Quot schemes and Hilbert schemes

§1.

In this section we construct Quot and Hilb schemes. These constructions are very basic and I shall assume that the reader is familiar with the construction of Grassmanians. We will show that Quot is a closed subscheme of a suitable Grassmanian.

Let \( X \) be a projective scheme with a very ample line bundle \( \mathcal{O}_X(1) \) and let \( \mathcal{E} \) be a coherent sheaf on \( X \). Let \( P \) be a polynomial in one variable over the rationals.

**Definition:** For any scheme \( S \) we define the functor 
\[ \text{Quot}_P^\mathcal{E}(S) = \{ \text{the set of all quotients } \mathcal{F} \text{ of the pull back of } \mathcal{E} \text{ to } X \times S \text{ which are flat over } S \text{ and the Hilbert polynomial of } \mathcal{F} \text{ at any point of } S \text{ is } P. \} \]

**Theorem.** The functor \( \text{Quot}_P^\mathcal{E} \) is representable by a projective scheme \( \mathcal{H} \), i.e. there exists a projective scheme \( \mathcal{H} \) and a quotient \( \mathcal{E} \xrightarrow{q} \mathcal{F} \) on \( X \times \mathcal{H} \) with \( \mathcal{F} \) flat over \( \mathcal{H} \) and has the Hilbert polynomial \( P \) at every point of \( \mathcal{H} \) with the following universal property: if \( \alpha \in \text{Quot}_P^\mathcal{E}(S) \) then there exists a unique morphism \( f : S \rightarrow \mathcal{H} \) such that the pull back of the above surjection to \( X \times S \) using \( f \) is \( \alpha \).

**Proof:** Step 1. There exists an integer \( N \) such that for any coherent sheaf \( \mathcal{F} \) on \( X \) with Hilbert polynomial \( P \) and any exact sequence
\[ 0 \rightarrow \mathcal{G} \xrightarrow{p} \mathcal{E} \xrightarrow{q} \mathcal{F} \rightarrow 0 \] \( \mathcal{F} \) and \( \mathcal{G} \) are \( N \)-regular.

We may as well assume that \( X = \mathbb{P}^r \) for some \( r \). Proof is by induction on the dimension of the support of \( \mathcal{E} \). If it is zero then there is nothing to prove. Choose a hyper plane section \( Y \) so that \( \text{Tor}^i(\cdot, \mathcal{O}_Y) = 0 \) for \( i > 0 \) and \( \cdot \) being any of the three sheaves occurring above. (One just needs to avoid the finitely many associated primes.) By induction one easily sees that there exists an integer \( M \) such that \( \mathcal{F} \otimes \mathcal{O}_Y \) and \( \mathcal{G} \otimes \mathcal{O}_Y \) are \( M \)-regular. We also may assume that \( \text{H}^i(\mathcal{E}(n)) = 0 \) for all \( n \geq M \) and for all \( i > 0 \). Also we may assume that \( \text{H}^1(\mathcal{E}(n)) = 0 \) for all \( n \geq M \). By the long exact sequences one gets that for \( i > 1 \) \( \text{H}^i(\mathcal{F}(n)) = 0 \) for \( n \geq M \). So to conclude our proof we need only to see when \( \text{H}^1 \) vanishes. The long exact sequence associated to (1) gives
\[ \text{H}^1(\mathcal{E}(n)) \rightarrow \text{H}^1(\mathcal{F}(n)) \rightarrow \text{H}^2(\mathcal{G}(n)) \]

to be exact. In the range \( n \geq M \) the extreme modules are zero and hence we get that \( \mathcal{F} \) is also \( M \)-regular. By the semicontinuity theorems one sees that \( \mathcal{G} \) is \( (M + \dim \text{H}^1(\mathcal{G}(M))) \)-regular. So we need to bound the above dimension. But since all higher cohomologies vanish we have
\[ \dim \text{H}^1(\mathcal{G}(M)) = \dim \text{H}^0(\mathcal{G}(M)) - \chi(\mathcal{G}(M)). \]

But from (1) we also have
\[ \dim \text{H}^0(\mathcal{G}(M)) \leq \dim \text{H}^0(\mathcal{E}(M)) \]
and thus we see that if we choose \( N = M + \dim \mathcal{H}^0(\mathcal{E}(M)) - \chi(\mathcal{G}(M)) = M + \mathcal{P}(M) \) we will be done.

**Step 2.** To prove the theorem we may twist all sheaves by \( N \) and thus assume that all the sheaves are 0-regular. Let \( V = \mathcal{H}^0(\mathcal{E}) \). By our choice, for any sheaf \( \mathcal{F} \) occurring in (1), we have \( \dim \mathcal{H}^0(\mathcal{F}) = \chi(\mathcal{F}) = \mathcal{P}(0) = \mathcal{d}(\text{say}) \). Also the map \( V \to \mathcal{H}^0(\mathcal{F}) \) is surjective. Thus given any such quotient we get a point in the Grassmanian \( G = G(d, V) \). Let

\[
0 \to A \to V \to B \to 0 \tag{2}
\]

be the universal exact sequence on \( G \) where \( B \) is a rank \( d \) vector bundle. Let \( p_1 : X \times G \to X \) and \( p_2 : X \times G \to G \) be the two projections. We have a canonical surjection from \( V \) to \( \mathcal{E} \). Using this we get a map \( A \to p_1^*\mathcal{E} \) and let \( Q \) be the quotient. By flattening stratification we have a locally closed subscheme of \( G \), which we call \( \mathcal{H} \) over which \( Q \) is flat and has the Hilbert polynomial \( \mathcal{P} \). We have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & p_2^*A & \to & V & \to & p_2^*B & \to & 0 \\
& & \downarrow \cong & \downarrow & \pi & \downarrow & \downarrow & \to & 0 \\
p_2^*A & \to & \mathcal{E} & \to & Q & \to & 0
\end{array}
\tag{3}
\]

and \( Q \) is flat over \( \mathcal{H} \) and has the right Hilbert polynomial.

**Step 3.** Finally we need to check that the \( \mathcal{H} \) constructed above has the required universal property. So let \( S \) be any scheme and let \( q : X \times S \to S \) be the projection. Assume that we are given a quotient \( \mathcal{E} \xrightarrow{u} \mathcal{M} \) on \( X \times S \) with \( \mathcal{M} \) flat over \( S \) and having the Hilbert polynomial \( \mathcal{P} \). Taking direct image and noting that the sheaves occurring have 0-regularity we get an exact sequence

\[
0 \to q_*(\ker(u)) \to V \to q_*(\mathcal{M}) \to 0. \tag{4}
\]

Since \( q_*(\mathcal{M}) \) is a vector bundle of rank \( d \) we get a map \( f \) from \( S \) to \( G \) so that (2) pulls back to (4) by \( f \). We will show that \( f \) factors through \( \mathcal{H} \). On \( X \times S \) we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & q^*f^*A & \to & q^*f^*V & \to & q^*f^*B & \to & 0 \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \to & 0 \\
0 & \to & \ker(u) & \to & \mathcal{E} & \to & \mathcal{M} & \to & 0
\end{array} \tag{5}
\]

The vertical arrows are all surjective by regularity. This implies that we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\pi} & Q \\
\downarrow & \cong & \downarrow \\
\mathcal{E} & \xrightarrow{u} & \mathcal{M}
\end{array} \tag{6}
\]

The universal property for flattening stratification does the trick now.

**Step 4.** Now we check that \( \mathcal{H} \) is projective. We will apply discrete valuative criterion for this since we already know that \( \mathcal{H} \) is quasi-projective. So let \( R \) be a discrete valuation ring and \( K \) its quotient field. Let \( i : \text{Spec} \ K \hookrightarrow \text{Spec} \ R \) the natural immersion. Assume we have a map \( \phi : \text{Spec} \ K \to \mathcal{H} \). We want to construct a map \( \varphi : \text{Spec} \ R \to \mathcal{H} \) extending
This map gives a sheaf \( M \) and a surjection \( i^*p^*E \to M \) on \( X \times \text{Spec} \, K \), where \( p \) is the first projection of \( X \times \text{Spec} \, R \). Using the natural map \( p^*E \) to \( i^*p^*E \) let the image of \( p^*E \) in \( M \) be \( K \). Since \( K \hookrightarrow M \) it is R-torsionfree it is R-flat. Then its Hilbert polynomial is the same as that of \( M \) since they agree generically. Now by the universal property of \( \mathcal{H} \) we get a map \( \phi \) as desired.

The above construction helps us to construct many other related schemes like \( \text{Hilb} \) and \( \text{Hom}(X,Y) \).

**Definition:** For any scheme \( S \) we define the functor \( \text{Hilb}^P(S) = \{ \text{the set of all subschemes of } X \times S \text{ which are flat over } S \text{ and have Hilbert polynomial } P \} \).

**Corollary.** \( \text{Hilb}^P \) is representable by a projective scheme.

**Proof:** The functor \( \text{Hilb}^P \) is just \( \text{Quot}^P_{\mathcal{O}_X} \).

**Corollary.** Let \( f : X \to Y \) be a morphism where \( X \) is quasi-projective and \( Y \) projective. Then the functor \( \Pi \) defined as follows is representable by a quasi-projective scheme. \( \Pi(S) = \{ \text{the set of all sections of the map } X \times S \to Y \times S \text{ induced by } f \} \).

**Proof:** First let us assume that \( X \) is projective. Then by the previous corollary, once we fix a polynomial there exists a projective scheme \( H \) and a universal subscheme \( Z \) in \( X \times H \) which parametrizes subschemes of \( X \). Using \( f \) we get a map \( g : Z \to Y \times H \). What we want to show is that the set \( \{ h \in H | g_h : Z_h \to Y \times \{ h \} \text{ is an isomorphism} \} \) is open in \( H \).

Consider \( \Delta \subset Z \times gZ = T \). We have \( \mathcal{O}_T \to \mathcal{O}_\Delta \) a surjective map. We are looking for points in \( H \) where the above map is an isomorphism. We have

\[
0 \to \mathcal{I} \to \mathcal{O}_T \to \mathcal{O}_\Delta \to 0.
\]

Let \( h \in H \) be such a point. Since \( \Delta \) is flat over \( H \) we get

\[
0 \to \mathcal{I} \otimes k(h) \to \mathcal{O}_{T_h} \to \mathcal{O}_{\Delta_h} \to 0
\]

to be exact. So \( \mathcal{O}_{T_h} \to \mathcal{O}_{\Delta_h} \) is an isomorphism if and only if \( \mathcal{I} \otimes k(h) = 0 \). Thus if \( W = \text{Supp} \mathcal{I} \) in \( Z \times gZ \) and if \( p : Z \times gZ \to H \) then \( h \notin p(W) \) is the set we want. But \( p \) is proper and since \( W \) is closed \( H \setminus p(W) \) is open.

For the general case note that since \( X \) is quasi-projective and \( Y \) is projective there exists a projective scheme \( \overline{X} \) containing \( X \) as an open subset and the map \( f \) extends to \( \overline{Y} \). From the previous part there exists an open subset \( U \) of \( \text{Hilb}^P(\overline{Y}) \) which parametrises sections of \( \overline{Y} \to X \). So we have a \( Z \hookrightarrow \overline{Y} \times U \to X \times U \) such that for every \( h \in U \) it is an isomorphism. We want to choose those \( h \) such that \( Z_h \subset Y \). So let \( T' = Z \cap (Y-\overline{Y}) \times U \). If \( W' \) is the image of \( T' \) in \( U \) then it is closed in \( U \) because \( \overline{Y} \) is projective. Let \( V = U-W' \). Then \( V \) has all the required properties.

**Corollary.** If \( X \) is projective and \( Y \) is quasi-projective then \( \text{Hom}(X,Y) \) exists and is open in \( \text{Hilb}^P(X \times Y) \).

**Proof:** It is obvious from the previous corollary.
We intend to make some infinitesimal calculations in this section. This will enable us to compute the Zariski tangent spaces of points in the various schemes that we have constructed.

**Definition.** Let $X$ be a scheme and $\mathcal{H}$ a sheaf of groups on $X$ acting on a sheaf $\mathcal{F}$. Then $\mathcal{F}$ is a formal principal homogeneous space with respect to $\mathcal{H}$ if

a) For every open subset $U$ of $X$, $\mathcal{F}(U)$ is either empty or a principal homogeneous space over $\mathcal{H}(U)$.

b) The natural map $\mathcal{H} \times \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is an isomorphism.

**Proposition.** Let $S$ be a scheme and $S_0$ a closed subscheme defined by an ideal sheaf $\mathcal{I}$ with $\mathcal{I}^2 = 0$. Let $X$ be an $S$-scheme and $\mathcal{F}$ a quasi-coherent sheaf on $X$. We will write a subscript $0$ for restrictions to $S_0$. Let $\mathcal{H}_0$ be a subsheaf of $\mathcal{F}_0$ and assume that the quotient $\mathcal{G}_0 = \mathcal{F}_0/\mathcal{H}_0$ is flat over $S_0$. Let $\mathcal{P}$ be the sheaf (of sets) which for any open set $U$ of $X$ associates all $S$-flat quotients of $\mathcal{F}$ which are lifts of $\mathcal{G}_0$ along with the quotient map $\mathcal{F}_0 \to \mathcal{G}_0$. Let $A = \text{Hom}(\mathcal{H}_0, \mathcal{G}_0 \otimes \mathcal{I})$. Then $\mathcal{P}$ is a formal principal homogeneous space over $A$.

**Proof:** Let $\mathcal{G}$ be any such lift. Then we have the following commutative diagram

$$
\begin{array}{c}
0 & \to & \mathcal{I} \mathcal{F} & \overset{i}{\to} & \mathcal{F} & \overset{p}{\to} & \mathcal{F}_0 & \to & 0 \\
0 & \to & \mathcal{I} \otimes \mathcal{G}_0 & \overset{j}{\to} & \mathcal{G} & \overset{q}{\to} & \mathcal{G}_0 & \to & 0 \\
\end{array}
$$

(1)

We may as well take first, the push-out by $Id \otimes \phi$ to get the following diagram

$$
\begin{array}{c}
0 & \to & \mathcal{I} \otimes \mathcal{G}_0 & \overset{i}{\to} & \mathcal{M} & \overset{p}{\to} & \mathcal{F}_0 & \to & 0 \\
0 & \to & \mathcal{I} \otimes \mathcal{G}_0 & \overset{j}{\to} & \mathcal{G} & \overset{q}{\to} & \mathcal{G}_0 & \to & 0 \\
\end{array}
$$

(2)

From this diagram it is clear that all such liftings correspond to all possible liftings $\alpha$ of $\beta$ which is precisely the sections of $A$. So we need to show only that all such liftings do give us $S$-flat sheaves assuming that there is one such flat lifting. Since for any such $\mathcal{G}$ we have $\mathcal{G} \otimes O_{S_0} = \mathcal{G}_0$ which is $S_0$-flat we need to check only that $\text{Tor}^1(\mathcal{G}, O_{S_0}) = 0$. Tensoring the exact sequence

$$
0 \to \mathcal{I} \to O_S \to O_{S_0} \to 0
$$

by $\mathcal{G}$ one gets that the above Tor is zero if and only if the natural map $\mathcal{I} \otimes \mathcal{G} \to \mathcal{G}$ is injective. $\mathcal{I} \otimes \mathcal{G} \cong \mathcal{I} \otimes \mathcal{G}_0$ and from (2) it is clear that this map is injective. Hence all such $\mathcal{G}$’s are $S$-flat proving the proposition.
Corollary. Let the notation be as in the proposition. Assume for every point \( x \in X \) there exists a neighbourhood \( U_x \) such that \( \mathcal{P}(U_x) \neq \emptyset \). Then there exists an element \( c(\mathcal{G}_0) \in H^1(X, A) \), the vanishing of which is necessary and sufficient for \( \mathcal{P}(X) \neq \emptyset \).

Proof: For any \( x, y \in X \) by the proposition we get an element in \( H^0(U_x \cap U_y, A) \) since we have local lifts. These give a 1-cocycle whose vanishing is necessary and sufficient for a global lift.

Let \( X \) be a projective scheme and let \( 0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0 \) (1) be an exact sequence of coherent sheaves on \( X \). Let \( V = H^0(\text{Hom}(\mathcal{F}, \mathcal{E})) \) and let \( R = k \oplus V^* \) be the \( k \)-algebra with \( V^{*2} = 0 \). Let \( S = \text{Spec}R \). Let \( X' = X \times S \) and \( p \) and \( q \) the two projections. Then we have

\[
0 \rightarrow p^* \mathcal{F} \rightarrow p^* \mathcal{E} \rightarrow p^* \mathcal{G} \rightarrow 0
\]

(2) to be exact. If we define \( \mathcal{P} \) as in the proposition for \( p^* \mathcal{E} \) and \( \mathcal{G} \) then (2) ensures that \( \mathcal{P}(X') \neq \emptyset \) and hence by the proposition we see that \( \mathcal{P}(X') \) is a principal homogeneous space over \( \text{Hom}(\mathcal{F}, \mathcal{G} \otimes V^*) = V \otimes V^* \). The canonical element in \( V \otimes V^* \) gives us an exact sequence

\[
0 \rightarrow \mathcal{F}' \rightarrow p'^* \mathcal{E} \rightarrow \mathcal{G}' \rightarrow 0
\]

(3).

It has the following universal property: Let \( R' = k \oplus W \) be any \( k \)-algebra with \( W^2 = 0 \) and \( T = \text{Spec}R' \). Let \( p' \) denote the first projection from \( X \times T \). Assume we have an exact sequence on \( X \times T \)

\[
0 \rightarrow \mathcal{F}'' \rightarrow p'^* \mathcal{E} \rightarrow \mathcal{G}'' \rightarrow 0
\]

(4) with \( \mathcal{G}'' \) flat over \( T \) and (4) reduces to (1) at the closed point. Then there exists a unique morphism \( f : T \rightarrow S \) so that (4) is the pull-back of (3) by \( f \).

We of course have an exact sequence

\[
0 \rightarrow p'^* \mathcal{F} \rightarrow p'^* \mathcal{E} \rightarrow p'^* \mathcal{G} \rightarrow 0
\]

and thus \( \mathcal{P}(X \times T) \neq \emptyset \) and then (4) corresponds to an \( \alpha \in \text{Hom}(\mathcal{F}', \mathcal{G} \otimes W) = V \otimes W \). Thus \( \alpha \) induces a map \( R \rightarrow R' \) and the rest is easy to check.

Corollary. The Zariski tangent space at \( q \in \text{Quot}^P_\mathcal{E}(X) \) is canonically isomorphic to \( \text{Hom}(\mathcal{F}, \mathcal{G}) \) where \( q \) corresponds to the exact sequence

\[
0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0.
\]

Proof: Let \( R' = \mathcal{O}_q, \text{Quot}^P_\mathcal{E} / \mathcal{M}^2 \) where \( \mathcal{M} \) is the maximal ideal. Let \( S \) be as in the above paragraph. Then by the universal property of \( \text{Quot}^P_\mathcal{E} \) one has a map \( g : S \rightarrow \text{Quot}^P_\mathcal{E} \) such that \( g(\text{closed point}) = q \) and the pull back of the universal exact sequence via \( g \) is (3). Since \( V^{*2} = 0 \) \( g \) factors through \( T = \text{Spec}R' \). By the previous universal property we also have a map \( f : T \rightarrow S \) such that the pull back of (3) by \( f \) gives the restriction of the universal exact sequence to \( T \). One checks by the universal properties that the two composites \( f \circ g \) and \( g \circ f \) are both identities and then the corollary is obvious.
Corollary. Let \( h \in \text{Hilb}^P(X) \) and \( Z \subset X \) the corresponding subscheme and \( J \) its ideal sheaf. If \( N = J/J^2 \) is the normal sheaf then the Zariski tangent space at \( h \) is canonically isomorphic to \( H^0(X, N) \).

Proof: We have an exact sequence, \( 0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 \), which corresponds to the point \( h \). So by the previous corollary the tangent space at \( h \) is, \( \text{Hom}(\mathcal{I}, \mathcal{O}_Z) = \text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z) = H^0(N_{Z/X}) \).

Corollary. Let \( h \in \text{Hilb}^P(X) \), be as above. Assume that \( Z \) is a local complete intersection in \( X \). Then \( \text{Hilb}^P \) is smooth at \( h \) if \( H^1(N_{Z/X}) = 0 \).

Proof: By smoothness criterion \( \mathcal{O}_h \) is regular if and only if for every \( A \to B \) surjective maps of artin local rings over \( k \) and morphism \( \mathcal{O}_h \to B \) there exists a lift \( \mathcal{O}_h \to A \). We may by inducting on length assume that \( \ker(A \to B) = I \cong k \), one dimensional and in particular square zero. Using \( f \) to pull back we get an exact sequence

\[
0 \to f^*\mathcal{I} \to \mathcal{O}_{X \times B} \to \mathcal{O}_Z \to 0,
\]

where \( Z \) at the closed point gives \( Z \subset X \). The fact that \( Z \) is a locally complete intersection and \( Z \) is flat over \( B \) implies that \( Z \) is also a local complete intersection. Thus for suitable open cover in \( X \times A \), we can lift \( Z \) to \( Z' \) locally. Now by the first corollary, since \( H^1(\text{Hom}(f^*\mathcal{I}, \mathcal{O}_Z \otimes k)) = H^1(\text{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)) = H^1(N_{Z/X}) = 0 \), we have a global lift and thus \( f \) lifts by the universal property of \( \text{Hilb}^P \).