On the Milnor Conjecture (Voevodsky's Theorem)

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Abstract

The Milnor Conjecture was posed by John Milnor in 1970 to give a description of the Milnor K-theory ring of a field F of characteristic different from 2 by means of the graded Witt ring of F, which is determined by the quadratic forms over F. The Milnor Conjecture was proved by Vladimir Voevodsky, for which he was awarded the Fields medal in 2002. In this article, we introduce the Milnor K-theory, starting with the classical algebraic K-theory. We then introduce the Witt ring of a field and describe the Milnor Conjecture and its various versions. This article is based on the talk given in the Mathematics Students' Seminar at TIFR on 12th August, 2011.

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1 Introduction

Algebraic K-theory deals with linear algebra over a general ring R and associates a sequence of groups $K_0(R), K_1(R), \ldots$ to R. The motivation for algebraic K-theory came from geometric topology. The main object of interest was a (finite) CW-complex or a manifold X and the ring associated to it was the integral group-ring $R = \mathbb{Z}[G]$, where $G := \pi_1(X)$ is the fundamental group of X. The study was initiated by J.H.C. Whitehead, who associated an element called *Whitehead torsion* $\tau(f) \in K_1(R)$ for a map $f : X \to Y$ of finite CW-complexes. He proved that a necessary condition for two homotopy equivalences $f, g : X \to Y$ to be homotopic is $\tau(f) = \tau(g)$. In particular, if $f : X \to Y$ is a homeomorphism, then $\tau(f) = 0$. Later, C.T.C. Wall defined a *Wall finiteness obstruction* as an element of $K_0(R)$, which measured the obstruction for a topological space X which is dominated by a CW-complex to be homotopy equivalent to a CW-complex. A few years later, Wagoner and Hatcher showed that the obstruction for a pseudo-isotopy of a closed manifold X to be deformed to an isotopy lives in $K_2(R)$. With this evidence, it was widely believed in 1950's and 60's that the gorups $K_i(R)$ should together carry a substantial information concerning the homeomorphism classification of manifolds homotopy equivalent to a given manifold. A comprehensive history of algebraic K-theory can be found in an article of Weibel [7].

One of the founders of algebraic K-theory of a ring R was Hyman Bass, who defined $K_0(R)$ and $K_1(R)$ and studied their relationship with projective modules. The *correct* definition of $K_2(R)$ was given by Milnor, which motivated him to define an infinte sequence of groups $K_i(F)$, for a field F, which got the name *Milnor K-theory* after Milnor. We shall study these K-groups in this expository article and describe the Milnor Conjecture.

2 Classical Algebraic K-theory

Throughout the article, a ring R will be assumed to be associative with unity and all R-modules will be left R-modules. A good reference for classical algebraic K-theory is Milnor's book [6].

2.1 $K_0(R)$

The most basic concept in algebraic K-theory is that of *group completion* of an abelian moniod. Let $(M, +)$ be an abelian monoid. Define an abelian group M^+ defined as follows:

$$M^+ := \frac{\text{Free abelian group on symbols } [m], \text{ where } m \in M}{\langle [m] + [n] - [m+n] \mid m, n \in M \rangle}.$$

There is a canonical monoid homomorphism $\gamma : M \to M^+$ given by $m \mapsto [m]$. A very basic example of this is that $\mathbb{Z}$ is a group completion of $\mathbb{N}$. The group completion M^+ of M satisfies the following universal propety: If A iss an abelian group and $\varphi : M \to A$ is a monoid homomorphism, then then there is a unique homomorphism $\varphi^+ : M^+ \to A$ making the following diagram commute:

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & A \\
M^+ \arrow[ur, dashed, "\varphi^+"'] &
\end{tikzcd}$$

Definition 2.1. Let R be a ring. A (left) R-module P is said to be *projective* if there exists a (left) R-module Q such that $P \oplus Q \simeq R^n$, for some $n \in N$.

Let $\mathscr{P}(R)$ denote the set of all the isomorphism classes of projective R-modules. $\mathscr{P}(R)$ forms an abelain monoid under $\oplus$ with the identity being the class of 0.

Definition 2.2. The group $K_0(R)$ is defined to be the group completion of the abelian monoid $\mathscr{P}(R)$.

Examples 2.3. (1) If $R = F$, a field, then the projective modules over F are just finite dimensional F-vector spaces, which are characterized by their dimension. Hence, $\mathscr{P}(F) \simeq \mathbb{N}$ and consequently, $K_0(F) = \mathbb{Z}$.
(2) If R is a local ring or a PID, then all the projective modules over R are free, and we have $K_0(R) = \mathbb{Z}$.
(3) Let F be a number field, that is, a finite extension of $\mathbb{Q}$. Let $\mathscr{O}_F$ be the ring of algebraic integers in F. Then $\mathscr{O}_F$ is a Dedekind domain and $K_0(\mathscr{O}_F) = \mathbb{Z} \oplus Cl(F)$, where $Cl(F)$ denotes the ideal class group of $\mathscr{O}_F$.

2.2 $K_1(R)$

For a ring R, the group $K_1(R)$ deals with a generalization of the determinant map in the case of matrices over a field. Observe that we have natural inclusions

$$R^\times = \mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_2(R) \hookrightarrow \cdots \hookrightarrow \mathrm{GL}_n(R) \hookrightarrow \cdots$$

where each inclusion $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$ is given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly, we have inclusions $\mathrm{SL}_1(R) \hookrightarrow \mathrm{SL}_2(R) \hookrightarrow \cdots$. Set

$$\mathrm{GL}(R) := \bigcup_{n \in \mathbb{N}} \mathrm{GL}_n(R),$$

and

$$\mathrm{SL}(R) := \bigcup_{n \in \mathbb{N}} \mathrm{SL}_n(R).$$

Definition 2.4. Let e_{ij}^{λ} denote the matrix in $\mathrm{GL}_n(R)$ with 1's on the diagonal, λ in the ijth places and 0's elsewhere, where $1 \leq i, j \leq n$, $i \neq j$ and $\lambda \in R$. Matrices of the form e_{ij}^{a} are called *elementary matrices*. Let $\mathrm{E}_n(R)$ denote the group generated by all the $n \times n$ elementary matrices. We define $\mathrm{E}(R) := \bigcup\limits_{n\in\mathbb{N}} \mathrm{E}_n(R)$.

Lemma 2.5 (Whitehead)**.** *Let R be a ring. Then $\mathrm{E}(R)$ is a perfect subgroup of $\mathrm{GL}(R)$, that is, $\mathrm{E}(R)$ is a normal subgroup of $\mathrm{GL}(R)$ and $\mathrm{E}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$.* □

Definition 2.6. For a ring R, define

$$K_1(R) := \mathrm{GL}(R)/\,\mathrm{E}(R).$$

If R is a commutative ring, then the determinant $\det : K_1(R) \to R^{\times}$ is a well-defined homomorphism, which is surjective. This map admits a section, namely, the composition given by

$$R^{\times} \simeq \mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}(R) \to K_1(R).$$

Define $\mathrm{SK}_1(R)$ to be the kernel of the determinant map det. We then have

$$K_1(R) = R^{\times} \oplus \mathrm{SK}_1(R).$$

Examples 2.7. (1) If $R = F$, a field, then the usual row- and column-reduction shows that det gives an isomorphism $K_1(F) \simeq F^{\times}$. Thus, $\mathrm{SK}_1(F) = 0$.
(2) Let F be a number field and $\mathcal{O}_F$ be the ring of algebraic integers in F. Then a theorem of Bass-Milnor-Serre asserts that $\mathrm{SK}_1(\mathcal{O}_F) = 0$. Therefore, $K_1(\mathcal{O}_F) = \mathcal{O}_F^{\times} = \mu(F) \oplus \mathbb{Z}^{r_{\mathbb{R}}+r_{\mathbb{C}}-1}$, where $r_{\mathbb{R}}$ is the number of distinct embeddings of F into $\mathbb{R}$ and $r_{\mathbb{C}}$ is the number conjugate pairs of distinct embeddings of F into $\mathbb{C}$.

2.3 $K_2(R)$

Recall that the group $\mathrm{E}_n(R)$ is generated by elementary matrices. The relations trivially satisfied by them have been abstractified in the *Steinberg group*, which we introduce next.

Definition 2.8. Let $n \geq 3$. The *Steinberg group* $\mathrm{St}_n(R)$ is defined by the following presentation:

$$\begin{aligned} \text{Generators: } & x_{ij}^{\lambda}, \text{ where } 1 \leq i, j \leq n, i \neq j, \lambda \in R. \\ \text{Relations: } & x_{ij}^{\lambda} \cdot x_{ij}^{\mu} = x_{ij}^{\lambda+\mu} \\ & [x_{ij}^{\lambda}, x_{jl}^{\mu}] = x_{il}^{\lambda\mu}, \text{ where } i \neq l \\ & [x_{ij}^{\lambda}, x_{kl}^{\mu}] = 1, \text{ where } i \neq l, j \neq k. \end{aligned}$$

Here $[x, y]$ denotes, as ususal, the commutator of x and y.

There is a canonical homomorphism

$$\phi_n : \mathrm{St}_n(R) \to \mathrm{GL}_n(R),$$

given by $x_{ij}^{\lambda} \mapsto e_{ij}^{\lambda}$. Clearly, $\mathrm{Im}(\phi_n) = \mathrm{E}_n(R)$. The maps ϕ_n toghether give a map

$$\phi : \mathrm{St}(R) \to \mathrm{GL}(R),$$

such that $\mathrm{Im}(\phi) = \mathrm{E}(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$.

Remark 2.9 (Steinberg)**.** $\mathrm{St}(R)$ is the universal central extension of $\mathrm{E}(R)$ with $\ker(\phi) = \mathrm{Centre}(\mathrm{St}(R))$. This means that for any group G and a map $\psi : G \to \mathrm{E}(R)$ such that $\ker(\psi) \subseteq \mathrm{Centre}(G)$, there exists a unique homomorphism $h : \mathrm{St}(R) \to G$ such that $\phi = \psi \circ h$.

Definition 2.10. For a ring R, we define

$$K_2(R) := \ker(\phi : \mathrm{St}(R) \to GL(R)) = \mathrm{Centre}(\mathrm{St}(R)).$$

We end this section by stating a theorem of Matsumoto on K_2 of a field, which is a starting point for Milnor K-theory.

Theorem 2.11 (Matsumoto). *Let F be a field. Then $K_2(F)$ has the following presentation by generators and relations:*

$$\begin{aligned} \text{Generators: } & \{x, y\}, \text{for all } x, y \in F. \\ \text{Relations: } & \{x, 1-x\}, \text{where } x \in F, x \neq 0, 1. \end{aligned}$$

□

3 Milnor K-theory

Henceforth, F will always denote a field of characteristic different form 2. We associate a graded ring

$$K_*(F) = K_0(F) \oplus K_1(F) \oplus \cdots \oplus K_n(F) \oplus \cdots$$

with F as follows. By definition, $K_0(F) = \mathbb{Z}$ and $K_1(F)$ is the set of symbols $\{l(a) \mid a \in F^\times\}$ with a structure of an abelian group given by defining $l(a) + l(b) = l(ab)$, for all $a, b \in F^\times$. Since $l(1) + l(1) = l(1)$, we have $l(1) = 0$, the identity of the group $K_1(F)$. Thus, $K_1(F)$ is just the abelian group $F^\times$, written additively. Let $T(K_1(F))$ denote the tensor algebra over $\mathbb{Z}$ of the abelian group $K_1(F)$. Let I denote the two-sided ideal of $T(K_1(F))$ generated by the symbols of the form $l(a) \otimes l(1-a)$, for all $a \in F^\times$, with $a \neq 0, 1$. We define the *Milnor ring of F* to be the graded ring

$$K_*(F) = \bigoplus_{n=0}^{\infty} K_n(F) := T(K_1(F))/I.$$

Thus for each $n \geq 2$, the group $K_n(F)$ is the quotient of the n-fold tensor product $\bigotimes_{i=1}^{n} K_1(F)$ by the subgroup generated by elements of the form $l(a_1) \otimes \cdots \otimes l(a_n)$, where $a_i + a_{i+1} = 1$ for some i. The class of a simple tensor $l(a_1) \otimes \cdots \otimes l(a_n)$ in $K_*(F)$ is denoted by $\{a_1, \ldots, a_n\}$, and is called a *symbol of length n*. The object of our interest, $K_2(F)$, is generated by the symbols $\{a, b\}$, for all $a, b \in F^\times$ that are subject to the following relations:

(1) $\{aa', b\} = \{a, b\} + \{a', b\}$, $\{a, bb'\} = \{a, b\} + \{a, b'\}$;

(2) $\{a, b\} = 0$, if $a + b = 1$.

We first establish some of the fundamentals properties of the Milnor ring $K_*(F)$.

Lemma 3.1. *For every $a \in F^\times$, we have $\{a, -a\} = 0$*

Proof. This is obvious if $a = 1$, so we consider the case $a \neq 1$. Then $-a = (1-a)/(1-a^{-1})$, and using the relations for $K_2(F)$, we obtain

$$\begin{aligned} \{a, -a\} &= \{a, (1-a)/(1-a^{-1})\} \\ &= \{a, 1-a\} - \{a, 1-a^{-1}\} \\ &= \{a^{-1}, 1-a^{-1}\} \\ &= 0. \end{aligned}$$

□

Lemma 3.2. *For every $\xi \in K_m(F)$ and every $\eta \in K_n(F)$, we have*

$$\eta\xi = (-1)^{mn}\xi\eta$$

in $K_{m+n}(F)$. Thus, the Milnor ring $K_*(F)$ is graded commutative.

Proof. Since any symbol is generated by symbols of length 1, it suffices to prove the lemma in the case $m = n = 1$. Since $\{a, -a\} = 0$ by Lemma 3.1, we have

$$\begin{aligned}\{a, b\} + \{b, a\} &= \{a, -a\} + \{a, b\} + \{b, a\} + \{b, -b\}\\ &= \{a, -ab\} + \{b, -ab\}\\ &= \{ab, -ab\}\\ &= 0,\end{aligned}$$

so $\{a, b\} = -\{b, a\}$, and the lemma follows. □

Lemma 3.3. *For every $a \in F^\times$, we have $\{a, a\} = \{a, -1\}$*

Proof. Since $\{a, -a\} = 0$, we have $\{a, a\} = \{a, (-1)(-a)\} = \{a, -1\} + \{a, -a\} = \{a, -1\}$. □

Another consequence of these arguments is the following:

Lemma 3.4. *If $a_1, \dots, a_n \in F^\times$ are such that $a_1 + \cdots + a_n = 0$ or 1, then $\{a_1, \dots, a_n\} = 0$.*

Proof. We prove this by induction on n. The assertion is true for $n = 1$ and 2, since $\{1\} = 0$, $\{a, -a\} = 0$, and $\{a, 1 - a\} = 0$, for all $a \in F^\times$. If $a_1 + a_2 = 0$, then by induction it follows that $\{a_1, \dots, a_n\} = \{a_1, a_2\} \cdot \{a_3, \dots, a_n\} = 0$. If $a_1 + a_2 \neq 0$, then we have

$$\frac{a_1}{a_1 + a_2} + \frac{a_2}{a_1 + a_2} = 1,$$

so that $\left\{\frac{a_1}{a_1 + a_2}, \frac{a_2}{a_1 + a_2}\right\} = 0$. Expanding, we see that

$$\{a_1, a_2\} - \{a_1, a_1 + a_2\} - \{a_1 + a_2, a_2\} + \{a_1 + a_2, a_1 + a_2\} = 0,$$

so

$$\{a_1, a_2\} = \{a_1, a_1 + a_2\} + \{a_1 + a_2, a_2\} - \{a_1 + a_2, a_1 + a_2\}.$$

Multiplying this by the symbol $\{a_3, \dots, a_n\}$, and observing that $\{a_1 + a_2, a_3, \dots, a_n\} = 0$ by induction, we obtain

$$\begin{aligned}\{a_1, \dots, a_n\} &= \{a_1, a_2\} \cdot \{a_3, \dots, a_n\}\\ &= \{a_1, a_1 + a_2, a_3, \dots, a_n\} - \{a_2, a_1 + a_2, a_3, \dots, a_n\} - \{a_1 + a_2, a_1 + a_2, a_3, \dots, a_n\}\\ &= 0.\end{aligned}$$

□

We end this section with the definition of Milnor K-theory modulo 2.

Definition 3.5. The Milnor K-ring modulo 2 is defined to be $k_*^M(F) = K_*^M(F)/2K_*^M(F)$. Note that $k_0(F) = \mathbb{Z}/2\mathbb{Z}$ and that $k_1(F) = F^\times/F^{\times 2}$. It is easy to see that $k_*^M(F)$ is a graded $\mathbb{Z}/2\mathbb{Z}$-algebra generated by $\{a\}$, for $a \in F^\times$ subject to the following relations:
1. $\{ab\} = \{a\} + \{b\}$,
2. $\{a, 1 - a\} = 0$,
3. $2\{a\} = 0$.

4 Quadratic Forms over a Field

In this section, we give an outline of results from the algebraic theory of quadratic forms over a field F of characteristic not 2.

Definition 4.1. An (n-ary) quadratic form over a field F is a homogeneous polynomial of degree 2 in n variables over F.

A quadratic form over F has the general form

$$q(x_1, \dots, x_n) = \sum_{1 \le i,j \le n} a_{ij} x_i x_j \in F[x_1, \dots, x_n].$$

Since we have $\operatorname{char} F \neq 2$, we can replace the coefficients a_{ij} by $\frac{a_{ij}+a_{ji}}{2}$ without changing the form. In this way, q uniquely determines a symmetric matrix $M_q = (a_{ij})$ and in matrix notations, we have

$$q(x_1, \dots, x_n) = x^t M_q x,$$

where x stands for $(x_1, \dots, x_n)$, viewed as a column vector.

Definition 4.2. Two n-ary quadratic forms q and q' are said to be equivalent if there exists a nonsingular linear transformation $A \in GL_n(F)$ such that $q'(x) = q(Ax)$ for all $x \in F^n$; in such a case, we write $q \cong q'$.

Equivalence of two quadratic forms is clearly an equivalence relation. Note that two quadratic forms q and q' are equivalent if and only if $M_{q'} = A^t M_q A$, for some $A \in GL_n(F)$.

Every quadratic form q determines a symmetric matrix M_q, which in turn, determines a symmetric bilinear form B_q on the F-vector space F^n. In terms of q, B_q can be given by

$$B_q(x, y) = \frac{1}{2}[q(x + y) - q(x) - q(y)].$$

In case M_q is a diagonal matrix with diagonal $(a_1, \dots, a_n)$, we abbreviate the form q by $\langle a_1, \dots, a_n \rangle$.

Definition 4.3. Let q be an n-ary quadratic form over a field F. Then the determinant of q is defined to be the determinant of the matrix M_q and is denoted by $\det q$. A quadratic form is said to be regular if $\det q \neq 0$.

Note that if two n-ary quadratic forms q and q' are equivalent, then $M_{q'} = A^t M_q A$ for some nonsingular matrix A. Hence, $\det q' = \det M_{q'} = \det M_q \cdot (\det A)^2 = \det q \cdot (\det A)^2$. This shows that an equivalence class $[q]$ of quadratic forms uniquely determines an element of F/F^2. If q is regular, then $[q]$ determines a unique element of F^*/F^{*2}.

Theorem 4.4. *Let F be a field and q be an n-ary quadratic form over F. Then q is equivalent to a diagonal form $\phi(x) = \sum_{i=1}^n a_i x_i^2$, for some $a_1, \dots, a_n \in F$.*

Proof. We use induction on n. If $q(x) = 0$ for all x, then $M_q = 0$ with respect to any basis. Assume $q(x_1) = a_1 \neq 0$ for some $x_1 \in F^n$ and consider the subspace

$$V = (Fx_1)^\perp = \{y \in F^n \mid B_q(y, x_1) = 0\}.$$

$q(x_1) \neq 0$ implies that $\dim V = n-1$, so we have $F^n = (Fx_1) \oplus V$ and $q = q_1 \oplus q_2$ with $q_1 = q|_{Fx_1} = \langle a_1 \rangle$ and $q_2 = q|_V$. By induction, $q_2 \cong \langle a_2, \dots, a_n \rangle$ for some $a_2, \dots, a_n \in F$, and the proof follows. □

Note. In the case $q \neq 0$ in the above proof, we can take for a_1 any nonzero element of F represented by q.

Notation. If q is a quadratic form over F equivalent to $\sum_{i=1}^{n} a_i x_i^2$, then we write $\langle a_1, \dots, a_n \rangle$ for q.

There are two basic operations on quadratic forms. If q is an n-ary quadratic form and q' an m-ary quadratic form, then we define their *orthogonal direct sum* $q \oplus q'$ to be $(n+m)$-ary quadratic form associated with the symmetric matrix

$$M_{q \oplus q'} := \begin{pmatrix} M_q & 0 \\ 0 & M_{q'} \end{pmatrix}.$$

The *tensor product* of q and q' is defined to be the the (nm)-ary quadratic form $q \otimes q'$ associated with the symmetric matrix

$$M_{q \otimes q'} := \begin{pmatrix} a_{11}M_{q'} & a_{12}M_{q'} & \cdots & a_{1n}M_{q'} \\ a_{21}M_{q'} & a_{22}M_{q'} & \cdots & a_{2n}M_{q'} \\ \vdots & \vdots & \ddots & M_{q'} \\ a_{n1}M_{q'} & a_{n2}M_{q'} & \cdots & a_{nn}M_{q'} \end{pmatrix}, \text{ where } M_q = (a_{ij}).$$

Thus, $M_{q \otimes q'}$ is just the *Kronecker product* of matrices M_q and $M_{q'}$. In particular, we have $\langle a \rangle \otimes \langle b \rangle \cong \langle ab \rangle$ and $\langle 1, a \rangle \otimes \langle 1, b \rangle \cong \langle 1, a, b, ab \rangle$ for all $a, b \in F$.

Lemma 4.5. *Let q be a regular quadratic form over F. Suppose there exists a nonzero $x = (x_1, \dots, x_n)$ such that $q(x) = 0$. Then there exists a quadratic form q' over F such that $q = \langle 1, -1 \rangle \oplus q'$.*

Proof. Let B_q be the bilinear form corresponding to q. Since B_q is nondegenerate, there exists a nonzero y such that $a = B_q(x, y) \neq 0$. Let $x' = a^{-1}x$ and $y' = y - a^{-1}q(y)x$. Now it is easy to see that $q = \langle 1, -1 \rangle \oplus q'$ by rewriting M_q in the basis in which x' and y' are the first two vectors. □

Definition 4.6. An n-ary quadratic form q over F is said to be *isotropic* if there exists a nonzero element $x \in F^n$ such that $q(x) = 0$. q is called *anisotropic* if it is not isotropic. A quadratic form of type $\langle 1, -1 \rangle \oplus \cdots \oplus \langle 1, -1 \rangle$ is called *hyperbolic*.

It is not difficult to verify that the operations $\oplus$ and $\otimes$ are preserved under equivalence of quadratic forms. This makes the set of equivalence classes of quadratic forms into a commutative semiring. The *Grothendieck completion* of this semiring, obtained by adding the additive inverses of elements to the semiring*formally* is a ring, called the *Grothendieck-Witt ring* of F and denoted by $\widehat{W}(F)$. The quotient ring of $\widehat{W}(F)$ by the hyperbolic forms is a ring with the form $\langle 1, -1 \rangle$ serving as the zero and the form $\langle 1 \rangle$ as the unity, called the *Witt ring* of F. The Witt ring of a field F is denoted by $W(F)$. Note that $W(F)$ is generated by one-dimensional forms $\langle a \rangle$, for $a \in F^\times$. Let $IF \subseteq W(F)$ denote the ideal of even-dimensional quadratic forms over F. Let I^nF denote the nth power $(IF)^n$.

Definition 4.7. The *Graded Witt ring* of F is defined to be

$$GW_*F := \bigoplus_{n \in \mathbb{N}} I^nF/I^{n+1}F$$

Note that GW_*F is a $\mathbb{Z}/2\mathbb{Z}$-algebra, with $GW_0F = WF/IF \simeq \mathbb{Z}/2\mathbb{Z}$.

The fundamental ideal IF is additively generated by the forms $\langle a, b \rangle$, for $a, b \in F^\times$. The following simple manipulation in W(F) shows that it is additively generated by the forms of type $\langle 1, -a \rangle$:

$$\begin{aligned} \langle a, b \rangle &= \langle 1, -1, a, b \rangle \\ &= \langle 1, b \rangle + \langle -1, a \rangle \\ &= \langle 1, b \rangle - \langle 1, -a \rangle. \end{aligned}$$

These special kind of quadratic forms and their tensor products were first studied by Albrecht Pfister and have many interesting properties; for instance, the set of values represented by them form a group.

Definition 4.8. A quadratic form of the type $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$, where $a_1, \ldots, a_n \in F^\times$ is called an *n-fold Pfister form*. We shall denote the form $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$ by $\langle\langle a_1, \ldots, a_n\rangle\rangle$.

The following proposition summerizes the main properties of Pfister forms.

Proposition 4.9. *Let $a_1, \ldots, a_n \in F^\times$ and let q denote the Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$.*
(1) *The set $G = \{q(x) \mid x \in F^{2^n}\}$ forms a group under multiplication.*
(2) *If q is isotropic, then q is hyperbolic.*
(3) *If $a \in F^\times$ is represented by q, then $q \simeq \langle a\rangle \otimes q$ in $W(F)$.* □

5 The Milnor Conjecture

In his 1970 paper [5], Milnor observed that the Hasse invariant of quadratic forms over a field F could be factored through Tate's norm residue symbol $K_2^M F/2K_2^M F \to Br(F)$. Since the Brauer group $Br(F)$ can be described as the second cohomology group of $Gal(F_{\text{sep}}/F)$ with $\mathbb{Z}/2\mathbb{Z}$-coefficients, this suggests a relation with Galois (and also, étale) cohomology. The Hasse invariant maps I^2F to $Br(F)$, and Milnor proved that it factors through an isomorphism $I^2F/I^3F \simeq K_2^M F/2K_2^M F$. This led him to introduce what we now call the Milnor K-groups of a field F we have seen above, namely, a graded ring $K_*^M F$ such that the groups $K_n^M F$ agree with the classical algebraic K-groups $K_n(F)$ for $n = 0, 1, 2$. Milnor constructed a canonical surjective map s_* from $K_n^M F/2K_n^M F$ onto I^n/I^{n+1} for each n, proved that it is an isomorphism for local and global fields, and asked if it is an isomorphism for every field F. If $\text{char}(F) = 2$, this was solved positively by Kato in 1981.

Milnor also constructed higher norm residue symbols $h^n : K_n^M F/2K_n^M F \to H^n_{\text{ét}}(F, \mathbb{Z}/2\mathbb{Z})$, and used Tate's results to prove that h^n is always an isomorphism for local and global fields. Milnor then stated that he did not know of any examples where it fails to be an isomorphism; this became known as the *Milnor Conjecture* after the case $n = 2$ was settled in 1981 by Merkurjev (see [3], also see [4]). The case $n = 3$ was solved in 1986 by Merkurjev-Suslin and by Rost. A proof that h^n is an isomorphism for all n was only discovered in the mid-1990's by Voevodsky.

In this section, we first describe the map s_* and then describe h^* and the relation with Galois cohomology.

Theorem 5.1. *There is a unique homomorphism*

$$s_* : k_*^M F \to GW_* F$$

sending $\{a\}$ to $\langle\langle a\rangle\rangle$. Moreover, s_* is surjective.

Proof. First, observe that existence of such an s_* implies its uniqueness. Since $I^n F$ is additively generated by n-fold Pfister forms, existence of s_* implies its surjectivity as well. Thus, it only remains to prove the existence of s_*.

Define a map s_* by $\{a\} \mapsto \langle\langle a\rangle\rangle$. We now verify that s_* respects the defining relations for $k_*^M F$. These are simple manipulations in the Witt ring W(F). Let $a, b \in F^\times$. We have

$$\langle\langle a\rangle\rangle + \langle\langle b\rangle\rangle - \langle\langle ab\rangle\rangle = \langle 1, -a, 1, -b, -1, ab\rangle = \langle 1, -a, -b, ab\rangle = \langle\langle a, b\rangle\rangle \in I^2F,$$

so $s_*(\{a\}) + s_*(\{b\}) - s_*(\{ab\}) = 0$ in $GW_* F$. For the second relation, we see that

$$\begin{aligned} s_*(\{a, 1-a\}) &= \langle\langle a, 1-a\rangle\rangle \\ &= \langle 1, -a\rangle \otimes \langle 1, a-1\rangle \\ &= \langle 1, -a, a-1, a(1-a)\rangle \\ &= 0, \end{aligned}$$

the form being isotropic, and hence, hyperbolic. The final relation also follows immediately:

$$2s_*(\{a\}) = 2\langle\langle a\rangle\rangle = \langle 1, -a, 1, -a\rangle = \langle\langle a, -1\rangle\rangle \in I^2F.$$

This completes the proof that s_* is homomorphism and hence, proves the theorem. □

Now a natural question arises: Is s_* injective? This was what Milnor conjectured.

The Milnor Conjecture. The map $s_* : k_*^M F \to GW_* F$ is an isomorphism. In particular, for all n, we have

$$K_n^M F / 2 K_n^M F \simeq I^n F / I^{n+1} F.$$

There were partial results towards this conjecture; we state the most notable one, proving the conjecture for the symbols, due to Arason and Pfister.

Theorem 5.2 (Arason-Pfister)**.** *Let* $a_1, \dots, a_n \in F^\times$. *Then* $s_*(\{a_1, \dots, a_n\}) = 0$ *if and only if the symbol* $\{a_1, \dots, a_n\} = 0$. □

Relation with Galois Cohomology

We now give another formulation of the Milnor Conjecture, in terms of Galois cohomology. Let G_{sep} denote the absolute Galois group $Gal(F_{\text{sep}}/F)$. A *Galois module* over F is an abelian group M with a continuous action of the profinite group G_{sep}. There are two Galois modules of our interest- one is $\mathbb{Z}/2\mathbb{Z}$ with the trivial action of G_{sep} and the other is $F_{\text{sep}}^\times$ with the natural action of G_{sep}. The first and a very basic result in Galois cohomology is *Hilbert's Theorem* 90:

Theorem 5.3. *Let L/F be a finite cyclic Galois extension, with $G := Gal(F_{\text{sep}}/F) = \langle\sigma\rangle$. For any $a \in L^\times$ such that $N_{L/F}(a) = 1$, there exists $b \in L^\times$ such that $a = b/\sigma(b)$.* □

Hilbert's Theorem 90 essentially says that $H^1(G, L^\times) = 0$. Observing that $H^n(G_{\text{sep}}, F_{\text{sep}}^\times) = \varinjlim H^n(Gal(L/F), L^\times)$, where the direct limit is taken over all finite cyclic Galois extensions, we see the following.

Theorem 5.4. $H^1(G_{\text{sep}}, F_{\text{sep}}^\times) = 0$. □

We have an exact sequence

$$1 \to \mu_2 \to F_{\text{sep}}^\times \to F_{\text{sep}}^\times \to 1,$$

where the map $F_{\text{sep}}^\times \to F_{\text{sep}}^\times$ is given by $x \mapsto x^2$ and where μ_2 denotes the multiplicative group $\{1, -1\}$. We write the corresponding long exact sequence in Galois cohomology:

$$0 \to H^0(G_{\text{sep}}, \mathbb{Z}/2\mathbb{Z}) \to H^0(G_{\text{sep}}, F_{\text{sep}}^\times) \to H^0(G_{\text{sep}}, F_{\text{sep}}^\times) \to H^1(G_{\text{sep}}, \mathbb{Z}/2\mathbb{Z}) \to H^1(G_{\text{sep}}, F_{\text{sep}}^\times) = 0$$

For simplicity, let us write $H^n(F)$ for $H^n(G_{\text{sep}}, \mathbb{Z}/2\mathbb{Z})$. Since $H^0(G, M) = M^G$, for any G-module M, we have an exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to F^\times \to F^\times \to H^1(F) \to 0,$$

where the map $F^\times \to F^\times$ is given by $x \mapsto x^2$. Therefore, it follows that $F^\times/F^{\times 2} \simeq H^1(F)$. But $k_1^M F = F^\times/F^{\times 2}$. Call this isomorphism $h^1 : k_1^M F \to H^1(F)$.

Theorem 5.5 (Bass-Tate)**.** *h^1 extends to a homomorphism $h^* : k_*^M F \to H^*(F)$, where the ring structure on $H^*(F)$ is given by the cup product.* □

Milnor conjectured that this map h^* is an isomorphism.

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