The Vaserstein symbol in dimension two

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Abstract

A beautiful theorem of L.N. Vaserstein in the mid-seventies, prior to the Serre's conjecture being solved in late seventies, describes an abelian group structure on the orbit space of unimodular rows of length three modulo elementary action over a two dimensional ring. We revisit this theorem, and explain why it is of importance even today.

Abstract

L.N. Vaserstein initiated the algebraic study of a group structure on orbit spaces of unimodular rows. Generalized Mennicke n-symbols were studied by Fossum-Foxby-Iversen, and A. Suslin related them with his completion of the "factorial powered" unimodular row $(a_0, a_1, a_2^2, \cdots, a_{n-1}^{n-1})$. W. van der Kallen combined these themes, with existing topological intuition, to get a universal weak Mennicke symbols interepretation of the group structure on orbit spaces of unimodular rows of size bigger than half the Krull dimension. We show why we feel that there is a a Witt group structure interpretation, as was shown by L.N.Vaserstein in dimension two.

Unimodular Rows

R will always be a commutative (noetherian) ring with 1.

A row $v = (v_1, \cdots, v_n) \in R^n$ is called **unimodular** (of length n) if there is a row $w = (w_n, \cdots, w_n)$ with the dot product

$$v.w^t = \sum_{i=1}^{n} v_i w_i = 1.$$

Examples:

Any row $e_i\sigma$ of an invertible matrix $\sigma \in Gl_n(R)$ is unimodular, via the Laplace expansion of the determinant with respect to that row. (Here $e_i = (0, \cdots, 1, 0, \cdots, 0)$ is the i-th coordinate vector.) Such a row is called a **completable** row.

A Non Completable Unimodular Row

Let

$$R = I\!\!R[x, y, z]/(x^2 + y^2 + z^2 - 1) = \Gamma(S^2)$$

be the coordinate ring of real valued polynomial functions on the real 2-sphere S^2. Then $(\overline{x}, \overline{y}, \overline{z}) \in Um_3(R)$, but is **not** completable:

Let $(\overline{x}, \overline{y}, \overline{z}) = e_1\sigma, \sigma \in Gl_3(R)$. Define $\phi : S^2 \to I\!\!R^3$ by

$$\phi(x) = (\sigma_{21}^{-1^t}(x), \sigma_{22}^{-1^t}(x), \sigma_{23}^{-1^t}(x))$$

ϕ is a nowhere vanishing vector field on S^2, which we know is not possible by topology.

THE ELEMENTARY SYMPLECTIC WITT GROUPS

We describe Witt's ideas in mid 1930's.

Given alternating matrices (of Pfaffian one) how to stably identify them, so as to get a nice group structure on these matrices.

(These ideas are prior to Grothendieck's ideas of constructing $G_0(R)$: category of finitely generated modules (upto isomorphism) over a ring R, with direct sum as addition, and where the sum of the end modules in an exact sequence of modules is identified with the central one.)

The 'addition' operation $\perp$: $\alpha \perp \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

The basic alternating matrix $\psi_r \in E_{2r}(\mathbb{Z})$:

$$\psi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \psi_r = \psi_{r-1} \perp \psi_1$$

<u>S</u>table equivalence of alternating matrices w.r.t. $E(A)$ - the stable elementary subgroup of $GL(A)$: Let $\phi \in M_{2r}(A)$, $\eta \in M_{2s}(A)$, then $\phi \simeq \eta$ iff

$$\phi \perp \psi_{s+t} = \varepsilon^t (\eta \perp \psi_{r+t}) \varepsilon,$$

for some $t \geq 0$, $\varepsilon \in E_{2(r+s+t)}(A)$.

$\simeq$ is an equivalence relation of the set of all matrices, and also on the set of all alternating matrices of Pfaffian one.

One can show that $\perp$ defines an addition on the equivalence classes of alternating matrices of Pfaffian one; and this is actually <u>a</u>n abelian group $W_E(A)$ - known as the *Elementary Symplectic Witt group.*

$[\psi_r]$ is the identity element, and $[\phi]^{-1} = [\phi^{-1}]$.

THE VASERSTEIN SYMBOL

The Vaserstein Symbol $V : Um_3(A) \longrightarrow W_E(A)$:

$$[(a,b,c)] \rightarrow \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c' & -b' \\ -b & -c' & 0 & a' \\ -c & b' & -a' & 0 \end{pmatrix} = V(a,b,c;a',b',c')$$

$$aa' + bb' + cc' = 1$$

The key point: Given a unimodular row of length three one can construct a natural alternating matrix of size four of Pfaffian one. (If you carry this idea out from this lecture then I have done my job!!)

Theorem: (L.N. Vaserstein)

V is an isomorphism if Krull dimension A is two.

We shall come back to this theorem if time permits. Let us first describe this group, and make some comments on above construction.

THE VASERSTEIN RULE: ADDITION OF ROWS IN DIMENSION TWO

Let $xx' + yy' = 1$ modulo (a), then

$$[(a,b,c)] * [(a,x,y)] = [(a,(b,c)\begin{pmatrix} x & y \\ -y' & x' \end{pmatrix})]$$

MENNICKE-NEWMANN LEMMA FOR UNIMODULAR ROWS

Let $v, w \in Um_n(A)$, $n \geq 3$, with $d = \dim(A) \leq 2n-3$. There there are $\varepsilon_1, \varepsilon_2 \in E_n(A)$, such that

$$v\varepsilon_1 = (x, a_2, \cdots, a_n)$$

$$w\varepsilon_2 = (y, a_2, \cdots, a_n)$$

Moreover, in fact, one can also arrange that $x + y = 1$.

COUNTER-EXAMPLE IN DIMENSION THREE

Let $A = I\!R[x,y,z,t]/(x^2+y^2+z^2+t^2-1)$ be the 3-dimensional coordinate ring of the real 3 sphere S^3. In 1992, W. van der Kallen and I observed that the Vaserstein symbol $V : Um_3(A)/E_3(A) \longrightarrow W_E(A)$ is **not** injective. We did this by finding two vectors v, w which were not in the same elementary orbit, but were equal in $W_E(A)$.

Let $v = (-t^2+x^2+y^2-z^2, -2tx+2yz, 2ty+2xz) \in Um_3(A)$. (In fact v is completable: Consider the three dimensional real vector space W of pure quaternions. Let $q = x+yi+zj+tk$. Then q is a unit quaternion, and acts on W by conjugation: $p \longrightarrow qpq^{-1}$. It can be checked that v is the first row of the matrix corresponding to this linear transformation. This row can be viewed as a map $h : S^3 \longrightarrow S^2$, and is known as the Hopf map - as this map generates $\pi_3(S^2)$: Verify that $(1,0,0)$ and $(-1,0,0)$ are regular values whose inverse images are two circles which are simply linked in S^3.

The second row w is got from v by substituting $-z$ for z. Since we are reversing the orientation on S^3, we replace the Hopf map by its negative in $\pi_3(S^2)$; which is different. Hence $[v] \neq [w]$. However, using Vaserstein's Rule it was possible to show that $V([v]) = V([w])$.

EXAMPLE IN DIMENSION THREE

In the same article, W. van der Kallen and I showed that if A is a three dimensional non-singular affine algebra over a perfect C_1 field k then the Vaserstein symbol is an isomorphism.

This theorem plays a crucial role in the recent result of Jean Fasel who showed that a unimodular row of length three over three dimensional algebras over an algebraically closed field can be completed to an invertible matrix of determinant one.

VAN DER KALLEN'S GROUP STRUCTURE

Using L.N.Vaserstein's group structure in dimension two as an inductive step W. van der Kallen first defined a group structure on $Um_{d+1}(A)$, when A had Krull dimension d as follows: Choose p_0 such that $a_0p_0 = 1$ modulo $(a_1, a_2, \cdots, a_d)$. Define

$[(a_0, a_1, \cdots, a_d)] * [(b_0, a_1, \cdots, a_d)] =$

$$[(a_0(b_0 + p_0) - 1, (b_0 + p_0)a_1, a_2 \cdots, a_d)]$$

Note: If $dim(A) = 2$, $a_0p_0 + a_1p_1 + a_2p_2 = 1$, then

$$(b_0, a_1)\begin{pmatrix} a_0 & a_1 \\ -p_1 & p_0 \end{pmatrix} = (a_0b_0 - a_1p_1, b_0a_1 + p_0a_1)$$

$$= (a_0(b_0 + p_0) - 1 + a_2p_2, (b_0 + p_0)a_1)$$

He constructs the group structure by induction. If modulo (a_d),
$[(a_0,a_1,\cdots,a_{d-1})] * [(b_0, a_1, \cdots, a_{d-1})] = [(c_0, c_1, \cdots, c_{d-1})]$, then
$[(a_0, a_1, \cdots, a_d)] * [(b_0, a_1, \cdots, a_d)] = [(c_0, c_1, \cdots, c_{d-1}, a_d)]$

GROUP STRUCTURE ON ORBITS

In a sense, topology played the important role of guiding us to these algebraic results on group structures on orbits. We describe some earlier results of L.N. Vaserstein regarding this, which were perhaps inspired by the earlier results of Atiyah-Swan, Whitehead, Adams, amongst others.

If $R = C(X)$ is the ring of continuous functions real valued functions on a topological space X then every unimodular row $v \in Um_n(C(X)), n \geq 2$, determines a map

$$arg(v) : X \longrightarrow \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$$

(The first is by evaluation, and the second is the standard homotopy equivalence.) We thus get an element $[arg(v)]$ of $[X, S^{n-1}]$. (As $n \geq 2$, we may ignore base points.) Clearly, vectors in the same elementary orbit define homotopic maps. Thus, we have a natural map

$$Um_n(C(X))/E_n(C(X)) \longrightarrow [X, S^{n-1}] = \pi^{n-1}(X).$$

Note that J.F. Adams has shown that S^{n-1} is not a H-space, unless $n = 1, 2, 4$, or 8. It is classically known that this is equivalent to saying that there is no suitable way to multiply the two projection maps $S^{n-1} \times S^{n-1}$ in $[S^{n-1} \times S^{n-1}, S^{n-1}]$. However, under suitable restrictions on the 'dimension' of X we may expect to define a product.

Henceforth, let X be a finite CW-complex of dimension $d \geq 2$. L.N. Vaserstein has shown that the ring $C(X)$ has stable dimension d. Now let $n \geq 3$, so that S^{n-1} will be atleast 1-connected. By the Suspension Theorem, the suspension map

$$S: \quad [X; S^{n-1}] \longrightarrow [SX; S^n]$$

is surjective if $d \leq 2(n-2)+1$, and bijective if $d \leq 2(n-2)$. Moreover, we know that $[SX, S^n]$ is an abelian group. Hence, the orbit space has a structure of an abelian group.

Inspired by the groups structures on orbits of unimodular vectors in the case of rings of continuous functions $C(X)$ on a CW-complex X, W. van der Kallen was able to obtain similar results algebraically, in the same range. He also showed that the above map is an isomorphism of groups in the topological situation when $R = C(X)$ is the ring of continuous functions.

The main reason this was possible is a natural extension of the Vaserstein Rule described earlier.

The SUSLIN MATRICES $S_r(v, w)$

Question: Is there a way to construct an invertible matrix of determinant one given a unimodular row v, or say a pair (v, w) with $\langle v, w \rangle = 1$? A.A. Suslin in his doctoral thesis answered this affirmatively, as follows. (However, these invertible matrices were not alternating!)

Let me interject with a description:

The magical radiance of Bhimpalasi inspired the following pictorial description in the *Sangita-Raga-Kalpadruma:*

With wide eyes and fragrant with celestial flowers, Bhimapalshri, the sages tell, sings with her deep voice to the lute. Her loverly form is the embodiment of art."

The lovely forms of the Suslin matrices evoke a similar sentiment!

The construction of the Suslin matrices $S_r(v,w)$ is possible once we have two row vectors v, w. It becomes more interesting if their dot product $vw^t = 1$. (The vectors are then automatically **unimodular rows**)

A.A. Suslin's inductive definition:

Let $v = (a_0, a_1, \cdots, a_r) = (a_0, v_1)$, with

$$v_1 = (a_1, \cdots, a_r), w = (b_0, b_1, \cdots, b_r) = (b_0, w_1),$$

with $w_1 = (b_1, \cdots, b_r)$. Set $S_0(v,w) = a_0$, and set

$$S_r(v,w) \quad = \quad \begin{pmatrix} a_0 I_{2^{r-1}} & S_{r-1}(v_1,w_1) \\ -S_{r-1}(w_1,v_1)^t & b_0 I_{2^{r-1}} \end{pmatrix}.$$

A. Suslin noted that

$$\begin{aligned} S_r(v,w)S_r(w,v)^t &= (v.w^t)I_{2^r} \\ &= S_r(w,v)^t S_r(v,w), \end{aligned}$$

and $\det S_r(v,w) = (v.w^t)^{2^{r-1}}$, for $r \geq 1$.

$$\begin{pmatrix} a_0 & 0 & 0 & 0 & a_1 & 0 & a_2 & a_3 \\ 0 & a_0 & 0 & 0 & 0 & a_1 & -b_3 & b_2 \\ 0 & 0 & a_0 & 0 & -b_2 & a_3 & b_1 & 0 \\ 0 & 0 & 0 & a_0 & -b_3 & -a_2 & 0 & b_1 \\ -b_1 & 0 & a_2 & a_3 & b_0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_3 & b_2 & 0 & b_0 & 0 & 0 \\ -b_2 & a_3 & -a_1 & 0 & 0 & 0 & b_0 & 0 \\ -b_3 & -a_2 & 0 & -a_1 & 0 & 0 & 0 & b_0 \end{pmatrix}$$

$S_3((a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3))$

To describe the nature of these matrices, A.A. Suslin describes a sequence of forms $J_r \in M_{2^r}(R)$ by the recurrence formulae:

$$J_r = \begin{cases} 1 & \text{for } r = 0 \\ J_{r-1} \perp -J_{r-1}, & \text{for } r \text{ even }, \\ J_{r-1} \top - J_{r-1}, & \text{for } r \text{ odd.} \end{cases}$$

(The English translation wrongly says $J_r = J_{r-1} \top J_{r-1}$ when r is odd.)

(Here $\alpha \perp \beta = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, while $\alpha \top \beta = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$).

It is easy to see that det $J_r = 1$, for all r, and that $J_r^t = J_r^{-1} = (-1)^{\frac{r(r+1)}{2}} J_r$. Moreover, J_r is antisymmetric if $r = 4k+1$ and $r = 4k+2$, whereas J_r is symmetric for $r = 4k$ and $r = 4k+3$.

A.A. Suslin noted that the following formulae are valid:

$$\begin{aligned} \text{for } r{=}4k : (S_r(v,w)J_r)^t &= S_r(v,w)J_r; \\ \text{for } r{=}4k+1 : S_r(v,w)J_rS_r(v,w)^t &= (v.w^t)J_r; \\ \text{for } r{=}4k+2 : (S_r(v,w)J_r)^t &= -S_r(v,w)J_r; \\ \text{for } r{=}4k+3 : S_r(v,w)J_rS_r(v,w)^t &= (v.w^t)J_r. \end{aligned}$$

Use of the Suslin matrices

The Suslin matrices have proved useful in several contexts. Some examples:

- The Suslin matrices were introduced by A. Suslin to show that a unimodular row of the form $(a_0, a_1, a_2^2, \cdots, a_r^r)$ can be completed to an invertible matrix. The actual completion can be got by doing a series of row and column operations to the matrix $S_r(v,w)$ to reduce it to size $(r+1)$. (This generalized the observation three years earlier by Swan-Towber that the square unimodular rows were completable – but their argument was based on a cancellation theorem for projective modules of rank 2 over an affine surface over an algebraically closed field!)

- A. Suslin showed that $SK_1\left(\dfrac{k[x_1, \cdots, x_n]}{(\sum_{i=1}^n x_iy_i - 1)}\right) \simeq \mathbb{Z}$, with generator $[S_{n-1}((x_1, \cdots, x_n), (y_1, \cdots, y_n))]$.

- Let $\sum_{i=1}^{n} x_i y_i = 1$. Let P, P^* be the projective modules corresponding to the unimodular rows $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ respectively. Then $P^* \simeq Hom_R(P, R)$, the dual of P. If n is even then $P \simeq P^*$. However, if $n > 1$ is odd then M.V. Nori, and R.G.Swan independently showed (using topological arguments) that P, P^* need not be isomorphic. This can also be shown using the Suslin matrices quite easily, as was done by me in our joint article.

- M. Boratynski showed that any ideal I in a polynomial ring R over a field can be generated upto radical by $m = \mu\left(\frac{I}{I^2}\right)$ elements, i.e. $\sqrt{I} = \sqrt{(f_1, \cdots, f_m)}$, for some $f_1, \cdots, f_m \in R$.

These details can be found in my book with F. Ischebeck titled Ideals and Reality.

One of my aims (joint work in progress with Selby Jose) is to show:

- The orbit spaces $Um_n(R)/E_n(R)$ have a nice Witt group structures, if R is a noetherian ring of Krull dimension d, and if $n \geq \max\{3, \frac{d+3}{2}\}$.

We have a nice definition of what the Witt group should be, viz. a suitable stable equivalence classes of Suslin matrices, and the obvious Vaserstein-Suslin symbol, taking an orbit class $[v]$ to the class of the Suslin matrix $[S_r(v, w)]$. We could show that this map is a surjective group homomorphism, but are falling short of showing it is injective in all cases. Help!!