Lemma 3.13. *Let D be a Dirichlet region for the Fuchsian group Γ.*

1. $\mathbb{H}/\Gamma$ is compact if and only if the Euclidean closure of D is a compact subset of $\mathbb{H}$.

2. If $\overline{D}$ is compact then D has only finitely many sides.

Theorem 3.14. *Let D be a Dirichlet region for the Fuchsian group Γ and assume that Λ_Γ has more than two points. The the following are equivalent:*

(1) *D has finitely many sides;*
(2) *the quotient $(\widehat{\mathbb{C}}\backslash\Lambda_\Gamma)/\Gamma$ is a union of compact Riemann surface with finitely many (possibly none) points removed;*
(3) *Γ is finitely generated (by the side-pairing transformations);*
(4) *the hyperbolic area of (convex hull(Λ_Γ))/Γ is finite.*

4. Kleinian Groups

4.1. **Basic results on Kleinian groups.** Let G denotes a subgroup of $\mathrm{PSL}(2,\mathbb{C})$. We say that G acts **properly discontinuously** at $z \in \widehat{\mathbb{C}}$ if there is a neighbourhood U of z such that the number of elements $g \in G$ with $g(U) \cap U \neq \emptyset$ if finite. If only the identity satisfies $g(U) \cap U \neq \emptyset$ then we say that G acts freely discontinuously at z. The set of points where G acts properly discontinuously (resp. freely discontinuously) is called the **regular set** (resp. **free regular set**) and it is denoted by $\Omega(G)$ or just Ω (resp. ${}^o\Omega(G)$ or ${}^o\Omega$). We say that G is **Kleinian** (or some times, **Kleinian of the second kind**) if ${}^o\Omega \neq \emptyset$.

A point $x \in \widehat{\mathbb{C}}$ is called a **limit point** if there exists $z \in {}^o\Omega$ and a sequence $\{g_m\}$ of distinct elements of G such that $g_m(z) \to x$. The set of limit points, called the **limit set** is denoted by $\Lambda(G)$ or simply Λ.

Proposition 4.1. *Λ is a closed, G-invariant, nowhere dense subset of $\widehat{\mathbb{C}}$. If it contains more than 2 points then it is a perfect set (and therefore, uncountable).*

Proposition 4.2. *$\widehat{\mathbb{C}}$ is the disjoint union of Ω and Λ.*

4.2. **Kleinian groups and Riemann surfaces.**

Theorem 4.3. *If G is a Kleinian group then Ω/G is a union of Riemann surfaces.*

If $\Delta \subset \Omega$ is a component of the regular set of G we denote by G_Δ its stabilizer in G, that is, $G_\Delta = \{g \in G;\ g(\Delta) = \Delta\}$. Two components Δ and Δ' are conjugate if there is an element $g \in G$ such that $g(\Delta) = \Delta'$. In that case we have that the stabilizers of the two components are conjugate subgroups of G, and therefore the quotient of the components by the stabilizers are the same Riemann surface. Thus we see that if Δ_1, Δ_2, ..., is a list of non-conjugate components of Ω, then $\Omega/G \cong \Delta_1/G_{\Delta_1} \cup \Delta_2/G_{\Delta_2} \cup \cdots$ (in particular, Ω/G consists of at most countably many Riemann surfaces).

Let $z_0 \in \Omega$ and suppose that it has non-trivial stabilizer, i.e. the subgroup $G_{z_0} = \{g \in G;\ g(z_0) = z_0\}$ is not the identity. It is not difficult to see that G_{z_0} must be a cyclic group of finite order, say n. The projection $\pi : \Omega \to \Omega/G$ is then like $z \mapsto z^n$ near z_0; we say that z_0 is a **ramification point** and π is ramifed over $p_0 = \pi(z_0)$.

The Kleinian group G is said to be of **finite type** over a component Δ if Δ/G_Δ is a compact surface of genus g, with $n \geq 0$ points removed (punctures), and the projection $\pi : \Delta \to \Delta/G_\Delta$ is ramified over finitely many points, say $m \geq 0$, with orders $\nu_1, \ldots, \nu_m$. Set $s = n + m$. Then (g, p) is called the type of Δ/G_Δ and $(g, p; \nu_1, \ldots, \nu_m, \infty, \infty)$ the **signature** (where the symbolc ∞ appears n times). We say that G is of **finite type** or **analytically finite** if it has only finitely many non-conjugate components (of Ω) and it is of finite type over each them.

Theorem 4.4 (Ahlfors Finiteness Theorem)**.** *A finitely generated Kleinian group is analytically finite.*

Remark: the definition of analytically finite requires to consider the whole regular set, otherwise the above result might not hold. For example, if G is the group of translations generated by $z \mapsto z + 1$, then G acts properly discontinuously on the upper half plane $\mathbb{H}$ with quotient the puncture disc $\mathbb{D}\backslash\{0\}$, which is not of finite type. However, the regular set of G is the whole complex plane and $\mathbb{C}/G$ is the punctured plane $\mathbb{C}\backslash\{0\} = \widehat{\mathbb{C}}\backslash\{0, \infty\}$, which is a surface with signature $(0, 2; \infty, \infty)$.

If Λ has more than 2 points then the group G is called **non-elementary**. In that case we have that each component of Ω has a hyperbolic metric (since the universal covering of each component is the upper half plane, by the Uniformization theorem done earlier).

Theorem 4.5 (Bers' First Area Theorem)**.** *If G is a finitely generated non-elementrary Kleinian group with n generators then area$(\Omega/G) \leq 4\pi(n-1)$.*

Theorem 4.6 (Bers' Second Area Theorem)**.** *If G is a finitely generated non-elementrary Kleinian group and Δ_0 is an invariant component (i.e., $g(\Delta_0) = \Delta_0$ for all g in G), then area$(\Omega/G) \leq 2\,$area(Δ/G).*

Conjucture 1 (Ahlfors zero measure conjecture)**.** *If G is a finitely generated discrete subgroup of $PSL(2, \mathbb{C})$ then either $\Lambda = \widehat{\mathbb{C}}$ or Λ has zero measure in S^2.*

This conjecture has been recently settled in the affirmative (2004) by Agol and Calegari-Gabai (independently) as a consequence of Marden's Tameness Conjecture: every hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold.

4.3. **Uniformization and Kleinian groups.** We have already seen that most Riemann surfaces can be expressed as the quotient of $\mathbb{H}$ by a Fuchsian group, as a consequence of the Uniformization Theorem. In particular if S is a compact surface of genus $g \geq 2$ then its universal covering is $\mathbb{H}$.

Let C_1, C_1', …, C_n, C_n' be $2n$ disjoint circles ($n > geq2$) in the Riemann sphere which bound a domain D. Let $g_1, \ldots, g_n$ be Möbius transformations such that $g_j(C_j) = C_j'$ and g_j sends the exterior of C_j to the interior of C_j'. The group generated by these transformation is called a **Schottky group** on $g_1, \ldots, g_n$. Such group is also called a classical Schottky group; if one takes Jordan curves instead of circles then one gets a (general) Schottky group.

Proposition 4.7. *Let G be a Schottky group on the generators $g_1, \ldots, g_n$. Then:*

1. *G is a function group (a Kleinian group with an invarian component Δ such that Δ/G is of finite type) with no non-invariant components.*
2. *G is free on $g_1, \ldots, g_n$.*
3. *G is purely loxodromic.*
4. *D is a fundamental doamin for G.*

Once we have that D is a fundamental domain for G we have Ω/G is a compact surface of genus g.

Theorem 4.8 (Schottky-Klein-Koebe Retrosection Theorem)**.** *Any compact surface of genus $g \geq 2$ can be represented by a Schottky group.*

Theorem 4.9. *Let G be a free, locally analytically finite (i.e., G is of finite type over every component of Ω), purely loxodromic Kleinian group. Then G is a Schottky group; in particular G is finitely generated.*

A **b-group** is a function group where the invariant component is simply connected. A **quasi-fuchsian** group is a b-group with exactly two components, both invariant.

Theorem 4.10 (Simultaneous Uniformization)**.** *Given two Riemann surfaces X and Y of genus $g \geq 2$ there exists a quasi-fuchsian group Γ, with invariant components Δ_1 and Δ_2 such that $\Delta_1/\Gamma \cong X$ and $\Delta_2/\Gamma \cong Y$.*

School of Mathematics, Tata Institute, Bombay 400 005, India
E-mail address: pablo@math.tifr.res.in