This talk is about some surprising applications of p-adic numbers. I'd present two problems. On the surface it doesn't look like they have anything to do with p-adic numbers but somehow they show up in the proof. The first one is at least about numbers but the second is about triangulations and areas which sounds quite far from p-adic numbers.

1. Skolem-Mahler-Lech \ Let \( \{a_n\}_{n \geq 0} \) be a sequence of integers given by a linear recurrence relation, i.e. 
\[ a_n = \lambda_1 a_{n-1} + \lambda_2 a_{n-2} + \cdots + \lambda_r a_{n-r} \]
for \( n \geq r \)
where \( \{\lambda_1, \lambda_2, \ldots, \lambda_r\} \subseteq \mathbb{Z} \).

**Ex 1.**
- \( a_0 = 0 \)
- \( a_1 = 1 \)
- \( a_n = a_{n-1} + a_{n-2} \)

\[ S = \{0, 2\} \]

**Ex 2.**
- \( a_0 = 0 \)
- \( a_1 = 1 \)
- \( a_2 = 0 \)
- \( a_3 = 2 \)
- \( a_4 = 0 \)
- \( a_5 = 3 \)
- \( a_6 = 0 \)
- \( a_7 = 5 \)

\[ S = \{1\} \cup \{\text{even numbers}\} \]

\[ a_n = a_{n-2} + a_{n-4} \]

**Ex 2**
- \( a_0 = 1 \)
- \( a_1 = -1 \)
- \( a_2 = 0 \)
- \( a_3 = 1 \)
- \( a_4 = -1 \)
- \( a_5 = 0 \)
- \( a_6 = 1 \)
- \( a_7 = -1 \)

\[ S = \{n \mid n \equiv 2 \pmod{3}\} \]

\[ a_n = a_{n-2} + a_{n-3} \]

**Ex 4**
- \( a_0 = 100 \)
- \( a_1 = 99 \)
- \( a_2 = 98 \)
- \( a_3 = 97 \)

\[ S = \{100\} \]

\[ a_n = 2a_{n-1} - a_{n-2} \]

Consider \( S = \{n \mid a_n = 0\} \).

**Thm:** The set \( S \) is eventually periodic.

Actually, \( S \) is the union of a finite set and a finite union number of residue classes.
\[ S = \text{finite set } U \{ n \mid n \equiv a \mod m \} \]

**Observation:** Fix a prime \( p \). The sequence \( \{a_n\} \) is periodic mod \( p \).

(only finitely many \( r \)-tuples modulo \( p \). You might be tempted to say it's periodic \( \mod \) every prime hence it's periodic but that's just false. Also we haven't used that the recurrence is linear)

\[
\begin{pmatrix}
A & \mathbf{v}_0 \\
\mathbf{v}_1 & \mathbf{y}
\end{pmatrix} =
\begin{pmatrix}
a_{r-1} & \mathbf{v}_0 \\
a_{r-2} & \mathbf{v}_1 \\
\vdots & \mathbf{y}
\end{pmatrix}
\]

\[
\mathbf{a}_n = \langle \mathbf{A}^n \mathbf{v}_0 + \mathbf{b}_1 \rangle
\]

**Obs:** \( A \) has finite order \( \mod p \). (Assume \( p \neq \det(A) \)).

(The sequence is periodic not just \( \mod p \) but \( \mod p^2, p^3 \) and so on. It's fruitful to look at modulo all powers of \( p \)).

\[
\mathbb{Z}_p = \lim_{\rightarrow n} \mathbb{Z}/p^n\mathbb{Z} = \left\{ \sum_{i=0}^{p-1} b_i p^i \mid b_i \in \{0, 1, \ldots, p-1\} \right\}
\]

\[
z = \sum_{i=0}^{p-1} b_i p^i
\]

\[
v_p(z) = \max \left\{ i \mid p^i \text{ divides } z \right\} = \max \left\{ i \left| \sum_{i=0}^{p-1} b_i p^i \right. \right\}
\]

\[
|z| = -v_p(z)
\]

\[
(0) = 0
\]
Recall \( |x|_r = |x|_r |y|_r \)

\[ \max(|x_1|, |y_1|) \leq |x+y| \leq \max(|x_1|, |y_1|) \]

In a metric space with \( d(x, y) = |x-y|_r \), and is compact and complete.

**Analysis on \( \mathbb{Z}_p \):**

**Fact:** \( \lim \sum_{i=0}^\infty |b_i| = 0 \)

**Definition:** Let \( B \) be an open ball around \( b_0 \) in \( \mathbb{Z}_p \). We say a function \( f : B \rightarrow \mathbb{Z}_p \) is analytic at \( b_0 \in B \) if there is a power series expansion

\[ f(z) = \sum_{k=0}^\infty a_k (z-b_0)^k \]

That converges in a has a positive radius of convergence.

**Fact:** Any point \( b' \) in \( B \) is a center of \( B \). We can recenter the power series expansion \( b' \) that converges on \( B \)

**Fact:** If \( f \) is not identically zero then we can write \( f(z) = (z-b_0)^n \cdot g(z) \)

where \( g \) is analytic on \( B \) and \( g(b_0) \neq 0 \)

**Corollary:** Zeros of \( f \) are isolated (if \( f \neq 0 \)).

**Corollary:** If \( f \neq 0 \) then \( f \) has finitely many zeros in \( B \).

**Remark:** Just like complex analysis.

**Back to the problem:** Fix residue class \( i \mod m \).

We'd prove that \( \sum_{n=1}^m q_{n+i} = 0 \) is either finite or all of \( \mathbb{Z}_{\geq 0} \).
key calculation: \( n \to a_{m_1i} \) is a
analytic function of \( n \) (that converges on all
of \( \mathbb{Z}_p \)). \( A^n = I + pB + B + \cdots \in \mathbb{M}_{m \times n} \mathbb{Z} \)

\[ a_{m_1i} = \langle A^m (v_1 v_0), e_i \rangle \]

\[ A^m = I + (p) pB + (p^2) p^2 B^2 + \cdots + (p^n) p^n B^n \]

\[ a_{m_1i} = \sum_{k \geq 0} p^k P_k(n) \]

where \( P_k(n) \) is a

polynomial with \( \mathbb{Z}_p \)-coeff 

of degree \( k \).

\[ a_{m_1i} = \sum_{k \geq 0} n^k c_k \]

where \( c_k \in \mathbb{Z}_p \)

\[ p^k \mid c_k \quad \text{(almost)} \]

(\( \forall p (p^k c_k) \to \infty \))

(2) A square cannot be divided into an odd number

of equal areas.

As of equal area,

can always be divided into even odd number of areas.

Assume the area of the square is 1.

Strategy: Color the vertices which appear in the

triangulation such that the coloring is a Sperner

colouring. (satisfies S1, S2)

S1, S2 \( \Rightarrow \) There exists a rainbow triangle.

Rainbow triangle cannot have area \( \frac{1}{n} \) with \( n \) odd.

S1: Every side of a triangle only uses two colors.

S2: Some side of the square uses has endpoints

blue and red. No other side of the square

contains both blue and red.
3) Claim: \( S_1, S_2 \implies \exists \) odd no. of rainbow triangles.

\[
\sum \frac{\text{# BR segments along } \Delta}{\text{BR}(\Delta)} = \text{odd}
\]

\( \Delta \)

\( \text{BR}(\Delta) = \text{even} \) if \( \Delta \) is not a rainbow

\( = \text{odd} \) if \( \Delta \) is rainbow.

Assume \( \Delta \) has vertices \((1,1), (2,1), (2,2), (1,2)\).

Assume vertices in the triangulation are rational.

\( v: \mathbb{Q}^* \to 1\mathbb{R}^* \) 2-adic valuation

\[
v(x/\beta) = v(x) - v(\beta)
\]

\[
v(a/\beta) = 1\beta, \quad 0, v(x), v(y)
\]

Blue: \( v(x) \leq 0, v(y) \)

Red: \( 0 < v(x), v(y) \)

Green: \( v(y) \leq 0, v(y) < v(x) \)

\[(0,1) \quad \text{blue} \quad (2,2) \quad (1,1) \quad \text{red} \]

Blue \((1,1)\) \quad \text{green} \quad \text{red} \((2,2)\)

\((0,1)\)
Claim: Area of a rainbow triangle cannot be \( \frac{1}{2} \) odd.

\[
\begin{align*}
  (x_1, y_1) & \quad B \\
  (x_2, y_2) & \quad C \\
  (x_3, y_3) & \quad R \\
\end{align*}
\]

\[
A = \frac{1}{2} \left| \begin{array}{ccc}
  x_1 & y_1 & 1 \\
  x_2 & y_2 & 1 \\
  x_3 & y_3 & 1 \\
\end{array} \right| = \frac{1}{2} \left[ x_1 y_2 - x_1 y_3 + x_2 y_3 - x_2 y_1 + x_3 y_1 - x_3 y_2 \right]
\]

\[
V(x_1, y_1) < V(m_1, y_1) \quad \text{for all} \quad (m_1) \neq (1, 2)
\]

\[
V(A) = -V \left( \sum x_i y_i \right)
\]

\[
\leq -1 + V(m_1, y_1)
\]

\[
\leq -1
\]

\[
V(A) \not= V_n \quad \text{where} \quad n \quad \text{is odd.}
\]

Some argument shows that a line \( L(\alpha^2) \)

can only get at most two colours.

If \( P_1, P_2, P_3 \) are coloured \( B, C, R \) then

\[
\det (A) \neq 0
\]

General case: let \( K \) be the extension of \( \mathbb{Q} \)
generated by the coordinates of the vertices.

\[
K = \left\{ L(\alpha^2) \mid \alpha \in \mathbb{Q} \right\}
\]

\[
L(\beta) = V(\mathbb{Q} a_1 t_1) = \min \{ \alpha a_i \}
\]

Rem: A hypercube of dim \( m \) can be divided into \( m \) simplices of equal volume if and only if \( m \) is divisible by \( n! \).

Rem: Stranger result: Not all areas can be rational numbers with odd denominators.