3. Fuchsian Groups

3.1. **Riemann surfaces and Fuchsian groups.** A **Riemann surface** S is a (connected) surface where the transition functions are holomorphic functions. It is easy to see that if $\pi : \tilde{S} \to S$ is the universal covering of S then it is possible to put on $\tilde{S}$ a (unique) Riemann surface structure so that π becomes a holomorphic function. Thus a natural question is to find what simply connected Riemann surface exists. The Riemann Mapping Theorem gives us a first result on the complex plane: any simply connected subset of $\mathbb{C}$ is either the complex plane of the unit disc, up to biholomorphisms.

Theorem 3.1 (Uniformization Theorem)**.** *Any simply connected Riemann surface is biholomorphic to one (and only one) of the following surfaces: the Riemann sphere* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$*, the complex plane* $\mathbb{C}$ *or the upper half plane* $\mathbb{H}$*.*

The unit disc $\mathbb{D}$ is conformally equivalent to the upper half plane; we can use for example the Cayley transform as done ealier in these notes.

We have then that any Riemann surface can be written as $\widetilde{S}/\Gamma$, where $\widetilde{S}$ is a subset of the Riemann sphere and Γ is the group of covering transformations. It is easy to see that the elements of Γ are biholomorphic mappings of $\widetilde{S}$. The groups of automorphisms (biholomorphic self-mappings) of the three simply connected surfaces $\widehat{\mathbb{C}}$, $\mathbb{C}$ and $\mathbb{H}$ are groups of Möbius transformations, so we conclude that any Riemann surface can be expressed as the quotient of an open subset of the Riemann sphere by a group of Möbius transformations.

The group of Möbius transformations can be identified with PSL$(2\mathbb{C})$, the quotient of SL$(2, \mathbb{C})$ by the complex numbers of absolute value 1. Thus we can identify a Möbius transformation with a matrix of complex coefficients and determinant 1, up to multiplication by a number in the unit circle. The classification of Möbius transformations, up to conjugacy, is well known; we have three different types of transformations:

1. the identity;
2. elliptic elements, conjugate to rotations $z \mapsto \lambda z$, with $|\lambda| = 1$; the trace of such elements satisfies $0 \leq tr^2 <$;
3. parabolic transformations, conjugate to $z \mapsto z + 1$ with $tr^2 = 4$;

4. loxodromic elements, conjugate to $z \mapsto \lambda^2 z$, where $|\lambda| > 1$. Among the loxodromic elements one has the hyperbolic transformations, where λ is real (and greater than 1) and satisfy $tr^2 > 4$.

Lemma 3.2. *A non-identity Möbius transformation has at least one fixed point and at most two in the Riemann sphere* $\widetilde{\mathbb{C}}$.

The group $Aut(\widehat{\mathbb{C}})$ is the group of Möbius transformations. Since covering transformations act fixed-points free we have that if $\widetilde{S} = \widehat{\mathbb{C}}$ then Γ must be trivial and so S should be the Riemann sphere too.

The group $Aut(\mathbb{C})$ consists of the Möbius transformations of the form $z \mapsto az + b$, where a and b are complex numbers and $a \neq 0$; such a transformation has a fixed point if and only $a \neq 1$, so if $S = \mathbb{C}/\Gamma$ then Γ consists of translations, $z \mapsto z + b$. It is easy to see that Γ can be only of three different types:

1. Γ is trivial and $S = \mathbb{C}$;
2. Γ is cyclic, thus $\Gamma \cong \mathbb{Z}$ generated by a single translation, say $z \mapsto z + 1$, and $S \cong \mathbb{C}\backslash\{0\}$ (use, for example, the exponential mapping as the covering map);
3. $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$, generated by two translations, $z \mapsto z + 1$ and $z \mapsto z + \tau$, where τ is a complex number with positive imaginary part; the surface S is a torus.

For all other Riemann surfaces we have that S is biholomorphic to $\mathbb{H}/\Gamma$, where Γ is a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ called a Fuchsian group.

3.2. **Basic properties of Fuchsian Groups.** We will now restric to the study of groups Γ for surfaces of the form $S = \mathbb{H}/\Gamma$ (or, equivalently, we can assume Γ is a subgroup of $Aut(\mathbb{D})$).

The group $\mathrm{PSL}(2, \mathbb{R})$ (or $\mathrm{PSL}(2, \mathbb{C})$) has a natural topology given by "coefficients convergence"; a sequence of Möbius transformations A_n converges to A is we can find representatives $A_n(z) = \frac{a_n z + b_n}{c_n z + d_n}$ and $A(z) = \frac{az+b}{cz+d}$ such that $a_n \to a, \ldots, d_n \to d$. A subgroup $G \leq \mathrm{PSL}(2, \mathbb{R})$ is **discrete** if there is no sequence of distinct elements $\{g_n\} \subset G$ such that $g_n \to A$, for some Möbius transformation A.

A subgroup Γ of $\mathrm{PSL}(2, \mathbb{C})$ is said to act **properly discontinuously** at a point $z_0 \in \widehat{\mathbb{C}}$ if there exists a neighbourhood U of z_0 such that the number of elements $g \in G$ with

$g(U) \cap U \neq \emptyset$ is finite. Clearly, since Γ is a covering group we have that it acts properly discontinuously on the upper half plane.

Lemma 3.3. *If $G \leq PSL(2, \mathbb{C})$ acts properly discontinuously on some non-empty set then G is discrete.*

The proof is simple: if G is not discrete then the images of any point z under the sequence $g_{n+1}^{-1} \circ g_n$ converge to z.
The converse of the above statement is false; the Picard group, consisting of Möbius transformations of the form $z \mapsto \frac{az+b}{cz+d}$, with $ad - bc = 1$ and a, b, c , d in $\mathbb{Z}[i]$ is clearly discrete (the entries are almost integers) but it does not act properly discontinuously anywhere on the Riemann sphere. However, for subgroups of PSL$(2, \mathbb{R})$ the two definitions are equivalent.

Proposition 3.4. *If $\Gamma \leq PSL(2, \mathbb{R})$ then the following are equivalent:*

(1) Γ *acts properly discontinuously on* $\mathbb{H}$*;*
(2) *there is a point* $z_0 \in \mathbb{H}$ *such that* Γ *acts properly discontinuously at* z_0*;*
(3) Γ *is discrete.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3). To prove the remaining implication we will work with $\Gamma \subset Aut(\mathbb{D})$. It is easy to see that if Γ does not act properly discontinuously on $\mathbb{H}$ then we can find $z_0 \in \mathbb{H}$ and sequences $\{z_n\} \subset \mathbb{H}$ of distinct points, all Γ-equivalent to z_0 and such that $z_n \to z_0$. Thus we have a sequence of elements $\{\gamma_n\} \subset \Gamma$ such that $\gamma_n(z_n) = z_0$; since the points z_n are distinct the transformations γ_n are also distinct.

Consider $\phi_n(z) = \frac{z - z_n}{1 - \bar{z_n} z}$ and set $C_n = \phi_{n+1} \circ \gamma_{n+1}^{-1} \circ \gamma_n \circ \phi_n^{-1}$. Then we have C_n are automorphisms of the unit disc with $C_n(0) = 0$, so $C_n(z) = \lambda_n\, z$, with $|\lambda_n| = 1$. Therefore there is a subsequence of λ_n that converges to λ_0 (let C_0 be the corresponding rotation). Since $\phi_n \to \phi_0$ where $\phi_0(z) = \frac{z - z_0}{1 - \bar{z_0} z}$ we have that $\gamma_{n+1}^{-1} \circ \gamma_n = \phi_{n+1}^{-1} \circ C_n \circ \phi_n \to \phi_0^{-1} \circ C_0 \circ \phi_0$ and thus Γ is not discrete. □

A **Fuchsian group** is a discrete subgroup of PSL$(2, \mathbb{R})$ or a group conjugate to one such subgroup. The above result is the motivation for the Thurston definition of **Kleinian group** as a discrete subgroup of PSL$(2, \mathbb{C})$. The reasons is that the action of Möbius transformations on $\widehat{\mathbb{C}}$ extends naturally to hyperbolic 3-space, $\mathbb{H}^3$, and one has that if G is a discrte group of Möbius transformations then it acts properly discontinuously on $\mathbb{H}^3$ and $\mathbb{H}^3/G$ is a manifold (see Maskit's book).

3.3. **Uniformization of compact surfaces.** The goal of this subsection is to show that if $S = \mathbb{H}/\Gamma$ is a compact surface then all non-trivial elements of Γ are hyperbolic. We will prove this statement with a series of lemmas.

Lemma 3.5. *If a Möbius transformation fixes a disc or a half space if and only if* $tr^2 \geq 0$.

One implication is simple (an elliptic, parabolic or hyperbolic transformation fixes a disc or half plane); for the other see Maskit's book.

Thus we have that loxodromic elements of a Fuchsian group are actually hyperbolic.

Lemma 3.6. *If A and B are Möbius transformations sharing exactly one fixed point and A has two fixed points then $ABA^{-1}B^{-1}$ is parabolic.*

Proof. Writing the transformation in matrix form we can assume that $A = \left[\begin{smallmatrix} k & 0 \\ 0 & k^{-1} \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right]$. Then $ABA^{-1}B^{-1} = \left[\begin{smallmatrix} 1 & ab(k^2-1) \\ 0 & 1 \end{smallmatrix}\right]$. Since $a \neq 0$, $b \neq 0$ and $k^2 \neq 1$ we have the result. □

Lemma 3.7. *If A is a loxodromic transformation and B is a Möbius transformation such that A and B share exactly one fixed point then the group generated by A and B is not discrete.*

Proof. By the previous lemma one can assume that A is parabolic. By conjugation we can write $A(z) = z + b$ and $B(z) = k^2 z$, with $|k| > 1$. An easy computation shows that $B^{-m}AB^m$ converges to the identity. □

Proposition 3.8 (Shimizu-Leutbecher Lemma)**.** *If G is a discrete subgroup of $PSL(2, \mathbb{C})$ containing $f(z) = z + 1$. If $g == \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is in G, with $ad - bc = 1$ then either $c = 0$ or $|c| \geq 1$.*

Corollary 3.9. *If G is a subgroup of $PSL(2, \mathbb{R})$ and $f(z) = z + 1$, and $\mathcal{H} = \{z \in \mathbb{H};\ Im(z) > 1\}$ then for any $g \in G$ either $g(\mathcal{H}) = \mathcal{H}$ or $g(\mathcal{H}) \cup \mathcal{H} = \emptyset$.*

Proof. If g fixes ∞ then it cannot be hyperbolic or elliptic. So g must be parabolic, of the form $g(z) = z + b$ ($b \in \mathbb{R}$), and thus $g(\mathcal{H}) = \mathcal{H}$. In case of g not fixing ∞ (that is, $c \neq 0$ in the above notation) we have that $g(\mathcal{H})$ is a disc bounded by a circle tangent to the ral axis at $g(\infty) = a/c$ and diameter $1/c^2$, achieved at the point $g(-d/c + i) = a/c + i/c^2$. Since $|c| \geq 1$ we get $g(\mathcal{H}) \cap \mathcal{H} = \emptyset$. □

Corollary 3.10. *If S is a compact surface written as $S = \mathbb{H}/\Gamma$ where Γ is a discrete subgroup of $PSL(2, \mathbb{R})$ then all non-trivial elements of Γ are hyperbolic transformations.*

3.4. **Dirichlet region.** A (convex) polygon D in $\mathbb{H}$ is defined as the intersection of countably many open half-planes (with respect to the hyperbolic metric), where only finitely many of the defining geodesics meet any compact subset of $\mathbb{H}$.

If Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and D is a polygon in the upper half plane we say that D is a **fundamental polygon** for Γ if the following conditions are satisfied:

(1) for all $\gamma \in \Gamma \backslash \{id\}$, $\gamma(D) \cap D = \emptyset$;
(2) for all $x \in \mathbb{H}$ there exists an $\gamma \in \Gamma$ such that $\gamma(x) \in \overline{D}$;
(3) the sides of D are paired by elements of Γ (that is, for each side s there is another side s' and an element γ_s such that $\gamma_s(s) = s'$; also $\gamma_{s'} = \gamma_s^{-1}$ and $(s')' = s$);
(4) any compact set of $\mathbb{H}$ meets only finitely many Γ-translates of D.

Construction on fundamental polygons (or more generally, fundamental regions) for Fuchsian groups is a key point in the theory, since it is eeasy to understand the action of a group on $\mathbb{H}$ if we know a fundamental polygon (where the action in the interior is trivial and on the boundary is given by the sides identification/pairing). We give here an example of one such construction. We begin with some tehcnical, simple result.

Lemma 3.11. *In the above conditions there is a point $z_0 \in \mathbb{H}$ such that $\gamma(z_0) = z_0$ if and only if γ is the identity.*

Choose one such point z_0 and for every non-trivial $\gamma \in \Gamma$ set $D_\gamma = \{z \in \mathbb{H};\ d(z, z_0) < d(z, \gamma(z_0))\}$. We have D_γ is a half-plane in $\mathbb{H}$. The **Dirichlet region** for Γ centered at z_0 is defined by $D = \cap D_\gamma$, where the intersection runs over all $\gamma \in \Gamma \backslash \{id\}$.

Theorem 3.12. *D is a fundamental polygon for Γ.*

Proof. Since there are only finitely many Γ translates of z_0 in any compact set we have that D is a convex polygon. Also, if K is compact, we can assume that K is the ball of radius ρ centered at z_0; only finitely many translates of z_0 meet the ball of radius 2ρ so if $d(\gamma^{-1}(z_0), z_0) > 2\rho$ we have that $\gamma(D) \cap K = \emptyset$, which proves property (4) above.

Since $\gamma(z_0) = z_0$ only for the identity eah side of D corresponds to a unique element of Γ.

To prove (1) consider γ non-trivial and $x \in D$. Then

$$d(\gamma(x), \gamma(z_0)) = d(x, z_0) < d(x, \gamma^{-1}(z_0)) = d(\gamma(x), z_0),$$

so $\gamma(x) \notin D$.

To show (2), let $x \in \mathbb{H}$; then there exists $\gamma \in \Gamma$ sich that $d(x, \gamma(z_0)) \leq d(x, h(z_0)$, for all $h \in \Gamma$, and thus

$$d(\gamma^{-1}(x), z_0) = d(x, \gamma(z_0)) \leq (x, \gamma \circ h(z_0)) = d(\gamma^{-1}(x), h(z_0));$$

since $\gamma \circ h$ covers Γ as h varies over Γ we get that $\gamma^{-1}(x) \in \overline{D_h}$ for all $h \in \Gamma$ and so $\gamma^{-1}(x) \in \overline{D}$.

Finally, for (3) suppose that x is in the (relative) interior of a side s; that means that there exists a unique $\gamma \in \Gamma$ such that $x \in \overline{D_\gamma}$. In other words,

$$\begin{cases} d(x, z_0) < d(x, h(z_0)), \ \forall\ h \neq \gamma, \\ d(x, z_0) = d(x, \gamma(z_0)). \end{cases}$$

These two statements imply

$$\begin{cases} d(\gamma^{-1}(x), h(z_0)) = d(x, \gamma \circ h(z_0)) > d(x, z_0) = d(\gamma^{-1}(x), z_0), \ \forall\ h \neq \gamma^{-1} \\ d(\gamma^{-1}(x), z_0) = d(x, z_0) = d(\gamma^{-1}(x), \gamma^{-1}(z_0)). \end{cases}$$

Thus $\gamma^{-1}(x)$ belongs to a side s' of D with $\gamma^{-1}(s) = s'$. □

The following two results (we do not include their proofs here) show useful applications of the Dirichlet region. We need a little notation first: if Γ is a Fuchsian group by Λ_Γ we mean the set of points in $\mathbb{R} \cup \{\infty\}$ where Γ does not act properly discontinuously. The statements are not as simple as they could be, but we do not want to introduce more concepts; if the reader knows about Kleinian groups then the statements can be simplied.

Lemma 3.13. *Let D be a Dirichlet region for the Fuchsian group Γ.*

1. $\mathbb{H}/\Gamma$ *is compact if and only if the Euclidean closure of D is a compact subset of* $\mathbb{H}$.

2. If $\overline{D}$ is compact then D has only finitely many sides.

Theorem 3.14. *Let D be a Dirichlet region for the Fuchsian group Γ and assume that Λ_Γ has more than two points. The the following are equivalent:*

(1) *D has finitely many sides;*
(2) *the quotient $(\widehat{\mathbb{C}}\backslash\Lambda_\Gamma)/\Gamma$ is a union of compact Riemann surface with finitely many (possibly none) points removed;*
(3) *Γ is finitely generated (by the side-pairing transformations);*
(4) *the hyperbolic area of $(convex\ hull(\Lambda_\Gamma))/\Gamma$ is finite.*

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