CHAPTER 3

Uniformization of Riemann surfaces

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One of the most important results in the area of Riemann surfaces is the Uniformization theorem, which classifies all simply connected surfaces up to biholomorphisms. In this chapter, after a technical section on the Dirichlet problem (solutions of equations involving the Laplacian operator), we prove that theorem. It turns out that there are very few simply connected surfaces: the Riemann sphere, the complex plane and the unit disc. We use this result in 3.2 to give a general formulation of the Uniformization theorem and obtain some consequences, like the classification of all surfaces with abelian fundamental group. We will see that most surfaces have the unit disc as their universal covering space, these surfaces are the object of our study in §§3.3 and 3.5; we cover some basic properties of the Riemannian geometry, automorphisms, Kleinian groups and the problem of moduli.

3.1. The Dirichlet Problem on Riemann surfaces

In this section we recall some result from Complex Analysis that some readers might not be familiar with. More precisely, we solve the Dirichlet problem; that is, to find a harmonic function on a domain with given boundary values. This will be used in the next section when we classify all simply connected Riemann surfaces.

Harmonic Functions and the Dirichlet Problem

3.1.1. Recall that a real-valued function $u : U \to \mathbb{R}$, with continuous second partial derivatives, is called harmonic if $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

**Lemma.** Let $U$ be an open subset of the complex plane and $F : U \to \mathbb{C}$ a holomorphic function. Then $\text{Re}(F)$ and $\text{Im}(F)$, the real and imaginary parts of $F$, are harmonic functions.

**Proof.** Write $F = u + iv$, where $u$ and $v$ are the real and imaginary parts of $F$ respectively. The Cauchy-Riemann equations says that $u_x = v_y$ and $u_y = -v_x$. So we have

$$\Delta u = u_{xx} + u_{yy} = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0,$$
In 3.1.3 we will show a local converse of this result: a harmonic function is locally the real part of a holomorphic function.

**3.1.2.** Let \( U \) be an open subset of \( \mathbb{C} \) and \( f : \partial U \to \mathbb{C} \) a continuous function defined on the boundary of \( U \). The Dirichlet problem with data \( U \) and \( f \) consists on finding a continuous function \( u : \overline{U} \to \mathbb{R} \), harmonic on \( U \) and such that \( u = f \) on \( \partial U \). As one might expect not every problem has a solution; we give one example in 3.1.23. The next result shows that we can always find a (unique) solution for the Dirichlet problem when the domain \( U \) is a disc.

For a complex number \( z_0 \) and a positive real number \( r \), we denote by \( D_r(z_0) \) the open disc of radius \( r \) and centre \( z_0 \), and by \( \overline{D}_r(z_0) \) the closed disc. We will write \( D_r \) for \( D_r(0) \).

**Theorem.** Let \( R \) a positive number and \( f : \partial\overline{D}_R \to \mathbb{R} \) a continuous function. Set

\[
 u(z) = \begin{cases} 
 \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} f(Re^{i\theta}) \, d\theta, & \text{for } |z| < R, \\
 f(z), & \text{for } |z| = R.
\end{cases}
\]

Then \( u \) solves the Dirichlet problem with data \( \overline{D}_R \) and \( f \).

**Proof.** For \( z \) and \( \xi \) complex numbers the function

\[
 P(z, \xi) = \frac{||\xi|^2 - |z|^2}{|\xi - z|^2}
\]

is the real part of the function

\[
 F(z, \xi) = \frac{\xi + z}{\xi - z},
\]

which is holomorphic for \( z \neq \xi \). The expression for \( u \) in \( \overline{D}_R \) can be rewritten as follows:

\[
 u(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, Re^{i\theta}) f(Re^{i\theta}) \, d\theta = \text{Re} \left( \frac{1}{2\pi} \int_0^{2\pi} F(z, Re^{i\theta}) f(Re^{i\theta}) \, d\theta \right) = \text{Re} \left( \frac{1}{2\pi i} \int_{|\xi| = R} F(z, \xi) f(\xi) \frac{1}{\xi} \, d\xi \right).
\]

If \( z \) is in \( \overline{D}_R \) then \( F \) is holomorphic since \( |z| < |\xi| = R \). Hence \( u \) is the real part of a holomorphic function and therefore harmonic.
Clearly $u$ is continuous in $D_R$. To complete the proof of the theorem we need to show that $u$ is continuous on the boundary of $D_R$. Let $G(z, \xi) = F(z, \xi)/\xi$. We have
\[ \frac{1}{2\pi} \int_0^{2\pi} P(z, Re^{i\theta}) \, d\theta = \text{Re} \left( \frac{1}{2\pi i} \int_{|\xi|=R} \frac{\xi + z}{\xi - z} \, d\xi \right) = \sum_{|\xi|<R} \text{res}_\xi G(z, \xi) = 1. \]

The function $G$ is considered as a function of $\xi$, where $z$ is a fixed point of $D_R$. If $z = 0$ then $G(\xi, 0) = 1/\xi$, so $G$ has only one pole at $\xi = 0$ with residue 1. On the other hand, if $z \neq 0$ we have that $G(\xi, z) = \frac{\xi + z}{\xi(\xi-z)}$; in this case, $G$ has two poles, at 0 and $z$, with residues $-1$ and 2 respectively. We see that the sum of the residues of $G$ is equal to 1.

Let $\xi_0$ be a point in $\partial D_R$, and $\epsilon > 0$. Since $f$ is continuous there exists a positive number $M$, such that $|f(\xi)| \leq M$, for all $\xi \in \partial U$. For $z \in D_R$ we have
\[ u(z) - u(\xi_0) = u(z) - f(\xi_0) = \frac{1}{2\pi} \int_0^{2\pi} P(z, Re^{i\theta}) \left( f(Re^{i\theta}) - f(\xi_0) \right) \, d\theta. \]

By the continuity of $f$ at $\xi_0$ there exists a $\delta_0 > 0$, such that $|f(\xi) - f(\xi_0)| < \epsilon$, if $\xi \in \partial U$ satisfies $|\xi - \xi_0| < \delta_0$. We partition the boundary of the disc $D_R$ into two disjoint sets, $A$ and $B$, where
\[ A = \{ \theta \in [0, 2\pi]; \ |Re^{i\theta} - \xi_0| < \delta_0 \}, \]
and $B = [0, 2\pi] \setminus A$. The set $A$ consists of the “angles” that are close to the point $\xi_0$ and $B$ is its complement in the unit circle. We have
\[ |u(z) - f(\xi_0)| \leq \left| \frac{1}{2\pi} \int_A P(z, \xi)(f(\xi) - f(\xi_0)) \, d\xi \right| + \left| \frac{1}{2\pi} \int_B P(z, \xi)(f(\xi) - f(\xi_0)) \, d\xi \right| \leq \epsilon + \frac{M}{\pi} \int_B P(z, Re^{i\theta}) \, d\theta. \]

The number $\epsilon$ in the above inequality comes from the fact that $f(\xi) - f(\xi_0)$ is small for “points in” $A$ and the total integral of $P$ over the boundary of $D_R$ is equal to 1. The bound of the second integral comes from the bound $M$ of $|f|$ and the fact that $P(z, \xi) > 0$, for $|\xi| > |z|$. Let now $z$ be in $D_R$ and close to $\xi_0$; that is, $|\xi_0 - z| < \delta \leq \delta_0/2$. For $\theta \in B$ we have that
\[ |Re^{i\theta} - z| \geq |Re^{i\theta} - \xi_0| - |\xi_0 - z| \geq \frac{\delta_0}{2}. \]
On the other hand,
\[ R - |z| = |\xi_0| - |z| \leq |\xi_0 - z| < \delta. \]

Using these inequalities we have
\[ P(z, Re^{i\theta}) = \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} \leq \frac{(R + |z|)(R - |z|)}{(\delta_0/2)^2} \leq \frac{8R\delta}{\delta_0^2}. \]

We can now bound the above integral as follows:
\[ \frac{M}{\pi} \int_B P(z, Re^{i\theta}) d\theta = \frac{M}{\pi} \frac{8R\delta}{\delta_0^2} \frac{2\pi}{2\pi} = \frac{16M\delta R}{\delta_0^2}. \]

For \( \epsilon \) given the value of \( \delta_0 \) is fixed, so we can make \( \delta \) small enough such that for we have \( |u(z) - u(\xi_0)| \leq 2\epsilon \), for \( z \) as above. This shows that \( u \) is continuous at \( \xi_0 \).

The function \( P \) is called the Poisson kernel.

### 3.1.3. Corollary

In the above proof we have shown the following result.

**Corollary.** If \( u \) is harmonic then \( u \) is locally the real part of a holomorphic function.

### 3.1.4. Corollary

Let \( u : D_R \to \mathbb{R} \) be a harmonic function. Then \( u \) satisfies
\[ u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta, \]
for \( 0 < r < R \).

**Proof.** Apply the above theorem on \( D_r \) for the boundary values given by \( u : \partial D_r \to \mathbb{R} \) and observe that \( P(z, 0) = 1 \).

### 3.1.5. Corollary (Mean Value Property)

Let \( u : D_R \to \mathbb{R} \) be a harmonic function. Let \( z_0 \) be a point in \( D_R \) and \( r > 0 \) a positive number such that \( \overline{D_r}(z_0) \) is contained in \( D_R \). Then
\[ u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta, \]
for \( 0 < r < R \).
3.1.6. From corollary 3.1.3 one expects that harmonic functions share some of the properties of holomorphic functions. In that sense one can consider the Mean Value Property as the analogy of Cauchy’s Integral Formula. In the next result we see that the Maximum Modulus Principle (1.1.9) is also satisfied by harmonic functions.

**Proposition (Maximum Modulus Principle).** Let \( u : D_R \to \mathbb{R} \) be a harmonic function. If there exists a point \( z_0 \in D_R \), such that \( u(z) \leq u(z_0) \) for all \( z \in D_R \), then \( u \) is constant.

**Proof.** The set 
\[
E = \{ z \in D_R ; \ u(z) = u(z_0) \} = u^{-1}(u(z_0))
\]
is closed since \( u \) is a continuous function. Let \( z \) be an arbitrary point of \( E \) and \( r > 0 \) such \( \overline{D}_r(z) \) is contained in \( D_R \). From the Mean Value Property we get 
\[
u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0) \, d\theta = u(z_0) = u(z).
\]
This implies that \( u(z + re^{i\theta}) = u(z_0) \) for all \( \theta \) in \( [0, 2\pi] \). Thus \( E \) is an open set. Since \( D_R \) is connected and \( E \) is not empty we have \( E = D_R \) and therefore \( u \) is constant on \( U \).

A similar result with minimum instead of maximum can be obtained from the fact that if \( u \) is a harmonic function then \( -u \) is also harmonic; we leave the details for the reader.

3.1.7. **Corollary.** If \( u : \overline{D}_R \to \mathbb{R} \) is harmonic on \( D_R \) and continuous on \( \overline{D}_R \) then its maximum value is attained in the boundary of \( D_R \); that is, there exists a point \( z_0 \in \partial D_R \), such that \( u(z) \leq u(z_0) \), for all \( z \in \overline{D}_R \).

**Proof.** Since \( \overline{D}_R \) is compact and \( u \) continuous there is a value \( u(z_1) \) where \( u \) attains its maximum. If \( z_1 \) is in \( \partial D_R \) there is nothing to prove. On the other hand, if \( z_1 \in D_R \), from the previous corollary we have that \( u \) is constant in \( D_R \) and therefore in \( \overline{D}_R \). In this case we can choose any point of \( \partial D_R \) as \( z_0 \).

3.1.8. **Corollary.** If the Dirichlet Problem has a solution on a bounded domain then the solution is unique.
Proof. Apply the Maximum Modulus Principle to \(u_1 - u_2\) and \(u_2 - u_1\), where \(u_1\) and \(u_2\) are two solutions of the (same) Dirichlet problem. Here the condition of the domain being bounded is necessary. Consider for example the Dirichlet problem on the upper half plane with boundary values given by the identically 0 function (on the real line). Then the constant function 0 and the function \(\text{Im}(z)\) are two distinct solutions of this Dirichlet problem.

3.1.9. Using this result we can show that the Mean Value Property (3.1.5) is also a sufficient condition for harmonicity.

**Proposition.** Let \(u : U \to \mathbb{R}\) be a continuous function on an open set \(U\) of the complex plane. Assume \(u\) satisfies the Mean Value Property, namely
\[
u(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + re^{i\theta}) \, d\theta,
\]
for all \(z_0 \in U\) and all positive \(r\) such that the closed disc of centre \(z_0\) and radius \(r\), is contained in \(U\). Then \(u\) is harmonic.

**Proof.** Let \(v\) be the solution of the Dirichlet problem on \(D_r(z_0)\) with values given by the function \(u\). Observe that the proof of the Maximum Modulus Principle uses only the Mean Value Property. Hence from 3.1.7 we have that \(v - u\) has its maximum on \(\partial D_r(z_0)\); that is \(v(z) - u(z) \leq 0\) for all \(z \in D_r(z_0)\). Applying the same argument to the function \(u - v\) we obtain that \(u(z) - v(z) \leq 0\) and this completes the proof.

3.1.10. Another similarity between harmonic and holomorphic functions is given by the following result.

**Corollary.** Let \(u : D_R \to \mathbb{R}\) be a sequence of harmonic functions which converges uniformly on compact subsets of \(D_R\) to a (continuous) function \(u : D_R \to \mathbb{R}\). Then \(u\) is harmonic.

**Proof.** For \(z_0\) in \(D_R\) let \(r > 0\) be such that \(\overline{D_r(z_0)} \subset D_R\). Then we have
\[
u(z_0) = \lim_n u_n(z_0) = \lim_n \frac{1}{2\pi} \int_{0}^{2\pi} u_n(z_0 + re^{i\theta}) \, d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + re^{i\theta}) \, d\theta.
\]
For the last equality we have used that \(u_n\) converges uniformly on compact sets to \(u\), so in particular on the circle of centre \(z_0\) and radius \(r\). It follows from 3.1.9 that \(u\) is harmonic.
3.1.11. The next lemma is needed to prove Harnack’s inequality.

**Lemma.** Let \( z \) be a complex number with \( |z| = s \) and \( r \) a positive number satisfying \( s < r \). Then

\[
\frac{r - s}{r + s} \leq \frac{r^2 - s^2}{|r e^{i\theta} - z|^2} \leq \frac{r + s}{r - s}
\]

for any real number \( \theta \).

**Proof.** To prove the left hand side inequality we use that

\[
|re^{i\theta} - z| \leq |re^{i\theta}| + |z| = r + s.
\]

The other inequality follows from

\[
|re^{i\theta} - z| \geq |re^{i\theta}| - |z| = r - s.
\]

These two inequalities, together with the expression \( r^2 - s^2 = (r - s)(r + s) \), prove the result.

Let \( u : D_R \to \mathbb{R}^+ \) be a positive harmonic function and \( z \in D_R \) a point with \( |z| = s \). Choose a positive real number \( r \) with \( s < r < R \); then

\[
u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - s^2}{|re^{i\theta} - z|^2} u(re^{i\theta}) \, d\theta \leq \frac{1}{2\pi} \frac{r + s}{r - s} \int_0^{2\pi} u(re^{i\theta}) \, d\theta = \frac{r + s}{r - s} u(0).
\]

**Proposition (Harnack’s inequality).** Let \( u : D_R \to \mathbb{R} \) be a positive harmonic function. Then, for all \( z_0 \in D_R \) with \( |z| = s \), and for all positive \( r \) such that \( s < r < R \), one has

\[
\frac{r - s}{r + s} u(0) \leq u(z) \leq \frac{r + s}{r - s} u(0).
\]

**Proof.** The right hand side inequality was proved before the statement of the proposition. The proof of the other inequality is similar.

3.1.12. The main application of the above inequality is the proof of the following theorem, which is similar to Montel’s theorem (1.1.13).

**Theorem (Harnack’s Principle).** Let \( M \) be a real number and \( \{u_n\}_{n=1}^\infty \) be a non-decreasing sequence of harmonic functions on \( D_R \) satisfying \( u_n \leq M \). Then the sequence \( \{u_n\} \) converges uniformly on compact subsets of \( D_R \) to a harmonic function \( u : D_R \to \mathbb{R} \).
Proof. The pointwise convergence follows from the fact that \( u_n(z) \) is a bounded, non-decreasing sequence of real numbers (for fixed \( z \)). Thus to complete the proof we only need to show that the convergence is uniform on compact subsets of \( D_R \). Let \( \epsilon > 0 \) be given; then there exists an \( n_0 \) such that \( u_n(0) - u_m(0) \leq \epsilon \), for \( n_0 \leq m \leq n \). Choose a real number \( s \) such that \( 0 < s < r < R \). We apply Harnack’s inequality to the non-negative function \( u_n - u_m \) on the disc \( D_s \):

\[
 u_n(z) - u_m(z) \leq \frac{r + s}{r - s} (u_n(0) - u_m(0)) \leq \frac{r + s}{r - s} \epsilon.
\]

Since \( \epsilon \) can be made arbitrarily small we get that \( \{u_n\}_n \) converges uniformly on \( D_s \). But any compact subset \( K \) of \( D_R \) is contained in a disc of the form \( D_s \). Thus \( u \) is harmonic (3.1.10).

3.1.13. The following more general result can be found in [1] (for our applications the previous version of Harnack’s Principle is enough).

Theorem. Consider a sequence of functions \( u_n(z) \), each defined and harmonic in certain region \( \Omega_n \). Let \( \Omega \) be a region such that every point in \( \Omega \) has a neighbourhood in all but a finite number of the \( \Omega_n \), and assume moreover that in this neighbourhood \( u_n(z) \leq u_{n+1}(z) \) for \( n \) sufficiently large. Then there are only two possibilities: either \( u_n(z) \) tends uniformly to \( \infty \) on every compact subset of \( \Omega \), or \( u_n(z) \) converges uniformly on compact subsets of \( \Omega \) to a harmonic function \( u : \Omega \to \mathbb{R} \).

Subharmonic functions

3.1.14. Finding non-trivial harmonic functions on domains is not an easy problem. What we will do is to consider a more general class of functions, called subharmonic functions, which are “close” enough to be harmonic; taking limits in this class we obtain harmonic functions. The precise definition we need is as follows.

Definition. A continuous function \( u : U \to \mathbb{R} \) on an open set \( U \) of the complex plane is said to be subharmonic if for every harmonic function \( h : U \to \mathbb{R} \), and every domain \( V \subset U \), the function \( u + h \) either is constant or has no maximum (on \( V \)).

Suppose \( V \) is a domain with compact closure \( \overline{V} \subset U \). Let \( h : \overline{V} \to \mathbb{R} \) be a continuous
function, harmonic on $V$. If $u$ is subharmonic (on $U$) then the maximum of $u + h$ on $\overline{V}$ is attained in the boundary of $V$. The proof is similar to the case of harmonic functions (3.1.7).

It is clear from the above definition that $u$ is subharmonic if and only if it is locally subharmonic; that is, every point of $U$ has a neighbourhood where $u$ is subharmonic.

3.1.15. Let $D$ be a disc with $\overline{D} \subset U$ and $u : U \to \mathbb{R}$ a subharmonic function. Denote by $P_{D,u}$ the function that is equal to $u$ on $U \setminus D$ and solves the Dirichlet problem on $D$ with boundary values given by $u|_{\partial D}$.

**Proposition.** A continuous function $u : U \to \mathbb{R}$ is subharmonic if and only if $u \leq P_{D,u}$ for every disc $D$ whose closure is contained in $U$.

**Proof.** Assume first that $u$ is subharmonic on $U$. For any disc $D$ with $\overline{D} \subset U$ we have that $u - P_{D,u}$ is equal to 0 on $\partial D$. Since $u - P_{D,u}$ is continuous on $\overline{D}$ and $u$ is subharmonic, either $u - P_{D,u}$ is identically 0 or it satisfies $u - P_{D,u} \leq 0$ on $D$.

To prove the converse let $h : U \to \mathbb{R}$ be a harmonic function and $V \subset U$ a domain. Assume that $u + h$ has a maximum value on $V$, say $m_0$. Set

$$C = \{ z \in V ; u(z) + h(z) = m_0 \}.$$ 

This set is a closed subset of $V$. Let $z_0$ be a point of $C$ and $D$ a disc of radius $r$ with centre at $z_0$ and such that $D \subset V$. Then we have

$$m_0 = u(z_0) + h(z_0) \leq P_{D,u}(z_0) + h(z_0) = \frac{1}{2\pi} \int_0^{2\pi} (u(z_0 + re^{i\theta}) + h(z_0 + re^{i\theta})) \, d\theta \leq m_0.$$ 

It follows that $u(z_0 + re^{i\theta}) + h(z_0 + re^{i\theta}) = m_0$ for all $\theta$; that is, $C$ is an open subset of $V$. Since $V$ is connected we have that $C = V$ and $u + h$ is constant on $V$. 

3.1.16. **Corollary.** Let $u : U \to \mathbb{R}$ be a continuous function. Then $u$ is subharmonic if and only if for every point $z_0 \in U$, and every positive number $r$ such that $\overline{D_r}(z_0)$ is contained in $U$, the following inequality holds:

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta.$$
**Proof.** If $D = \overline{D}_r(z_0)$ then the right hand side of the above inequality is simply $P_{D,u}(z_0)$.  

**Corollary (Maximum Modulus Principle).** *Subharmonic functions satisfy the Maximum Modulus Principle.*

**Proof.** The proof is similar to the case of harmonic functions.  

**3.1.17. Proposition.** *Let $u, v : U \to \mathbb{R}$ be subharmonic functions, $c$ a positive real number, and $D \subset U$ a disc. Then the functions $cu, u + v, \max(u, v)$ and $P_{D,u}$ are subharmonic (on $U$).*

**Proof.** The fact that $cu$ and $u + v$ are subharmonic follows from the above corollary.

To show that the maximum of two subharmonic functions is subharmonic consider a point $z_0$ of $U$ and assume that $\max(u, v)(z_0) = u(z_0)$; then we have

$$\max(u, v)(z_0) = u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r e^{i\theta}) \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \max(u, v)(z_0 + r e^{i\theta}) \, d\theta.$$  

It follows from the previous result that $\max(u, v)$ is subharmonic.

Consider now the function $P_{D,u}$. Clearly this function is subharmonic on $U \setminus \overline{D}$ (since it is equal to $u$ on this set) and on $D$ (because it is harmonic). So we need to check subharmonicity only at the points on the boundary of $D$. Let $z_0$ be one such point; using the inequality $u \leq P_{D,u}$ (3.1.15) we have

$$P_{D,u}(z_0) = u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r e^{i\theta}) \, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} P_{D,u}(z_0 + r e^{i\theta}) \, d\theta.$$  

We can now apply the previous corollary.  

**3.1.18.** *For functions of class $C^2$ (that is, with continuous partial derivatives of second order) we have another characterisation of subharmonicity as follows.*

**Proposition.** *Let $u : U \to \mathbb{R}$ be a $C^2$ function. Then $u$ is subharmonic if and only if $\Delta u \geq 0$ on $U$.*

This result is taken some times as the definition of subharmonic functions. However a function does not need to have partial derivatives in order to satisfy definition 3.1.14.
3.1.19. Harmonic functions on Riemann surfaces were defined in 1.4.11. Since a harmonic function \( u : U \rightarrow \mathbb{R} \), defined on a domain of the complex plane, is locally the real part of a holomorphic function one sees that harmonicity is preserved under changes of coordinates (recall that to compute the partial derivatives of a function on Riemann surface we need to take local coordinates). However, to define subharmonic functions we need a little more of extra work. We begin with a definition.

**Definition.** A **disc** on a Riemann surface \( X \) is a domain \( D \) such that there exists a local coordinate patch \((U,z)\) with \( \overline{D} \subset U \) and \( z(\overline{D}) \) is a closed disc on \( \mathbb{C} \).

Given a disc \( D \) on \( X \), and a continuous function \( u : X \rightarrow \mathbb{R} \), we can define \( P_{D,u} : X \rightarrow \mathbb{R} \) as in 3.1.15.

**Definition.** A continuous function \( u : X \rightarrow \mathbb{R} \) defined on a Riemann surface \( X \) is called **subharmonic** if for every disc \( D \) of \( X \), and every harmonic function on \( D \) satisfying \( u \leq h \) one has that \( u \equiv h \) of \( u < h \) (on \( D \)).

It is easy to show that for the case of \( X = \mathbb{C} \) this definition is equivalent to 3.1.14. Proposition 3.1.15 and corollary 3.1.16 extend to Riemann surfaces with similar proofs. Therefore we see that talking of the Dirichlet problem and solutions of it on Riemann surfaces makes sense.

In a more invariant way we have the following definition.

**Definition.** Let \( X \) be a Riemann surface, \( V \subset X \) an open set of \( X \) and \( u : V \rightarrow \mathbb{R} \) a real-valued function defined on \( V \). We say that \( u \) is **harmonic** (respectively **subharmonic**) if for any local patch \((U,z)\) with \( U \cap V \neq \emptyset \), the function

\[
(u \circ z^{-1}) : z(U \cap V) \rightarrow \mathbb{R}
\]

is harmonic (respectively, subharmonic).

**Perron’s method**

3.1.20. The idea of Perron’s method to find harmonic functions consists of taking a family of subharmonic functions that satisfy certain conditions and then show that the pointwise supremum of such a family must be harmonic.
Theorem (Perron’s method). Let $U$ be a domain of the complex plane and $f : \partial U \to \mathbb{R}$ a bounded function. Denote by $\mathcal{M}$ the family of subharmonic functions $u : U \to \mathbb{R}$ satisfying $\limsup_{z \to z_0} u(z) \leq f(z_0)$, for all $z \in \partial U$. Then the supremum of the family $\mathcal{M}$ is a harmonic function.

Proof. First of all, observe that if $|f(z)| \leq K$ for $z$ in $\partial U$ then $v(z) \leq K$ for all $z$ in $U$ and all $v$ in $\mathcal{M}$ (this is simply a consequence of the Maximum Modulus Principle for subharmonic functions (3.1.16)).

It is easy to see that the family $\mathcal{M}$ has the following properties:

1. If $u_1$ and $u_2$ belong to $\mathcal{M}$ so does $\max(u_1, u_2)$.
2. If $u \in \mathcal{M}$, and $D$ is a disc contained in $U$ then $P_{D,u}$ is in $\mathcal{M}$.

Fix a point $z_0 \in U$ and let $D$ be a disc containing $z_0$ and satisfying $\overline{D} \subset U$. Then there exists a sequence of functions $\{u_n\}_n$ in $\mathcal{M}$ (the sequence may depend on the point $z_0$) such that $\limsup_{n \to \infty} u(z)_n = u(z_0)$. Let us define functions $v_n$ by $v_n = \max(u_1, \ldots, u_n)$. The sequence $\{v_n\}_n$ is clearly non-decreasing and contained in $\mathcal{M}$ (because of property 1 above). If we set $w_n = P_{D,v_n}$ we have that $w_n$ belongs to $\mathcal{M}$ (property 2). Moreover the following inequalities hold:

$$u_n(z_0) \leq v_n(z_0) \leq w_n(z_0) \leq u(z_0).$$

So $\lim_{n \to \infty} w_n(z_0) = u_n(z_0)$. Let $w$ be the limit of the sequence $\{w_n\}_n$. Then $w$ is harmonic on $D$ by Harnack’s principle and $w \leq u$ with $w(z_0) = u(z_0)$.

Consider now another point of $U$, say $z_1$, and let let $u'_n$ be a sequence and $D'$ a disc similar to the ones considered above. We set $\tilde{u}'_n = \max(u_n, u'_n)$ and repeat the above process to obtain a function $w'$ satisfying $w \leq w' \leq u$, and $w(z_1) = w'(z_1)$. But then $w(z_0) = w'(z_0)$ (since $w(z_0) \leq w'(z_0) \leq u(z_0) = w(z_0)$). So $w \equiv w'$ on $D'$. Thus $w$ is harmonic on the domain $U$. 

3.1.21. Lemma. Let $U$ be a domain in the complex plane and $z_0$ a point of $\partial U$. Assume that there exists a continuous function $\omega : \partial U \to [0, +\infty)$ such that $\omega(z_0) = 0$ and $\omega(z) > 0$, for all $z \in \partial U \setminus \{z_0\}$. If $f : \partial U \to \mathbb{R}$ is a bounded function, continuous at $z_0$ and $\mathcal{M}$ is as in theorem 3.1.20, then $\lim_{z \to z_0} u(z) = f(z_0)$, for $z \in U$.

Proof. It suffices to show that
\[ \limsup_{z \to z_0} u(z) \leq f(z_0) + \epsilon, \quad \text{and} \quad \liminf_{z \to z_0} u(z) \geq f(z_0) - \epsilon, \]

for \( \epsilon > 0 \) arbitrary, \( z_0 \in \partial U \) and \( z \in U \).

Let \( W \) be a neighbourhood of \( z_0 \) such that \( |f(z) - f(z_0)| < \epsilon \) for \( z \in W \). Consider the function
\[
g(z) = f(z_0) + \epsilon + \frac{\omega(z)}{\omega_0} (K - f(z_0)),
\]
where \( \omega_0 > 0 \) is the minimum of the harmonic function \( u \) on \( U \setminus (W \cap U) \). For \( z \) in \( W \) we have \( g(z) \geq f(z_0) + \epsilon \), while for \( z \) not in \( W \) we see that \( g(z) \geq K + \epsilon > f(z) \). By the Maximum principle we have that if \( v \in M \) then \( v < g \). Thus \( u \leq g \), which implies that \( \limsup_{z \to z_0} u(z) \leq g(z_0) \leq f(z_0) + \epsilon \).

The second inequality is proven in a similar way by using the function
\[
h(z) = f(z_0) - \epsilon - \frac{\omega(z)}{\omega_0} (K + f(z_0)).
\]

\[ \square \]

3.1.22. The function \( \omega \) in the above lemma is called a barrier at \( z_0 \). It is clear that if every point of the boundary of \( U \) has a barrier then we can solve the Dirichlet problem for that domain. One would like to have geometric conditions on a domain so that we can easily see the existence of barriers at its boundary points. An easy example is given by the upper half plane \( U = \mathbb{H} = \{ z \in \mathbb{C}; \text{Im}(z) > 0 \} \). Take any point, say \( z_0 = 0 \). Then \( \omega(z) = \text{Im}(e^{i\pi/2}z) \) is a barrier at the origin. More generally, let \( z_0 \in \partial U \) and let \( z_1 \) denote a point not in \( U \). Denote by \([z_0, z_1]\) the segment joining these two points and assume that \([z_0, z_1] \cap \overline{U} = \{z_0\} \). Then the function
\[
\omega(z) = \text{Im} \left( \sqrt{\frac{z - z_0}{z - z_1}} \right)
\]
is a barrier at \( z_0 \), for a proper choice of the square root.

3.1.23. We end this section with an example of a Dirichlet problem that has no solution. Consider the open set \( U = \mathbb{D}^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \), the punctured unit disc, and the function \( f \) defined on \( \partial U = S^1 \cup \{0\} \) by \( f(0) = 1 \), \( f(z) = 0 \), for \(|z| = 1 \). If \( u \) were a solution for the Dirichlet problem with this data, then \( u \) would have its maximum (it must have a maximum since \( \overline{U} \) is compact) at the boundary of
the disc (0 is an interior point of $\overline{U}$). But this would imply that $u \equiv 0$, contradicting the fact that $u(0) = 1$.

### 3.2. Uniformization of simply connected Riemann surfaces

We have seen that a compact, simply connected Riemann surface is biholomorphic to the Riemann sphere. By the Riemann mapping theorem we have that any simply connected open subset of the Riemann sphere is biholomorphic to either the complex plane or the unit disc. In this section we show that these three surfaces are the only simply connected Riemann surfaces, up to biholomorphisms. The proof assumes only a couple of results from Complex Analysis (that we state at the beginning) and the theory of harmonic functions; it is based on a paper of R.R. Simha [23].

**3.2.1. Theorem (Koebe).** Let $\mathcal{A}$ be the class of one-to-one holomorphic functions defined on the unit disc $f : \mathbb{D} \to \mathbb{C}$ and satisfying $f(0) = 0$, $f'(0) = 1$. Then $\mathcal{A}$ is normal and compact in the topology of uniform convergence on compact subsets of the disc.

**3.2.2. Theorem (Riemann Mapping Theorem).** If $A$ is a simply connected open subset of the complex plane, with $\mathbb{C}\backslash A$ not empty, then $A$ is biholomorphic to the unit disc.

**3.2.3. Lemma.** Let $h : \Omega \to \mathbb{R}$ be a harmonic function defined on an open, connected set $\Omega$ of the complex plane. If there exists an open subset $U$ of $\Omega$, such that $h|_U$ is constant, then $h$ is constant (on $\Omega$).

**Proof.** Let $p_0$ be a fixed point of $U$. For any point $p$ of $\Omega$ consider a path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = p_1$, $\gamma(1) = p$. A harmonic function is locally the real part of a holomorphic function (3.1.3); that is, for every point $q$ of $\Omega$ there exists a neighbourhood $V$ of $q$, and a holomorphic function $f_V : V \to \mathbb{C}$, such that $h = \text{Re}(f_V)$ on $V$. Since the image of $\gamma$ is compact we can find connected open sets, $U_0, \ldots, U_n$, satisfying the following properties:

1. there exist holomorphic functions $f_j : U_j \to \mathbb{C}$, such that $h$ is the real part of
1. $f_j$ on $U_j$, for $j = 0, \ldots, n$;
2. $U_j \cap U_{j+1} \neq \emptyset$, for $j = 0, \ldots, n - 1$.

Since $U_0 \cap U$ is not empty, and $h$ is constant on $U$, we have that the real part of $f_0$ is constant on $U_0$. But then $f_0$ must be constant. Similarly we get that $f_1$ must be constant on $U_1$; in particular, the real part of $f_1$, which is equal to $h|_{U_1}$, is constant. By a finite number of steps we get that $h|_{U_n}$ is constant and therefore $h(p) = h(p_0)$; that is, $h$ is constant on $\Omega$.

### 3.2.4. Lemma

*Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a connected open subset $\Omega$ of the complex plane. Assume that $f$ is a (branched) covering map of degree $d$ onto its image $f(\Omega)$. Then*

$$\int_{\Omega} f \wedge d\bar{f} = d \int_{f(\Omega)} dz \wedge d\bar{z}.$$  

*Proof.* Assume first that $\Omega = f(\Omega) = \mathbb{D}$ and $f$ is given by $f(z) = z^n$. The proof in this case is an easy calculation. If we write $z = re^{i\theta}$ we have $dz \wedge d\bar{z} = -2iri \, dr \wedge d\theta$ and therefore

$$\int_{\mathbb{D}} df \wedge d\bar{f} = \int_{\mathbb{D}} nz^{n-1} \, dz \wedge dz^{n-1} \, d\bar{z} = -2in^2 \int_0^{2\pi} \int_0^1 r^{2n-1} \, dr \wedge d\theta = -2i\pi n.$$  

On the other hand

$$\int_{\mathbb{D}} dz \wedge d\bar{z} = -2i \int_{\mathbb{D}} r \, dr \wedge d\theta = -2\pi i.$$  

To prove the general case use the fact that $f$ is a (branched) covering, and therefore it behaves locally as the function $z \mapsto z^n$ studied above.

*Remark.* The lemma, in a non-formal language, says that the area of $f(\Omega)$ counted with “multiplicity” is equal to the “true” area of $f(\Omega)$ multiplied by the degree of $f$.

### 3.2.5. We now prove that an annulus on a Riemann surface is always conformally equivalent to a standard annulus on the complex plane.

**Theorem (The Annulus Theorem).** Let $U$ be an open subset of $\mathbb{R}^2$ containing the closed annulus $\{z \in \mathbb{C}; 1 \leq |z| \leq 2\}$. Suppose that there exists a Riemann
surface structure on $U$ such that the holomorphic functions (in that structure) are smooth functions of $\mathbb{R}^2$. Then the open annulus

$$A = \{z \in \mathbb{C}; 1 < |z| < 2\},$$

with the complex structure induced from $U$, is biholomorphic to a unique annulus

$$A_R = \{z \in \mathbb{C}; 1 < |z| < R\}$$

with the standard Riemann surface structure induced from $\mathbb{C}$.

**Proof.** It is easy to see (use 3.1.22) that there exists a barrier at every point of $\partial A$ so the Dirichlet problem has solution on $A$. For $c$ a positive real number let $h_c$ be the unique solution of the Dirichlet problem with boundary values 0 in $\{z; |z| = 1\}$ and $c$ in $\{z; |z| = 2\}$. Observe that $h_c$ is linear on $c$: if $c, d$ and $\lambda$ are positive numbers, then $h_{c+d} = h_c + h_d$ and $h_{\lambda c} = \lambda h_c$. It is also easy to see that $h_c$ is a proper function.

By Sard’s theorem (1.4.23) the set of points where the function $h$ does have zero derivative has measure zero. Let $t$ be a regular value (the image of a point where $h$ has non-zero derivative); then $h^{-1}(t)$ is a collection of 1 dimensional closed manifolds. By 1.4.24 these manifolds must be curves diffeomorphic to circles. By the maximum modulus theorem for harmonic functions we have that none of this circles can enclosed a disc in $A$. Otherwise the maximum and minimum of $h$ in that disc will be achieved in the boundary, where $h$ is constant (with value $t$), and thus $h$ will be constant in the whole disc; but then, $h$ will be constant on $A$ by 3.1.7. Similarly, we cannot have two (disjoint) curves in $h^{-1}(t)$ that bound an annulus inside $A$. So
we see that $h^{-1}(t)$ consists of a single curve, diffeomorphic to $S^1$, and homotopic to the boundary curves of $A$, as in figure 12. We simplify notation and write $C_t$ for this curve. We also have that if $t < s$ then $C_t$ and $C_s$ bound a cylinder in $A$, with $C_t$ closer to $S^1$ than $C_s$ (see the remark after the proof for a formal definition of “closer” in this setting).

Consider the integral $\lambda_c = \int_{C_t} *dh_c$ where $0 < t < c$. From $d * dh_c = \Delta h_c$ we see that $d * h_c$ is a closed form and therefore $\lambda_c$ is independent of $t$ (1.4.20). Since $h_c$ depends linearly on $c$ so does $\lambda_c$. In particular we have that either $\lambda_c = 0$ for all $c > 0$, or $\lambda_c \to +\infty$ as $c \to +\infty$. If we write $dh_c = (\partial h_c/\partial x)dx + (\partial h_c/\partial y)dy$, we have (1.4.7) $dh_c \land *dh_c = ((\partial h_c/\partial x)^2 + (\partial h_c/\partial y)^2)dx \land dy$. By the monotone convergence theorem we see that

$$0 < \int_A dh_c \land *dh_c = \lim_{\epsilon \to 0} \int_{\epsilon \leq h_c(z) \leq c-\epsilon} dh_c \land dh_c,$$

where we choose $\epsilon$ a regular value and such that $2 - \epsilon$ is is also a regular value. By Stokes’ theorem this last integral is equal to

$$\lim_{\epsilon \to 0} \int_{C_{c-\epsilon}} d(h_c(*dh_c)) = (c - \epsilon) \int_{C_{c-\epsilon}} *dh_c - (\epsilon) \int_{C_{c-\epsilon}} *dh_c = c\lambda_c.$$  

This implies that $\lambda_c \neq 0$. So there exists a unique value of $c$ such that $\lambda_c = 2\pi$. Set $R = e^c$, and define a holomorphic function on $A$ by the expression

$$f(z) = \exp \left( h_c(z_0) + \int_{z_0}^z (dh_c + i(*dh_c)) \right),$$

where $z_0$ is an arbitrary (but fixed) point of $A$. By our choice of $c$ the periods of the 1-form $dh_c + i(*dh_c)$ are integer multiples of $2\pi i$, so $f$ is well-defined. Observe that $|f| \to 1$ as $|z| \to 1$, and $|f| \to R$ when $|z| \to 2$. It follows that $f : A \to A_R$ is onto and proper. The surjectivity of $f$ is a consequence of the fact that $f$ is an open mapping (it is holomorphic). To see that $f$ is proper let $K \subset A_R$ be a compact set. We can assume that $K = \{z \in A_R; r_1 \leq |z| \leq r_2\}$, for $1 < r_1 < r_2 < R$, since any compact subset of $A_R$ is contained in one such annulus. Let $\epsilon > 0$ be such that $1 + \epsilon < r_1$ and $r_2 < R - \epsilon$. Then we have that there exists a $\delta > 0$, such that $|f(z)| \leq 1 + \epsilon$, for $z$ with $|z| < 1 + \delta$, and $|f(z)| \geq R - \epsilon$, for $z$ satisfying $|z| > 2 - \delta$. Hence $f^{-1}(K)$ is contained in the annulus $\{z \in A; 1 + \delta \leq |z| \leq 2 - \delta\}$. Since $f^{-1}(K)$ is closed it follows that it is also compact, and therefore $f$ is proper, as we
From 1.3.11 and exercise 21 we have that $f$ is a (possibly branched) covering map, of degree $d \geq 1$. We need to show that $d$ is precisely equal to 1. To see this we use 3.2.4; first of all, a simple computation gives

$$\int_A df \wedge d\bar{f} = \lim_{\epsilon \to 0} \int_{c - \epsilon \leq h(z) \leq c} d(f \bar{d}) = \lim_{\epsilon \to 0} -i \int_{c - \epsilon \leq h(z) \leq c} d(|f|^2(* dh_c)) =$$

$$= -2\pi i (R^2 - 1) = \int_{A_R} d\omega \wedge d\bar{\omega}.$$ 

Here $\omega = dx \wedge dy$ is the standard area form in the plane. The above computation simply shows that the area of $f(A_R)$, counted with “multiplicity”, is the same as the area of $A_R$. Thus $f$ has to be one-to-one. 

\[\square\]

Remark. The formal way of saying that $C_t$ is closer to $S^1$ than $C_s$ is by saying that $C_t$ lies in the annulus bounded by $C_s$ and $S^1$.

3.2.6. Definition. A Riemann surface $X$ is called \textbf{planar} is every smooth closed 1-form on $X$ with compact support is exact.

It is clear that any simply connected Riemann surface is planar: if $\omega$ is a form on $X$, and $p_0$ is a fixed point of $X$, the expression $f(p) = \int_{p_0}^p \omega$ defines a function of $X$ such that $df = \omega$. Here the integration is done on a path from $p_0$ to $p$; since $X$ is simply connected we have that the value of this integral does not depend of the path. It is also clear that any open subset of a planar Riemann surface is planar.

\textbf{Theorem.} Let $X$ be a Riemann surface, $K$ a compact subset of $X$. Then there exists a connected open subset $U$ of $X$, with $K \subset U$, and a compact Riemann surface $Y$ such that $U$ is biholomorphic to an open subset of $Y$. Moreover, if $X$ is planar then $Y$ can be chosen to be planar.

\textbf{Proof.} Without loss of generality we can assume that $K$ is connected. Choose a smooth function with compact support, $\varphi : X \to \mathbb{R}$, such that $\varphi(p) > 0$ for all $p \in K$. Let $V = \varphi^{-1}((0, +\infty))$, and $r = \inf\{\varphi(p); p \in K\}$. Observe that $r > 0$ because $K$ is compact. We have that $\varphi : V \to \mathbb{R}^+$ is proper. Let $E$ be the set of critical points of $\varphi$ in $V$. By Sard’s theorem (1.4.23) $\varphi(E)$ has zero measure in $\mathbb{R}^+$; and since $\varphi$ is proper on $V$, this set $\varphi(E)$ is closed in $\mathbb{R}^+$. Therefore there exist two
positive numbers, 0 < r_1 < r_2 < r, such that \( \varphi([r_1, r_2]) \cap E = \emptyset \). Let \( c \) be a point in the interval \((r_1, r_2)\), and \( U \) the connected component of \( \varphi^{-1}((c, +\infty)) \) that contains \( K \). We will show that \( U \) satisfies the conditions in the statement of the theorem.

First of all the boundary of \( U \) is a collection of components of \( \varphi^{-1}(c) \). Since \( \varphi \) is proper, and \( c \) is not a critical value of \( \varphi \), it follows from 1.4.24 that \( \partial U \) is a finite collection of curves, \( \{C_i\}_{i=1}^n \), where each curve \( C_i \) is diffeomorphic to the unit circle \( S^1 \).

For each \( i = 1, \ldots, n \) choose one such diffeomorphism, \( \phi : C_i \to S^1 \), and extend it to a smooth function \( \psi_i : V_i \to S^1 \), where \( V_i \) is a neighbourhood of \( C_i \). One can easily check that the Jacobian of the mapping

\[
g_i = (\psi_i, \varphi) : V_i \to S^1 \times \mathbb{R}
\]

is never zero. Therefore there exists a neighbourhood \( T_i \) of \( C_i \), \( T_i \subset V_i \), and a positive number \( \epsilon_i \), such that \( g_i : T_i \to S^1 \times (c - \epsilon_i, c + \epsilon_i) \) is a diffeomorphism. Fix \( \delta \) in \((0, c)\); by the Annulus theorem (3.2.5) we have that there exists a biholomorphic mapping \( h_i : g_i(S^1 \times (c - \delta, c + \delta)) \to A_{R_i} \), where \( A_{R_i} = \{z \in \mathbb{C}; 1 < |z| < R_i\} \).

We can further assume that \(|h_i| \to R_i\) near \( g_i^{-1}(S^1 \times \{c - \delta\}) \) and \(|h_i| \to 1\) near \( g_i^{-1}(S^1 \times \{c + \delta\}) \). If that were not the case we only need to compose \( h_i \) with the mapping \( z \mapsto R_i/z \), which interchanges the two components of the annulus \( A_{R_i} \).

We can thus use \( h_i \) to attach (smoothly) the disc \( D_i = \{z \in \mathbb{C}; |z| < R_i\} \) to \( U \), obtaining in this way a compact surface \( Y \), that clearly contains a biholomorphic copy of the set \( K \).

To complete the proof of the theorem we need to show that \( Y \) can be chosen to be planar when \( X \) is a planar surface. Let \( \omega \) be a closed 1-form with compact support on \( Y \). Since \( D_i \) is simply connected, closed forms are exact, and therefore there exists a smooth function \( f_i \) on \( D_i \), such that \( w|_{D_i} = df_i \). Let \( a_i \) be a positive number; if \( a_i \) is small enough we can choose a smooth function \( \chi_i \), with compact support on \( D_i \), such that \( \chi_i \equiv 1 \) on the disc of radius \( R_i - a_i \) (in \( D_i \)), and \( \omega' = \omega - \sum_i d(\chi_i f_i) \) is a closed 1-form with compact support on \( X \). Therefore we have that there exists a smooth function \( g \) on \( X \), with \( \omega' = dg \) on \( U \). Consider the form \( \omega'' = \omega' - d((1 - \sum_i \chi_i)g) \).

We have now that \( \omega'' \) is a closed form with compact support in the disjoint union \( \bigcup_i D_i \); hence \( \omega'' = \sum_i \omega''_i \), where \( \omega''_i \) is a closed form with compact support on \( D_i \).
Using again the fact that $D_i$ is simply connected we get functions $F_i$ such that $\omega_i'' = dF_i$ on $D_i$, where $F_i$ are smooth and have compact support on $D_i$. Therefore the form $\omega$ is exact, as we wanted to show.

3.2.7. Theorem. Any planar connected Riemann surface is biholomorphic to an open subset of $\hat{\mathbb{C}}$.

Proof. If $X$ is compact then we have that all forms have compact support. The planarity condition implies that the space of holomorphic 1-forms have zero dimension; that is, the surface has genus 0. We have already seen (2.3.5) that $X$ must be biholomorphically equivalent to the Riemann sphere $\hat{\mathbb{C}}$.

Suppose now that $X$ is not compact. Since $X$ is metrizable [3] we can write it as an increasing union of connected open subsets $U_n$, with compact closure. By theorem 3.2.6 we have that each $U_n$ is biholomorphic to an open subset of a planar compact Riemann surface $Y_n$, which by the above remarks should be biholomorphic to $\hat{\mathbb{C}}$. So we have a set of holomorphic, one-to-one (not necessarily surjective) mappings $f_n : U_n \to \mathbb{C}$. Choose a point $p$ in $U_1$, and a holomorphic chart $(U, z)$ around $p$, with $z(p) = 0$ and $U \subset U_1$. By replacing $f_n$ by $a_nf_n+b_n$, where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$, we can assume that, for all $n$:

1. $f_n(p) = 0$;  
2. $f_n(p) = dz(p)$.

Let $K_n$ be the set of one-to-one holomorphic functions on $U_n$ satisfying (1) and (2). We have $K_n$ are non-empty sets, and by Koebe’s theorem (3.2.1) each $K_n$ is compact. Hence the product $K = \prod_n K_n$ is compact.

The sets $E_m = \{(g_1, g_2, \ldots) \in K; g_m|_{U_n} = g_n, \text{for } n < m\}$ are non-empty and closed. Since $E_{m+1} \subset E_m$, we have that the intersection of all the $E_m$’s is non-empty. In other words, there exist holomorphic functions $g_m$, defined on $U_m$, such that $g_{m+1}|_{U_m} = g_m$, on $U_m$. So we have a holomorphic function $g$ on $X$ such that $g|_{U_m} = f_m$, for all $m$. Clearly $g$ is on-to-one, and therefore it defines a biholomorphic mapping from $X$ onto an open subset of $\mathbb{C}$.

3.2.8. Theorem (Poincaré-Koebe Uniformization Theorem). Any simply connected Riemann surface is biholomorphically equivalent to one (and only one) of the following three surfaces: the Riemann sphere, the complex plane or the unit disc (with their standard structures).
Proof. If \( X \) is simply connected then it satisfies the planarity condition, and therefore \( X \) is biholomorphically equivalent to \( \hat{\mathbb{C}} \) or a simply connected open subset of the complex plane. Using Riemann’s Mapping theorem, we have that, in the latter case, \( X \) is biholomorphic to either \( \mathbb{C} \) or the unit disc \( \mathbb{D} \).

It is not difficult to show that the Riemann surfaces \( \hat{\mathbb{C}}, \mathbb{C} \) and \( \mathbb{D} \) are not biholomorphic: first of all, the Riemann sphere is compact but the complex plane and the unit disc are not. The complex plane and the unit disc are not biholomorphic since any holomorphic function \( f : \mathbb{C} \to \mathbb{D} \) must be constant, by Liouville’s theorem. \( \square \)

3.3. Uniformization of Riemann surfaces and Kleinian groups

In this section we show that any Riemann surface can be written as a quotient \( \tilde{X}/G \), where \( \tilde{X} \) is a simply connected surface (studied in the previous section). The elements of the group \( G \) are Möbius transformations; we study some properties of these groups, which are elements of a big class of groups known as Kleinian groups.

3.3.1. Let \( X \) be a Riemann surface and \( \pi : \tilde{X} \to X \) a universal covering. From Topology (1.1.21) we have that \( X \) is homeomorphic to the quotient \( \tilde{X}/G \), where \( G \) is the group of deck transformations of the covering. The elements of \( G \) are homeomorphisms of \( \tilde{X} \), and non-identity transformations do not have fixed points in \( \tilde{X} \). In exercise 7 we asked the reader to show that \( \tilde{X} \) is a manifold. But we have more than that: there is a (unique) Riemann surface structure on \( \tilde{X} \) such that \( \pi \) becomes a holomorphic mapping. We have left the proof of this fact to the reader, but we include it here because of its importance. Let \( p \) be a point of \( X \) and \( U \) an evenly covered neighbourhood of \( p \). By shrinking \( U \) if necessary we can assume that there is a local coordinate defined on it, say \( z : U \to z(U) \subset \mathbb{C} \). Write \( \pi^{-1}(U) = \bigsqcup_j V_j \) as a disjoint union of open sets, where \( \pi|_{V_j} : V_j \to U \) is a homeomorphism. The mapping \( w_j = z \circ \pi|_{V_j} : V_j \to z(U) \) is a homeomorphism. We take on \( \tilde{X} \) the atlas consisting of all local coordinates of this form, \( \{(V_j, w_j)\} \).

To show that \( \tilde{X} \) is a Riemann surface we only need to check that the changes of coordinates are holomorphic mappings. But this is clear since changes of coordinates
on $\tilde{X}$ are equal to changes of coordinates on $X$. More precisely, if $(\tilde{U}, \tilde{z})$ is another local coordinate on $X$, with $\tilde{U}$ evenly covered, let $(\tilde{V}, \tilde{w})$ be a local coordinate on $\tilde{X}$ constructed as above. Assume $\tilde{V} \cap V_j \neq \emptyset$; then we have

$$w_j \circ \tilde{w}^{-1} = z \circ (\pi|_{V_j}) \circ (\pi|_{\tilde{V}})^{-1} \circ \tilde{z}^{-1} = z \circ \tilde{z}^{-1},$$

which is holomorphic since it is a change of coordinates on the Riemann surface $X$. Observe that we have taken restrictions of the mapping $\pi$ to sets where it is a homeomorphism, so we can consider the inverse function ($\pi$ in general will not have a global inverse).

The expression of $\pi$ in the above coordinates $(U, z)$ and $(V_j, w_1)$ is given by

$$z \circ \pi \circ w_1^{-1} = z \circ (\pi|_{V_j}) \circ (\pi|_{V_j})^{-1} \circ z^{-1} = Id : z(U) \to z(U),$$

which shows that $\pi$ is a holomorphic mapping.

The elements of $G$ are homeomorphisms; moreover, they are biholomorphic mappings in the above Riemann surfaces structure. To prove this statement consider $p$ a point of $\tilde{X}$ and $g \in G$. Let $p_1 = g(p)$ and denote by $q$ the point $q = \pi(p) = \pi(p_1)$.

Choose a local coordinate $(U, z)$ defined in a neighbourhood of $q$, with $U$ evenly covered, and let $V_0$ and $V_1$ the components of $\pi^{-1}(U)$ to which $p$ and $p_1$ belong, respectively. We have then local coordinates around these points given by $(V_0, w_0 = z \circ (\pi|_{V_0}))$ and $(V_1, w_1 = z \circ (\pi|_{V_1}))$. Observe that $g(V_0) = V_1$ and $\pi|_{V_1} \circ g = \pi|_{V_0}$. To see then that $g$ is holomorphic we need to compose it with these local coordinates:

$$w_1 \circ g \circ w_0^{-1} = z \circ (\pi|_{V_1}) \circ g \circ (\pi|_{V_0})^{-1} \circ z^{-1} = Id_{z(U)},$$

which proves our claim.

By the Uniformization theorem for simply connected surfaces (3.2.8) we have that there exists a biholomorphic mapping $f : Y \to \tilde{X}$, where $Y$ is the Riemann sphere, the complex plane or the unit disc. The mapping $\pi : \tilde{X} \to X$ is a covering if and only if $f \circ \pi : Y \to X$ is a covering, so we can assume that $\tilde{X}$ is one of the three mentioned surfaces. Putting all these facts together we obtain the general form of the Uniformization theorem.
Theorem (Uniformization theorem for Riemann surfaces). Let $X$ be a Riemann surface. Then $X$ is biholomorphic to $\tilde{X}/G$, where $\tilde{X}$ is the Riemann sphere, the complex plane or the unit disc, and $G$ is a group of biholomorphisms of $\tilde{X}$, isomorphic to the fundamental group of $X$.

Proof. We only need to show that $X$ is biholomorphic to $\tilde{X}/G$. From Topology we have that there exists a homeomorphism between these two surfaces; the proof that such mapping is actually holomorphic is similar to the above computations so we leave it to the reader. \hfill \Box

3.3.2. Let $p_0 \in X$ and choose $x_0 \in \tilde{X}$ satisfying $\pi(x_0) = p_0$. Let $U$ be an evenly covered neighbourhood of $x_0$ and $\pi^{-1} = \bigsqcup_j V_j$ as above. We have that $\pi^{-1}(p_0) = \{g(x_0); \ g \in G\}$, and if $g$ and $h$ are distinct elements of $G$ then $g(x_0)$ and $h(x_0)$ belong to different $V_j$'s. Since these sets are disjoint we have that $\pi^{-1}(p_0)$ is a discrete subset of $\tilde{X}$ (it does not have accumulation points). More precisely, if there is a sequence of transformations, say $\{g_n\}_n$ with $g_n(x_0) \to x_1$, then $\pi(x_1) = \lim_n \pi(g_n(x_0)) = \lim_n \pi(x_0) = p_0$. The point $x_1$ belongs to one of the sets $V_j$, say $V_{j_1}$. But then we will have that $g_n(x_0) \in V_{j_1}$ for $n \geq n_0$, a contradiction with the fact that $\pi$ restricted to $V_{j_1}$ is a homeomorphism.

3.3.3. The group of automorphisms (biholomorphic self-mappings) of the Riemann sphere, $Aut(\hat{C})$, is the group of Möbius transformations, as we have seen in corollary 1.3.14. We can identified $Aut(\hat{C})$ with a group of matrices (or rather, equivalence classes of matrices) as follows. Let $GL(2, \mathbb{C})$ denote the group of square matrices of order 2 with complex coefficients and non-zero determinant (equivalently, the group of invertible linear mappings of $\mathbb{C}^2$). We define an equivalence relation $\sim$ in this group by identifying $M_1$ and $M_2$ if there is a non-zero complex number, say $\lambda$, such that $M_2 = \lambda M_1$. The quotient space $PGL(2, \mathbb{C}) = GL(2, \mathbb{C})/\sim$ is known as the projective general linear group. If we consider the subgroup $SL(2, \mathbb{C})$ of $GL(2, \mathbb{C})$ of matrices of determinant equal to 1, and restrict the relation $\sim$ to complex numbers $\lambda$ with $|\lambda| = 1$, we obtain a quotient group $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\sim$, known as the special projective linear group. Since any non-zero number has a
square root in \( \mathbb{C} \) it is not difficult to see that \( \text{PGL}(2, \mathbb{C}) \) and \( \text{PSL}(2, \mathbb{C}) \) are isomorphic groups. The identification between \( \text{Aut}(\hat{\mathbb{C}}) \) and \( \text{PSL}(2, \mathbb{C}) \) is given by

\[
\text{Aut}(\hat{\mathbb{C}}) \to \text{PSL}(2, \mathbb{C})
\]

\[
A(z) = \frac{az + b}{cz + d} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix},
\]

where we use square brackets to denote equivalence classes of matrices in \( \text{PSL}(2, \mathbb{C}) \).

It is easy to see that this mapping is a group homomorphism. From now onwards we will freely interchange Möbius transformations with (classes of) matrices; for example we will write the composition of two transformations as \( AB \) instead of the more complicated notation \( A \circ B \).

3.3.4. Consider the map \( j : \text{PSL}(2, \mathbb{C}) \to \mathbb{P}^3 \) defined by \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto [a : b : c : d] \).

We can use \( j \) to put a topology on the group \( \text{Aut}(\hat{\mathbb{C}}) \): a sequence of Möbius transformation \( \{A_n\}_n \) converges to the transformation \( A \) if and only if \( j(A_n) \) converges to \( j(A) \). It is easy to see that this is equivalent to require that there exist elements of \( \text{PSL}(2, \mathbb{C}) \), \( \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \) and \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), corresponding to \( A_n \) and \( A \) respectively, such that \( a_n \to a, b_n \to b \) and so on.

Although this is the most natural topology of \( \text{Aut}(\hat{\mathbb{C}}) \) it does not behave nicely with respect to the “character” of the transformations. For example, the sequence of mappings \( \{A_n(z) = (1 + \frac{1}{n}) i z\}_n \) converges to \( A(z) = i z \); the transformation \( A \) is a rotation around the origin, it preserves the circles centred at that point, but the mappings \( A_n \) do not preserve any circle in the complex plane. Another example is provided by the sequence of transformations \( \{A_n\}_n \) given by

\[
A_n = \begin{bmatrix}
(n + 1)/n & 1 \\
0 & n/(n + 1)
\end{bmatrix}.
\]

Each of these mappings has two fixed points in \( \hat{\mathbb{C}} \), \( z_n = -\frac{n(n + 1)}{2n+1} \) and \( \infty \). The limit of this sequence is \( A(z) = z + 1 \), which has only one fixed point, namely \( \infty \) (the sequence of fixed points \( z_n \) converges to the point \( \infty \), so in the limit all fixed points “collapse”.)
3.3.5. The number of fixed points can be used to classify Möbius transformations. We start with an easy lemma.

**Lemma.** A non-identity Möbius transformation has at least one and at most two fixed points in \( \hat{\mathbb{C}} \).

**Proof.** The fixed points of the transformation \( A(z) = \frac{az+b}{cz+d} \) are given by the solutions of the equation \( A(z) = z \) in the Riemann sphere. If \( c \neq 0 \) we have a second degree equation, \( az+b = cz^2 + d \), which can have at most two distinct roots (and it has at least one). On the other hand, if \( c = 0 \) we can write \( A \) as \( A(z) = \lambda z + \mu \), where \( \lambda \neq 0 \). If \( \lambda = 1 \) the transformation \( A \) fixes only the point \( \infty \); in the case of \( \lambda \neq 1 \) the points \( \infty \) and \( \mu/(1-\lambda) \) are fixed by \( A \). \( \square \)

**Corollary.** If a Möbius transformation has three fixed points then it must be the identity.

Assume \( A \) has only one fixed point, say \( z_0 \). If \( z_0 = \infty \), then \( A \) is of the form \( A(z) = z + \mu \). Let \( S \) be the transformation \( S(z) = \frac{1}{\mu} z \) (since we are assuming that \( A \) has only one fixed point we have \( \mu \neq 0 \)); then \( SAS^{-1} \) is given by \( z \mapsto z + 1 \). If \( z_0 \neq \infty \), the transformation \( S_1(z) = \frac{1}{z-z_0} \) satisfies \( S_1AS_1^{-1}(z) = z + 1 \).

If \( A \) has two fixed points, say \( z_0 \) and \( z_1 \), let \( S_2(z) = \frac{z-z_0}{z-z_1} \), where we substitute a factor (numerator or denominator) by 1 if the corresponding fixed point is the point \( \infty \). It is easy to see that \( (S_2AS_2^{-1})(z) = \lambda z \) for some complex number \( \lambda \), with \( \lambda \neq 0, 1 \).

We can now give a classification of Möbius transformations.

**Definition.** Let \( A \) be a non-identity Möbius transformation. Then \( A \) is called

1. **parabolic**, if it is conjugate to \( z \mapsto z + 1 \);
2. **elliptic**, if it is conjugate to \( z \mapsto \lambda z \), where \( |\lambda| = 1 \) but \( \lambda \neq 1 \);
3. **loxodromic**, if it is conjugate to \( z \mapsto \lambda z \), where \( \lambda \neq 0, 1 \). If \( \lambda \) is real and positive \( A \) is called **hyperbolic**.

The above classification can be given in terms of the trace of the transformation as the following lemma shows. The proof is an easy exercise left to the reader.

**Lemma.** Let \( A(z) = \frac{az+b}{cz+d} \) be a Möbius transformation with \( ad - bc = 1 \). Assume
A is not the identity transformation. Then:

1. \( A \) is parabolic if and only if \((a + d)^2 = 2\);
2. \( A \) is elliptic if and only if \((a + d)^2 < 4\);
3. \( A \) is loxodromic if and only if \((a + d)^2\) does not belong to the interval \([0, 4]\).

In particular \( A \) is hyperbolic if and only if \((a + d)^2 > 4\).

Observe that \( A \) has order 2 if and only if \(a + d = 0\).

### 3.3.6.

We can now look with more detail to some particular cases of the Uniformization theorem. We start with the easiest situation, when the universal covering is the sphere.

**Proposition.** If \( X \) is a Riemann surface whose universal covering space is (biholomorphic to) the Riemann sphere then \( X \) is (biholomorphic to) the Riemann sphere.

**Proof.** Non-identity covering transformations do not have fixed points, but any Möbius transformation has at least two fixed points, so the covering group of \( \hat{\mathbb{C}} \to X \) must be trivial.

### 3.3.7.

The next case we consider is that of surfaces covered by the plane. For a biholomorphic mapping \( A : \mathbb{C} \to \mathbb{C} \) we have that the point \( \infty \) is a removable singularity when we consider \( A \) as a mapping defined on the Riemann sphere (take local coordinates and write \( A \) as a mapping from the punctured unit disc to itself). If we extend \( A \) to \( \hat{\mathbb{C}} \) we have that \( A \) is a Möbius transformation fixing the point \( \infty \) so it must be of the form \( A(z) = \lambda z + \mu \), with \( \lambda \neq 0 \) (see also [4, theorem 11.4, pg. 33]). In other words,

\[ Aut(\mathbb{C}) = \{g(z) = \lambda z + \mu; \lambda, \mu \in \mathbb{C}, \lambda \neq 0\}. \]

Assume that \( X \) is a Riemann surface covered by \( \mathbb{C} \) and let \( G \) be the group of covering transformations. Since the elements of \( G \), other than the identity mapping, cannot have fixed points, all transformations of \( G \) must then be of the form \( z \mapsto z + \mu \).

If \( G \) is the trivial group then clearly \( X \) is the complex plane. Assume now that \( G \) is cyclic; that is, it is of the form \( G = \{A_n(z) = z + n\mu; n \in \mathbb{Z}\} \). If we conjugate \( G \) by an automorphism of \( \mathbb{C} \), say \( S \), we obtain that \( \mathbb{C}/G \) and \( \mathbb{C}/SGS^{-1} \) are biholomorphic
surfaces. Thus we can assume \( \mu = 1 \). It is easy to see then that \( X \) is the punctured plane, \( \mathbb{C}^* = \{ z \in \mathbb{C}; \ z \neq 0 \} \), and the covering mapping \( \pi : \mathbb{C} \to \mathbb{C}^* \) is given by the exponential mapping, \( \pi(z) = e^{2\pi i z} \).

Suppose now that \( G \) has two generators; by a conjugation we can assume that \( A(z) = z + 1 \) is an element of \( G \). Let \( B(z) = z + \mu \) be another element of \( G \), not in the subgroup generated by \( A \). If \( \mu = p/q \) is rational we can assume that \( p \) and \( q \) are positive integers, with \( 0 < p < q \) and relatively prime. Let \( r \) and \( s \) be integers such that \( rp + sq = 1 \). Then \( (A^sB^r)(z) = z + (1/q) \) and \( G \) will be cyclic. If \( \mu \) is real but not rational we can write \( \mu = m + \epsilon \), for some integer \( m \) and a positive \( \epsilon \) in \( (0, 1) \). Since the pair \( \{ A(z) = z + 1, \ (A^{-m}B)(z) = z + \epsilon \} \) also generates \( G \) we can assume that \( m = 0 \). For each positive integer \( n \), there exists an integer \( p_n \), and a non-rational number \( \epsilon_n \) in \( (0, 1) \), such that \( n\epsilon = p_n + \epsilon_n \). Consider the elements \( C_n \) of \( G \) given by \( C_n = A^{-p_n}B^n \); these transformations are of the form \( C_n(z) = z + \epsilon_n \). We claim that the numbers \( \epsilon_n \) are all distinct: if \( \epsilon_n = \epsilon_m \) we will have \( \epsilon_n = n\epsilon - p_n \) and thus \( m\epsilon = p_m + n\epsilon - p_n \), which would imply that \( \epsilon \in \mathbb{Q} \). Since all the \( \epsilon_n \) are distinct we can get a subsequence, say \( \{ \epsilon_{n_j} \}_j \), converging to some point of \([0, 1]\). In such case the transformations \( C_{n_{j+1}}^{-1}C_{n_j} \) are all distinct and converge to the identity. But then \( (C_{n_{j+1}}^{-1}C_{n_j})(z) \to z \) for all \( z \in \mathbb{C} \), a contradiction with the definition of covering space (see also 3.3.11). Hence we have that \( B(z) = z + \mu \), with \( \mu \) not real; we can assume that \( \text{Im}(\mu) > 0 \) (take \( B^{-1} \) if necessary), and obtain that \( G \) is of the form \( G_\tau \), as in example 1.3.6, so \( X \) is a torus.

We claim that these three cases, the complex plane, the punctured plane and tori, are all the possibilities of Riemann surfaces covered by \( \mathbb{C} \). To prove the claim, let \( X \) be a Riemann surface of the form \( \mathbb{C}/G \). All the transformations of \( G \) are of the form \( T_\lambda : z \mapsto z + \lambda \). Let \( r = \min \{|\lambda|; \ T_\lambda \in G\} \). Observe that we take \( r \) to be a minimum, not the infimum: if \( T_r \) were not in \( G \) we could construct a sequence of distinct elements of \( G \) converging to the identity, using a trick similar to the above one (we leave the proof to the reader). By a conjugation we can assume that \( r = 1 \). Let \( \mu \) be such that \( T_\mu \in G \) and \( |\mu| \) is minimum among the transformations in \( G \) which are not of the form \( T_n(z) = z + n \), for \( n \) integer. If the group \( G \) is cyclic then \( X \) should be either the complex plane or the punctured plane. On the other hand,
if $G$ is not cyclic the argument above applies and we see that $\mu$ cannot be a real number. Thus $G$ contains a subgroup of the form $G_\mu$. We claim that $G = G_\mu$. If not, let $T_\lambda$ be an element in $G$ but not in $G_\mu$. Since $\{1, \mu\}$ are linearly independent over $\mathbb{R}$ we can write $\lambda = r + s\mu$, where $r$ and $s$ are real numbers, but not integers. Let $m_1$ and $m_2$ be two integers such that $|r - m_1| \leq 1/2$ and $|s - m_2| \leq 1/2$; the number $\lambda' = \lambda - m_1 - m_2\mu$ satisfies

$$|\lambda'| < \frac{1}{2} + \frac{1}{2} |\mu| \leq |\mu|,$$

where the first inequality is strict since $\mu$ is not a real number. But this contradicts the choice of $\mu$. These computations complete the proof of the following theorem.

**Theorem.** If $X$ is a Riemann surface whose universal covering space is $\mathbb{C}$, then $X$ is (biholomorphic to) $\mathbb{C}$, $\mathbb{C}^*$ or a torus.

It follows from this theorem and 3.3.6 that "most" surfaces are covered by the unit disc. In particular any compact surface of genus greater than 1 has $\mathbb{D}$ as its universal covering. We will see some applications of this fact in the next section (for example, the Riemann-Hurwitz theorem 3.4.20).

**3.3.8.** The next two results, which are easy consequences of Schwarz lemma, characterise the automorphisms of the unit disc.

**Lemma.** If $f : \mathbb{D} \to \mathbb{D}$ is a biholomorphism of the unit disc with $f(0) = 0$. Then $f$ is a rotation around the origin; that is, $f(z) = \lambda z$, for some complex number $\lambda$ of absolute value 1.

**Proof.** This result is part of Schwarz lemma (1.1.7).

**Proposition.** The automorphisms of the unit disc $\mathbb{D}$ are the Möbius transformations of the form $T_{w, \lambda}(z) = \frac{z - w}{1 - \bar{w}z}$, where $w \in \mathbb{D}$ and $|\lambda| = 1$.

**Proof.** We first need to show that these transformations are automorphisms of $\mathbb{D}$. Let $e^{i\theta}$ be a point of $\partial \mathbb{D}$ (the boundary of the unit disk); then we have

$$|T_{w, \lambda}(e^{i\theta})| = |\lambda| \left| \frac{e^{i\theta} - w}{1 - \bar{w}e^{-i\theta}} \right| = |\lambda| \left| \frac{e^{i\theta} - w}{e^{-i\theta} - \bar{w}} \right| = 1,$$

since the denominator of the last fraction is the complex conjugate of its numerator. This computation only shows that $T_{w, \lambda}(S^1) \subset S^1$. Since $T_{w, \lambda}^{-1}(z) = T_{-w, \lambda}$ we have
that $T_{w,\lambda}(S^1) = S^1$, and therefore $T(D)$ is equal to either $\mathbb{D}$ of $\hat{\mathbb{C}} \setminus \mathbb{D}$. But since the image of the origin is given by $T_{w,\lambda}(0) = -\lambda w$, which is a point in $\mathbb{D}$, we have that $T_{w,\lambda}(\mathbb{D}) = \mathbb{D}$.

Let $f : \mathbb{D} \to \mathbb{D}$ be an arbitrary automorphism of the unit disc. Write $w_0 = f(0)$. Then $T_{w_0,1} \circ f$ fixes the origin, so it must be a rotation by the previous lemma; that is $(T_{w_0,1} \circ f)(z) = \lambda z$ ($|\lambda| = 1$). A simple computation shows that $f(z) = T_{w_0,1}^{-1}(\lambda z) = T_{\lambda w,\lambda}(z)$.

3.3.9. From the Riemann Mapping theorem (or the Uniformization theorem) we have that the upper half plane $\mathbb{H}$ is biholomorphic to the unit disc $\mathbb{D}$; the M"obius transformation $T(z) = \frac{z+i}{z+i} : \mathbb{H} \to \mathbb{D}$ gives one such identification. To see this observe that for $x$ real we have that $|T(x)|$ is the ratio of the distance from $x$ to $i$ to the distance from $x$ to $-i$ and therefore $|T(x)| = 1$. By topological arguments we get that $T(\mathbb{R} \cup \{\infty\}) = S^1$ and $T(\mathbb{H})$ must be either the unit disc or its exterior. Since $T(i) = 0$ we have that $T(\mathbb{H}) = \mathbb{D}$. The advantage of using the upper half plane over the unit disc is that many computations are easier. For example, the next proposition shows that the automorphisms of $\mathbb{H}$ are just M"obius transformations with real coefficients and positive determinant, certainly simpler expressions than those of elements of $\text{Aut}(\mathbb{D})$.

**Proposition.** The automorphisms of $\mathbb{H}$ are the M"obius transformations of the form $A(z) = \frac{az+b}{cz+d}$, where $a, b, c, d$ are real numbers satisfying $ad - bc > 0$ (or equivalently $ad - bc = 1$). The group $\text{Aut}(\mathbb{H})$ acts transitively on $\mathbb{H}$; that is, for any two points $w_0$ and $w_1$ of $\mathbb{H}$, there exists an element $T \in \text{Aut}(\mathbb{H})$, such that $T(w_0) = w_1$.

**Proof.** If $A : \mathbb{H} \to \mathbb{H}$ is an automorphism of the upper half plane, then $TAT^{-1}$ is an automorphism of the unit disc, where $T(z) = \frac{z+i}{z+i}$. By 3.3.8 we have that $TAT^{-1}$ is a M"obius transformation and therefore $A$ is also a M"obius transformation. This shows that $\text{Aut}(\mathbb{H})$ is a group of M"obius transformations.

Let $G$ denote the group of M"obius transformations of the form given in the statement of the proposition; we want to show that $G$ is the full group of automorphisms
of \( \mathbb{H} \). If \( A \in G \) we have that
\[
A(z) = \frac{az + b}{cz + d} = \frac{a\bar{z} + b}{c\bar{z} + d} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz + d|^2},
\]
so
\[
\text{Im}(A(z)) = \frac{(ad - bc)\text{Im}(z)}{|cz + d|^2}.
\]
This shows that \( G \) is a subgroup of \( \text{Aut}(\mathbb{H}) \) (if \( z \) has positive imaginary part so does \( A(z) \)).

Let \( w_0 = x_0 + iy_0 \) be a point of the upper half plane. The transformation \( M(z) = \frac{z - x_0}{y_0} \) satisfies \( M(w_0) = i \) (since \( w_0 \in \mathbb{H} \) we have \( y_0 > 0 \)). We can write \( M \) as
\[
M(z) = \frac{z}{\sqrt{y_0}} - \frac{x_0}{\sqrt{y_0}},
\]
so \( M \) belongs to \( G \) (in the above expression we have taken the positive square root of \( y_0 \), which is possible since \( y_0 \) is a positive real number). Therefore \( G \) acts transitively on \( \mathbb{H} \) (map \( w_0 \) to \( i \) and then \( i \) to \( w_1 \)), and consequently \( \text{Aut}(\mathbb{H}) \) too.

If \( B \) is an element of \( \text{Aut}(\mathbb{H}) \) fixing the point \( i \) the transformation \( R = TBT^{-1} \) is an automorphism of \( \mathbb{D} \) that fixes the origin. Hence \( R(z) = \lambda^2 z \), for \( \lambda \) a complex number of absolute value 1. The matrices corresponding to \( R \) and \( T \) are
\[
R = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \quad \text{and} \quad T = \frac{1}{\sqrt{2i}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix},
\]
respectively. If we write \( \lambda = \cos(\theta) + i\sin(\theta) \) an easy calculation shows that \( B \) is given by
\[
B(z) = \frac{\cos(\theta) z + \sin(\theta)}{-\sin(\theta) z + \cos(\theta)},
\]
which belongs to \( G \).

Let now \( C \) denote any automorphism of \( \mathbb{H} \); we have \( C(w_0) = i \) for some point \( w_0 \) in the upper half plane. Let \( M \) be as above. Then \( MC^{-1} \) fixes the point \( i \), so it follows from the above computation that \( MC^{-1} \in G \). Since \( M \in G \) we have that \( \text{Aut}(\mathbb{H}) = G \).

In a way similar to the identification of \( \text{Aut}(\hat{\mathbb{C}}) \) with \( \text{PSL}(2, \mathbb{C}) \) we can give an isomorphism between \( \text{Aut}(\mathbb{H}) \) and \( \text{PSL}(2, \mathbb{R}) \), where this last group consists of equivalence classes of matrices with real coefficients and determinant 1. In this case
we do not have that \( \text{PSL}(2, \mathbb{R}) \) and \( \text{PGL}(2, \mathbb{R}) \) are isomorphic, since a matrix in \( \text{GL}(2, \mathbb{R}) \) with negative determinant cannot be equivalent to a matrix with positive determinant (negative numbers do not have square roots in \( \mathbb{R} \)).

3.3.10. What elements of \( \text{Aut}(\mathbb{H}) \) have fixed points in \( \mathbb{H} \)? First of all, if \( A \in \text{PSL}(2, \mathbb{R}) \) fixes the point \( z_0 \in \hat{\mathbb{C}} \), then \( A \) must also fix its conjugate \( \overline{z_0} \), since the coefficients of \( A \) are real (we understand \( \overline{\infty} = \infty \)). If \( A \) is parabolic its fixed point must be in \( \hat{\mathbb{R}} = \mathbb{R} \cup \{ \infty \} \). If \( A \) is elliptic, with \( ad - bc = 1 \), the solutions of the equation \( A(z) = z \) are given by

\[
z = \frac{a - d \pm \sqrt{(d - a)^2 + 4bc}}{2c} = \frac{a - d \pm \sqrt{d^2 + a^2 - 2ad + 4bc}}{2c} = \frac{a - d \pm \sqrt{(a + d)^2 - 4}}{2c}.\]

Since \( c \neq 0 \) and \( 0 \leq (a + d) < 4 \) the transformation \( A \) must have a fixed point in \( \mathbb{H} \). If \( A \) is loxodromic it must be hyperbolic and it is easy to see that its fixed points are both in \( \hat{\mathbb{R}} \). It follows from this computations that if \( X \) is a Riemann surface of the form \( X = \mathbb{H}/G \) then \( G \) does not have elliptic elements.

3.3.11. We next define a general class of groups of Möbius transformations and reformulate the Uniformization theorem.

**Definition.** A group of Möbius transformation \( G \) is said to act properly discontinuously at a point \( z \in \hat{\mathbb{C}} \) if there exists an open neighbourhood \( U \) of \( z \), such that the subgroup of \( G \) given by

\[
\{ g \in G; g(U) \cap U \neq \emptyset \}
\]

is finite. We denote by \( \Omega(G) \) the (open) set of points of the Riemann sphere where \( G \) acts properly discontinuously; this set is called the region of discontinuity of \( G \). The group \( G \) is called Kleinian if \( \Omega(G) \neq \emptyset \).

We can now rewrite the Uniformization theorem in terms of Kleinian groups.

**Theorem (Uniformization theorem).** Any (connected) Riemann surface \( X \) is biholomorphic to a quotient of the form \( \hat{X}/G \), where \( \hat{X} \) is the Riemann sphere, the complex plane or the unit disc, and \( G \) is a Kleinian group satisfying \( \hat{X} \subset \Omega(G) \).

**Remark.** Observe that in this above result we do not claim that \( \hat{X} \) is equal
to the region of discontinuity $\Omega(G)$ of $G$. There are cases when these two sets are different. For example, if $X$ is the punctured unit disc $\mathbb{D}^*$, then $\tilde{X} = \mathbb{H}$ and $G = \{z \mapsto z + n; \ n \in \mathbb{Z}\}$. But the region of discontinuity of $G$ is the whole complex plane and $\mathbb{C}/G$ is the punctured plane $\mathbb{C}^*$ (the upper half plane covers the punctured unit, the lower half plane the exterior of the unit disc and $\mathbb{R}$ covers $S^1$).

3.3.12. Proposition. Kleinian groups are discrete.

Proof. By discrete we mean that $G$ does not have accumulation points in $\text{PSL}(2, \mathbb{C})$, with the topology described in 3.3.4. Assume that $G$ is not discrete; then there exists a sequence of distinct elements of $G$, say $\{A_n\}_n$, such that $A_n \to A$, where $A$ is a Möbius transformation, not necessarily in $G$. The sequence of Möbius transformations $\{B_n = A_{n+1}^{-1}A_n\}_n$ has infinitely many distinct elements and converges to the identity. But then $B_n(z) \to z$, for all $z$ in $\hat{\mathbb{C}}$, and therefore $G$ cannot be Kleinian.

Let $G$ be a Kleinian group, $A$ an elliptic transformation of $G$. By conjugating with an element of $\text{PSL}(2, \mathbb{C})$ if necessary we can assume that $A$ is of the form $A(z) = e^{i\theta}z$. It is easy to see that $A$ has finite order if and only if $\theta$ is a rational number. Assume now that $\theta \notin \mathbb{Q}$, and define a mapping $j : <A> \to S^1$, where $<A> = \{A^n; \ n \in \mathbb{Z}\}$ is the subgroup of $G$ generated by $A$, by the expression $j(A^n) = e^{in \theta}$. Since $A$ does not have finite order we get that the image of $j$ is an infinite set of $S^1$ and therefore it has an accumulation point, say $e^{i\theta_0}$. Let $\{n_j\}_j$ be a sequence of integers such that $e^{in_j \theta} \to e^{i\theta_0}$. Then the transformations $A_{n_j}$ converge to $z \mapsto e^{i\theta_0}z$, so $G$ cannot be Kleinian. We have proved the following result.

Proposition. If $G$ is a Kleinian group and $A$ is an elliptic element of $G$ then $A$ has finite order.

3.3.13. Another interesting property of Kleinian groups is given in the following proposition.

Proposition. A Kleinian group is either finite or infinite countable.

Proof. Let $z_0$ be a point in the region of discontinuity of $G$ and $H$ the stabiliser of $z_0$ in $G$; that is,

$$H = \{g \in G; \ g(z_0) = z_0\}.$$
Since $G$ acts properly discontinuously at $z_0$ we have that $H$ is finite. Let $G(z_0)$ denote the orbit of $z_0$ under the given group, $G(z_0) = \{g(z_0); g \in G\}$. We have that $G(z_0)$ is a discrete set of $\C$ and it must then countable (exercise 83). On the other hand, it is easy to see that there is a bijection between $G/H$ and $G(z_0)$, given by $[g] \mapsto g(z_0)$. It follows that $G$ is either finite or infinite countable.

3.3.14. Our next application of the Uniformization theorem is to determine all surfaces with abelian fundamental group.

**Lemma.** Let $A$ and $B$ be two Möbius transformations, neither of them equal to the identity. Assume that $AB = BA$. Then one and only one of the following cases is satisfied:

1. if $A$ is parabolic then $B$ is also parabolic and they have the same fixed points;
2. if $A$ is not parabolic then $B$ is not parabolic and either they have the same fixed points, or both transformations have order 2, and each of them interchanges the fixed points of the other.

**Proof.** First of all, the results of the lemma are invariant under conjugation, so we can choose the fixed points of the transformations in a way that computations are easy. Observe that if $A$ fixes a set $W$ pointwise (that is, $A(w) = w$ for all $w$ in $W$), then $B$ fixes $W$ as a set, $B(W) = W$, although $B$ does not need to fix each point of $W$. Clearly this statement holds if we interchange $A$ and $B$.

Assume first that $A$ is parabolic, say $A(z) = z + 1$ (remember that we are free to conjugate $A$ and $B$ for our computations). Then $B(\infty) = B(A(\infty)) = A(B(\infty))$ so $B(\infty)$ must be a fixed point of $A$, which implies that $B(\infty) = \infty$, and hence $B$ is of the form $B(z) = \lambda z + \mu$. If $B$ fixes a point $z_0$ in $\C$, from the above remark we see that $A$ must fix the point $z_0$, which cannot happen by hypothesis. Hence $\lambda = 1$ and $B$ is a parabolic transformation with fixed point $\infty$.

Assume now that neither of the transformation is parabolic (by the above computation, if one transformation is parabolic so is the other). Let $A$ be of the form $A(z) = \lambda z$. From the above computations we have that $B(\{0, \infty\}) = \{0, \infty\}$, so there are two possible cases:

1. $B(\infty) = \infty$ and $B(0) = 0$. Then $A$ and $B$ have the same fixed points.
2. $B(\infty) = 0$ and $B(0) = \infty$. In this case $B(z) = \mu/z$. By a conjugation
that does not change $A$ we can assume that $\mu = 1$. Then $B$ fixes $\pm 1$. Since $A(\{1, -1\}) = \{1, -1\}$ we see that $A(z) = -z$ (the possibility of $A$ being the identity does not occur by hypothesis), so $A$ has order 2 and this completes the proof. \qed

3.3.15. **Theorem.** Suppose $X$ is a Riemann surface with abelian fundamental group. The one (and only one) of the following cases occurs:

1. $X$ is simply connected and $X$ is $\hat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{D}$;
2. $\pi_1(X, x_0) \cong \mathbb{Z}$ and $X$ is $\mathbb{C}^*$, $\mathbb{D}^*$ or $A_r = \{z \in \mathbb{C}; \ r < |z| < q\}$, for some real number $r \in (0, 1)$;
3. $\pi_1(X, x_0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $X$ is a torus.

**Remark.** In the above theorem all statements have to be understood “up to biholomorphisms”.

**Proof.** The first case is the Uniformization theorem for simply connected surfaces; the surfaces with universal covering space the complex plane have been studied in 3.3.7. Thus we have only to study surfaces with abelian fundamental group and the upper half plane as the universal covering space. Moreover, we can assume that the fundamental group is not trivial.

If $\pi_1(X, x_0)$ is cyclic, then $X = \mathbb{H}/ < A>$, where $A$ is an element of $\text{PSL}(2, \mathbb{R})$. Since non-trivial deck transformations do not have fixed points $A$ will be either parabolic or hyperbolic. In the first case we can assume that $A(z) = z + 1$, after a conjugation and taking inverses if necessary. Then $X = \mathbb{D}^*$ and the covering mapping is $z \mapsto \exp(2\pi iz)$. In the case of $A$ hyperbolic we have $A(z) = \lambda z$, for some number $\lambda > 1$; we get that $X$ is an annulus, with covering mapping

$$z \mapsto \exp\left(2\pi i \frac{\log z}{\log \lambda}\right),$$

where $\log$ is the principal branch of the logarithm, $\log(re^{i\theta}) = \log r + i\theta$. The radius of the annulus is given by $r = \exp\left(\frac{-2\pi^2}{\log^2 \lambda}\right) \in (0, 1)$.

To complete the proof of the theorem we need to show that there are no Riemann surfaces with abelian fundamental group of rank greater than 1 and universal covering space the upper half plane. If one of the generators of $G$, say $A$, is parabolic, we can assume that $A(z) = z + 1$. By lemma 3.3.14 all elements of $G$ are also parabolic, and because they are automorphisms of $\mathbb{H}$, they must be of the form $z \mapsto z + t,$
for $t$ real. But when we studied surfaces covered by $\mathbb{C}$ we saw that in that case $G$ would not be discrete. If $A$ is hyperbolic, we have that all elements of $G$ are also hyperbolic; a similar proof shows that this case cannot occur.

3.4. Hyperbolic Geometry, Fuchsian Groups and Hurwitz’s Theorem

In this section we will study some properties of groups of automorphisms of the upper half plane. We show that there exists a natural metric on $\mathbb{H}$, called the hyperbolic metric, for which the elements of $\text{Aut}(\mathbb{H})$ are isometries. It follows from this that we can put a metric on compact surfaces (of genus greater than 1). A somehow surprising result is that the area of a surface does not depend of the Riemann surface structure. We will also prove that the group of automorphisms of compact surfaces (covered by $\mathbb{H}$) is finite.

3.4.1. If $\gamma : [a, b] \rightarrow \mathbb{R}^2$ is a piecewise smooth curve (1.4.18) its length in Euclidean geometry is given by the integral

$$||\gamma||_E = \int_a^b |\gamma'(t)| \, dt.$$  

(Since $\gamma$ is piecewise smooth the integral is finite.) This statement is usually formulated by saying that the infinitesimal length element (the length of the tangent vector $\gamma'$) is $|dz|$. The distance $d_E$ between two points $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1)$ is the length of the segment joining them, that is:

$$d(p_0, p_1) = ||s||,$$

where

$$s(t) = (tx_0 + (1-t)x_1, ty_0 + (1-t)y_1), \quad \text{for } 0 \leq t \leq 1.$$  

One can check that the length of this curve is minimum among the lengths of the curves joining $p_1$ and $p_2$; that is,

$$d_E(p_0, p_1) = ||s|| = \inf\{||\gamma||; \gamma(a) = p_1, \gamma(b) = p_2\}.$$
The segments are called geodesics. It can be easily verified that the distance between any two points in a geodesic is given by the length of the piece of the geodesic joining them.

3.4.2. By the above argument we see that to define a metric on the upper half plane it suffices to give its infinitesimal length element. We set this to be equal to \( ds = \frac{|dz|}{\text{Im}(z)} \). As explained in the case of the Euclidean metric, this simply means that the length of a (piecewise smooth) curve \( \gamma : [a, b] \to \mathbb{H} \) is given by the following expression:

\[
||\gamma|| = \int_a^b \frac{|\gamma'(t)|}{\text{Im}(\gamma(t))} \, dt.
\]

This integral is finite because the curve is assumed to be piecewise smooth. Similarly one defines the distance \( d \) between two points \( z_0 \) and \( z_1 \) of \( \mathbb{H} \) by

\[
d(z_0, z_1) = \inf \{ ||\gamma||; \gamma(a) = z_0, \gamma(b) = z_1 \}.
\]

We will use \( d \) for this new distance and \( d_E \) for the (standard) Euclidean distance. \( ds \) and \( d \) are called the hyperbolic metric and hyperbolic distance, respectively. We need to prove that \( d \) is indeed a distance, but before that we will study some properties of the metric \( ds \) and its relation with Möbius transformations.

3.4.3. Before proceeding further we recall some results from Complex Analysis.

Definition. Let \( z_j, j = 1, \ldots, 4 \) be four distinct points in \( \hat{\mathbb{C}} \); the cross ratio of these points is defined by

\[
(z_1, z_2; z_3, z_4) = \frac{z_4 - z_2}{z_4 - z_1} \frac{z_3 - z_1}{z_3 - z_2},
\]

where we delete the corresponding terms (or we take limits) if one of the points is the point \( \infty \).

Observe that in the case of one of the four points being \( \infty \) there will be two terms in the above expression with \( \infty \) in them, one in the numerator and the other in the denominator, so after removing those terms we are left with a well defined fraction. Some authors change the order of the factors in the definition of cross ratio. However, for the applications all definitions are equivalent. It can be easily proved that of the possible 24 definitions of cross ratio (there are 24 permutations of four letters) there are only 6 different values.
If $z_1 = \infty$, $z_2 = 0$ and $z_3 = 1$ then $(z_1, z_2; z_3, z_4) = z_4$. More generally, an easy computation shows that $S(z) = (z_1, z_2; z_3, z)$ is the value (at $z$) of the unique Möbius transformation $S$ that takes $z_1$, $z_2$ and $z_3$ to $\infty$, 0 and 1, respectively. This remark will be useful in the proof of the following result.

**3.4.4. Lemma.** Möbius transformations preserve cross ratios. More precisely, if $T$ is a Möbius transformation, and $z_j$, $j = 1, \ldots, 4$ four distinct points in the Riemann sphere, then $(T(z_1), T(z_2); T(z_3), T(z_4)) = (z_1, z_2; z_3, z_4)$.

**Proof.** The proof can be done with an easy (but long) direct calculation; however, with the last remark in the above subsection we can get a short and elegant proof as follows. Let $S$ be the Möbius transformation that takes $z_1$, $z_2$ and $z_3$ to $\infty$, 0 and 1, respectively (this $S$ is given by $S(z_4) = (z_1, z_2; z_3, z_4)$). Then $ST^{-1}$ takes $T(z_1)$, $T(z_2)$ and $T(z_3)$ to $\infty$, 0 and 1 respectively and therefore we have

$$(T(z_1), T(z_2); T(z_3), T(z_4)) = ST^{-1}(T(z_4)) = S(z_4) = (z_1, z_2; z_3, z_4).$$

\(\square\)

**3.4.5. Lemma.** For distinct points in the Riemann sphere lie on a line or circle if and only if their cross ratio is real.

**Proof.** We will show first that the image of $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ under a Möbius transformation is a line or a circle. If $S(z) = (z_1, z_2; z_3, z)$ is given by $S(z) = \frac{az + b}{cz + d}$, then $S(z)$ is real if and only if $S(z) = \overline{S(z)}$; that is,

$$\frac{az + b}{cz + d} = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}.$$ 

From this expression we obtain

$$(a\bar{c} - \bar{a}c)|z|^2 + (a\bar{d} - \bar{b}c)z + (b\bar{c} - \bar{a}d)\bar{z} + (b\bar{d} - \bar{b}d)z = 0. \quad (6)$$

If $a\bar{c} - \bar{a}c = 0$ then we must have $a\bar{d} - \bar{b}c \neq 0$. Otherwise we get the following pair of equations

$$a\bar{c} = \bar{a}c, \quad a\bar{d} = \bar{b}c.$$ 

If $a \neq 0$ we have $\bar{d} = \frac{\bar{b}}{\bar{a}}$, and thus $ad - bc = a\frac{\bar{b}}{\bar{a}} = a\frac{\bar{b}}{\bar{a}} - bc = 0$, which is not possible. On the other hand, if $a = 0$ we get $\bar{b}c = 0$, which again gives us $ad - bc = 0$. So
we see that \( ad - bc \neq 0 \). Write \( ad - bc = u + iv, \, z = x + i \) and \( b \bar{d} = r + is \); then equation (6) becomes

\[
v x + u y + s = 0.
\]

Since \( u \) and \( v \) cannot be simultaneously equal to 0 we get that this is the equation of a line.

If \( a \bar{c} - \bar{a}c \neq 0 \) equation (6) is equivalent to

\[
|z + \frac{\bar{a}d - b\bar{c}}{\bar{a}c - a\bar{c}}| = \frac{|ad - bc|}{|\bar{a}c - a\bar{c}|},
\]

which is the equation of a circle.

To complete the proof of the lemma we argue as follows. If \((z_1, z_2; z_3, z_4)\) is a real number then \( z_j \) lies in \( S^{-1}(\hat{\mathbb{R}}) \), where \( S \) is the Möbius transformation that defines the cross ratio (i.e. \( S(z) = (z_1, z_2; z_3, z) \)). By the first part of the proof we have that \( S^{-1}(\hat{\mathbb{R}}) \) is either a line or a circle.

Suppose now that \((z_1, z_2; z_3, z_4)\) lie in a line or circle, say \( C \). If we consider the transformation \( S \) once more we have that \( S^{-1}(0), \, S^{-1}(1) \) and \( S^{-1}(\infty) \) are in \( C \), so \( S^{-1}(\hat{\mathbb{R}}) = C \), and therefore \( S(C) = \hat{\mathbb{R}} \). Thus \( S(z_4) = (z_1, z_2; z_3, z_4) \) is a real number (it cannot be \( \infty \) by the definition of cross ratio).

**Corollary.** If \( C \) is the family of lines and circles in \( \hat{\mathbb{C}} \) and \( A \) is a Möbius transformation, then \( A(C) = C \).

**3.4.6.** The Möbius transformation \( T(z) = \frac{z - i}{z + i} : \mathbb{H} \to \mathbb{D} \) can be used to define a hyperbolic metric on the unit disc such that \( T \) becomes an isometry. This means that if the metric on \( \mathbb{D} \) is given by \( \lambda(z) |dz| \), where \( \lambda \) is a positive function, then we must have

\[
\lambda(T(z)) |T'(z)| = \frac{1}{\text{Im}(z)},
\]

for \( z \) in \( \mathbb{H} \). If that is the case a simple use of the change of variables theorem for integrals shows that \( ||\gamma|| = ||T(\gamma)|| \) for a piecewise smooth curve on \( \mathbb{H} \). It is easy to see that \( \lambda \) is given by the expression

\[
\lambda(z) = \frac{2}{1 - |z|^2}.
\]
3.4.7. **Proposition.** $\text{Aut}(\mathbb{H})$ acts by isometries with respect to the hyperbolic metric: for any piecewise smooth curve $\gamma : [a,b] \to \mathbb{H}$ and any Möbius transformation $A \in \text{Aut}(\mathbb{H})$, one has $||\gamma|| = ||A(\gamma)||$.

**Proof.** Let $A$ be given by $A(z) = \frac{az+b}{cz+d}$, where the coefficients are real and satisfy $ad - bc = 1$. We have $A'(z) = \frac{\text{Im}(z)}{(cz+d)^2}$. In 3.3.8 we computed that

$$\text{Im}(A(z)) = \frac{\text{Im}(z)}{|cz+d|^2}.$$ 

Using these expressions we get

$$\frac{|A'(z)|}{\text{Im}(A(z))} = \frac{1}{\text{Im}(z)},$$

so

$$||A(\gamma)|| = \int_a^b \frac{|A'(t)||\gamma'(t)|}{\text{Im}(A(\gamma(t)))} \, dt = \int_a^b \frac{|\gamma'(t)|}{\text{Im}(\gamma(t))} \, dt = ||\gamma||.$$  

\[\square\]

3.4.8. **Theorem.** $d$ is a distance in $\mathbb{H}$. The topology induced by it is the standard Euclidean topology.

**Proof.** It is easy to see that $d$ is symmetric, non-negative and satisfies the triangle inequality. We will show that $d(w_0, w_1)$ is strictly positive for $w_0 \neq w_1$. We will work in the unit disc, since the computation is easier in this case, and in the process we will obtain a formula for the distance of a point in $\mathbb{D}$ to the origin that will be useful later.

Let $w_0$ and $w_1$ be two distinct points in $\mathbb{D}$. Using the Möbius transformation

$$M(z) = \frac{z - w_0}{1 - \overline{w_0}z},$$

we can assume that $w_0 = 0$. By a rotation we can further assume that $w_1 = t$, where $t$ is a point in the open interval $(0, 1)$. Consider a path $\gamma : [0, 1] \to \mathbb{D}$ joining 0 and $t$. If we write $\gamma(t) = x(t) + iy(t)$, we have that

$$||\gamma|| = \int_0^1 \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} \, dt \geq \int_0^1 \frac{2|x'(t)|}{1 - x(t)^2} \, dt \geq \int_0^{\frac{2t}{1-t^2}} \frac{1}{1 - t} \, dt = \log \left( \frac{1 + t}{1 - t} \right).$$
In the second inequality we have used that the function \( f(s) = \frac{1}{1-s^2} \) is negative for \( s < 0 \) and increasing for \( s \geq 0 \). Observe that the hyperbolic length of the path \( \gamma(s) = st, \ s \in [0, 1] \), is precisely the above expression, \( \log \left( \frac{1+t}{1-t} \right) \); thus \( d(0, t) = d(w_0, w_1) > 0 \). This completes the proof of the fact that \( d \) is a distance.

To prove that the topology induced by \( d \) is the standard topology of \( \mathbb{D} \) we use the above computations. We have that the hyperbolic disc \( D_h(0, r) \), of centre 0 and radius \( r > 0 \), is given by

\[
D_h(0, r) = \{ z \in \mathbb{D}; |z| < \frac{e^r - 1}{e^r + 1} \};
\]

that is, an Euclidean disc of centre 0 and different radius. This shows that the neighbourhoods of 0 in the hyperbolic and Euclidean topologies are the same. Since the group of M"obius transformations acts transitively by homeomorphisms in \( \mathbb{D} \) (3.3.8) we have that both topologies are the same.

As a corollary of the above computations we get that the distance from 0 to any point \( w \in \mathbb{D} \) is given by

\[
d(0, w) = \log \left( \frac{1 + |w|}{1 - |w|} \right).
\]

In particular, as \( w \) approaches \( S^1 \) (in the Euclidean distance) we have that \( d(0, w) \) goes to infinity. This shows that the unit circle is at infinity distance of any point in the unit disc (apply the triangle inequality). Similarly the real axis is at infinite (hyperbolic) distance from points in the upper half plane. Thus the hyperbolic metric is the natural one if we want to study properties of \( \mathbb{H} \) (or \( \mathbb{D} \)) on its own, rather than considering it as a subset of the Riemann sphere.

### 3.4.9. A geodesic

A geodesic is a (smooth) curve that minimises the distance locally between points in it (its image). More precisely, if \( \gamma \) is a geodesic defined on the interval \((a, b)\) and \( t_0 \in (a, b) \), then there is a neighbourhood \( U \) of \( \gamma(t_0) \), such that the distance between any two points in \( \gamma((a, b)) \cap U \) is given by the length of \( \gamma \) between those two points. For example, if \( \gamma \) is a geodesic in the upper half plane, with the same notation we have that if \( \gamma(t_j) \in U \), for \( j = 1, 2 \),

\[
d(\gamma(t_1), \gamma(t_2)) = \int_{t_1}^{t_2} \frac{\gamma'(t)}{\Im(\gamma(t))} \, dt.
\]
However a geodesic does not need to minimise distances globally. Consider the case of the sphere $S^2$ where geodesics are given by great circles, that is, the intersection of planes through the origin with $S^2$. If $p_1$ and $p_2$ are two points in the sphere, not diametrically opposed, then there are two geodesics joining them, one of which will realize the distance between $p_1$ and $p_2$ while the other hand will have longer length. We also have that there could be more than one geodesic between two points. In the same example of the sphere, any two points diametrically opposed are joined by infinitely many different geodesics. And there are spaces where some points cannot be joined by geodesics. The space $\mathbb{R}^2\setminus\{(0,0)\}$ with the Euclidean metric is an example; the points $(1,0)$ and $(-1,0)$ are at distance $2$, but there is no geodesic between them realizing that distance. In the case of the hyperbolic metric we are in the best possible situation: any two points can be joined by a unique geodesic that realizes the distance between them.

3.4.10. Proposition. The hyperbolic geodesics of $\mathbb{D}$ are the circles and lines perpendicular to $S^1$. In the case of the upper half plane, the geodesics are the circles and lines perpendicular to the real axis. Given any two points in $\mathbb{D}$ (or $\mathbb{H}$) there exists a unique geodesic between them; moreover, such geodesic realizes the distance between any two points in its image.

Proof. In the proof of 3.4.8 we have obtained that the segment $(0,1)$ is a geodesic in $\mathbb{D}$. It is not difficult to see that the full diameter $(-1,1)$ is a geodesic. If $C$ is a circle (or line) perpendicular to $S^1$ it is possible to find an automorphism of the unit disc, say $A$, such that $A(C \cap \mathbb{D}) = (-1,1)$ (exercise 77). It follows from this that any circle or line orthogonal to $S^1$ is a geodesic. Conversely, if $\gamma$ is a geodesic in the unit disc, consider two points $w_j = \gamma(t_j)$, $j = 0, 1$, close enough so that $\gamma$ realizes the distance between them. By an automorphism of $\mathbb{D}$, say $A$, we can map $w_0$ to $0$ and $w_1$ to a point in $(0,1)$. Since the automorphisms of $\mathbb{D}$ are hyperbolic isometries we have that $\gamma$ should be contained in the image of $(-1,1)$ under $A^{-1}$ and thus it is a circle or line orthogonal to $S^1$ (to be more precise, the image of $(a,b)$ under $\gamma$ is contained on a circle or line orthogonal to $S^1$).
To show that any two points of $\mathbb{D}$ lie in a unique geodesic it suffices to consider that case of one point being the origin and the other a point $t$ in the interval $(0,1)$. But again this is part of the proof of 3.4.8.

3.4.11. **Theorem.** *The hyperbolic metric on $\mathbb{D}$ ($\mathbb{H}$) is complete.*

**Proof.** Let $\{z_n\}$ be a Cauchy sequence in the unit disc with respect to the hyperbolic metric. Given $\epsilon > 0$ there exists an $n_0$ such that $d(z_n, z_m) < \epsilon$, for $n, m \geq n_0$. Therefore $d(0, z_n) \leq d(0, z_{n_0}) + d(z_{n_0}, z_n)$, and thus $d(0, z_n)$ is bounded. The distance $d(0, z_n)$ is given by $\log\left(\frac{1+|z_n|}{1-|z_n|}\right)$, so there exists a number $0 < R < 1$, such that $|z_n| \leq R$, for all $n$. Since the set $\{z; |z| \leq R\}$ is compact in the Euclidean topology there exists a convergent subsequence. But the topologies induced by the hyperbolic and Euclidean metrics are equivalent, so that subsequence converges (to the same limit point) with respect to the hyperbolic metric. The triangle inequality shows that the full sequence $\{z_n\}_n$ converges in the hyperbolic metric.

3.4.12. **The angle** between two lines or circles in $\mathbb{H}$ meeting at a point $z_0$ is defined as the angle formed by the tangent lines to the curves at $z_0$. For simplicity we say that two lines or circles meeting at a point of $\mathbb{R}$ do it with angle equal to 0. A triangle is the portion of $\mathbb{H}$ enclosed by three distinct geodesic that meet pairwise. A triangle is called ideal if the geodesics meet in a point in the (extended) real axis. The hyperbolic area of a region $D$ of $\mathbb{H}$ is given by the integral

$$\text{Area}(D) = \int_D \frac{1}{y^2} \, dx \, dy.$$

**Theorem** (Gauss-Bonet for triangles). *The hyperbolic area of a triangle with angles $\alpha$, $\beta$ and $\gamma$ is equal to $\pi - (\alpha + \beta + \gamma)$.

**Proof.** Consider first the case of a triangle $T$ with two angles equal to 0. By using a Möbius transformation and the reflection $r(z) = -\bar{z}$ (which is also a hyperbolic isometry and preserve angles, see exercise 78) we can assume that $T$ is as in figure 13. In this case we can compute the area directly as follows:

$$\text{Area}(T) = \iint_T \frac{1}{y^2} \, dx \, dy = \int_0^d \int_0^\infty \frac{1}{y^2} \, dx \, dy = \int_0^d \frac{1}{\sqrt{c^2 - (x-c)^2}} \, dx = \int_0^\alpha -d\theta = \pi - \alpha.$$
If $T$ has only one angle equal to 0 we can compute its area as the difference of the area of two triangle, each of them with two zero angles, as in the figure 14. The general case follows in a similar way.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{triangle.png}
\caption{Triangle with two zero angles.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{gauss-bonet.png}
\caption{Gauss-Bonet.}
\end{figure}

3.4.13. We want now to apply some of these results to compact surfaces covered by the unit disc. We start with a definition.

\textbf{Definition.} A Kleinian group $G$ is called \textbf{Fuchsian} if there exists a disc or half plane $H$ which is invariant under the elements of $G$, that is, $g(H) = H$ for all $g \in G$.

When talking about Fuchsian groups we will use the word disc to mean a disc or a half plane.

A striking fact of Fuchsian groups is that discreteness and properly discontinuous action are almost equivalent.

\textbf{Proposition.} Let $G$ be a Fuchsian group with invariant disc $H$. The following are equivalent:
1. $G$ acts discontinuously on $H$ (i.e. $H \subset \Omega(G)$);

2. $G$ is discrete.

**Proof.** We have seen that Kleinian groups are discrete, so we only need to prove “$2 \Rightarrow 1$”; we will show this by contradiction. By a conjugation we can assume that $\mathbb{D}$ is the unit disc and $G$ does not act properly discontinuously at the origin. This means that for any neighbourhood $U$ of 0, the set $\{g \in G; g(U) \cap U \neq \emptyset\}$ is infinite (in particular $G$ is not finite, which is obvious since finite groups acts properly discontinuously on the whole Riemann sphere). Let $r_1$ be a positive number and define $U = D(0, r_1)$, in the hyperbolic metric. Let $z_1$ and $w_1$ be points in $U \setminus \{0\}$, such that $z_1 = g_1(w_1)$, for some transformation $g_1 \in G$. Choose a positive number $r_2$ satisfying $r_2 < \min\{d(z_1, 0), d(w_1, 0)\}$. We can find $z_2$ and $w_2$ in $U \setminus \{0\}$, such that $z_2 = g_2(w_2)$, for $g_2 \in G$, and $g_2 \neq g_1$. Continuing this process we find sequences of positive numbers $\{r_n\}_n$, points in the unit disc $\{z_n\}$ and distinct elements of $G \{g_n\}_n$, such that:

1. $r_n$ is a decreasing sequence converging to 0;
2. $d(0, z_n) < r_n$;
3. $d(g_n^{-1}(z_n), 0) < r_n$.

Since $G$ acts by isometries on the hyperbolic distance we have

$$d(0, g_n(0)) \leq d(0, z_n) + d(z_n, g_n(0)) = d(0, z_n) + d(g_n^{-1}(z_n), 0) < 2r_n.$$  

Let $w_n = g_n(0)$. By proposition 3.3.8 we have that $g_n$ is of the form

$$g_n(z) = \lambda_n \left( \frac{z + w_n}{1 + \overline{w}_n z} \right),$$

where $|\lambda| = 1$. Choose a subsequence $\{\lambda_{n_j}\}_j$ with $\lambda_{n_j} \to j \lambda_0$. Since $w_n \to 0$ we have that the transformations $g_{n_j}$ converge to the rotation $R(z) = \lambda_0 z$, and thus $G$ is not discrete.

**3.4.14.** The action of a Kleinian group on its region of discontinuity (or a part of it) is better understood by taking a set that contains one element of each orbit. For example, if $G$ is the group of translations $G = \{T_n(z) = z + n; n \in \mathbb{Z}\}$, acting on $\mathbb{H}$, we have that every point of $\mathbb{H}$ can be mapped by an element of $G$ to a point $z$, with $0 \leq \text{Re}(z) < 1$. If we consider the vertical strip $S = \{z \in \mathbb{H}; 0 \leq \text{Re}(z) \leq 1\},$
we have that $G$ identifies the two vertical lines in the boundary of this strip. The quotient space $\mathbb{H}/G$ is equivalent to $S/G$; it is easy to see (geometrically) that $S/G$ is a cylinder, which is clearly homeomorphic to the punctured disc. We have indeed proved that $\mathbb{H}/G$ is the punctured disc; this discussion might help us to understand why. The next definition generalises this situation to Fuchsian (or Kleinian) groups.

**Definition.** Let $G$ be a Fuchsian group acting on $\mathbb{H}$. A fundamental domain of $G$ for its action on $\mathbb{H}$ is a connected open set $D$ satisfying the following conditions:

FD1: for every element $g$ of $G$, not equal to the identity, $g(D) \cap D = \emptyset$;

FD2: for every $z$ in $\mathbb{H}$ there exists a transformation $g$ of $G$, such that $g(z)$ belongs to $\overline{D}$, the closure of $D$ in $\mathbb{H}$;

FD3: the boundary of $D$ in $\mathbb{H}$ consists of a countable number of smooth curves, called the sides of $D$. For every side $s$ there exists another side, say $s'$, not necessarily distinct from $s$, and an element $g$ of $G$, such that $g(s) = s'$ and $(s')' = s$;

FD4: (local finiteness) for every compact set $K$ of $\mathbb{H}$, the group

$$\{g \in G; \ g(K) \cap K \neq \emptyset\},$$

is finite.

**3.4.15.** The following result is needed to prove the existence of fundamental domains.

**Lemma.** Let $G$ be a non-cycle Fuchsian group with invariant disc $\Delta$. Then there exists a point $z_0 \in \Delta$ that is not fixed by any non-trivial element of $G$.

**Proof.** Assume $\Delta = \mathbb{D}$ and that 0 is fixed by some non-trivial element of $G$; let $H$ be the subgroup of $G$ consisting of the elements that fix 0. By lemma 3.3.8 all elements of $H$ are rotations around the origin. If $G = H$ by discreteness we have that $G$ must be cyclic. On the other hand, if $H$ is a proper subgroup, since $G$ acts properly discontinuously at 0, we can find a positive number $r$, such that $g(U) \cap U = \emptyset$, for all $g \notin H$, where $U$ is the disc of centre 0 and radius $r$. Any point in $U \setminus \{0\}$ will satisfy the conditions of the lemma.

**3.4.16.** Let $G$ be a Fuchsian group that leaves the upper half plane invariant and choose $p \in \mathbb{H}$ satisfying the conditions of the above lemma. For $g \in G$, not
equal to the identity, define
\[ H_g = \{ z \in \mathbb{H}; \ d(p, z) < d(g(p), z) \}, \]

where we use the hyperbolic metric to measure distances. Thus \( H_g \) consists of the points in \( \mathbb{H} \) that are closer to \( p \) than to \( g(p) \). Geometrically one can obtain \( H_g \) by considering the hyperbolic geodesic segment that joins \( p \) and \( g(p) \), say \( L \), and then taking the geodesic \( L' \) orthogonal to \( L \) on its midpoint; \( H_g \) will be the half plane determined by \( L' \) containing \( p \). The **Dirichlet region** \( D_p(G) \) of \( G \) (relative to \( p \)) is defined as the intersection of all such hyperplanes:
\[ D_p(G) = \bigcap_{g \neq Id} H_g. \]

For example, if \( G \) is the group of translations \( G = \{ T_n(z) = z + n; \ n \in \mathbb{Z} \} \), and \( p = (1/2) + i \), the Dirichlet region \( D_p(G) \) is precisely the (open) vertical strip we considered in 3.4.14: \( D_p(G) = \{ z \in \mathbb{H}; \ 0 < \text{Re}(z) < 1 \} \).

**Proposition.** The Dirichlet region is a fundamental domain for the action of \( G \) on \( \mathbb{H} \).

**Proof.** To simplify notation we will write \( D \) for the Dirichlet region \( D_p(G) \) (relative to some point \( p \)).

**Condition FD1.** Let \( g \neq Id \) be an element of \( G \) and \( z \) a point of \( D \). Since \( z \in H_{g^{-1}} \) we have
\[ d(g(p), g(z)) = d(p, z) < d(g^{-1}(p), z) = d(p, g(z)), \]
which implies that \( g(z) \) does not belong to \( D \).

**Condition FD2.** Let \( z \) be a point in the upper half plane. By the discontinuous action of \( G \) we have that there exists an element \( g \in G \) (not necessarily unique), such that \( d(g(p), z) \leq d(h(p), z) \) for all \( h \in G \). If we write the elements of \( G \) as \( g \circ h \) we have
\[ d(g^{-1}(z), p) = d(z, g(p)) \leq d(z, (g \circ h)(p)) = d(g^{-1}(z), h(p)). \]
This means that \( g^{-1}(z) \) belongs to the closure of \( H_h \) for all \( h \in G \) and therefore to the closure of \( D \).

**Condition FD3.** Since \( G \) is countable it is clear that the boundary of \( D \) has at
most countably many sides. Let z be a point in the relative interior of a side s. This is equivalent to say that there exists a unique element $g \in G$, such that

$$d(z, p) < d(z, h(p)), \quad \forall h \neq g, \quad Id \quad \text{and} \quad d(z, p) = d(z, g(p)).$$

Hence $d(g^{-1}(z), p) = d(g^{-1}(z), g^{-1}(p)) = d(z, p)$, and for $h \neq g^{-1}, Id$ we have

$$d(g^{-1}(z), h(p)) = d(z, (g \circ h)(p)) > d(z, p) = d(g^{-1}(z), p).$$

Thus $g^{-1}(z)$ belongs to the side $s'$ with $g^{-1}(s') = s$.

**Condition FD4.** Let $K \subset \mathbb{H}$ be compact. Without loss of generality we can assume that $K$ is the closed disc centred at $p$ and of hyperbolic radius $r$ (any compact subset of the upper half plane is contained in one such disc). We have that there are only finitely many images of $p$ (under transformations of $G$) in the disc of radius $2r$ centred at $p$. From this it follows that if $d(g^{-1}(p), p) > 2r$ then $g(D) \cap K = \emptyset$.

3.4.17. The following lemma is easy to prove; it shows that the Dirichlet region is a “good” choice of fundamental domain.

**Lemma.** Let $G$ be a Fuchsian group acting on $\mathbb{H}$ and $D$ a Dirichlet region of $G$.

1. The quotient surface $\mathbb{H}/G$ is compact if and only if $\overline{D}$ is compact in $\mathbb{H}$.
2. If $\overline{D}$ is compact then $D$ has only finitely many sides.

3.4.18. Assume that $X$ is a Riemann surface given by $\mathbb{H}/G$; that is, $X$ is biholomorphic to the quotient surface $\mathbb{H}/G$. We can put a metric on $X$ by using the natural projection $\pi : \mathbb{H} \to X$ similar to the way we calculated the metric on $\mathbb{D}$ from the mapping $T : \mathbb{H} \to \mathbb{D}$. Although $\pi$ does not need to be globally one-to-one, it is so locally, and that is all we need. More precisely, for a point $p_0$ in $X$, let $U$ be an evenly covered neighbourhood of $p_0$, and write $\pi^{-1}(U) = \bigsqcup V_j$ for the decomposition of the preimage of $U$ in disjoint open sets of $\mathbb{H}$, each homeomorphic via $\pi$ to $U$. The functions $z_j(\pi(p)) = p$, for $p \in V_j$, serve as local coordinates on $X$.

Thus to define a metric on $X$ all we need to do is to find expressions of the form $\lambda_j(\pi(p)) |dz_j|$, such that $\lambda_j = \lambda_k \left| \frac{dz_k}{dz_j} \right|$. We then set $\lambda_j(\pi(p)) = \frac{1}{\text{Im}(p)}$, for $p \in V_j$. For any other set $V_k$ as above, there exists an element $g \in G$, such that $g(V_j) = V_k$. The value of $\lambda_k$ is given by $\lambda_k(\pi(q)) = \frac{1}{\text{Im}(q)}$, and $z_k(\pi(q)) = q = g(p) = g(z_j(p))$, for
q ∈ V_k, so p = g^{-1}(q) belongs to V_j. Under these circumstances we have that
\[
\frac{1}{\text{Im}(q)} = \frac{1}{\text{Im}(g(p))} = \frac{|g'(p)|}{|\text{Im}(p)|} \left| \frac{dz_k}{dz_j} \right|
\]

We call this metric the hyperbolic metric of the surface. Observe that the metric depends on the complex structure of X; however, the area of X does not, as the following result shows.

**Theorem** (Gauss-Bonet for compact surfaces). *If X is a compact surface of genus g ≥ 2 then the hyperbolic area of X is equal to 2π(2g − 2).*

**Proof.** Let D be a Dirichlet region for G. By 3.4.17 we know that D has only finitely many sides, so its boundary has zero area. On the other hand, by property FD1 we have that \( \pi : D \to X \) is one-to-one, so \( \text{Area}(D) = \text{Area}(X) \). Choose a point \( p_0 \) in the interior of D and join it to the (finitely many) vertices of the boundary of D by geodesics. This is possible, since D is a finite intersection of convex sets, and thus it is convex. In this way we obtain a triangulation of D that projects to a triangulation of X. Assume that the sides and vertices of \( \partial D \) project to E sides and V vertices on X. It is not difficult to see, using the Euler-Poincaré formula, that \( V - E + 1 = 2 - 2g \). On the other hand, by the Gauss-Bonet theorem for triangles we have

\[
\text{Area}(D) = 2E \pi - 2\pi - \sum \text{(interior angles at the vertices)}.
\]

The term \(-2\pi\) comes from the sums of the angles at \( p_0 \). Since the vertices of \( \partial D \) project to V points in X, we have that the sum of the interior angles is \( 2\pi V \), and therefore

\[
\text{Area}(X) = \text{Area}(D) = 2\pi(2g - 1 + V) - 2\pi = 2\pi(2g - 2).
\]

\[\square\]

**3.4.19.** Our next goal is to study automorphisms of compact surfaces. Let \( \pi : \mathbb{H} \to X \) be the universal covering of a compact surface of genus \( g \geq 2 \), with covering group \( G \). If \( f : X \to X \) is an automorphism we can lift it to a biholomorphic mapping \( A : \mathbb{H} \to \mathbb{H} \); in particular \( A \) is a Möbius transformation. The
transformation $A$ satisfies $\pi A = f\pi$; for any element $g \in G$ we have

$$\pi A g A^{-1} = f \pi g A^{-1} = f \pi A^{-1} = f f^{-1} \pi = \pi,$$

so there exists an element $h \in G$, such that $A g A^{-1} = h$. In other words, $A$ belongs to $N(G)$, the normaliser of $G$ in $\text{Aut}(\mathbb{H})$. (The group $N(G)$ is the biggest subgroup of $\text{Aut}(\mathbb{H})$ on which $G$ is normal; it consists of the element $B \in \text{Aut}(\mathbb{H})$, such that $B G B^{-1} = G$). The converse statement is also true; namely, if $B \in N(G)$, then the expression $h(\pi(z)) = \pi(B(z))$ defines an automorphism, $h$, of $X$. Since the elements of $G$ will project to the identity mapping of $X$, we can identified $\text{Aut}(X)$ with $N(G)/G$. The following result guarantees that (under mild conditions on $G$) the group $N(G)$ is Fuchsian.

**Proposition.** If $G$ is a torsion-free non-cyclic Fuchsian subgroup of $\text{PSL}(2, \mathbb{R})$, then $N(G)$ is also Fuchsian.

Recall that a group is said to be torsion-free if there are no non-trivial elements of finite order.

**Proof.** All we need to show is that $N(G)$ is discrete (3.4.13). Assume that there exists a sequence $\{h_n\}$ of distinct elements of $N(G)$ converging to the identity. For all $g \in G$ we have that $\{h_n g h_n^{-1}\}_n$ is a sequence of elements of $G$ converging to $g$. Since $G$ is Fuchsian we get that $h_n g h_n^{-1} = g$, for $n > n_0$, for some positive integer $n_0$. By lemma 3.3.14 we get that $h_n$ and $g$ must have the same fixed points (since we are assuming that $G$ is torsion-free, the situation where $h_n$ and $g$ have order 2 does not occur).

If all elements of $G$ have the same fixed points we would have that $G$ consists of only parabolic or hyperbolic transformations. In either case $G$ would be cyclic, against the hypothesis. Let us choose $g_1$ and $g_2$ in $G$ with at least three distinct fixed points, say $z_1$, $z_2$ and $z_3$. Then every element $h$ of $N(G)$ will fix $z_j$, for $j = 1, 2, 3$. By 3.3.5 we get $h = Id$.

**3.4.20.** The automorphisms group of the Riemann sphere is the group of Möbius transformations. For the case of a torus $T_\tau$, any translation of the form $T(z) = z + c$, with $c \in \mathbb{C}$, induces an automorphism on $T_\tau$. Thus in these two cases
we have that the automorphisms group is not only infinite, but it is not discrete either. In the case of compact surfaces this cannot happen.

**Theorem (Hurwitz).** Let \( X \) be a compact Riemann surface of genus \( g \geq 2 \). Then \( \text{Aut}(X) \) has at most \( 84(g - 1) \) elements.

**Proof.** If \( X = \mathbb{H}/G \), since \( G \) is torsion-free and non-cyclic we know that \( N(G) \) is Fuchsian (3.4.19), and thus \( Y = \mathbb{H}/N(G) \) is a Riemann surface. The covering \( \mathbb{H} \to Y \) clearly factors through \( X \), so \( Y \) must be compact. The mapping \( q : X \to Y \) has degree equal to the order of \( H = \text{Aut}(X) \), say \( n \); thus \( H \) is a finite group (remarks 1 and 2 below). To find the bound on \( n \) we make a detailed study of the Riemann-Hurwitz formula.

The set of points of an automorphism of \( X \) (other than the identity) are finite, and since \( H \) is finite as well, we have that the set of points of \( X \) fixed by some non-trivial element of \( H \) is a finite set. Let \( p_1, \ldots, p_r \) be a maximal set of inequivalent fixed points of non-trivial elements of \( H \). That is, each \( p_j \) is fixed by some automorphism of \( X \) not equal to the identity; and if \( j \neq k \), we have that \( h(p_j) \neq p_k \) for all \( h \in H \). Thus these points project under \( q \) to different points of \( Y \). For each \( j \), let \( \nu_j \) be the order of the subgroup \( H_j \) of \( H \) of automorphisms of \( X \) fixing \( p_j \). We have that there are \( n/\nu_j \) distinct points in \( X \) that project to the same point, \( q(p_j) \) of \( Y \), and each such point is fixed by a subgroup of \( H \) of order \( \nu_j \). Thus we obtain that the total branching number \( B \) of the mapping \( q \) is given by

\[
B = \sum_{j=1}^{r} \frac{n}{\nu_j} (\nu_j - 1) = n \sum_{j=1}^{r} \left( 1 - \frac{1}{\nu_j} \right).
\]

Observe that \( \nu_j \geq 2 \), so \( 1/\nu_j \geq 1/2 \). The Riemann-Hurwitz formula in this setting gives us

\[
2g - 2 = n \left( 2\gamma - 2 + \sum_{j=1}^{r} \left( 1 - \frac{1}{\nu_j} \right) \right),
\]

where \( \gamma \) is the genus of \( Y \) (and \( g \geq \gamma \)). If \( g = \gamma \) then \( n = 1 \) (recall that \( n \) is the order of \( \text{Aut}(X) \), which we are trying to bound). In the case of \( g > \gamma \) we have the following cases:

- \( \gamma \geq 2 \). Then \( 2(g - 1) \geq 2n \) implies that \( n \leq g - 1 \).
\item $\gamma = 1$. In this case we have a value $\nu_j \geq 2$, so the right hand side of the Riemann-Hurwitz relation is, at least, equal to $n/2$, or equivalently, $n \leq 4(g - 1)$.
\item $\gamma = 0$ and $r \geq 5$. This cases gives $n \leq 4(g - 1)$.
\item $\gamma = 0$ and $r = 4$. Since the right hand side of Riemann-Hurwitz relation must be positive we get that at least one $\nu_j \geq 3$, and $n \leq 12(g - 1)$.
\item $\gamma = 0$, $r = 3$. We can assume that $2 \leq \nu_1 \leq \nu_2 \leq \nu_3$. Then $\nu_3 \geq 3$ and $\nu_2 \geq 3$. There are several cases to study:
  a. If $\nu_3 \geq 7$ we get $n \leq 84(g - 1)$, with equality in the case of $\nu_1 = 2$, $\nu_2 = 3$ and $\nu_3 = 7$.
  b. $\nu_3 = 6$, $\nu_1 = 2$. Then $\nu_2 \geq 4$ and $n \leq 24(g - 1)$.
  c. $\nu_3 = 6$, $\nu_1 \geq 3$. Then $n \leq 12(g - 1)$.
  d. $\nu_3 = 5$, $\nu_1 = 2$. Then $\nu_2 \geq 4$ and $n \leq 40(g - 1)$.
  e. $\nu_3 = 5$, $\nu_1 \geq 3$. Then $n \leq 15(g - 1)$.
  f. $\nu_3 = 4$, $\nu_1 \geq 3$. Then $n \leq 24(g - 1)$.

\textbf{Remarks.} 1. If $G$ is a Fuchsian group then $\mathbb{H}/G$ is a Riemann surface. 2. It is easy to show that, in the situation of the above proof, there exists a point $p \in X$ which is not fixed by any non-identity element of $\text{Aut}(X)$. This shows that the order of the covering $X \to Y = X/\text{Aut}(X)$ is equal to the order of the group $\text{Aut}(X)$.

\section*{3.5. Moduli spaces}

\subsection*{3.5.1.}
So far in this book we have studied properties of a fixed Riemann surface. The problem of moduli spaces deals with the study of varying Riemann surface structures on a fixed topological surface. More precisely, two surfaces $X$ and $Y$ are said to be conformatly equivalent (or simply equivalent) if there exists a biholomorphic mapping between them, $f : X \to Y$. Our goal is to know under what conditions $X$ and $Y$ are equivalent. An example of this type of problem is given by the Uniformization theorem (3.2.8); it classifies all simply connected surfaces up to biholomorphisms. The general problem is difficult and the study of it constitutes
a whole area of research on its own, with new mathematical tools. In this section we will give a couple of examples of how this problem can be treated; the reader interested on more results can find a nice introductory text in [18].

3.5.2. Before we get to explain our examples we need to make a few general remarks on the relation between conformally equivalent surfaces and their universal coverings and covering groups. Let $X$ and $Y$ be two surfaces, with universal coverings $\tilde{X}$ and $\tilde{Y}$, and covering mappings $\pi_X$ and $\pi_Y$ respectively. Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the lift of $f$ to the universal covering spaces; the following diagram will be then commutative:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & \tilde{Y} \\
\pi_X & \downarrow & \pi_Y \\
X & \xrightarrow{f} & Y
\end{array}
$$

In particular we have that $\tilde{f}$ is a Möbius transformation (see §§ 1.3.14, 3.3.3 and 3.3.9). The spaces $\tilde{X}$ and $\tilde{Y}$ are homeomorphic; we will then identify them and consider $\tilde{X}$ as the universal covering space of both $X$ and $Y$.

We can also give a more algebraic statement, in terms of the covering groups. Let $G_X$ and $G_Y$ be the covering groups of $X$ and $Y$ respectively. Since we have identified the universal covering spaces of $X$ and $Y$ we can consider these two groups as subgroups of $\text{Aut}(\tilde{X})$. Then the mapping $\tilde{f}$ will satisfy $\tilde{f}G_X\tilde{f}^{-1}$; that is, $G_X$ and $G_Y$ are conjugate subgroups of $\text{Aut}(\tilde{X})$. We will use this formulation of the problem in our examples since it make many of the computations easier.

3.5.3. Our first example consists on the study of equivalence classes of annuli. Let $r_1$, $r_2$ and $r$ be real numbers satisfying $r_1 < r_2$ and $1 < r$; we denote by $A(z_0, r_1, r_2)$ the annulus $\{ z \in \mathbb{C}; r_1 < |z - z_0| < r_2 \}$ and by $A_r$ the annulus $A(0, 1, r)$. Clearly $A(z_0, r_1, r_2)$ is equivalent to $A_{r_2/r_1}$ by the transformation $z \mapsto \frac{1}{r_1} (z - z_0)$. So every annulus is conformally equivalent to one of the form $A_r$, which means that the space of equivalence classes of annuli is contained in the interval $(1, +\infty)$. To fully determine conformal equivalence of annuli we need to find under what conditions $A_r$ and $A_s$ (with $r$ and $s$ real numbers greater than 1) are equivalent. The universal covering of $A_r$ is given by $\pi_r : \mathbb{H} \to A_r$, where $\pi_r(z) = \exp (2\pi i \log z / \log \lambda)$, and $r$
and $\lambda$ are related by the expression $r = \exp\left(-2\pi^2/\log \lambda\right)$ (3.3.15). Here log is the principal branch of the logarithm defined on $\mathbb{C}\setminus[0, +\infty)$. The covering group $G_r$ is generated by the transformation $g_r(z) = \lambda z$; that is, $G_r = \{z \mapsto \lambda^n z; \ n \in \mathbb{Z}\}$. Let $G_s$ denote the covering group of the annulus $A_s$, and let $g_s(z) = \mu z$ be a generator of $G_s$. If $f : A_r \to A_s$ is a biholomorphism, and $\tilde{f} : \mathbb{H} \to \mathbb{H}$ a lift to the universal covering space, then $\tilde{f}$ will be an element of $\text{SL}(2, \mathbb{C})$. Since $\tilde{f}$ conjugates $G_r$ into $G_s$ we have that $\tilde{f}g_r\tilde{f}^{-1}$ is equal to either $g_s$ or $g_s^{-1}$. The transformation $M(z) = -1/z$ is an automorphism of $\mathbb{H}$ that conjugates $g_s$ into $g_s^{-1}$, so we can assume that $\tilde{f}g_r\tilde{f}^{-1} = g_s$ (otherwise we consider $M\tilde{f}$, which is also a lift of $f$). The fixed points of $g_r$ and $g_s$ are 0 and $\infty$, so $\tilde{f} \{0, \infty\} = \{0, \infty\}$. But since $g^n_s(z_0) \to \infty$ for $z_0$ in $\mathbb{H}$ and $n \to +\infty$ we must have $\tilde{f}(\infty) = \infty$, and therefore $\tilde{f}(0) = 0$. So $\tilde{f}$ is of the form $\tilde{f} = kz$, for $k$ a positive real number. A simple computation shows that, with this expression of $\tilde{f}$, we have $\tilde{f}g_r\tilde{f}^{-1} = g_r$, which mean that $r = s$. Thus we have proved the following result.

**Theorem.** Any annulus $A(z_0, r_1, r_2)$ is conformally equivalent to one and only one annulus of the form $A_r$, for $r > 1$. More precisely we can take $r = r_2/r_1$.

**Remark.** See [22, pg. 291] for a purely analytic proof of the above theorem.

### 3.5.4.

Consider now the case when $X$ and $Y$ are surfaces of genus 1. By the Abel-Jacobi theorem (2.9.13) they must be of the form $\mathbb{C}/G_\tau = T_\tau$, for some $\tau$ with positive imaginary part. The classification of tori is given by the next result.

**Theorem.** Two tori $T_\tau$ and $T_\eta$ are conformally equivalent if and only if

\[
\tau = \frac{a\eta + b}{c\eta + d},
\]

where $a, b, c$ and $d$ are integer numbers satisfying $ad - bc = 1$ (that is, $\tau$ and $\eta$ are related by an element of $\text{SL}(2, \mathbb{Z})$).

**Proof.** Let us denote by $T^\tau_{n,m}$ the transformation $z \mapsto z + n + m\tau$ (and similarly for $\eta$). Suppose $\tilde{f}$ is a mapping satisfying $\tilde{f}G_\tau\tilde{f}^{-1} = G_\eta$. Since $\tilde{f}$ is an automorphism of $\mathbb{C}$ it must be of the form $\tilde{f}(z) = \lambda z + \mu$. It is easy to check the following identities:

\[
\tilde{f}T^\tau_{1,0}\tilde{f}^{-1}(z) = z + \lambda
\]

\[
\tilde{f}T^\tau_{0,1}\tilde{f}^{-1}(z) = z + \lambda\tau.
\]
We must then have \( \lambda = c\eta + d \) and \( \lambda \tau = a\eta + b \), for some integers \( a, b, c \) and \( d \); that is, \( \tau \) and \( \eta \) satisfy the relation 7. Since \( \tilde{f}^{-1}G_{\eta}\tilde{f} = G_{\tau} \), the transformation \( z \mapsto \frac{az+b}{cz+d} \) must be invertible; that is, \( ad - bc = \pm 1 \). The imaginary part of \( \frac{a\eta+b}{c\eta+d} \) is equal to \( ad - bc \). Since \( \tilde{f}^{-1}G_{\eta}\tilde{f} = G_{\tau} \), the transformation \( z \mapsto \frac{az+b}{cz+d} \) must be invertible; that is, \( ad - bc = \pm 1 \). The imaginary part of \( \frac{a\eta+b}{c\eta+d} \) is equal to \( ad - bc \). Since \( \tilde{f}^{-1}G_{\eta}\tilde{f} = G_{\tau} \), the transformation \( z \mapsto \frac{az+b}{cz+d} \) must be invertible; that is, \( ad - bc = \pm 1 \). The imaginary part of \( \frac{a\eta+b}{c\eta+d} \) is equal to \( ad - bc \).

Assume now that \( \tau \) and \( \eta \) are related by an element of \( \text{SL}(2, \mathbb{Z}) \), as in the theorem. Let \( S \) be the Möbius transformation given by \( S(z) = cz + d \). If \( c = 0 \) then \( a = d = \pm 1 \), which implies that \( \tau = \eta \) so there is nothing to prove. Thus we can assume that \( c \neq 0 \). It is not difficult to see that

\[
ST^r_{n,m}S^{-1} = T^n_{nd+mb,nc+ma};
\]

this equation implies that \( SG_{\tau}S^{-1} \subset G_{\eta} \). Choosing \( (n, m) = (a, -c) \) and \( (n, m) = (-b, d) \) we get that \( SG_{\tau}S^{-1} \) contains the transformations \( T^n_{1,0} \) and \( T^n_{0,1} \), and thus \( SG_{\tau}S^{-1} \supset G_{\eta} \). So the transformation \( S \) conjugates \( G_{\tau} \) into \( G_{\eta} \) and therefore the tori \( T_{\tau} \) and \( T_{\eta} \) are conformally equivalent.

3.5.5. From the above theorem we have that the space of equivalence classes of tori, denoted by \( \mathcal{M}_1 \) can be identified with \( \mathbb{H}/\text{SL}(2, \mathbb{Z}) \). To study this space we can follow the techniques of 3.4. It is not difficult, for example, to find a fundamental domain for the action of \( \text{SL}(2, \mathbb{Z}) \) on \( \mathbb{H} \). Let \( P \) be the open polygon bounded by the geodesics:

\[
L_1 = \{ z \in \mathbb{H}; \, \text{Re}(z) = 1/2 \},
\]

\[
L_2 = \{ z \in \mathbb{H}; \, \text{Re}(z) = -1/2 \},
\]

\[
L_2 = \{ z \in \mathbb{H}; \, |z| = 1 \}.
\]

Claim. \( P \) is the Dirichlet region (for the action of \( \text{SL}(2, \mathbb{Z}) \) on \( \mathbb{H} \)) centred at the point 2i.

Proof of the claim. We first prove that \( L_1 \) is contained in the set of points equidistant from 2i and 2i + 1; that is, \( L_1 \subset D_A \) for \( A(z) = z + 1 \), in the notation of 3.4.16. Observe that since \( d(z, 2i) = d(z, 2i + 1) \) we have \( d(R(z), R(2i)) = d(R(z), R(2i + 1)) \) for \( R(z) = \frac{1}{2} + iz \). But then we have

\[
d(R(z), 2i + 1) = d(R(z), R(2i)) = d(R(z), R(2i + 1)) = d(R(z), 2i).
\]
In other words, if \( d(z, 2i) = d(z, 2i + 1) \) then \( d(R(z), 2i) = d(R(z), 2i + 1) \). Since \( R(z) = z \) if and only if \( \text{Re}(z) = \frac{1}{2} \) we have that \( L_1 \subset D_A \) (the geodesic \( D_A \) is the “full” vertical line containing \( L_1 \)).

Similarly one can prove that \( L_2 \subset D_{A^{-1}} \) and \( L_3 \subset D_B \), where \( B(z) = -1/z \). These three statements show that \( D_{2i} \) is contained in \( P \).

Assume now that \( D_{2i} \) is a proper subset of \( P \). Then there exists a point \( z_0 \in P \) and a non-trivial element \( h \in \text{SL}(2, \mathbb{C}) \), such that \( h(z_0) \in P \). Write \( h(z) = \frac{az + b}{cz + d} \).

We have (see 3.3.9) \( \text{Im}(h(z_0)) = \frac{\text{Im}(z_0)}{|cz_0 + d|^2} \). Write \( z_0 = x_0 + iy_0 \). Since \( z_0 \) is in \( P \) we have

\[
|z_0|^2 = x_0^2 + y_0^2 > 1 \quad \text{and} \quad -\frac{1}{2} < x_0 < \frac{1}{2}.
\]

Using these inequalities one can easily prove the following:

\[
|cz_0 + d|^2 = c|z_0|^2 + 2cdx_0 + d^2 > |c|^2 + |d|^2 - |cd| = (|c| - |d|)^2 + |cd|.
\]

The last term in the above displayed formula is a positive integer (it cannot be 0 since \( ad - bc \neq 0 \)). Thus it is at least equal to 1, which implies that \( |cz_0 + d|^2 > 1 \). So we have that if \( z_0 \) and \( h(z_0) \) are both in \( P \) then \( \text{Im}(h(z_0)) < \text{Im}(z_0) \). We can apply the same argument to \( h(z_0) \) and \( z_0 = h^{-1}(h(z_0)) \), to get \( \text{Im}(z_0) < \text{Im}(h(z_0)) \).

This contradiction shows that \( P \) is indeed equal to \( D_{2i} \).

To have a picture of the space of (conformal equivalence classes) of tori we just need to consider \( \overline{P} \), the closure of \( P \) in the hyperbolic plane, by the action of \( \text{SL}(2, \mathbb{Z}) \) (observe that the point of infinity is not a part of \( \overline{P} \)). The transformation \( A \) identifies \( L_2 \) and \( L_1 \), while \( B \) fixes the side \( L_3 \) (as a set, not necessarily pointwise). Thus we can think of \( \partial P \) as consisting of four sides, \( L_1, L_2, s = \{ z \in \mathbb{H}; \ |z| = 1, \ 0 \leq \text{Re}(z) < \frac{1}{2} \} \) and \( s' = \{ z \in \mathbb{H}; \ |z| = 1, \ -\frac{1}{2} < \text{Re}(z) \leq 0 \} \), with \( B(s) = s' \) (condition FD3 in definition 3.4.14). Any torus will be conformally equivalent to one torus of the form \( T_{\tau} \) where \( \tau \) belongs to \( \overline{P} \). The precise formulation is given in the next result.

**Theorem.** Any torus is conformally equivalent to one and only one torus \( T_{\tau} \) with \( \tau \) satisfying the following conditions:

1. \(|\tau| \geq 1\);
2. \(-\frac{1}{2} < \text{Re}(\tau) \leq \frac{1}{2}\);
3. If \(|\tau| = 1\) then \( \text{Re}(\tau) \geq 0\).

The boundary of $P$ has three vertices: $i$, $\frac{1+i\sqrt{3}}{2}$ and $\frac{-1+i\sqrt{3}}{2}$ (the point $i$ is the meeting point of the sides $s$ and $s'$). The tori corresponding to these points are special in the sense that we explain next.

An automorphism $M : \mathbb{C} \to \mathbb{C}$ of the complex plane induces an automorphism on a torus $T_\tau$ if and only if $MG_\tau M^{-1} = G_\eta$, where $\tau$ and $\eta$ are related by an element of $\text{SL}(2, \mathbb{Z})$. We observe that if $M$ is of the form $M(z) = z + \mu$, for $\mu$ a complex number, then $MG_\tau M^{-1}$ is actually equal to $G_\tau$. Moreover, if $\mu$ is not of the form $n + m \tau$ (for $n$ and $m$ integers), then $M$ induces a non-trivial automorphism of $T_\tau$. Thus we have that any torus has a group of automorphism with “many” elements (see the remarks before Hurwitz’s theorem in 3.4.20). The mapping $M(z) = -z$ also conjugates $G_\tau$ into itself, so it will give another automorphism of $T_\tau$. The fixed points of $M$ are given by points $z_0$ of $\mathbb{C}$ satisfying $M(z_0) = z_0 + n + m \tau$, for some integers $n$ and $m$. It is easy to see that there are only four possible points, up to equivalence by elements of $G_\tau$: $0$, $1/2$, $\tau/2$ and $(1 + \tau)/2$. So if $f : T_\tau \to T_\tau$ denotes the automorphism of the torus $T_\tau$ induced by $M$, we see that $f$ has four fixed points on $T_\tau$, corresponding to the above four points. In particular one can prove that $f$ is the “hyperelliptic involution” of $T_\tau$ (we use quotation marks since we have defined hyperelliptic involutions only for surfaces of genus at least 2).

Suppose $\tau \in \overline{P}$ corresponds to a torus with some automorphism different from the ones in the previous paragraph. Then there exists a Möbius transformation $S(z) = \frac{az+b}{cz+d}$, in $\text{SL}(2, \mathbb{Z})$, such $S(\tau) = \tau$. Solving this equation we get that $\tau$ should be of the form

$$
\tau = \frac{a - d}{2c} + i \frac{\sqrt{4 - (a + d)^2}}{2c}.
$$

We have that $|a + d| < 2$ (otherwise $\tau$ would be real) and $c \neq 0$ (since $z \mapsto z + \mu$ does not have fixed points on $\overline{P}$). Since $|\tau| \geq 1$ we get $\frac{1-\text{sgn}d}{c^2} \geq 1$. This inequality gives us the following different options:

- $a = 0, c = 1, d = 0$, $\tau_1 = i$;
- $a = 1, c = 1, d = 0$, $\tau_2 = \frac{1 + i\sqrt{3}}{2}$;
- $a = 1, c = -1, d = 0$, $\tau_3 = \frac{-1 + i\sqrt{3}}{2}$;
Consider the torus $T_{\tau_1}$, and let $T_1$ be the Möbius transformation $T_1(z) = \tau_1 z (= iz)$. We have $T_1^4 = Id$ and $T_1 T_{n,m}^i T_1^{-1} = T_{m,-n}^i$. Thus $T_1$ induces an automorphism of order 4 on $T_{\tau_1}$. The tori corresponding to $\tau_2$ and $\tau_3$ are conformally equivalent (by the transformation $z \mapsto z - 2$), so we consider only one of them, say $T_{\tau_2}$. Using the identity $\tau_3^2 = -1$ we see that $T_2(z) = \tau_2 z$ satisfies $T_2 T_{n,m}^\tau T_2^{-1} = T_{m,n-m}^{\tau_2}$. Since the mapping $(p, q) \mapsto (q, p - q)$ of $\mathbb{Z}^2$ is invertible we have that $T_2$ induces an automorphism on $T_{\tau_2}$. It is easy to check that the order of that automorphism is 6. We obtain that the tori corresponding to the vertices of $P$ are precisely those with some “extra automorphisms”.

3.5.7. Topologically all annuli are “the same”, that is, homeomorphic, and similarly all tori. If one takes two annuli and identifies the boundaries (“glue” them by their boundaries) one gets a torus. This particular surface has an order 2 mapping interchanging the two annuli. Such mapping cannot be holomorphic (in the torus) since it has many fixed points, namely the two curves that formed the boundaries of the two annuli. However, it is possible to show that this mapping is anti-holomorphic (that is, its conjugate is holomorphic). More precisely, let $\tau = it$ be a complex number with $t > 1$, and consider the symmetry (anti-holomorphic mapping of order 2) of the complex plane given by $\sigma : z \mapsto -\bar{z}$. We have

$$\sigma T_{n,m}^\tau \sigma^{-1}(z) = \sigma T_{n,m}^\tau (-\bar{z}) = \sigma (-\bar{z} + n + it) = z - n - it = T_{-n,-m}^\tau(z).$$

Thus $\sigma$ induces an automorphism of $T_{\tau}$, say $R$. It is easy to check that $R$ is anti-holomorphic and has order 2. What are the fixed points of $R$? If a point $p$ in $T_{\tau}$ is fixed by $R$, then there exist integers $n$ and $m$ such that $z_0 = -\bar{z}_0 + n + mti$, where $z_0$ is a point of $\mathbb{C}$ that projects to $p$ under the natural quotient map. The solutions of this equation are given by the lines $L_n = \{z \in \mathbb{C}; \text{Re}(z) = \frac{n}{2}\}$. Since $L_n$ is equivalent under elements of $G_\tau$ to $L_{n\pm 2n'}$, we have only two set of solutions, the imaginary axis and $L_1$. These two lines project to two closed curves on $T_{\tau}$ (since $z \mapsto z + ti$ belongs to $G_\tau$), the ones corresponding to the boundaries of the annuli above explained.

Consider now the transformation $\rho(z) = -\bar{z}$; it is clear that $\rho$ induces an anti-holomorphic involution on $\mathbb{H}$. If $S$ is an element of $\text{SL}(2, \mathbb{C})$, then $\rho S \rho = S$, so
\( \rho \) induces a mapping on \( \mathcal{M}_1 \), the space of tori. If \( \tau \) is fixed point of \( \rho \) then there exists an element \( T \in \text{SL}(2, \mathbb{C}) \) such that \( \rho(\tau) = T(\tau) \). But then \( T \rho T^{-1} \) will be an anti-holomorphic mapping of \( \mathbb{H} \) fixing \( \tau \). So, without loss of generality we can assume that \( \rho(\tau) = \tau \). The solutions of this equation are given by \( \tau \in \mathbb{H} \) satisfying \( \text{Re}(\tau) = 0 \); that is, the tori “built by gluing two annuli”.