Contents

1 Basics 1
2 The Transversality Theorem 1
3 Transversality and Homotopy 2
4 Intersection Number Mod 2 4
5 Degree Mod 2 4

1 Basics

Definition. Let $f : M \to N$ be a smooth map from a manifold $M$ with or without boundary to a manifold $N$. Let $S$ be an embedded submanifold of $N$. We say that $f$ is transversal to $S$ if

$$df_p(T_pM) + T_{f(p)}S = T_{f(p)}N$$

for all $p \in f^{-1}(S)$.

Theorem 1.1. [GP10, pg. 28] Let $f : M \to N$ be a smooth map from a smooth manifold $M$ to a smooth manifold $N$. If $f$ is transversal to an embedded submanifold $S$ of $N$, then $f^{-1}(S)$ is an embedded submanifold of $M$. Furthermore, the codimension of $f^{-1}(S)$ in $M$ is same as the codimension of $S$ in $N$.

Theorem 1.2. [GP10, pg. 60] Let $f : X \to N$ be a smooth map from a smooth manifold $X$ with boundary to a smooth manifold $N$. Let $S$ be an embedded submanifold of $N$ such that $f : X \to N$ and $\partial f : \partial X \to N$ are both transversal to $S$. Then $f^{-1}(S)$ is an embedded submanifold with boundary of $X$. Further, $\partial(f^{-1}(S)) = f^{-1}(S) \cap \partial X$, and the codimension of $f^{-1}(S)$ in $X$ is same as the codimension of $S$ in $N$.

2 The Transversality Theorem

Theorem 2.1. The Transversality Theorem. Let $M$, $A$ and $N$ be smooth manifolds, and $F : M \times A \to N$ be a smooth map. If $F$ is transversal to an embedded submanifold $S$ of $N$, then the map $F_a : M \to N$ is transversal to $S$ for almost all $a \in A$, where $F_a$ is the map $M \to N$ which sends $p \in M$ to $F(p, a)$.

Proof. We know by Theorem 1.1 that $W := F^{-1}(S)$ is a submanifold of $M \times A$. Let $\pi_A : M \times A \to A$ and $\pi_M : M \times A \to M$ be the natural projections, and let $\rho : W \to A$ denote the restriction of $\pi_A$ to $W$. We will show that whenever $a \in A$ is a regular value of $\rho$, then $F_a$ is transversal to $S$. 
Claim. If $a$ is a regular value of $\rho$, and $(p, a) \in W$, then $T_{(p, a)}(M \times A) = T_{(p, a)}W + T_{(p, a)}(M \times \{a\})$.

Proof. Let $v \in T_{(p, a)}(M \times A)$ be arbitrary. Let $v_A = d\pi_A|_{(p, a)}(v)$ and $v_M = d\pi_M|_{(p, a)}(v)$. Then $v = v_A + v_M$. The assumption that $a$ is a regular value of $\rho$ implies that there is $w \in T_{(p, a)}W$ such that $d\rho_{(p, a)}(w) = v_A$. But $d\rho_{(p, a)}(w) = d\pi_A|_{(p, a)}(w)$. Therefore $v = d\pi_A|_{(p, a)}(w) + v_M$. Split $w = w_A + v_M$, just like we had split $v$, and note that $d\pi_A|_{(p, a)}(w_M) = 0$ and $d\pi_A|_{(p, a)}(w_A) = w_A$. Thus we have

$$v = w_A + v_M = (w_A + w_M) + (v_M - w_M) = w + (v_M - w_M)$$

But $v_M - w_M \in T_{(p, a)}(M \times \{a\})$, and the claim is proved.

Now from the fact that $F$ is transversal to $S$, for any $(p, a) \in F^{-1}(S)$, we have

$$dF_{(p, a)}(T_{(p, a)}(M \times A)) + T_sS = T_sN$$

where $s = F(p, a)$. This gives

$$dF_{(p, a)}(T_{(p, a)}W) + dF_{(p, a)}(T_{(p, a)}(M \times \{a\})) + T_sS = T_sN$$

where we have used the claim. Now since $F(W) \subseteq S$, we have $dF_{(p, a)}(T_{(p, a)}W) \subseteq T_sS$, which leads to

$$dF_{(p, a)}(T_{(p, a)}(M \times \{a\})) + T_sS = T_sN$$

But $dF_{(p, a)}(T_{(p, a)}(M \times \{a\})) = dF_a|_p(T_pM)$. Thus

$$dF_a|_p(T_pM) + T_sS = T_sN$$

This shows that $F_a$ is transversal to $S$. Now by Sard’s theorem, almost all $a \in A$ are regular values of $\rho$, and thus we have $F_a$ is transversal to $S$ for almost all $a \in A$, and we are done.

By a similar reasoning as in the above, we can prove

**Theorem 2.2. The Transversality Theorem (Boundary Version).** Let $X$ be a smooth manifold with boundary, and $A$ and $N$ be smooth manifolds, and $F : X \times A \to N$ be a smooth map. If both $F$ and $\partial F$ are transversal to an embedded submanifold $S$ of $N$, then the maps $F_a : X \to N$ and $\partial F_a$ are transversal to $S$ for almost all $a \in A$.

**Corollary 2.3. General Position Lemma.** Let $M$ and $S$ be smooth submanifolds of $\mathbb{R}^n$. Then for almost all $a \in \mathbb{R}^n$, we have that the manifold $M_a := \{p + a : p \in M\}$ is transversal to $S$.

**Proof.** Consider the map $F : M \times \mathbb{R}^n \to \mathbb{R}^n$ defined as $F(p, a) = p + a$. Then $F$ is a submersion, and is hence transversal to $S$. Thus, by Theorem 2.1, we have $F_a : M \to \mathbb{R}^n$ is transversal to $S$ for almost all $a \in \mathbb{R}^n$. This is same as saying that $M_a$ is transversal to $F_a$ for almost all $a \in \mathbb{R}^n$ and we are done.

**3 Transversality and Homotopy**

**Theorem 3.1. $\varepsilon$-Neighborhood Theorem.** Let $N$ be a compact manifold in $\mathbb{R}^n$ and let $\varepsilon > 0$. Let $N^\varepsilon$ be the set of all the points in $\mathbb{R}^n$ which are at a distance less than $\varepsilon$ from $N$. If $\varepsilon$ is sufficiently small, then there is a submersion $\pi : N^\varepsilon \to N$ such that $\pi$ restricts to the identity on $N$.

**Proof.** Let $\rho : E \to N$ denote the normal bundle of $N$ in $\mathbb{R}^n$. Define a map $f : E \to \mathbb{R}^n$ as $f(y, v) = y + v$ for all $(y, v) \in E$. It is clear that $df$ is an isomorphism at each point $(y, 0) \in E$. Further, $f$ maps $N \times \{0\}$ diffeomorphically onto $N$. Thus, by the inverse function theorem, there is a neighborhood of $N \times \{0\}$ in $E$ which maps, under $f$, diffeomorphically onto a neighborhood of $N$ in $\mathbb{R}^n$. By compactness of $N$, we

---

1Here $F_a$ is the map $X \to N$ which sends $p \in X$ to $F(p, a)$ and similarly for $\partial F_a$. 

can choose this neighborhood to be of ‘uniform thickness’\(^2\). So there exists \( \varepsilon > 0 \) small enough such that a neighborhood \( U \) of \( N \times \{0\} \) in \( E \) maps diffeomorphically onto \( N^\varepsilon \). Write \( h \) to denote \( f|_U : U \to N^\varepsilon \). Since \( \rho : U \to N \times \{0\} \) is a submersion, we have \( \rho \circ h^{-1} : N^\varepsilon \to N \times \{0\} \) is also a submersion. Identifying \( N \times \{0\} \) with \( N \), we see that \( \pi \) is the required submersion.

\[ \text{Theorem 3.2. Transversality Homotopy Theorem.} \]

\( f : M \to N \) be a smooth map between smooth manifolds and \( S \) be an embedded submanifold of \( N \). Then there is a smooth map \( g : M \to N \) homotopic to \( f \) such that \( g \pitchfork S \).

**Proof.** We may assume that \( N \) is embedded in \( \mathbb{R}^n \). Since \( N \) is compact, by the \( \varepsilon \)-Neighborhood Theorem there is \( \varepsilon > 0 \) small enough such that \( \pi : N^\varepsilon \to N \) is a smooth submersion which restricts to the identity on \( N \). Let \( B \) be the unit ball in \( \mathbb{R}^n \), and define a map \( F : X \times B \to N \) as \( F(x,b) = \pi(f(x) + \varepsilon b) \).

Since \( F \) is the composite of two submersions, we see that \( F \) itself is a submersion, and is therefore transversal to \( S \). Thus, by Transversality Theorem, there is a \( b \in B \) such that \( g : M \to N \) defined as \( g(x) = F(x,b) \) is transversal to \( S \). Finally, the map \( H : X \times I \to N \) defined as \( H(x,t) = F(x, tb) \) is a homotopy between \( f \) and \( g \) and we are done.

\[ \text{Lemma 3.3.} \]

\( f : X \to N \) be a smooth map, where \( X \) is a smooth manifold with boundary and \( N \) is a smooth manifold. Let \( S \) be a closed embedded submanifold of \( N \). Then the set of points \( x \in X \) where \( f \) is transversal to \( S \) is an open set of \( X \).

**Proof.** Let \( x \in X \) be such that \( f \) is transversal to \( S \) at \( x \). There are two cases. Assume first that \( x \notin f^\leftarrow(S) \). Since \( S \) is closed in \( N \), \( f^\leftarrow(S) \) is closed in \( X \), and thus there is a neighborhood of \( x \) in \( X \) which avoids \( f^\leftarrow(S) \), and \( f \) is vacuously transversal to \( S \) on this neighborhood.

Now assume that \( x \in f^\leftarrow(S) \). Consider a chart \( (V,\psi) \) on \( N \) centered at \( f(x) \) such that \( \psi \) maps \( S \cap V \) to a slice in \( \mathbb{R}^n \), where \( n \) is the dimension of \( N \). Compose \( \psi \) by an appropriate projection \( \pi \) which collapses this slice to a point. Thus \( \pi \circ \psi \circ f \) is a submersion at \( x \), and therefore it remains a submersion in a neighborhood \( U \) of \( x \). It is easy to see that \( f \) is transversal to \( S \) on \( U \), and we are done.

\[ \text{Theorem 3.4. Transversality Homotopy Theorem (Boundary Version).} \]

\( f : X \to N \) be a smooth map from a smooth manifold with boundary \( X \) to a smooth manifold \( N \). Assume \( N \) is compact. Let \( S \) be a closed embedded submanifold of \( N \). If \( \partial f : \partial X \to N \) is transversal to \( S \), then there is a smooth map \( g : X \to N \) homotopic to \( f \) such that \( g \pitchfork S \) and \( \partial g = \partial f \).

**Proof.** By Lemma 3.3 we see that there is a neighborhood \( U \) of \( \partial X \) in \( X \) such that \( f|_U \) is transversal to \( S \). Let \( \gamma : [0,1] \to [0,1] \) be a smooth map which is identically 1 outside \( U \) and is identically 0 in a neighborhood of \( \partial X \). Define \( \tau : X \to [0,1] \) by setting \( \tau = \gamma^2 \).

We may assume that \( N \) is embedded in \( \mathbb{R}^n \). Since \( N \) is compact, there is \( \varepsilon > 0 \) small enough such that \( \pi : N^\varepsilon \to N \) is a smooth submersion. Let \( B \) be the unit ball in \( \mathbb{R}^n \), and define a map \( F : X \times B \to N \) as \( F(x,b) = \pi(f(x) + \varepsilon b) \). Further define \( G : X \times B \to N \) as \( G(x,b) = F(x, \tau(x)b) \).

We show that \( G \) is transversal to \( S \). Let \( (x,b) \in G^{-1}(S) \). If \( \tau(x) \neq 0 \), then \( G \) is a submersion at \( (x,b) \), because the map \( B \to N \) defined as \( b \mapsto G(x, \tau(x)b) \) is a submersion, being the composite of the submersions \( b \mapsto \tau(x)b : B \to B \) and \( b \mapsto F(x,b) : B \to Y \). Thus \( G \) is transversal to \( S \) at \( (x,b) \). So we may assume \( \tau(x) = 0 \). Thus \( d\tau_x = 0 \)\(^3\). Define \( \mu : X \times B \to X \times B \) as \( \mu(x,b) = (x, \tau(x)b) \). Then we have \( d\mu(x,b)(u,v) = (v, \tau(x)w + d\tau_x(v)b) \). By the chain rule applied to \( F \circ \mu \), we have \( dG(x,b)(v,w) = dF(x,0)(v,0) \), which is same as \( dF(x,0)(v) \). But since \( \tau(x) = 0 \), we have \( x \in U \), and since \( f|_U \) is transversal to \( S \), we have conclude that \( G \) is transversal to \( S \) at \( (x,b) \). This completes the proof that \( G \) is transversal to \( S \).

Therefore, in particular, \( G \) restricted to \( (X \times S) \setminus \partial(X \times S) \) is transversal to \( S \). Note that \( (X \times S) \setminus \partial(X \times S) = (X \setminus \partial X) \times S \). Thus, by Theorem 2.1, there is a \( b \in B \) such that \( : X \setminus \partial X \to N \) defined as \( g(x) = G(x,b) \) is transversal to \( S \). Consider the extension \( \tilde{g} : X \to N \) of \( g \) defined as \( \tilde{g}(x) = G(x,b) \) for

\(^2\)This makes sense because the normal bundle is naturally equipped with a Riemannian metric.

\(^3\)This is the reason to consider the square of \( \gamma \).
all $x \in X$. Since $\tau$ vanishes identically at $\partial X$, we see that $\partial \tilde{g} = \partial f$. Thus $\tilde{g}$ is transversal to $S$. The map $H : X \times I \to N$ defined as $H(x, t) = G(x, tb)$ is a homotopy between $f$ and $g$ and we are done. □

4 Intersection Number Mod 2

Let $M$ and $N$ be smooth manifolds and $S$ be a submanifold of $N$. We say that $M$ and $S$ are of complementary dimension if $\dim M + \dim S = \dim N$. Now assume that $S$ is closed in $N$, $M$ is compact, and let $f : M \to N$ be a smooth map which is transversal to $S$. Then $f^{-1}(S)$ is a 0-dimensional closed submanifold of $M$, and is hence finite. We write $I_2(f, S)$ to denote $|f^{-1}(S)|$ mod 2. We call $I_2(f, S)$ the mod 2 intersection number of $f$ with $S$.

Theorem 4.1. Let $f_0, f_1 : M \to N$ be smooth maps between smooth manifolds, both transversal to a closed submanifold $S$ of $N$. Assume $M$ is compact. If $f_0$ and $f_1$ are homotopic then $I_2(f_0, S) = I_2(f_1, S)$.

Proof. Let $F : M \times I \to N$ be a homotopy between $f_0$ and $f_1$. Note that $\partial F$ is transversal to $S$. By the Transversality Homotopy Theorem (boundary version) we have a map $G : M \times I \to N$ homotopic to $F$ such that $G$ is transversal to $S$ and $\partial G = \partial F$. Thus, by Theorem 1.2, $G^{-1}(S)$ is a compact 1-manifold $K$ of $M \times I$ with boundary, such that

$$\partial K = G^{-1}(S) \cap \partial (M \times I) = (f_0^{-1}(S) \times \{0\}) \cup (f_1^{-1}(S) \times \{1\})$$

By classification of 1-manifolds, if $G^{-1}(S)$ has $k$-components, then the cardinality of $\partial G^{-1}(S)$ is $2k$, which is even. Therefore $|f_0^{-1}(S)| \equiv |f_1^{-1}(S)| \pmod{2}$. □

The above theorem allows us to define the mod 2 intersection number of an arbitrary smooth map $f : M \to N$ with $S$, where $M$ is compact and $S$ is a closed submanifold of $N$. For by the Transversality Homotopy Theorem, there is a smooth map $g : M \to N$ homotopic to $f$ which is transversal to $S$. We define $I_2(f, S) := I_2(g, S)$. The above theorem guarantees that this is well defined.

Corollary 4.2. Let $f_0, f_1 : M \to N$ be any two homotopic maps from a compact smooth manifold $M$ to a smooth manifold $N$. Let $S$ be a closed submanifold of $N$. Then $I_2(f_0, S) = I_2(f_1, S)$.

Theorem 4.3. Boundary Theorem. Let $f : M \to N$ be a smooth map between smooth manifolds and assume that $M$ is the boundary of some compact manifold $X$. If $f$ can be extended smoothly to all of $X$ then $I_2(f, S) = 0$ for any closed submanifold $S$ in $N$ of dimension complementary to $M$.

Proof. Let $F : X \to N$ be an extension of $f$. Let $S$ be a closed embedded submanifold of $N$ of complementary dimension to $M$. By the Transversality Theorem (Boundary Version) there is a map $G : X \to N$ homotopic to $F$ such that both $G$ and $g := \partial G$ are transversal to $S$. Thus we have $f$ is homotopic to $g$, so we need to show that $I_2(g, S) = 0$. But by Theorem 1.2 we have $G^{-1}(S)$ is a 1-dimensional submanifold with boundary in $X$ whose boundary is same as $G^{-1}(S) \cap M$, which is same as $g^{-1}(S)$. But the boundary of $G^{-1}(S)$ has an even number of points, and thus $I_2(g, S) = 0$, and we are done. □

5 Degree Mod 2

Theorem 5.1. Let $M$ and $N$ be smooth manifolds where $M$ is compact and $N$ is connected. Let $f : M \to N$ be a smooth map. If $\dim M = \dim N$, then $I_2(f, \{q\})$ is same for all $q \in N$. This common value is termed as the mod 2 degree of $f$, and we denote it by $\deg_2(f)$.

Proof. Assume $\dim M = \dim N > 1$. Let $q$ and $q'$ be two distinct points in $N$, and let $S$ be the image of an embedding of a circle in $N$ which passes through both $q$ and $q'$. Let $g : M \to N$ be a map homotopic to $f$ which is transversal to $S$, and well as both $q$ and $q'$. Since $M$ and $N$ are of the same dimension, by Theorem 1.1 $g^{-1}(S)$ is an embedded submanifold of $M$ of dimension 1. Thus $g^{-1}(S)$ is a disjoint union of
finitely many submanifolds of $M$, each diffeomorphic to $S^1$. This reduces the problem to the case where both $M$ and $N$ are $S^1$, in which case the proof is easy. An alternate proof can be found in [GP10] pg. 80].

**Theorem 5.2.** Homotopic maps have the same mod two degree.

**Proof.** Immediate from Theorem 4.1.

**Theorem 5.3.** Let $M$ and $N$ be smooth manifolds of the same dimension, where $M$ is compact and $N$ is connected. Assume that $M$ is the boundary of a manifold $X$. Let $f : M \to N$ be a smooth map. If $f$ is smoothly extendible to all of $X$, then $\deg_2(f) = 0$.

**Proof.** Immediate from the Boundary Theorem.

**Corollary 5.4.** $S^1$ is not simply-connected.

**Theorem 5.5.** No compact manifold other than the one point space is contractible.

**Proof.** Let $M$ be a compact connected manifold of dimension at least 1. We want to show that $M$ is not contractible. Suppose not. Fix a point $p \in M$. The contractibility of $M$ implied that the identity map $\text{Id} : M \to M$ is homotopic to the constant map $c : M \to M$ whose image is $\{p\}$. Let $q \in M$ be different from $p$. Clearly, both the identity map and the constant map $c$ are transversal to $\{q\}$. Thus we have $\deg_2(\text{Id}) = \deg_2(c)$. But $\deg_2(\text{Id}) = I_2(\text{Id}, \{q\}) = 1$, and $\deg_2(c) = I_2(c, \{q\}) = 0$. Thus we arrive at a contradiction, and therefore we must have the manifold $M$ is not contractible.

**Note.** For any questions or comments please write to me at khetan@math.tifr.res.in

**References**