\mathbb{D}^*$ *and* $\mathbb{A}_r$ *are not biholomorphic (for any radius* $0 < r < 1$*).*

Uniformization of tori

By Abel's theorem a torus (Riemann surface of genus 1) is of the form $S_\tau = \mathbb{C}/\mathcal{L}_\tau$, where $\mathcal{L}_\tau$ is the lattice $\{z \mapsto z + m + n\tau;\ m, n \in \mathbb{Z}\}$. Here $\tau \in \mathbb{H}$.

Theorem

Two tori S_τ and S_σ are biholomorphic if and only if there is an $M \in SL(2, \mathbb{Z})$ such that $M(\tau) = \sigma$.

Let S_i denote the fixed torus determined by i. A **marked torus** is a triple $(S_i, f\tau, S_\tau)$, where $f_\tau : S_i \to S_\tau$ is a homeomorphism. Two marked tori $(S_i, f\tau, S_\tau)$ and $(S_i, f\sigma, S_\sigma)$ are considered equivalent if there is a biholomorphic mapping $\varphi : S_\tau \to S_\sigma$ such that $f_\sigma^{-1} \circ \varphi \circ f_\tau$ is homotopic to the identity (on S_i). It is not too difficult to see that two marked tori are equivalent if and only if $\tau = \sigma$.Thus

$$\mathbb{H} = \{\text{marked tori}\}, \quad \mathbb{H}/\mathrm{SL}(2, \mathbb{Z}) = \{\text{biholomorphic tori}\};$$

moreover, $\mathrm{SL}(2, \mathbb{Z})$ acts properly discontinuously on $\mathbb{H}$.

Complex Dilatation

Let $w : D \to D'$ be an orientation-preserving differential mapping between two domains of $\mathbb{C}$. Define two differential operators:

$$w_z = \frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) \quad \text{and} \quad w_{\overline{z}} = \frac{\partial w}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right).$$

The **complex dilatation** of w at z_0 (assume w has non-zero Jacobian at z_0) is given by $\mu_w(z_0) = w_{\overline{z}}(z_0)/w_z(z_0)$. We have $|\mu_w(z_0)| < 1$.

For a point $z \in D$ set $L_w(z,r) = max_\zeta\{|w(\zeta) - w(z)|;\ |\zeta - z| = r\}$ and $l_w(z,r) = min_\zeta\{|w(\zeta) - w(z)|;\ |\zeta - z| = r\}$.

If w is a C^1-function then

$$\lim_{r\to 0} \frac{L_w(z,r)}{l_w(z,r)} = \frac{1+|\mu_w(z)|}{1-|\mu_w(z)|}.$$

Quasiconformal Mappings

Definition

An orientation-preserving homeomorphism $w : D \to D'$ is **quasiconformal (qc)** *if the circular dilatation*

$$H(z) = \limsup_{r \to 0} \frac{L_w(z,r)}{l_w(z,r)}$$

is bounded on D. If $H(z) \le K$ a.e. on D we say that w is K-qc.

An alternative, more analytic, definition of qc is the following.

Definition

An orientation-preserving homeomorphism $w : D \to D'$ is **qc** *if w has distributional derivatives in L^p_{loc}, for some $p \ge 1$ and $|w_{\bar{z}}(t)| \le k\,|w_z(t)|$, a.e. on D, for some k satisfying $0 \le k < 1$. The mapping is called K-qc for any K with $K \ge (1+k)/(1-k)$.*

Properties of qc mappings

- The two definitions above are equivalent.
- The requirement of distributional derivatives in L^p_{loc} can be replaced by w being ACL (absolutely continuous on lines).
- A 1-qc mapping is holomorphic.
- The inverse of a K-qc mapping is K-qc. The composition of a K_1-qc with a K_2-qc is $K_1 K_2$-qc.
- If w is K-qc and ϕ and φ are holomorphic then $\phi \circ w \circ \varphi^{-1}$ is K-qc.

Ahlfors-Bers Theorem

Theorem (Existence of qc mappings)

Given any μ *with* $||\mu||_\infty < 1$ *on* $\mathbb{C}$, *there exists a unique qc-homeomorphism of* $\widehat{\mathbb{C}}$ *with dilatation* μ *and fixing* ∞, 0 *and* 1.

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Dependency on parameters is given by the next result, also known as the measurable Riemann Mapping Theorem.

Theorem (Ahlfors-Bers)

For every fixed $z \in \mathbb{C}$, the mapping $\mu \to w^\mu(z)$ is holomorphic; that is, if μ depends holomorphically on $(t_1, \ldots, t_k) \in \mathbb{C}^k$, then $w^\mu(z)$ is a holomorphic function of the variables $(t_1, \ldots, t_k)$.

An orientation preserving homeomorphism between two Riemann surfaces is K-qc if and only if it is K-qc when expressed in local coordinates.

Let $f : X \to X$ be a qc-mapping on a surface. The complex dilation μ behaves like a $(-1, 1)$ tensor; that is, when written on local coordinates z and w we have

$$\mu(z) \left(\frac{dz}{dw}\right)^{-1} \overline{\left(\frac{dz}{dw}\right)} = \mu(w).$$

We denote by $L^{\infty}_{(-1,1)}(X)_1$ the space of $(-1, 1)$ tensors with norm less than 1 (also called Beltrami coefficients).

If $\mu \in L^{\infty}_{(-1,1)}(X)_1$ then on each coordinate (U, z) we can find a qc-mapping w_μ with dilatation $\mu(z)$. The pair $(U, w_\mu(z))$ is a local coordinate on X. The atlas constructed gives a Riemann surface structure to X that we will denote by X_μ.

Proposition

The identity map $Id_X : X \to X_\mu$ is a qc homeomorphism of X with complex dilatation given by the $(-1, 1)$ tensor μ.

Teichmüller spaces

Let X be a Riemann surface. A **marked Riemann surface** is a triple (X, f, X_1), where X_1 is a Riemann surface and $f : X \to X_1$ is a qc homeomorphism. Let Marked(X) denote the space of marked Riemann surface (modelled on X).

- Two marked surfaces $(X, f, X_1) \sim (X, g, X_2)$ are **Teichmüller equivalent** if there exists a biholomorphic mapping $\varphi : X_1 \to X_2$ such that $g^{-1} \circ \varphi \circ f$ is homotopic to the identity mapping of X.
- The **Teichmüler space** of X is the quotient $T(X) = \text{Marked}(X)/\sim$.

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- The **Teichmüler space** of X is the quotient $T(X) = \text{Marked}(X)/\sim$.
- $(X, f, X_1) \sim_R (X, g, X_2)$ are **Riemann equivalent** if there exists a biholomorphic mapping $\varphi : X_1 \to X_2$.
- The **Riemann or moduli space** of X is the quotient $R(X) = \text{Marked}(X)/\sim_R$.

The mapping class group

Let $Q(X)$ denote the set of qc homeomorphisms of X onto itself, and $Q_0(X)$ those homotopic to the identity. The quotient group is called the **mapping class group** or **modular group**, $Mod(X) = Q(X)/Q_0(X)$.

$Mod(X)$ acts naturally on $T(X)$ by $\varphi^*([X, f, X_1]) = [X, f \circ \varphi^{-1}, X_1]$.

Theorem

$T(X)/Mod(X)$ can be naturally identified with $R(X)$.

$T(g, n)$ denotes the space of compact surfaces of genus g with n punctures.

Proposition

Assume $2g - 2 + n > 0$. The modular group acts effectively on $T(g, n)$ except for the following cases: $(0, 3)$, $(0, 4)$, $(1, 1)$, $(1, 2)$ and $(2, 0)$.

The Teichmüller space of Fuchsian group

Let G we a Kleinian group and w a qc homeomorphism of $\widehat{\mathbb{C}}$ with dilatation μ. The group wGw^{-1} is again a Kleinian group if and only if $(\mu \circ g)\overline{g'}/g = \mu$ a.e. on $\widehat{\mathbb{C}}$, for all $g \in G$. We say μ is compatible with G.

For a Fuchsian group G let $B(L^{\infty}(\mathbb{H}, G))$ denote the dilatations defined on $\mathbb{H}$, compatible with G and of L^{∞}-norm less than 1. We will assume that G is finitely generated, torsion free and the limit set of G is $\mathbb{R} \cup \{\infty\}$.

If μ is compatible with G on $\mathbb{H}$ we can make it compatible on the whole sphere $\widehat{\mathbb{C}}$ in two ways:

- by reflection, setting for z in the lower half-plane, $\mu(z) = \overline{\mu(\overline{z})}$. The corresponding qc mapping is denoted by w_μ and $G_\mu = w_\mu G w_\mu^{-1}$ is a Fuchsian group preserving $\mathbb{H}$.
- Extending μ by 0 on the lower half-plane. The corresponding qc mapping is denoted by w^μ. We have w^μ is holomorphic on the lower half plane. The group $G^\mu = w^\mu G (w^\mu)^{-1}$ is called a **quasifuchsian** group.

Proposition

Let μ and ν be elements of $B(L^\infty(\mathbb{H}, G))$ so that $G_\mu = G_\nu$. Let $X = \mathbb{H}/G$ and $Y = \mathbb{H}/G_\mu$, and $f_\mu, f_\nu : X \to Y$ the induced mappings. Then the following are equivalent:

- *f_μ is homotopic to f_ν;*
- *$w_\mu|_\mathbb{R} = w_\nu|_\mathbb{R}$;*
- *$w_\mu g(w_\mu)^{-1} = w_\nu g(w_\nu)^{-1}$ for all $g \in G$.*

We define the **Teichmüller** $T(G)$ and **Riemann** $R(G)$ spaces of G as the quotients of $B(L^\infty(\mathbb{H}, G))$ by the following relations:

- $\mu \sim \nu$ if $w_\mu|_\mathbb{R} = w_\nu|_\mathbb{R}$;
- $\mu \sim_R \nu$ if G_μ and G_ν are conjugate subgroups of $\mathrm{PSL}(2, \mathbb{R})$.

Proposition

If G is a Fuchsian group and $X = \mathbb{H}/G$ then the spaces $T(G)$ and $T(X)$ are naturally isomorphic.

The Bers Embedding

Proposition

For a Fuchsian group G and μ and ν in $B(L^\infty(\mathbb{H}, G))$ the following are equivalent:

- *μ is Teichmüller equivalent to ν;*
- *$w^\mu|_{\mathbb{R}} = w^\nu|_{\mathbb{R}}$;*
- *$w^\mu|_{\mathbb{L}} = w^\nu|_{\mathbb{L}}$, where $\mathbb{L}$ is the lower half-plane, $\mathbb{L} = \{Im(z) < 0\}$.*

For G Fuchsian let $B_2(\mathbb{L}, G)$ be the space of functions σ defined on the lower half plane and satisfying:

- σ is holomorphic;
- $(\sigma \circ g)(g')^2 = \sigma$ on $\mathbb{L}$, for all $g \in G$;
- $||\sigma||_{B_2} = ||\lambda_{\mathbb{L}}^{-2}(z)\,\sigma(z)||_\infty$ is bounded, where $\lambda_{\mathbb{L}}(x+iy) = 1/(2|y|)$ is the Poincaré metric of the lower half-plane.

If w is a locally one-to-one holomorphic function then we define the **Schwarzian derivative** of w by

$$S(w) = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2.$$

Theorem (Nehari)

If w is a one-to-one holomorphic function on the lower half plane then $||Sw||_{B_2} \leq 6$.

Proposition

$S(w^\mu|_{\mathbb{L}}) = S(w^\nu|_{\mathbb{L}})$ *if and only if μ and ν are Teichmüller equivalent.*

The above results allow to define the following mapping, called the **Bers embedding**:

$$\beta : T(G) \to B_2(\mathbb{L}, G)$$
$$[\mu] \mapsto S(w^\mu|_{\mathbb{L}}).$$

Properties of the Bers embedding

The Bers embedding gives a complex structure to $T(G)$. Let $T_\beta(G)$ denote the image of the Bers embedding. We have natural inclusions $T_\beta(G) \subset T_\beta(\{1\})$ and $B_2(\mathbb{L}, G) \subset B_2(\mathbb{L})$.

Proposition

$T_\beta(G)$is the component containing zero in the intersection of $T_\beta(\{1\})$ with $B_2(\mathbb{L}, G)$.

Let $\mathfrak{S}$ be the (closed and bounded) subset of $B_2(\mathbb{L})$ consisting of the Schwarzian derivatives of all one-to-one holomorphic functions defined on $\mathbb{L}$.

Theorem (Gehring)

The closure of $T_\beta(\{1\})$ in $B_2(\mathbb{L})$ is a proper subset of $\mathfrak{S}$. The interior of $\mathfrak{S}$ is precisely $T_\beta(\{1\})$.

The boundary of Teichmüller space

Any point $\varphi \in B_2(\mathbb{L}, G)$ induces a homomorphism $\sigma^\varphi : G \to \text{Möb}(\mathbb{C})$ as follows: let f be a meromorphic function on $\mathbb{L}$ satisfying $Sf = \varphi$. Then for any $g \in G$ there is a unique Möbius transformation h such that $f \circ g = h \circ f$; we set $\sigma^\varphi(g) = h$.

If $\varphi = S(w^\mu|_\mathbb{L}) \in T_\beta(C)$ then $\sigma^\varphi(g) = w^\mu \circ g \circ (w^\mu)^{-1}$; this homomorphism is type preserving.

We say that a point $\varphi \in \partial T_\beta(G)$ is a **cusp** if there is a hyperbolic transformation $g \in G$ such that $\sigma^\varphi(g)$ is parabolic.

Theorem

The subset of cusps in the boundary of Teichmuüller space is of first category.

Theorem (McMullen)

Cusps are dense in $\partial T_\beta(G)$.

Teichmüller distance

If w is a qc mapping with dilatation μ we write $K(w) = \frac{1+||\mu||_\infty}{1-||\mu||_\infty}$.
The following expression defines a distance on $T(X)$:

$$d_T([X, f, X_1], [X, g, X_2]) = \inf_{\varphi} \frac{1}{2} \log K(\varphi),$$

where φ varies over all qc mappings from X_1 to X_2 homotopic to $g^{-1} \circ f$.
This function is called the **Teichmüller distance**.

Proposition

d_T is a complete metric.

Theorem

The modular group $Mod(X)$ acts properly discontinuously by isometries on $T(X)$.

Theorem (Royden)

The group of isometries of $T(g,0)$ with respect to the Teichmüller metric is the modular group if $g > 2$, and a $\mathbb{Z}_2$-quotient of $Mod(2,0)$ for the case of $g = 2$.

A homeomorphism f of X, induces a mapping f^* on $T(X)$; the set of fixed points of f^* are the Riemann surfaces, homeomorphic to X, for which f is holomorphic. Since every Riemann surface of genus 2 is hyperelliptic the hyperelliptic involution acts trivially on $T(2,0)$.

Theorem

Assume $2g - 2 + n > 0$; the group of biholomorphisms of $T(g,n)$ is

- $Mod(g,n)$ *if* $2g - 2 + n > 2$;
- $Mod(2,0)/\mathbb{Z}_2$, *for* $(2,0)$ *and* $(0,6)$;
- $Mod(0,5)$ *for* $(1,2)$;
- $\mathrm{PSL}(2,\mathbb{R})$ *in the cases* $(1,1)$ *and* $(0,4)$;
- *trivial in the case* $(0,3)$.