REPRESENTATION OF S_n

1. Definition

Let G be a finite group.

Definition 1. *A G representation is a finite dimensional complex vector space V and a group homomorphism $\rho_V : G \to GL(V)$.*

Definition 2. *A homomorphism of G representations is a complex linear map $T : V \to W$ such that the following diagram commutes for each $g \in G$*

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\
V \arrow[r, "T"'] & W
\end{tikzcd}$$

The set of such homomorphisms will be denoted by $Hom_G(V, W)$.

Let $T : V \to W$ be a homomorphism of G representations. There is a unique structure of a G representation on the kernel and cokernel such that all morphisms in the following sequence are homomorphisms of G representations

$$0 \to Ker(T) \to V \to W \to coker(T) \to 0$$

2. New representations from old ones

2.1. **Dual Representation.**

Given a G representation V, we can make V^* (the dual vector space consisting of $\mathbb{C}$ linear maps $V \to \mathbb{C}$) into a G representation as follows. Let $f \in V^*$, then for any $g \in G$, consider the composite

$$V \xrightarrow{\rho_V(g)} V \xrightarrow{f} \mathbb{C}$$

Using this composite, we get a map

$$\tau : G \to GL(V^*) \qquad\qquad g \mapsto (f \mapsto f \circ \rho_V(g))$$

However, this is not a group homomorphism, as

$$\begin{aligned}
\tau(gh)(f) &= f \circ \rho_V(gh) \\
&= f \circ \rho_V(g) \circ \rho_V(h) \\
&= \tau(g)(f) \circ \rho_V(h) \\
&= \tau(h)(\tau(g)(f)) = \tau(h) \circ \tau(g)(f)
\end{aligned}$$

In order to get a group homomorphism we define

$$\rho_{V^*} : G \to GL(V^*) \qquad\qquad g \mapsto (f \mapsto f \circ \rho_V(g^{-1}))$$

Exercise 3. *Check that the above defines a group homomorphism.*

Exercise 4. *By choosing a basis for V we can write $\rho_V(g)$ as a matrix. Show that the matrix of $\rho_{V^*}(g)$ in the dual basis is given by $(\rho_V(g^{-1}))^t = (\rho_V(g)^{-1})^t$.*

2.2. **Tensor product.**

Definition 5. *The tensor product, written $V \otimes W$, is "the" vector space which has the universal property that for every vector space M, there is a natural isomorphism*

$$B(V \times W, M) \to Hom(V \otimes W, M)$$

The object on the left is the vector space of bilinear maps $V \times W \to M$.

Construction: Let $\{e_i, i \in I\}$ be a basis for V and $\{f_j, j \in J\}$ be a basis for W. Let $V \otimes W$ be the vector space with basis $\{e_i \otimes f_j, (i,j) \in I \times J\}$. A general element of this vector space looks like

$$\sum_{i,j} a_{ij} e_i \otimes f_j$$

We may express some of the elements of the following type as

$$\sum_{i,j} (a_i b_j) e_i \otimes f_j = (\sum_i a_i e_i) \otimes (\sum_j b_j f_j)$$

This shows that the vector space $V \otimes W$ is generated by elements of the form $v \otimes w$ where $v \in V, w \in W$, satisfying the following relations

(1) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
(2) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$
(3) $\lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$

A general element in $V \otimes W$ may be written as $\sum_i v_i \otimes w_i$.

Properties:

(1) There is a natural bilinear map $V \times W \to V \otimes V$ given by $(v, w) \mapsto v \otimes w$
(2) There is a natural isomorphism $V^* \otimes W \to Hom(V, W)$ given by $f \otimes w \mapsto \{f \otimes w(v) = f(v)w\}$
(3) There is a natural isomorphism $Hom(V \otimes W, M) \to Hom(V, Hom(W, M))$, which follows from the previous point.

Definition 6. *Given G representations V and W we define a representation on the tensor product $V \otimes W$ by*

$$\rho_{V \otimes W}(g)(v \otimes w) = \rho_V(g)v \otimes \rho_W(g)w$$

We can make $Hom(V, W)$ into a G representation using the isomorphism with $V^* \otimes W$.

$$\begin{aligned}
[\rho_{V^* \otimes W}(g)(f \otimes w)](v) &= [\rho_{V^*}(g)f \otimes \rho_W(g)w](v) \\
&= [(f \circ \rho_V(g)^{-1}) \otimes \rho_W(g)w](v) \\
&= f \circ \rho_V(g)^{-1}(v)\rho_W(g)w \\
&= \rho_W(g) \circ f \otimes w \circ \rho_V(g)^{-1}(v)
\end{aligned}$$

The above calculation amounts to saying that if $\phi \in Hom(V, W)$, then

$$\rho(g)\phi = \rho_W(g) \circ \phi \circ \rho_V(g)^{-1}$$

For $\phi \in Hom(V, W)$, we have $\rho(g)(\phi) = \phi$ if and only if

$$\rho_W(g) \circ \phi = \phi \circ \rho_V(g)$$

i.e., if and only if $\phi \in Hom_G(V, W)$.

Definition 7. *For a G representation V, let* $V^G = \{v \in V \mid \rho_V(g)v = v \ \forall g \in G\}$

We have just seen that

$$Hom_G(V, W) = Hom(V, W)^G$$

3. Complete Reducibility

Let $W \to V$ be an inclusion of G representations. Then there is a G invariant subspace W' of V such that the natural homomorphism of G representations

$$W \oplus W' \to V$$

is an isomorphism. To construct W' one defines a G invariant positive hermitian inner product on V. Choose any positive hermitian inner product, call it $<,>: V \times V \to \mathbb{C}$ and then define $(,)$ by averaging this over G,

$$(v, w) := \sum_{g \in G} < \rho(g)v, \rho(g)w >= (\rho(h)v, \rho(h)w)$$

Define

$$W' = \{w' \in V \mid (w', w) = 0 \qquad \forall w \in W\}$$

Clearly W' is G invariant and we have the natural G homomorphism $W \oplus W' \to V$ is an isomorphism.

Definition 8. *We say a representation is irreducible if the only G invariant subspaces are 0 and the whole space.*

Theorem 9. *Any representation is a direct sum of irreducible representations.*

Proof. Follows from the discussion in the preceding paragraph. ☐

Question: How do we find the irreducible representations of G?

Lemma 10 (Schur's Lemma)**.** *Let V and W be irreducible G representations, then $Hom_G(V, W) = 0$ if V and W are not isomorphic and $Hom_G(V, W) = \mathbb{C}$ if V and W are isomorphic.*

Proof. Let $\phi \in Hom_G(V, W)$, $\phi \neq 0$. Then $Ker\phi = 0$ as V is irreducible, similarly, $Im\phi = W$ as W is irreducible, thus proving that ϕ is an isomorphism. Let $\psi \in Hom_G(V, W)$ be another morphism. Consider $\eta = \psi \circ \phi^{-1} : W \to W$. Since we are working over $\mathbb{C}$, η has an eigenvector $w \in W$, i.e., $(\eta - \lambda)w = 0$. This forces $\eta - \lambda = 0$ since W is irreducible. Thus, $\psi = \lambda\phi$. ☐

4. CHARACTERS

Definition 11. *Let* V *be a* G *representation. Define a map* $\chi_V : G \to \mathbb{C}$ *by*

$$\chi_V(g) = trace(\rho_V(g))$$

Remark 12. For example, choose a basis for V and then compute the trace of the matrix $\rho(g)$. This is independent of the basis chosen as $tr(ABA^{-1}) = tr(B)$. This also shows that $\chi_V(g) = \chi_V(hgh^{-1})$.

We observe that

(1) $\chi_{V \oplus W} = \chi_V + \chi_W$
(2) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

Definition 13. *A function* $f : G \to \mathbb{C}$*, which is called a class function if* $f(hgh^{-1}) = f(g)\ \forall g, h \in G$*. The space of all class functions will be denoted* $\mathscr{C}$*.*

The above remark shows that χ_V is in $\mathscr{C}$ for every G representation.

Definition 14. *We define a positive hermitian inner product on* $\mathscr{C}$ *by*

$$(f, h) := \frac{1}{|G|} \sum_{g \in G} f(g)\overline{h(g)}$$

Let V be a G representation. We define a G morphism $\phi : V \to V$ by

$$v \mapsto \frac{1}{|G|} \sum_{g \in G} \rho(g)v$$

From the definition it is clear that $\phi(v)$ is a G invariant vector. Thus, we conclude that $\phi(V) = V^G$. In particular, if V is irreducible and nontrivial, then $\phi = 0$. Thus, for a nontrivial irreducible representation we have

$$\sum_{g \in G} \chi_V(g) = 0$$

This brings us to the following lemma

Lemma 15. *If* V *is irreducible then* $(\chi_V, \chi_V) = 1$*. If* V *and* W *are irreducible and nonisomorphic then* $(\chi_V, \chi_W) = 0$*.*

Proof. As in the proof of complete reducibility, we first put a G invariant positive inner product on V. For an orthonormal basis with respect to this inner product, the image of the representation $\rho : G \to GL_n$ lands in the unitary group. Thus,

$$\begin{aligned}(\chi_V, \chi_W) &= \frac{1}{|G|}\sum_{g\in G}\chi_V(\rho(g))\overline{\chi_W(\rho(g))} \\ &= \frac{1}{|G|}\sum_{g\in G}\chi_V(\rho(g))\chi_W(\overline{\rho(g)}) \\ &= \frac{1}{|G|}\sum_{g\in G}\chi_V(\rho(g))\chi_W(\overline{\rho(g)^t}) \\ &= \frac{1}{|G|}\sum_{g\in G}\chi_V(\rho(g))\chi_W(\rho(g)^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G}\chi_{V\otimes W^*}(\rho(g))\end{aligned}$$

In the second last inequality we have used that $\rho(g)$ is a unitary matrix. By complete reducibility we know that

$$V \otimes W^* \cong \bigoplus_i V_i^{n_i}$$

with each V_i irreducible. Thus,

$$\chi_{V\otimes W^*} = \sum_i n_i \chi_{V_i}$$

Let us denote the trivial representation by $\rho_{\mathbb{1}}$. We conclude that

$$\begin{aligned}\frac{1}{|G|}\sum_{g\in G}\chi_{V\otimes W^*}(g) &= \sum_i n_i \frac{1}{|G|}\sum_{g\in G}\chi_{V_i}(g) \\ &= n_{\rho_{\mathbb{1}}}\end{aligned}$$

Thus, we get that (χ_V, χ_W) is the number of times the trivial representation occurs in $V \otimes W^*$. This is 0 if V and W are nonisomorphic. By Schur's lemma, this is 1 if they are isomorphic. ☐

Let V and W be irreducible G representations such that $\chi_V = \chi_W$, then $(\chi_V, \chi_W) > 0$ and from the previous lemma we conclude that $(\chi_V, \chi_W) = 1$ and $V \cong W$.

Definition 16. *Left multiplication given by $g \cdot e_h = e_{gh}$ makes $\mathbb{C}[G]$ into G representation. This is called the regular representation and denote $R(G)$.*

Lemma 17. *The functions χ_V, V irreducible forms an orthonormal basis for $\mathscr{C}$.*

Proof. Let $f \in \mathscr{C}$ be such that $(\chi_V, f) = 0$ for every irreducible representation V. Define an endomorphism of a representation V by

$$\phi = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}g$$

This is actually a G endomorphism since as f as follows (using f is a class function)

$$\begin{aligned} h \circ \phi \circ h^{-1} &= \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}hgh^{-1} \\ &= \frac{1}{|G|}\sum_{g\in G}\overline{f(h^{-1}gh)}g \\ &= \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}g = \phi \end{aligned}$$

If V is irreducible, then we know by Schur's lemma that this is a scalar whose value is $\frac{1}{dim(V)}Trace(\phi)$.

$$Trace(\phi) = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}Trace(g) = (\chi_V, f) = 0$$

Thus, for any irreducible representation

$$\phi = 0$$

By complete reducibility we know that any representation $V = \oplus_i V_i$, with V_i irreducible. It is clear that $\phi_V = \oplus_i \phi_{V_i}$ and so we conclude that $\phi = 0$ for any representation, in particular, the regular representation. For the regular representation, $\phi(e) = \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}e_g = 0$. This means that $f(g) = 0$, i.e., $f = 0$. □

We next compute the inner product $(\chi_{R(G)}, \chi_V)$ where V is an irreducible representation. By taking the basis e_g for $R(G)$, it is easily computed that $\chi_{R(G)}(g) = 0$ for $g \neq e$ and it is $|G|$ for $g = e$. Thus, we get

$$(\chi_{R(G)}, \chi_V) = \frac{1}{|G|}\chi_{R(G)}(e)\chi_V(e) = dim(V)$$

Theorem 18. *Let G be a finite group. Then*

1. *The number if irreducible representations of G is equal to the number of conjugacy classes in G.*
2. *There is an isomorphism of G representations $R(G) \cong \oplus V_i^{dim(V_i)}$*

Proof. To prove the first statement, we know that the irreducible representations are in 1-1 correspondence with their characters. Since the characters are orthogonal to each other, there are as many irreducible representations as the dimension of the space spanned by the characters. But we know that this is the space of class functions. The dimension of this space is clearly the number of conjugacy classes in G.

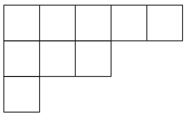
To prove the second statement, write $R(G) = \oplus_i V_i^{n_i}$. Then $\chi_{R(G)} = \sum_i n_i \chi_{V_i}$ and $n_i = (\chi_{R(G)}, \chi_{V_i}) = dim(V_i)$. □

5. THE CASE $G = S_n$

In this section we construct the representations of S_n, the symmetric group on n letters. We start with the following observations,

1. Every element of S_n can be written ”uniquely“ as a product of cycles. A cycle is an element of the type $(i_1 i_2 \dots i_k)$ with all entries distinct.
2. If $\gamma \in S_n$, then $\gamma(i_1 i_2 \dots i_k)\gamma^{-1} = (\gamma(i_1)\gamma(i_2)\dots\gamma(i_k))$.

The second observation shows that the conjugacy classes are in 1-1 correspondence with tuples $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$ with $\sum_i \lambda_i = n$, here the λ_i correspond to the length of the cycles. These are represented by Young diagrams



where the length of the i-th row is λ_i. Given a Young diagram, we may fill it with numbers from 1 to n to get a Young tableau, for example,

$$\begin{array}{|c|c|c|c|c|} \hline 3 & 4 & 7 & 8 & 9 \\ \hline 1 & 2 & 3 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}$$

A young tableau is called standard if the numbers in the rows and columns are increasing, for example,

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 6 \\ \hline 2 & 8 & 9 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array}$$

Given a standard Young tableau, we define two groups

$$P = P_\lambda = \{g \in S_n \mid g \text{ leaves the rows invariant}\}$$
$$Q = Q_\lambda = \{g \in S_n \mid g \text{ leaves the columns invariant}\}$$

Consider the following two elements in the group ring $\mathbb{C}[S_n]$

$$a = a_\lambda = \sum_{g \in P_\lambda} e_g$$
$$b = b_\lambda = \sum_{g \in Q_\lambda} sgn(g) e_g$$

Define $c = c_\lambda = a_\lambda b_\lambda$. Consider the map $\mathbb{C}[S_n] \to \mathbb{C}[S_n]$ given by right multiplication by c. This is clearly a homomorphism of S_n representations. The main assertion we want to show is that the image of this homomorphism is an irreducible S_n representation.

We observe that $P \cap Q = \{1\}$, this is because given any integer d, it is in a unique row and column and so if g preserves the rows and columns, it is forced to be the identity. This shows that any element of S_n can be written in atmost one way as $p \cdot q, p \in P, q \in Q$. Thus,

$$c = \sum_{pq \in PQ} sgn(q) e_{pq} \tag{1}$$

Lemma 19. (1) *For $p \in P$ we have $p \cdot a = a \cdot p = a$*
(2) *For $q \in Q$ we have $(sgn(q)q) \cdot b = b \cdot (sgn(q)q) = b$*
(3) *For $p \in P, q \in Q$ we have $p \cdot c \cdot (sgn(q)q) = c$ and upto scalar multiplication c is the only such element in $\mathbb{C}[S_n]$.*

Proof. The first two assertions are clear from the definitions of a and b. The first part of the third assertion follows from the previous two parts. It only remains to prove the second part of the third assertion. Suppose $\alpha \in \mathbb{C}[S_n]$ is such that for every $p \in P, q \in Q$ we have $p \cdot \alpha \cdot (sgn(q)q) = \alpha$. Let us write $\alpha = \sum_{g \in S_n} \alpha_g e_g$, then

$$\begin{aligned} p \cdot \alpha \cdot (sgn(q)q) &= \sum_{g \in S_n} sgn(q) \alpha_g p \cdot e_g \cdot q \\ &= \sum_{g \in S_n} sgn(q) \alpha_g e_{pgq} = \alpha = \sum_{g \in S_n} \alpha_g e_g \end{aligned}$$

This shows that $\alpha_g = sgn(q)\alpha_{pgq}$, in particular, taking $g = 1$, we get that $\alpha_1 = sgn(q)\alpha_{pq}$. Thus, if we show that $\alpha_g = 0$ for $g \notin PQ$, then we would get that $\alpha = \alpha_1 c$ using equation (1), which would complete the proof of the lemma.

Let $\sigma \in S_n$ such that $\sigma \notin PQ$. Suppose there is a transposition $(ij) \in P$ such that $\sigma^{-1}(ij)\sigma \in Q$. Then taking $q = \sigma^{-1}(ij)\sigma$, we get that

$$\alpha_\sigma = sgn(\sigma^{-1}(ij)\sigma)\alpha_{p\sigma q} = -\alpha_\sigma$$

which implies that $\alpha_\sigma = 0$. Let us assume the contrary, i.e., for every transposition $t = (ij) \in P$, we have $\sigma^{-1} t \sigma \notin Q$. This means that if the numbers i and j are in the same row, then the numbers $\sigma^{-1}(i)$ and $\sigma^{-1}(j)$ are in different columns, or else the transposition $(\sigma^{-1}(i)\, \sigma^{-1}(j)) \in Q$, contrary to our assumption. This means that there is an element $q \in Q$ such that $q(\sigma^{-1}(i))$ and i are in the same row for every i. Now we can find an element $p \in P$ such that $p(q(\sigma^{-1}(i))) = i$, which means that $pq\sigma^{-1} = 1$, i.e. $\sigma = pq$, which is a contradiction. □

We order the partitions lexicographically,

$$\lambda > \mu \text{ if the first nonvanishing } \lambda_i - \mu_i \text{ is positive}$$

Lemma 20. *Let $\lambda > \mu$ be Young tableau, then there is a transposition $t \in P_\lambda \cap Q_\mu$.*

Proof. Suppose that for every transposition $t = (ij) \in P_\lambda$ we have $t \notin Q_\mu$. This means that for any two integers in the same row of λ, they are in different columns in μ. Consider the first row of λ. Since all the elements of this row are in different columns of μ, by applying an element of Q_μ we can bring them all into the first row of μ. Now consider the second row of λ and we can do the same. Proceeding in this manner, it is clear that we would get that $\lambda_i \leq \mu_i$ for every i, which contradicts the assumption that $\lambda > \mu$. □

Lemma 21. *Let λ and μ be Young tableau.*

(1) *If $\lambda > \mu$, then for all $x \in \mathbb{C}[S_n]$, $a_\lambda \cdot x \cdot b_\mu = 0$. In particular, if $\lambda > \mu$, then $c_\lambda \cdot c_\mu = 0$.*
(2) *For all $x \in \mathbb{C}[S_n]$, $c_\lambda \cdot x \cdot c_\lambda$ is a scalar multiple of c_λ. In particular $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$.*

Proof. To prove the first assertion, it suffices to prove this for $x = e_g$. Further, it suffice to show that

$$a_\lambda \cdot e_g \cdot b_\mu \cdot e_{g^{-1}} = 0$$

It is easy to see that $e_g \cdot b_\mu \cdot e_{g^{-1}} = b_{g\mu}$, where $g\mu$ is the Young tableau obtained by applying g to μ. Thus, it suffices to show that $a_\lambda \cdot b_\mu = 0$. From the previous lemma we know that there is a transposition $t \in P_\lambda \cap Q_\mu$. Thus, using lemma 19 we get that

$$a_\lambda \cdot b_\mu = a_\lambda \cdot t \cdot t \cdot b_\mu = -a_\lambda \cdot b_\mu$$

The second assertion follows from lemma 19 part (3). □

Remark 22. If $\lambda < \mu$, then there is a transposition $t \in P_\mu \cap Q_\lambda$. Thus,

$$b_\lambda \cdot a_\mu = b_\lambda \cdot t \cdot t \cdot a_\mu = -b_\lambda \cdot a_\mu$$

and so $b_\lambda \cdot a_\mu = 0$.

Let V_λ denote the subspace $\mathbb{C}[S_n]c_\lambda \subset \mathbb{C}[S_n]$. Clearly this is invariant under left multiplication.

Lemma 23. (1) *Each V_λ is an irreducible representation of $\mathbb{C}[S_n]$.*
(2) *For $\lambda \neq \mu$, V_λ and V_μ are nonisomorphic.*

Proof. Let $W \subset V_\lambda$ be an invariant subspace for $\mathbb{C}[S_n]$, then $c_\lambda W \subset c_\lambda V$ and so it is either $\mathbb{C}c_\lambda$ or 0. If $c_\lambda W = \mathbb{C}c_\lambda$, then this means that $c_\lambda \in W$ and so

$$V = \mathbb{C}[S_n]c_\lambda \subset \mathbb{C}[S_n]W \subset W$$

If $c_\lambda W = 0$, then we have

$$0 = \mathbb{C}[S_n]c_\lambda W = V_\lambda W$$

Consider the projection followed by inclusion

$$\phi : \mathbb{C}[S_n] \to W \to \mathbb{C}[S_n]$$

Let us say that $1 \mapsto \beta$, then ϕ is given by right multiplication by β. Since $\phi^2 = \phi$ as it is a projection, we get that $\beta^2 = \beta \neq 0$, which means that $WW \neq 0$ since it contains β^2. But $WW \subset V_\lambda W = 0$ which is a contradiction. Thus, $W = 0$. This also shows that multiplication by c_λ is a scalar multiple of the projection onto V_λ. Thus, $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$ with $n_\lambda \neq 0$.

To prove part (2), let us assume that $\lambda > \mu$. We know that right multiplication by c_λ is an isomorphism on V_λ, however, it is 0 on V_μ since $c_\mu \cdot c_\lambda = 0$ by remark 22. □

Lemma 24. *For any λ, $c_\lambda \cdot c_\lambda = n_\lambda c_\lambda$, where $n_\lambda = n!/dim(V_\lambda)$.*

Proof. We know that

$$\mathbb{C}[S_n] = \mathbb{C}[S_n]c_\lambda \oplus Ker(c_\lambda)$$

Since right multiplication by c_λ is 0 on $Ker(c_\lambda)$ and it is the scalar n_λ on $V_\lambda = \mathbb{C}[S_n]c_\lambda$, we get that the trace of right multiplication by c_λ is $n_\lambda dim(V_\lambda)$. Let us compute the trace of right multiplication by c_λ on $\mathbb{C}[S_n]$. Recall that $c_\lambda = \sum_{g \in PQ} sgn(g)e_g$. For any $h \in G$, we have

$$e_h \cdot c_\lambda = \sum_{g \in PQ} sgn(g)e_{hg}$$

Coefficient of e_h on the right is 1. Thus, the trace is $n!$, and so $n_\lambda = n!/dim(V_\lambda)$. □

In what follows, λ will denote a Young diagram and T will denote a Young tableau. Recall that a Young tableau is a Young diagram filled with numbers. Recall that T is called standard if the rows and columns are in increasing order. Above, we had put a total order on the collection of Young diagrams. Now we proceed to put a total order on the collection of standard Young tableau corresponding to a Young diagram.

Let T and U be standard Young tableaux corresponding to a Young diagram λ. Let i be an integer, then we define the tableau $T(i)$ as the one obtained from boxes occupyied by the numbers $\{1, 2, \ldots, i\}$. For example, if

$$T = \begin{ytableau} 1 & 3 & 4 & 5 & 6 \\ 2 & 8 & 9 \\ 7 \end{ytableau}$$

then we have

$$T(1) = \begin{ytableau} 1 \end{ytableau} \qquad T(2) = \begin{ytableau} 1 \\ 2 \end{ytableau} \qquad T(3) = \begin{ytableau} 1 & 3 \\ 2 \end{ytableau} \qquad T(4) = \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}$$

$$T(7) = \begin{ytableau} 1 & 3 & 4 & 5 & 6 \\ 2 \\ 7 \end{ytableau} \qquad T(8) = \begin{ytableau} 1 & 3 & 4 & 5 & 6 \\ 2 & 8 \\ 7 \end{ytableau} \qquad T(9) = T$$

Since T is standard, the rows and columns are increasing and from this it is easily seen that $T(i+1)$ is obtained by attaching a box to one of the rows of $T(i)$, further, it is clear that $T(i)$ is a Young tableau for i, in particular, a Young diagram for i.

Definition 25. *Define $T < U$ if for some $s > 1$ we have the following. For each $i < s, T(i) = U(i)$ and $T(s) < U(s)$.*

The following lemma is clear from the above discussion.

Lemma 26. *If $T \neq U$, then $T > U$ or $T < U$, i.e., we have a total order.*

Lemma 27. *If $T < U$, then there are integers $j < s$ such that j and s are in the same row of U and in the same column of T.*

Proof. By definition, there is some $s > 1$ such that $T(s) < U(s)$ and $T(s-1) = U(s-1)$. This means that the box containing s is attached to the k-th row of U and to the l-th column of T. Let j be the entry in the k-th row and l-th column of T. Then $j < s$ and both are in the same row of U and in the same column of T. The proof is best illustrated by an example. Let

$$U = \begin{ytableau} 1 & 3 & 4 & 5 & 6 \\ 2 & 7 & 9 \\ 8 \end{ytableau} \qquad T = \begin{ytableau} 1 & 3 & 4 & 5 & 6 \\ 2 & 8 & 9 \\ 7 \end{ytableau} \qquad T(6) = U(6) = \begin{ytableau} 1 & 3 & 4 & 5 & 6 \\ 2 \end{ytableau}$$

$$U(7) = \begin{ytableau} 1 & 3 & 4 & 5 & 6 \\ 2 & 7 \end{ytableau} \qquad T(7) = \begin{ytableau} 1 & 3 & 4 & 5 & 6 \\ 2 \\ 7 \end{ytableau}$$

Thus, in this example the pair is 2,7. ☐

Proposition 28. *When $T < U$ are standard Young tableau corresponding to λ, then $c_{\lambda,T} \cdot c_{\lambda,U} = 0$*

Proof. If t denotes the transposition (js), then we have $b_{\lambda,T} \cdot a_{\lambda,U} = b_{\lambda,T} \cdot t \cdot t \cdot a_{\lambda,U}$. But $t \in Q_{\lambda,T} \cap P_{\lambda,U}$ and so $b_{\lambda,T} \cdot t = -b_{\lambda,T}$ and $t \cdot a_{\lambda,U} = a_{\lambda,U}$. Thus, we get that $b_{\lambda,T} \cdot a_{\lambda,U} = -b_{\lambda,T} \cdot a_{\lambda,U}$, which does the job. □