Towards the Kobayashi-Hitchin correspondence*

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Abstract

The Kobayashi-Hitchin correspondence states that a holomorphic vector bundle on a compact Kähler manifold admits a Hermitian-Einstein metric if and only if it is polystable in the sense of Mumford-Takemoto. We introduce the relevant notions around this statement in some detail, concluding with a precise formulation of the correspondence and a very brief overview of its proof.

1 Overview

Throughout the text, let (X, g) be a compact connected Kähler manifold of complex dimension n. Let E be a holomorphic vector bundle on X. We want to define a notion of "good" Hermitian metrics on E, the so-called Hermitian-Einstein metrics. For this we review some basic facts about connections on vector bundles and their curvature. Then we define Hermitian-Einstein metrics and give some brief remarks about their uniqueness. After that we introduce the degree of torsion-free coherent sheaves on X for later use. Regarding the existence of Hermitian-Einstein metrics on E, it turns out that there is a necessary condition involving the degree of coherent subsheaves of the sheaf of holomorphic sections of E, which is called polystability. Finally, we are able to formulate the Kobayashi-Hitchin correspondence, whose essential statement is that the condition of polystability is already sufficient for the existence of Hermitian-Einstein metrics.

Good general references for this subject are the books by Lübke and Teleman [LT95] and Kobayashi [Ko87] and Siu's lecture notes [Siu87]. References for individual results can be found in the text.

2 Hermitian-Einstein metrics

Let E be a holomorphic vector bundle on X. We write A^k resp. $A^{p,q}$ for the space of smooth k-forms resp. (p, q)-forms on X and $A^k(E)$ resp. $A^{p,q}(E)$ for the space of smooth k-forms resp. (p, q)-forms on X with values in E. If not specified, all sections are assumed to be smooth.

Definition 2.1. A *connection* on E is a $\mathbb{C}$-linear map

$$D : A^0(E) \longrightarrow A^1(E)$$

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satisfying the Leibniz rule

$$D(f \cdot \sigma) = df \otimes \sigma + f \cdot D\sigma \quad \text{for all } f \in A^0 \text{ and } \sigma \in A^0(E).$$

By using appropriate cutoff functions, D can be seen to act on local sections as well. Furthermore, D also acts as an operator

$$D : A^k(E) \longrightarrow A^{k+1}(E)$$

for any $k \geqslant 1$ by

$$D(u \otimes \sigma) = du \otimes \sigma + (-1)^k u \wedge D\sigma$$

for all $u \in A^k$ and $\sigma \in A^0(E)$.

Given two connections D_0 and D_1, we have

$$(D_1 - D_0)(f \cdot \sigma) = f \cdot (D_1 - D_0)(\sigma),$$

so $D_1 - D_0$ is defined by an element of $A^1(\operatorname{End} E)$, where $\operatorname{End} E$ denotes the bundle of endomorphisms of E. This means that the space of connections on E is an affine space modelled on the vector space $A^1(\operatorname{End} E)$.

Let D be a connection on E. In a local trivialization of E, we have another connection given by the exterior derivative d, so there is a locally defined 1-form A with values in $\operatorname{End} E$, the *connection form*, such that

$$D\sigma = d\sigma + A \wedge \sigma$$

for local sections σ.

Definition 2.2. The map

$$R := D \circ D : A^0(E) \longrightarrow A^2(E)$$

is called the *curvature* of the connection D.

Note that we have

$$R(f \cdot \sigma) = D(df \otimes \sigma + f \cdot D\sigma) = d(df) \otimes \sigma - df \wedge D\sigma + df \wedge D\sigma + f \cdot D(D\sigma) = f \cdot R(\sigma),$$

which means that R is defined by a global section $F \in A^2(\operatorname{End} E)$, the *curvature form* of D. Since locally we have

$$D(D\sigma) = D(d\sigma + A \wedge \sigma) = d(d\sigma) + dA \wedge \sigma - A \wedge d\sigma + A \wedge d\sigma + A \wedge A \wedge \sigma = (dA + A \wedge A) \wedge \sigma,$$

the connection and curvature forms are locally related by

$$F = dA + A \wedge A. \tag{1}$$

We have a decomposition

$$D = D^{1,0} + D^{0,1} \quad \text{where } D^{1,0} : A^0(E) \to A^{1,0}(E) \text{ and } D^{0,1} : A^0(E) \to A^{0,1}(E).$$

Definition 2.3. D is said to be *compatible with the holomorphic structure* of E if

$$D^{0,1}\sigma = 0$$

for every holomorphic section σ of E.

Note that in this case the local connection form A must be of type $(1,0)$. Now let h be a Hermitian metric on E. Then h defines maps

$$A^k(E) \times A^l(E) \longrightarrow A^{k+l},$$

which will also be denoted by h.

Definition 2.4. D is said to be *compatible with the Hermitian metric* h if

$$dh(\sigma, \tau) = h(D\sigma, \tau) + h(\sigma, D\tau)$$

for all sections σ and τ of E.

Lemma 2.5 ([Ko87, Proposition I.4.9]). *Given a Hermitian holomorphic vector bundle (E, h), there is one and only one connection D_h on E that is compatible with both the holomorphic structure of E and the Hermitian metric h.*

This connection is called the *Chern connection* for h. We can express it in terms of h as follows. In a local holomorphic frame $(e_1, \ldots, e_r)$ for E, let A_h be the corresponding connection form for D_h, write $A_h = (A_{\alpha\beta})_{\alpha,\beta}$ as a matrix of 1-forms and set

$$h_{\alpha\beta} := h(e_\alpha, e_\beta).$$

Since D is compatible with the holomorphic structure of E, we have

$$dh(e_\alpha, e_\beta) = h(De_\alpha, e_\beta) + h(e_\alpha, De_\beta).$$

Comparing the $(1,0)$ parts of both sides of the equation and using that A_h is of type $(1,0)$, we obtain

$$\partial h_{\alpha\beta} = h\left(\sum\nolimits_\gamma A_{\alpha\gamma} e_\gamma, e_\beta\right) = \sum\nolimits_\gamma A_{\alpha\gamma} h_{\gamma\beta},$$

which means that we must have

$$A_h = (\partial h) h^{-1}$$

as matrices. (In fact, this is how one proves the lemma.)

Denote by F_h the curvature form of D_h. Since A_h is of type $(1,0)$, from (1), it follows that the $(0,2)$-part of F_h vanishes. Since D_h is also compatible with the Hermitian metric h, it turns out that F_h is skew-Hermitian with respect to h, and so the $(2,0)$-part of F_h vanishes, too. Consequently, the 2-form F_h is of type $(1,1)$ and we have

$$F_h = \bar{\partial} A_h = \bar{\partial}((\partial h) h^{-1}).$$

Denote by ω_g the fundamental form of the Kähler metric g, which can be expressed in local holomorphic coordinates $(z_1, \ldots, z_n)$ as

$$\omega_g = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}.$$

Let Λ_g be the adjoint operation of forming the $\wedge$-product with ω_g, i. e.

$$\langle \omega_g \wedge u, v \rangle_g = \langle u, \Lambda_g v \rangle_g$$

for $u \in A^{p,q}$ and $v \in A^{p+1,q+1}$. For $p = q = 0$, we have

$$\sqrt{-1} \Lambda_g v = \sum_{i,j} g^{\bar{j}i} v_{i\bar{j}},$$

where

$$v = \sum_{i,j} v_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$$

and $(g^{\bar{j}i})_{j,i}$ is the inverse matrix of $(g_{i\bar{j}})_{i,j}$. Of course, Λ_g is also defined for forms with values in a vector bundle.

Definition 2.6. A Hermitian metric h on a holomorphic vector bundle E is called a *Hermitian-Einstein metric* (with respect to g) if the curvature form F_h of the Chern connection for h satisfies

$$\sqrt{-1}\Lambda_g F_h = \lambda \operatorname{id}_E$$

for some $\lambda \in \mathbb{R}$, which is then called the *Einstein factor*.

This notion was introduced by Kobayashi in [Ko80] as a generalization of Kähler-Einstein metrics on the tangent bundle of a Kähler manifold. We will later see that there is only one possible value for the Einstein factor λ, which is a topological invariant of the vector bundle E.

For a certain class of holomorphic vector bundles, Hermitian-Einstein metrics are essentially unique:

Definition 2.7. A holomorphic vector bundle E is called *simple* if the only global holomorphic sections of $\operatorname{End} E$ are the multiples of the identity automorphism.

Proposition 2.8 ([Ko87, Proposition VI.3.37 (d)])**.** *If E is a simple holomorphic vector bundle and h and $\tilde{h}$ are Hermitian-Einstein metrics on E, then there is a positive constant c such that $\tilde{h} = c \cdot h$.*

For the existence of Hermitian-Einstein metrics, the Hermitian-Einstein condition can be weakened as follows.

Proposition 2.9 ([Ko87, Proposition IV.2.4])**.** *If a holomorphic vector bundle E admits a Hermitian metric h_0 such that*

$$\sqrt{-1}\Lambda_g F_{h_0} = \phi \operatorname{id}_E$$

with a function $\phi : X \to \mathbb{R}$ (in which case one says that h_0 satisfies the weak Hermitian-Einstein condition*), then there is a Hermitian-Einstein metric h on E.*

Proof. Let

$$\lambda = \left(\int_X \phi \frac{\omega_g^n}{n!} \right) \Big/ \left(\int_X \frac{\omega_g^n}{n!} \right)$$

be the average of ϕ over X. Then

$$\int_X (\lambda - \phi) \frac{\omega_g^n}{n!} = 0.$$

Since X is compact, there is a smooth function $u : X \to \mathbb{R}$ satisfying

$$\Delta u = \lambda - \phi, \quad \text{where } \Delta = -\sum_{i,j} g^{\bar{j}i} \frac{\partial^2}{\partial z^i \partial z^{\bar{j}}} \text{ is the Laplacian in local coordinates.}$$

Now set $h := e^u h_0$. Then h is a Hermitian metric on E and one can calculate

$$\sqrt{-1}\Lambda_g F_h = \sqrt{-1}\Lambda_g F_{h_0} + (\Delta u) \operatorname{id}_E = \lambda \operatorname{id}_E .$$ □

Corollary 2.10.

(1) *Every line bundle admits a Hermitian-Einstein metric.*

(2) *If $\dim X = 1$, the existence of a Hermitian-Einstein metric on E does not depend on the choice of Kähler metric.*

Conditions for the existence of Hermitian-Einstein metrics will be discussed in the subsequent sections.

3 The degree

The sheaf $\mathcal{E} := \mathcal{O}_X(E)$ of holomorphic sections of a holomorphic vector bundle E on X is a locally free coherent analytic sheaf. In particular, it is torsion-free, so every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ is also torsion-free. In the following, we consider torsion-free coherent analytic sheaves $\mathcal{E}$ on X.

We want to define the degree of $\mathcal{E}$. For a holomorphic line bundle L on X, the *Chern class* of L is defined as follows. Consider the exponential sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 1,$$

where $\mathbb{Z}$ denotes the sheaf of locally constant $\mathbb{Z}$-valued functions on X and $\mathcal{O}_X$ and $\mathcal{O}_X^*$ are the sheaves of holomorphic and non-vanishing holomorphic functions on X, respectively. Here, ι is the inclusion and the morphism exp is given by $f \mapsto e^{2\pi\sqrt{-1}f}$. This is an exact sequence of sheaves, and so the associated long exact sequence of cohomology groups yields a morphism

$$c_1 : \mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}),$$

where $\mathrm{Pic}(X)$ is the Picard group of X. Then $c_1(L) \in H^2(X, \mathbb{Z})$ is called the *Chern class* of L. It is a topological invariant of L. Note that there are natural maps

$$H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{C}) \xrightarrow{\sim} H^2_{\mathrm{dR}}(X),$$

where $\mathbb{C}$ is the sheaf of locally constant $\mathbb{C}$-valued functions on X, $H^2_{\mathrm{dR}}(X)$ is the second De Rham cohomology of X and the second map above is the isomorphism from De Rham's theorem. By abuse of notation, we also write $c_1(L) \in H^2_{\mathrm{dR}}(X)$ for its image under the above maps. Given a Hermitian metric h on L, $c_1(L)$ is represented by a multiple of the curvature form of h:

$$c_1(L) = \left[\frac{1}{2\pi\sqrt{-1}}\,\partial\bar{\partial}\log h\right].$$

More generally, for a holomorphic vector bundle E of rank r, its Chern class is defined to be

$$c_1(E) := c_1(\det E),$$

where $\det E = \Lambda^r E$ is the determinant line bundle of E. Given a Hermitian metric h on E, it is represented as

$$c_1(E) = \left[\frac{\sqrt{-1}}{2\pi}\,\mathrm{tr}(F_h)\right]$$

since for the curvature form of the induced metric $\det h$ on $\det E$, we have

$$\frac{1}{2\pi\sqrt{-1}}\,\partial\bar{\partial}\log\det h = \frac{\sqrt{-1}}{2\pi}\,\mathrm{tr}(F_h).$$

The Chern class of a coherent sheaf $\mathcal{E}$ can be defined in the same way once we have defined a determinant line bundle $\det\mathcal{E}$. This can be done as follows. If

$$0 \longrightarrow E_m \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow 0$$

is an exact sequence of vector bundles, one can show that

$$\bigotimes_{i=0}^{m} (\det E_i)^{\otimes(-1)^i}$$

is isomorphic to the trivial line bundle. Given a coherent sheaf $\mathcal{E}$, there are local resolutions

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E}|_U \longrightarrow 0$$

of $\mathcal{E}$ on open sets $U \subset X$ by locally free coherent sheaves $\mathcal{E}_0, \dots, \mathcal{E}_n$. Let E_i be the vector bundle corresponding to the sheaf $\mathcal{E}_i$. It turns out that

$$(\det \mathcal{E})|_U := \bigotimes_{i=0}^{n} (\det E_i)^{\otimes(-1)^i}$$

is independent of the choice of local resolution and defines a line bundle $\det \mathcal{E}$ on X, the *determinant line bundle* of $\mathcal{E}$. Then the Chern class of $\mathcal{E}$ is again defined to be

$$c_1(\mathcal{E}) := c_1(\det \mathcal{E}).$$

Definition 3.1. The *g-degree* of a coherent sheaf $\mathcal{E}$ is defined as

$$\begin{aligned} \deg_g(\mathcal{E}) &:= (c_1(\mathcal{E}) \cup [\omega_g]^{n-1}) \cap [X] \\ &= \int_X c_1(\mathcal{E}) \wedge \omega_g^{n-1} \in \mathbb{R}. \end{aligned}$$

Remark 3.2. This also works in the more general setting of a *semi-Kähler manifold*, i. e. with a metric g satisfying $d(\omega_g^{n-1}) = 0$. By representing $c_1(\mathcal{E})$ using the curvature form of a Hermitian metric on $\det \mathcal{E}$, it can even be generalized to *Gauduchon manifolds*, i. e. where g satisfies $\partial\bar{\partial}(\omega_g^{n-1}) = 0$. The advantage of the latter is that Gauduchon metrics always exist on compact complex manifolds [Ga77]. In this case, however, the degree does not only depend on $c_1(\mathcal{E})$ and is, in fact, only a holomorphic instead of a topological invariant.

Let us consider some special cases:

- $\dim X = 1$: In this case, the degree is obviously independent of the Kähler metric g.
- X is projective: Choose an ample line bundle H on X and let g be the Kähler metric induced by the polarization H. Then we have $[\omega_g] = c_1(H)$. In this case, the g-degree of $\mathcal{E}$, which is then called the H-degree, is an integer and can be seen as an intersection product of line bundles:

$$\deg_H(\mathcal{E}) = \det(\mathcal{E}) \cdot H^{n-1} \in \mathbb{Z}.$$

4 Stability

Suppose we are given a holomorphic vector bundle E on X together with a Hermitian-Einstein metric h, i. e. satisfying

$$\sqrt{-1}\Lambda_g F_h = \lambda \operatorname{id}_E$$

for some $\lambda \in \mathbb{R}$. Taking the trace and integrating over (X, g), we obtain

$$\int_X \sqrt{-1} \operatorname{tr}(\Lambda_g F_h) \frac{\omega_g^n}{n!} = \lambda \operatorname{rk}(E) \operatorname{vol}_g(X),$$

where $\operatorname{rk}(E)$ is the rank of E and $\operatorname{vol}_g(X)$ denotes the volume of X with respect to g. The left-hand side equals

$$2\pi \int_X \frac{\sqrt{-1}}{2\pi} \operatorname{tr}(F_h) \wedge \frac{\omega_g^{n-1}}{(n-1)!} = \frac{2\pi}{(n-1)!} \deg_g(E).$$

This means that we must have

$$\lambda = \frac{2\pi \deg_g(E)}{(n-1)! \operatorname{rk}(E) \operatorname{vol}_g(X)}. \tag{2}$$

In particular, since this expression is independent of the Hermitian-Einstein metric h, there is only one possible value for λ, which is a topological invariant of E.

Now suppose we are given a non-trivial coherent subsheaf $\mathcal{F}$ of the sheaf $\mathcal{E} = \mathcal{O}_X(E)$ of holomorphic sections of E. The g-degree of $\mathcal{F}$ can be computed using any Hermitian metric h on E (not necessarily Hermitian-Einstein) as follows. Since $\mathcal{F}$ is torsion-free (see above), there is a closed analytic subset $S \subset X$ of codimension $\geqslant 2$ such that $\mathcal{F}$ is locally free on $X \setminus S$. This means that there is a holomorphic subbundle F of $E|_{X\setminus S}$ such that

$$\mathcal{F}|_{X\setminus S} = \mathcal{O}(F).$$

Let π be the orthogonal projection of E onto F over $X \setminus S$ with respect to h. Then we have the following formula.

Proposition 4.1 (Chern-Weil formula, e. g. [Si88, Lemma 3.2])**.**

$$\deg_g(\mathcal{F}) = \frac{\sqrt{-1}}{2\pi n}\int_{X\setminus S} \operatorname{tr}(\pi\Lambda_g F_h)\omega_g^n - \frac{1}{2\pi n}\int_{X\setminus S} |\bar{\partial}\pi|_h^2\omega_g^n.$$

Now if h is a Hermitian-Einstein metric as above, the Chern-Weil formula yields

$$\deg_g(\mathcal{F}) \leqslant \frac{\lambda \operatorname{rk}(\mathcal{F})(n-1)! \operatorname{vol}_g(X)}{2\pi},$$

where $\operatorname{rk}(\mathcal{F}) := \operatorname{rk}(F)$ is defined to be the rank of the vector bundle corresponding to $\mathcal{F}$ on $X \setminus S$. Using (2), it follows that

$$\frac{\deg_g(\mathcal{F})}{\operatorname{rk}(\mathcal{F})} \leqslant \frac{\deg_g(E)}{\operatorname{rk}(E)}.$$

For a non-trivial torsion-free coherent sheaf $\mathcal{E}$, we call

$$\mu_g(\mathcal{E}) := \frac{\deg_g(\mathcal{E})}{\operatorname{rk}(\mathcal{E})}$$

the *g-slope* of $\mathcal{E}$. We have the following definition.

Definition 4.2. Let $\mathcal{E}$ be a torsion-free coherent sheaf.

(1) $\mathcal{E}$ is called *semistable* (with respect to g) if $\mu_g(\mathcal{F}) \leqslant \mu_g(\mathcal{E})$ holds for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rk}(\mathcal{F})$.

(2) If the strict inequality holds for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rk}(\mathcal{F}) < \operatorname{rk}(\mathcal{E})$, then $\mathcal{E}$ is called *stable* (with respect to g).

A holomorphic vector bundle E is called *(semi)stable* if its sheaf $\mathcal{E}$ of holomorphic sections is (semi)stable. This notion is commonly referred to as *stability in the sense of Mumford-Takemoto* (see also [Ta72]) or *slope-stability.* Making use of this definition, we can summarize the above discussion as follows.

Lemma 4.3. *Every holomorphic vector bundle admitting a Hermitian-Einstein metric must be semistable.*

Corresponding to Corollary 2.10, we have the following trivial remark.

Remark 4.4.

(1) Every line bundle is stable.

(2) If $\dim X = 1$, the stability of $\mathcal{E}$ does not depend on the choice of Kähler metric. (This is because the degree of sheaves is independent of the Kähler metric.)

Now the question arises whether the existence of a Hermitian-Einstein metric already implies the stability of the bundle. This is easily seen to be false by using the following observations.

Lemma 4.5 ([Ko87, Corollary V.7.14])**.** *Every stable holomorphic vector bundle is simple (see Definition 2.7).*

This implies that every stable holomorphic vector bundle is *irreducible*, i. e. it cannot be decomposed as a non-trivial direct sum of holomorphic subbundles. On the other hand, given non-trivial holomorphic vector bundles $E_1, \dots, E_m$ with $m \geqslant 2$ of the same g-slope, each admitting a Hermitian-Einstein metric, their direct sum also admits a Hermitian-Einstein metric while not being stable. Thus the existence of a Hermitian-Einstein metric cannot imply stability in general. However, such direct sum decompositions turn out to be the only obstructions to stability of Hermitian-Einstein bundles, a fact which can be made precise as follows.

Definition 4.6. A torsion-free coherent sheaf $\mathcal{E}$ is said to be *polystable* (with respect to g) if it decomposes as a direct sum

$$\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m$$

of stable coherent subsheaves of the same g-slope $\mu_g(\mathcal{E}_1) = \cdots = \mu_g(\mathcal{E}_m)$.

Again, a holomorphic vector bundle E is called *polystable* if the sheaf $\mathcal{E} = \mathcal{O}_X(E)$ is polystable.

Remark 4.7. Given $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m$ as above, it follows that $\mathcal{E}$ is semistable and its g-slope equals

$$\mu_g(\mathcal{E}) = \mu_g(\mathcal{E}_1) = \cdots = \mu_g(\mathcal{E}_m).$$

Now there is a stronger version of Lemma 4.3.

Proposition 4.8 ([Ko82, Theorem 2.4], [Lue83, p. 245])**.** *Every holomorphic vector bundle E admitting a Hermitian-Einstein metric must be polystable. In particular, if E is irreducible, then it is stable.*

Proof. By Lemma 4.3, E is semistable. Suppose that E is not stable. Then there is a coherent subsheaf $\mathcal{F}$ of $\mathcal{E} = \mathcal{O}_X(E)$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$ such that $\mu_g(\mathcal{F}) = \mu_g(\mathcal{E})$. From the Chern-Weil formula (Proposition 4.1), it follows that

$$\int_{X \setminus S} |\bar{\partial}\pi|_h^2 \omega_g^n = 0,$$

where the notation is as above. This means that the projection π is holomorphic, i. e. we have a decomposition

$$E = F \oplus F^\perp$$

on $X \setminus S$, where F is the holomorphic vector bundle on $X \setminus S$ corresponding to $\mathcal{F}$ and $F^\perp$ is its orthogonal complement in E with respect to the Hermitian-Einstein metric h. A theorem of Lübke (see [Lue83]) guarantees that this decomposition extends to X. Restricting h to F resp. $F^\perp$, we obtain a Hermitian-Einstein metric on F resp. $F^\perp$ with the same Einstein factor λ. This implies that

$$\mu_g(E) = \mu_g(F) = \mu_g(F^\perp)$$

and if one of the bundles F and $F^\perp$ is not stable, we iterate the argument, which eventually yields the desired decomposition of E. □

Examples 4.9. According to [Ko87], Examples IV.6.3 (see also V.8.7), the following irreducible holomorphic vector bundles on compact Kähler manifolds admit Hermitian-Einstein metrics and are therefore stable.

(1) The tangent and cotangent bundles of a compact irreducible Hermitian symmetric space.

(2) The symmetric tensor power $S^kT\mathbb{P}^n$ of the tangent bundle of the complex projective space $\mathbb{P}^n$.

(3) The exterior power $\Lambda^kT\mathbb{P}^n$ of the tangent bundle of the complex projective space $\mathbb{P}^n$.

Now we can formulate the Kobayashi-Hitchin correspondence, which clarifies the relationship between stability and the existence of Hermitian-Einstein metrics by giving a converse to Proposition 4.8.

Theorem 4.10. *Let E be a holomorphic vector bundle on a compact Kähler manifold (X,g). If E is stable, then E admits a Hermitian-Einstein metric, which is unique up to a constant multiple.*

By the above considerations, one can deduce the following corollary.

Corollary 4.11 (Kobayashi-Hitchin correspondence)**.** *Let E be a holomorphic vector bundle on a compact Kähler manifold (X,g). Then E admits a Hermitian-Einstein metric if and only if E is polystable.*

The proof of Theorem 4.10 was initiated by Donaldson, who treated the case of compact Riemann surfaces [Do83] by giving a new proof of the famous theorem of Narasimhan and Seshadri [NS65] on the relation between polystable holomorphic vector bundles of degree 0 and finite-dimensional unitary representations of the fundamental group. It was Donaldson who attributed the problem to Kobayashi and Hitchin. Later he gave proofs for projective-algebraic surfaces [Do83] and projective-algebraic manifolds of arbitrary dimension [Do87]. The proof for the general case of compact Kähler manifolds is due to Uhlenbeck and Yau [UY86], [UY89].

We give a very brief overview of a method of proof developed by Simpson [Si88], which is a combination of the methods of Donaldson and Uhlenbeck-Yau. Let E be a stable holomorphic vector bundle on X. Choose some Hermitian metric h_0 on E, the *background metric*, and consider the evolution equation

$$\frac{dh_t}{dt}\,h_t^{-1} = -(\sqrt{-1}\Lambda_g F_{h_t} - \lambda\,\mathrm{id}_E)$$

for smooth families $(h_t)_t$ of Hermitian metrics on E over a real interval, where λ is as above. Using the theory of parabolic partial differential equations and a-priori estimates, one can show that there is a solution defined on the interval $[0,\infty)$, see [Do85].

Up to this point, the stability of E is not needed. If the solution converges to a Hermitian metric h_∞ in an appropriate sense as $t\to\infty$, the limit can be seen to be a Hermitian-Einstein metric. Otherwise, one has to produce a contradiction to the stability of E, i. e. one has to find a coherent subsheaf $\mathcal{F}$ of $\mathcal{E}=\mathcal{O}_X(E)$ with

$$0<\mathrm{rk}(\mathcal{F})<\mathrm{rk}(\mathcal{E}) \quad \text{and} \quad \mu_g(\mathcal{F})\geqslant\mu_g(\mathcal{E}). \tag{3}$$

This is called a *destabilizing subsheaf.* In the course of the proof (see [Si88]), it is first produced in the following weak sense.

Definition 4.12. A *weakly holomorphic subbundle* of E is an L^2 section π of $\mathrm{End}\,E$ with L^2 first-order weak derivatives satisfying

$$\pi=\pi^*=\pi^2 \quad \text{and} \quad (\mathrm{id}_E-\pi)\circ\bar\partial\pi=0,$$

where π^* is the adjoint of π with respect to h_0.

The conditions (3) can be explained for such objects. It then remains to show that this produces a coherent subsheaf of $\mathcal{E}$ in the ordinary sense, which is due to a regularity theorem of Uhlenbeck-Yau [UY86], an alternative proof of which can be found in [Po05].

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