SEMINAR

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1. Automorphisms of Surfaces

1.1. **Background.** The classification of homeomorphisms of surfaces is related to the topology and geometry of 3-manifolds in the following way. In his paper in the Bull. A.M.S. 1982, Thurston conjectured that the interior of every compact 3-manifold has a canonical decoposition into pieces which have geometric structures. He established the following result: let S be a surface and $\phi : S \to S$ a diffeomorphism; let M_ϕ be the 3-manifold defined by $M_\phi = S \times [0,1]/\sim_\phi$, where the relation is given by $(x,0) \sim_\phi (\phi(x),1)$; then M_ϕ has a hyperbolic structure if and only if ϕ is pseudo-Anosiov. This last term denotes one of the three types of automorphisms of surfaces, according to Thurston's classification; the other two types are homeomoprhisms of finite order and homeomorphisms that leave invariant a finite, pairwise disjoint, set of curves. The classification is up to homotopy, as explained in the next paragraph.

Given an oriented, smooth surface S we denote by $Hom(S)$, $Hom^+(S)$ and $Hom_0(S)$ the groups of homeomorphisms of S, orientation-preserving homeomorphisms and homeomorphisms homotopic to the identity, respectively. The extended mapping class group is the quotient group $Mod^*(S) = Hom(S)/Hom_0(S)$; the mapping class group is the index 2 subgroup given by $Mod(S) = Hom^+(S)/Hom_0(S)$.

Two remarks: first, a homeomorphism on S is homotopic to a diffeomorphism, so one can work in the smooth category. Secondly, by a result of Baer, Mangler and Epstein, two elements in $Hom^+(S)$ are homotopic if and only if they are isotopic, if S is of finite topological type (a compact surface minus a finite number of points and discs). References: [1] D.B.A. Epstein, "Curves on 2-manifolds and isotopies", Acta Math. 115 (1966), 83-107 and [2] W. Mangler, "Die Klassen von topologischen Abbildungen einer geschlossene Fläche auf sich", Math. Z. 44 (1939), 541-554.

Date: September 22, 2008.

The Dehn-Nielsen-Baer theorem states that $Mod^*(S)$ is isomorphic to $Out(\pi_1(S)) = Aut(\pi_1(S))/Inn(\pi_1(S))$. Here $Inn(\pi_1(S))$ denotes the group of inner automorphisms of the fundamental group of S, that is, automorphisms given by conjugation by elements of $\pi_1(S)$. Strictly speaking we should write the base point in the above notation, but since change of base point is given by a conjugation, and we are taking quotients by inner automorphisms, the base point does not matter (as long as the same base point is used in the definition, of course). References: [1] B. Farb and D. Margalit, "A primer on mapping class groups" and [2] N. Ivanov, "Mapping Class Groups".

1.2. **Automorphisms of the torus.** Let T denote a torus, that is, a compact surface of genus 1. We identify T with the quotient $\mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ acts by translations, for example, $(x, y) \mapsto (x + m, y + n)$, with m and n integers. Since we are interested on a topological object, the mapping class group, the particular action of $\mathbb{Z}^2$ on the plane is not important; however, when we want to give some examples we might change this action to suit our needs.

Since $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$ is abelian we have that the inner automorphisms group is trivial. Thus $Mod^*(T)$ is isomorphic to $GL(2, \mathbb{Z})$, the group of rank 2 matrices with integer coefficients and determinant ± 1. The subgroup $Mod(T)$ is isomorphic to $SL(2, \mathbb{Z})$, that is, those matrices with determinant equal to 1.

If h is a mapping class then we have a unique element α of $SL(2, \mathbb{Z})$ associated to it by the above isomorphism. If $\alpha = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right)$ then its characteristic polynomial is $t^2 - (tr\alpha)\, t + 1$, where tr denotes the trace. Let λ and $\lambda^{-1} = 1/\lambda$ be the eigenvalues of α. There are three different possibilities:

(1) λ and λ^{-1} are complex but not real; in this case they must have absolute value 1 and an easy computation shows that $tr(\alpha)$ can take only the values 0, 1 and -1. We have that h has finite order equal to 2, 3, 4 or 6. To construct one example take $\tau = \exp(\pi i/3)$, and let us consider T as the quotient of the complex plane by the group of transformations $z \mapsto z + m + n\tau$. Then T has an automorphism given by $h(z) = \tau\, z$, with associated matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. It is easy to check that h has order 6.

(2) $\lambda = \lambda^{-1} = \pm 1$, then $tr(\alpha) = \pm 2$. We have that α has an integral eigenvector that projects to an invariant closed curve on the torus. One possible way of looking at this is, for example, considering α as a Möbius transformation acting on the complex plane

by $z \mapsto \frac{pz+q}{rz+s}$. Then if $\binom{x}{y}$ is an eigenvector for α if and only if the real number x/y is a fixed point of the corresponding Möbius transformation. In the particular case of $\lambda = \lambda^{-1} = \pm 1$ we have that the fixed point in the (extended) real line is the number $\frac{p-s}{2r}$, which is rational. So the eigenvector induces a closed curve on the torus. Conversely, if p/q is a rational number the transformation $\begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}$ has eigenvector $\binom{p}{q}$. The maps of this type are Dehn twists (or power of such maps) as the one given in the figure.

(3) $|\lambda| > 1 > |\lambda^{-1}|$; in this case, by considerations of Kleinian groups the associated Möbius transformation cannot have a rational fixed point (see Maskit: II.C.6), so the invariant curved induced by the eigenvalue λ is not closed. One has that the mapping h "stretches" in the direction of the eigenvector corresponding to λ and "shrinks" in the direction of the other eigenvector, corresponding to $1/\lambda$. These transformations are called Anosov.

1.3. **Automorphisms of the torus and Riemann surfaces.** The above discussion suggest that one looks at $SL(2,\mathbb{Z})$ as a group acting on the space of tori as follows: from the complex point of view, different actions of $\mathbb{Z}^2$ on the complex plane give different tori. From the theory of elliptic functions we have that any torus can be representedas $\mathbb{C}/\mathcal{L}_\tau = T_\tau$, where $\mathcal{L}_\tau$ is the group of translations of the form $z \mapsto z + m + n\tau$, with m and n integers. One has that τ is a complex number with positive imaginary part, so we have a parameter space for torus given by the upper half plane $\tau \in \mathbb{H} = \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$. Elements of the modular group $SL(2,\mathbb{Z})$ act as Möbius transformations on $\mathbb{H}$; they are of three types: elliptic, parabolic and hyperbolic. Elliptic transformations have a fixed point in $\mathbb{H}$ and correspond to automorphisms of finite order in a torus. Parabolic transformations fix a rational point on the real line (or the point of infinity) and correspond to the transformations of type (2) above, that have a fixed closed curve. Finally hyperbolic mappings correspond to Anosov homeomorphisms. Observe that the following two transformations, which are Dehn twists, generate $SL(2,\mathbb{Z})$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}.$$

1.4. **Generalization.** Thurston's classification of automorphisms of surfaces generalizes to higher genus in a way similar to what we explained in the last paragraph. There is a a parameter space of a surface S, of genus at least 2, called **Teichmüller space** of S, $T(S)$. Any element of the mapping class group $Mod(S)$ acts naturally on $T(S)$ in a continuous way. The space $T(S)$ is an open ball in $\mathbb{R}^{6g-6}$; the first step in Thurston's classification is to compactify $T(S)$ by means of some classes of foliations, so $\overline{T(S)}$ is homeomorphic to the closed ball in $\mathbb{R}^{6g-6}$. The action of $Mod(S)$ on $T(S)$ extends to a continuous action on the compactified space. Thus, by Brouwer's fixed point, any element ϕ of $Mod(S)$ has a fixed point in $\overline{T(S)}$. There are three possibilities: (1) ϕ fixes a point in the interior: $T(S)$; it is easy to see that ϕ acts as a holomorphic (invertible) mapping on some surface S' and, by Hurwitz's theorem, ϕ must have finite order. (2) If ϕ has only one fixed point in the boundary of $T(S)$ then that point corresponds to a collection of closed, pairwise disjoint curves, and ϕ leaves those curves invariants; in this case we talk of **reducible** mappings. (3) Finally, ϕ can have two fixed points on the boundary of $T(S)$, these points corresponds to foliations on S, and ϕ stretches in one direction by a factor of λ and shrinks in a "complementary" direction by a factor of λ; the mapping (or its class) is called **pseudo-Anosov**.

2. Hyperbolic Geometry

2.1. **Basic Hyperbolic Geometry.** Let $\mathbb{D}$ denote the unit disc on the complex plane, that is $\mathbb{D} = \{z \in \mathbb{C};\ |z|, 1\}$. For an open subset of $\mathbb{C}$ we denote by $Aut(G)$ the group of biholomorphic mappings of G (bijective holomorphisms from G to G).

An easy consequence of Schwarz's lemma is that the group $Aut(\mathbb{D})$ consists of the Möbius transformations of the form $g(z) = \frac{az+\bar{c}}{cz+\bar{a}}$; where a and c are complex number satisfying $|a|^2 - |c|^2 = 1$. An easy computation shows that, for such mapping g, one has

$$\frac{|g'(z)|}{1-|g(z)|^2} = \frac{1}{1-|z|^2},$$

for all z in $\mathbb{D}$. This means that the metric $\lambda(z)|dz| = \frac{2}{1-|z|^2}\,|dz|$ is invariant under $Aut(\mathbb{D})$. More precisely, if $\gamma : [a,b] \to \mathbb{D}$, is a smooth curve, we define the length of γ by $l(\gamma) = \int_\gamma \lambda(z)|dz| = \int_a^b \lambda(\gamma(t))\,|\gamma'(t)|\,dt$, then $l(\gamma) = l(g\circ\gamma)$. We define the **hyperbolic distance** on $\mathbb{D}$ by $d(z,w) = \inf l(\gamma)$, where the infimum is taken over all (smooth) curves joining z and w.