a a parameter space of a surface S, of genus at least 2, called **Teichmüller space** of S, $T(S)$. Any element of the mapping class group $Mod(S)$ acts naturally on $T(S)$ in a continuous way. The space $T(S)$ is an open ball in $\mathbb{R}^{6g-6}$; the first step in Thurston's classification is to compactify $T(S)$ by means of some classes of foliations, so $\overline{T(S)}$ is homeomorphic to the closed ball in $\mathbb{R}^{6g-6}$. The action of $Mod(S)$ on $T(S)$ extends to a continuous action on the compactified space. Thus, by Brouwer's fixed point, any element ϕ of $Mod(S)$ has a fixed point in $\overline{T(S)}$. There are three possibilities: (1) ϕ fixes a point in the interior: $T(S)$; it is easy to see that ϕ acts as a holomorphic (invertible) mapping on some surface S' and, by Hurwitz's theorem, ϕ must have finite order. (2) If ϕ has only one fixed point in the boundary of $T(S)$ then that point corresponds to a collection of closed, pairwise disjoint curves, and ϕ leaves those curves invariants; in this case we talk of **reducible** mappings. (3) Finally, ϕ can have two fixed points on the boundary of $T(S)$, these points corresponds to foliations on S, and ϕ stretches in one direction by a factor of λ and shrinks in a "complementary" direction by a factor of λ; the mapping (or its class) is called **pseudo-Anosov**.

2. Hyperbolic Geometry

2.1. **Basic Hyperbolic Geometry.** Let $\mathbb{D}$ denote the unit disc on the complex plane, that is $\mathbb{D} = \{z \in \mathbb{C};\ |z|, 1\}$. For an open subset of $\mathbb{C}$ we denote by $Aut(G)$ the group of biholomorphic mappings of G (bijective holomorphisms from G to G).

An easy consequence of Schwarz's lemma is that the group $Aut(\mathbb{D})$ consists of the Möbius transformations of the form $g(z) = \frac{az+\bar{c}}{cz+\bar{a}}$; where a and c are complex number satisfying $|a|^2 - |c|^2 = 1$. An easy computation shows that, for such mapping g, one has

$$\frac{|g'(z)|}{1-|g(z)|^2} = \frac{1}{1-|z|^2},$$

for all z in $\mathbb{D}$. This means that the metric $\lambda(z)|dz| = \frac{2}{1-|z|^2}\,|dz|$ is invariant under $Aut(\mathbb{D})$. More precisely, if $\gamma : [a,b] \to \mathbb{D}$, is a smooth curve, we define the length of γ by $l(\gamma) = \int_\gamma \lambda(z)|dz| = \int_a^b \lambda(\gamma(t))\,|\gamma'(t)|\,dt$, then $l(\gamma) = l(g\circ\gamma)$. We define the **hyperbolic distance** on $\mathbb{D}$ by $d(z,w) = \inf l(\gamma)$, where the infimum is taken over all (smooth) curves joining z and w.

Theorem 2.1. *The hyperbolic distance d above defined is a distance; it is invariant under $Aut(\mathbb{D})$ (equivalently, the elements of $Aut(\mathbb{D})$ are isometries with respect to d) and given any two points in $\mathbb{D}$ there is a (smooth) curve joining them whose length is the distance between the points ($(\mathbb{D}, d)$ is a geodesic path space).*

The standard proof of this theorem is starts from the last statement as follows: by an automorphism of the disc we can assume that $z = 0$; composing with a rotation then we can further assume w is a point in the open interval $(0, 1)$, say t_0. Let $\gamma : [0, 1] \to \mathbb{D}$ be a (smooth) curve with $\gamma(0) = 0$ and $\gamma(1) = t_0$; write $\gamma(t) = x(t) + iy(t)$. Then

$$l(\gamma) = \int_0^1 \frac{2\sqrt{x'(t)^2 + y'(t)^2}}{1 - (x(t)^2 + y(t)^2)}\, dt \geq \int_0^1 \frac{2\sqrt{x'(t)^2}}{1 - x(t)^2}\, dt \geq \int_0^1 \frac{2x'(t)}{1 - x(t)^2}\, dt = \log \frac{1 + t_0}{1 - t_0}.$$

Equality occurs only if $x'(t) > 0$ and $y(t) = 0$ for all t, that is, γ is the line segment from 0 to t_0 without *backtracking*.

Thus we have $d(0, z) = \log \frac{1+|z|}{1-|z|}$; applying a Möbius transformation on $Aut(\mathbb{D})$ we get

$$d(z, w) = \log \frac{|1 - z\bar{w}| + |z - w|}{|1 - z\bar{w}| - |z - w|}.$$

From the above construction and the fact that Möbius transformations take lines and circles to lines and circles we have that the geodesics in the hyperbolic metric are the lines and circles orthogonal to S^1. Moreover we see that geodesics are global in the sense that they realized the distance between any two points on their images.

From the above formula of the hyperbolic distance we get that the circle of center the origin and (hyperbolic) radius r, C_r, is just the Euclidean circle of center the origin and radius R where $r = \log \frac{1+R}{1-R}$. Thus hyperbolic circles (and discs) are Euclidean circles (and discs), with different radius and, generally, different center. Hence we have that the topology induced by the hyperbolic metric on the unit disc coincides with the topology induced by the Euclidean metric.

2.2. **Length, Area and the Gauss-Bonnet Theorem.** It is easy to compute by direct integration that $l(C_r) = 2\pi \sinh r$. Here $\sinh r = (e^r - e^{-r})/2$; thus $l(C_r) \sim e^r$ as $r \to \infty$ (in the hyperbolic metric, or in Euclidean terms $r \to 1$), as opposed to the Euclidean metric where the length of a circle grows linearly with the radius.

The (hyperbolic) area of a subset $E \in \mathbb{D}$ is given by $Area(E) = \int_E \lambda^2(z)\, dx\, dy$; in particular, for a disc of radius r, say D_r, we have $Area(D_r) = 4\pi \sinh^2(r/2) = 2\pi(\cosh r -$

1). So $l(C_r) \sim Area(D_r)$ as $r \to \infty$, another difference with respect to the Euclidean metric.

A triangle is the region enclosed by three geodesics. We allow the vertices of a triangle to be in the unit circle. The angles in the hyperbolic plane are measured as in the Euclidean case (that is, the angles between the tangents for the curves); if two geodesics meet on a point on S^1 we declare the angle to be zero.

The Gauss-Bonnet Theorem for triangle states that a triangle T with (interior) angles α, β and γ has area equal to $\pi - (\alpha + \beta + \gamma)$. The proof for the case of a triangle with two zero angles is just by a simple integration; the rest of the cases (one angle equal to zero and no zero angles) can be deduced from this case by expressing a triangle as the differences of two triangles in the previous cases. See the figures at the end of the section (where the triangles are in the upper half plane, rather than in the unit disc). For a polygon with n sides and angles $\alpha_1, \ldots, \alpha_n$ we get that the area is equal to $(n-2)\pi - (\alpha_1 + \cdots + \alpha_n)$.

2.3. **The Upper Half Plane Model.** Let $\mathbb{H}$ denote the upper half plane in the complex plane, $\mathbb{H} = \{z \in \mathbb{C};\ Im(z) > 0\}$; the Cayley transform $T(z) = \frac{z-i}{z+i} : \mathbb{H} \to \mathbb{D}$ is a biholomorphic mapping between the upper half plane and the unit disc, so we can put a hyperbolic metric on $\mathbb{H}$ using T. The expression for this metric is $\frac{1}{Im(z)}\,|dz|$. The properties of the metric on $\mathbb{D}$ are satisfied for this metric on $\mathbb{H}$. For example, the figures below show the proof (up to an integration) of the Gauss-Bonnet theorem on $\mathbb{H}$.

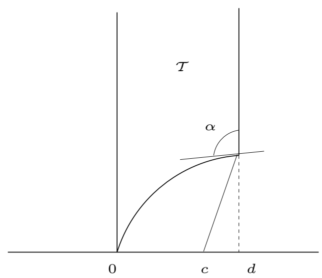

FIGURE 1. Triangle with two zero angles.

2.4. **Gromov's hyperbolicity.** A **geodesic** on a metric space X is a curve $g : [a,b] \to X$ such that $d(g(t), g(s)) = |t - s|$, for all t, s in $[a,b]$. A geodesic triangle is the region of X enclosed by three geodesics. A (geodesic) triangle is called δ-thin (for some $\delta > 0$) if any side is contained in the δ-neighborhood of the other two sides. A metric space is called

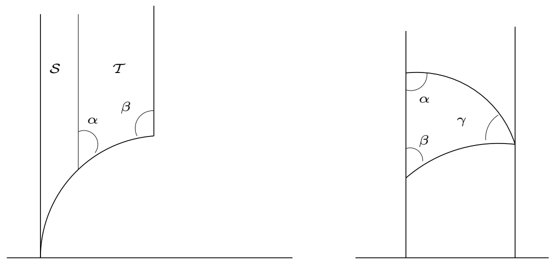

FIGURE 2. Gauss-Bonet Theorem.

δ-hyperbolic if all geodesic triangles are δ-thin for some fixed δ. In that case X is alse said to be **Gromov-hyperbolic**.

Hyperbolic space (either the unit disc model or the upper half plane one) is δ-hyperbolic. One can easily see that as follows: if a point p in a side of a triangle T is not in the *delta*-neighborhood of the other two sides then one can put a half disc of radius δ inside T; see figure below. From the area formulæof the disc and the triangle we get

$$\pi(\cosh\delta - 1) \leq \pi - (\alpha+\beta+\gamma) \leq \pi \Rightarrow \cosh\delta - 1 \leq 1 \Rightarrow \delta \leq \log(2+\sqrt{3}).$$

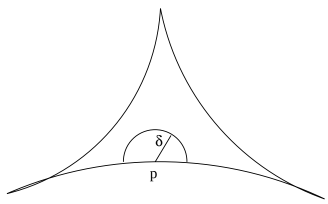

FIGURE 3. Gromov's hperbolicity.

2.5. **Isometries of Hyperbolic Space.** Every isometry of $\mathbb{D}$ (with the hyperbolic metric) is of the form $z \mapsto g(z)$ or $z \mapsto g(\bar{z})$, where $g \in Aut(\mathbb{D})$. To see this, let h be an isometry; composing with an element of $Aut(\mathbb{D})$ we can assume that $h(0) = 0$; by composing with a rotation we can further assume that $h(1/2) = 1/2$. But then h must fix every point in $(-1, 1)$ (by a simple distance calculation) and from that it is easy to see that for every $z \in \mathbb{D}$ we have $h(z)$ is either z or $\bar{z}$. A simple continuity argument shows that $h(z) = z$ or $h(z) = \bar{z}$ for **all** z in the unit disc.

Thus we have that the orientation-preserving isometris of $\mathbb{D}$ or $\mathbb{H}$ are Möbius transformations; moreover, this subgroup coincides with the group of biholomorphic mappings of

$\mathbb{D}$ or $\mathbb{H}$, making thus the hyperbolic metric natural from the Complex Analysis point of view.

As we mentioned ealier, an element of $Aut(\mathbb{D})$ is of the form $g(z) = \frac{az+\bar{c}}{cz+\bar{a}}$; the elements of $Aut(\mathbb{H})$ are of the form $g(z) = \frac{az+b}{cz+d}$, where a, b, c and d are real numbers satisfying $ad - bc = 1$. In any case we can assign to an element of $Aut(\mathbb{D})$ or $Aut(\mathbb{H})$ an element of $\mathrm{PSL}(2, \mathbb{C})$ (or $\mathrm{PSL}(2, \mathbb{R})$ in the case of the upper half plane), say $\left[\begin{smallmatrix} a & \bar{c} \\ c & \bar{a} \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ respectively.

From Complex Analysis we have the following classification of $Aut(\mathbb{H})$, up to conjugacy, that is, any element of $Aut(\mathbb{H})$ is conjugate to one of the following:

1. Identity.
2. Elliptic transformations: conjugate to rotations, with corresponding matrices of the form $\left[\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right]$, for $0 < \theta < \pi$;
3. Parabolic transformations. These are conjugate to the translations $z \mapsto z + 1$ or $z \mapsto z - 1$;
4. Hyperbolic transformations, conjugate to $z \mapsto k^2 z$, for k real and $k > 1$.

In terms of their fixed points, elliptic elements have one fixed point in the hyperbolic plane, parabolics have one fixed point on the boundary (S^1 or $\mathbb{R} \cup \{\infty\}$) and hyperbolic transformations have two fixed points on the boundary.

Given an isometry g of the hyperbolic plane we define the **translation length**, $\tau(g)$ by

$$\tau(g) = \inf\{d(z, g(z));\ z \in \mathbb{D}\,(\mathbb{H})\}.$$

The identity has $\tau(g) = 0$ but it is not of interest. Elliptic elements have $\tau(g) = 0$ and that value is realized at the fixed point of the transformation. Parabolics have $\tau(g) = 0$ (consider the translation $z \mapsto z + 1$ and make $Im(z) \to +\infty$) but this value is not realized for any point of the hyperbolic plane. Hyperbolic isometris satisfy $\tau(g) > 0$, with the value realized on a geodesic, called the axis of the transformation A_g. Moreover, we have

$$\sinh(d(z, g(z))/2) = \sinh(\tau(g)/2)\,\cosh(d(z, A_g));$$

so the further z is from A_g the more it is translated by g.