Towards the Kobayashi-Hitchin correspondence*

Matthias Stemmler

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Abstract

The Kobayashi-Hitchin correspondence states that a holomorphic vector bundle on a compact Kähler manifold admits a Hermitian-Einstein metric if and only if it is polystable in the sense of Mumford-Takemoto. We introduce the relevant notions around this statement in some detail, concluding with a precise formulation of the correspondence and a very brief overview of its proof.

1 Overview

Throughout the text, let \((X, g)\) be a compact connected Kähler manifold of complex dimension \(n\). Let \(E\) be a holomorphic vector bundle on \(X\). We want to define a notion of “good” Hermitian metrics on \(E\), the so-called Hermitian-Einstein metrics. For this we review some basic facts about connections on vector bundles and their curvature. Then we define Hermitian-Einstein metrics and give some brief remarks about their uniqueness. After that we introduce the degree of torsion-free coherent sheaves on \(X\) for later use. Regarding the existence of Hermitian-Einstein metrics on \(E\), it turns out that there is a necessary condition involving the degree of coherent subsheaves of the sheaf of holomorphic sections of \(E\), which is called polystability. Finally, we are able to formulate the Kobayashi-Hitchin correspondence, whose essential statement is that the condition of polystability is already sufficient for the existence of Hermitian-Einstein metrics.

Good general references for this subject are the books by Lübke and Teleman [LT95] and Kobayashi [Ko87] and Siu’s lecture notes [Siu87]. References for individual results can be found in the text.

2 Hermitian-Einstein metrics

Let \(E\) be a holomorphic vector bundle on \(X\). We write \(A^k\) resp. \(A^{p,q}\) for the space of smooth \(k\)-forms resp. \((p,q)\)-forms on \(X\) and \(A^k(E)\) resp. \(A^{p,q}(E)\) for the space of smooth \(k\)-forms resp. \((p,q)\)-forms on \(X\) with values in \(E\). If not specified, all sections are assumed to be smooth.

Definition 2.1. A connection on \(E\) is a \(\mathbb{C}\)-linear map

\[
D : A^0(E) \rightarrow A^1(E)
\]

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satisfying the Leibniz rule

\[ D(f \cdot \sigma) = df \otimes \sigma + f \cdot D\sigma \] for all \( f \in A^0 \) and \( \sigma \in A^0(E) \).

By using appropriate cutoff functions, \( D \) can be seen to act on local sections as well. Furthermore, \( D \) also acts as an operator

\[ D : A^k(E) \rightarrow A^{k+1}(E) \]

for any \( k \geq 1 \) by

\[ D(u \otimes \sigma) = du \otimes \sigma + (-1)^k u \wedge D\sigma \]

for all \( u \in A^k \) and \( \sigma \in A^0(E) \).

Given two connections \( D_0 \) and \( D_1 \), we have

\[ (D_1 - D_0)(f \cdot \sigma) = f \cdot (D_1 - D_0)(\sigma), \]

so \( D_1 - D_0 \) is defined by an element of \( A^1(\text{End } E) \), where \( \text{End } E \) denotes the bundle of endomorphisms of \( E \). This means that the space of connections on \( E \) is an affine space modelled on the vector space \( A^1(\text{End } E) \).

Let \( D \) be a connection on \( E \). In a local trivialization of \( E \), we have another connection given by the exterior derivative \( d \), so there is a locally defined 1-form \( A \) with values in \( \text{End } E \), the connection form, such that

\[ D\sigma = d\sigma + A \wedge \sigma \]

for local sections \( \sigma \).

**Definition 2.2.** The map

\[ R := D \circ D : A^0(E) \rightarrow A^2(E) \]

is called the curvature of the connection \( D \).

Note that we have

\[ R(f \cdot \sigma) = D(df \otimes \sigma + f \cdot D\sigma) = d(df) \otimes \sigma - df \wedge D\sigma + df \wedge D\sigma + f \cdot D(D\sigma) = f \cdot R(\sigma), \]

which means that \( R \) is defined by a global section \( F \in A^2(\text{End } E) \), the curvature form of \( D \). Since locally we have

\[ D(D\sigma) = D(d\sigma + A \wedge \sigma) = d(d\sigma) + dA \wedge \sigma - A \wedge d\sigma + A \wedge d\sigma + A \wedge A \wedge \sigma = (dA + A \wedge A) \wedge \sigma, \]

the connection and curvature forms are locally related by

\[ F = dA + A \wedge A. \] (1)

We have a decomposition

\[ D = D^{1,0} + D^{0,1} \] where \( D^{1,0} : A^0(E) \rightarrow A^{1,0}(E) \) and \( D^{0,1} : A^0(E) \rightarrow A^{0,1}(E) \).

**Definition 2.3.** \( D \) is said to be compatible with the holomorphic structure of \( E \) if

\[ D^{0,1} \sigma = 0 \]

for every holomorphic section \( \sigma \) of \( E \).
Note that in this case the local connection form $A$ must be of type $(1,0)$. Now let $h$ be a Hermitian metric on $E$. Then $h$ defines maps

$$A^k(E) \times A^l(E) \longrightarrow A^{k+l},$$

which will also be denoted by $h$.

**Definition 2.4.** $D$ is said to be **compatible with the Hermitian metric** $h$ if

$$dh(\sigma, \tau) = h(D\sigma, \tau) + h(\sigma, D\tau)$$

for all sections $\sigma$ and $\tau$ of $E$.

**Lemma 2.5** ([Ko87, Proposition I.4.9]). Given a Hermitian holomorphic vector bundle $(E, h)$, there is one and only one connection $D_h$ on $E$ that is compatible with both the holomorphic structure of $E$ and the Hermitian metric $h$.

This connection is called the **Chern connection** for $h$. We can express it in terms of $h$ as follows. In a local holomorphic frame $(e_1, \ldots, e_r)$ for $E$, let $A_h$ be the corresponding connection form for $D_h$, write $A_h = (A_{\alpha\beta})_{\alpha,\beta}$ as a matrix of 1-forms and set $h_{\alpha\beta} := h(e_\alpha, e_\beta)$.

Since $D$ is compatible with the holomorphic structure of $E$, we have

$$dh(e_\alpha, e_\beta) = h(De_\alpha, e_\beta) + h(e_\alpha, De_\beta).$$

Comparing the $(1,0)$ parts of both sides of the equation and using that $A_h$ is of type $(1,0)$, we obtain

$$\partial h_{\alpha\beta} = h\left(\sum_\gamma A_{\alpha\gamma} e_\gamma, e_\beta\right) = \sum_\gamma A_{\alpha\gamma} h_{\gamma\beta},$$

which means that we must have

$$A_h = (\partial h) h^{-1}$$

as matrices. (In fact, this is how one proves the lemma.)

Denote by $F_h$ the curvature form of $D_h$. Since $A_h$ is of type $(1,0)$, from (1), it follows that the $(0,2)$-part of $F_h$ vanishes. Since $D_h$ is also compatible with the Hermitian metric $h$, it turns out that $F_h$ is skew-Hermitian with respect to $h$, and so the $(2,0)$-part of $F_h$ vanishes, too. Consequently, the 2-form $F_h$ is of type $(1,1)$ and we have

$$F_h = \bar{\partial} A_h = \bar{\partial}((\partial h) h^{-1}).$$

Denote by $\omega_g$ the fundamental form of the Kähler metric $g$, which can be expressed in local holomorphic coordinates $(z_1, \ldots, z_n)$ as

$$\omega_g = \sqrt{-1} \sum_{i,j} g_{ij} dz^i \wedge \bar{dz}^j.$$ 

Let $\Lambda_g$ be the adjoint operation of forming the $\wedge$-product with $\omega_g$, i.e.

$$\langle \omega_g \wedge u, v \rangle_g = \langle u, \Lambda_g v \rangle_g$$

for $u \in A^{p,q}$ and $v \in A^{p+1,q+1}$. For $p = q = 0$, we have

$$\sqrt{-1} \Lambda_g v = \sum_{i,j} g^{ij} v_{ij},$$

where

$$v = \sum_{i,j} v_{ij} dz^i \wedge \bar{dz}^j.$$
and \((g^i_j)^{-1}\) is the inverse matrix of \((g_{i,j})_{i,j}\). Of course, \(\Lambda_g\) is also defined for forms with values in a vector bundle.

**Definition 2.6.** A Hermitian metric \(h\) on a holomorphic vector bundle \(E\) is called a **Hermitian-Einstein metric** (with respect to \(g\)) if the curvature form \(F_h\) of the Chern connection for \(h\) satisfies
\[
\sqrt{-1} \Lambda_g F_h = \lambda \text{id}_E
\]
for some \(\lambda \in \mathbb{R}\), which is then called the **Einstein factor**.

This notion was introduced by Kobayashi in [Ko80] as a generalization of Kähler-Einstein metrics on the tangent bundle of a Kähler manifold. We will later see that there is only one possible value for the Einstein factor \(\lambda\), which is a topological invariant of the vector bundle \(E\).

For a certain class of holomorphic vector bundles, Hermitian-Einstein metrics are essentially unique:

**Definition 2.7.** A holomorphic vector bundle \(E\) is called **simple** if the only global holomorphic sections of \(\text{End} E\) are the multiples of the identity automorphism.

**Proposition 2.8 ([Ko87, Proposition VI.3.37 (d)])**. If \(E\) is a simple holomorphic vector bundle and \(h\) and \(\tilde{h}\) are Hermitian-Einstein metrics on \(E\), then there is a positive constant \(c\) such that \(\tilde{h} = c \cdot h\).

For the existence of Hermitian-Einstein metrics, the Hermitian-Einstein condition can be weakened as follows.

**Proposition 2.9 ([Ko87, Proposition IV.2.4])**. If a holomorphic vector bundle \(E\) admits a Hermitian metric \(h_0\) such that
\[
\sqrt{-1} \Lambda_g F_{h_0} = \phi \text{id}_E
\]
with a function \(\phi : X \to \mathbb{R}\) (in which case one says that \(h_0\) satisfies the weak Hermitian-Einstein condition), then there is a Hermitian-Einstein metric \(h\) on \(E\).

**Proof.** Let
\[
\lambda = \left(\int_X \phi \frac{\omega^n_g}{n!}\right) / \left(\int_X \frac{\omega^n_g}{n!}\right)
\]
be the average of \(\phi\) over \(X\). Then
\[
\int_X (\lambda - \phi) \frac{\omega^n_g}{n!} = 0.
\]
Since \(X\) is compact, there is a smooth function \(u : X \to \mathbb{R}\) satisfying
\[
\Delta u = \lambda - \phi, \quad \text{where } \Delta = -\sum_{i,j} g^{i,j} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \text{ is the Laplacian in local coordinates.}
\]
Now set \(h := e^u h_0\). Then \(h\) is a Hermitian metric on \(E\) and one can calculate
\[
\sqrt{-1} \Lambda_g F_h = \sqrt{-1} \Lambda_g F_{h_0} + (\Delta u) \text{id}_E = \lambda \text{id}_E .
\]

**Corollary 2.10.**

1. Every line bundle admits a Hermitian-Einstein metric.
2. If \(\dim X = 1\), the existence of a Hermitian-Einstein metric on \(E\) does not depend on the choice of Kähler metric.

Conditions for the existence of Hermitian-Einstein metrics will be discussed in the subsequent sections.
3 The degree

The sheaf $\mathcal{E} := \mathcal{O}_X(E)$ of holomorphic sections of a holomorphic vector bundle $E$ on $X$ is a locally free coherent analytic sheaf. In particular, it is torsion-free, so every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ is also torsion-free. In the following, we consider torsion-free coherent analytic sheaves $\mathcal{E}$ on $X$.

We want to define the degree of $\mathcal{E}$. For a holomorphic line bundle $L$ on $X$, the Chern class of $L$ is defined as follows. Consider the exponential sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \rightarrow 1,$$

where $\mathbb{Z}$ denotes the sheaf of locally constant $\mathbb{Z}$-valued functions on $X$ and $\mathcal{O}_X$ and $\mathcal{O}_X^*$ are the sheaves of holomorphic and non-vanishing holomorphic functions on $X$, respectively. Here, $i$ is the inclusion and the morphism exp is given by $f \mapsto e^{2\pi\sqrt{-1}f}$. This is an exact sequence of sheaves, and so the associated long exact sequence of cohomology groups yields a morphism

$$c_1 : \text{Pic}(X) = H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z}),$$

where Pic($X$) is the Picard group of $X$. Then $c_1(L) \in H^2(X, \mathbb{Z})$ is called the Chern class of $L$. It is a topological invariant of $L$. Note that there are natural maps

$$H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C}) \xrightarrow{\sim} H^2_{\mathrm{dR}}(X),$$

where $\mathbb{C}$ is the sheaf of locally constant $\mathbb{C}$-valued functions on $X$, $H^2_{\mathrm{dR}}(X)$ is the second De Rham cohomology of $X$ and the second map above is the isomorphism from De Rham’s theorem. By abuse of notation, we also write $c_1(L) \in H^2_{\mathrm{dR}}(X)$ for its image under the above maps. Given a Hermitian metric $h$ on $L$, $c_1(L)$ is represented by a multiple of the curvature form of $h$:

$$c_1(L) = \left[ \frac{1}{2\pi\sqrt{-1}} \partial \bar{\partial} \log h \right].$$

More generally, for a holomorphic vector bundle $E$ of rank $r$, its Chern class is defined to be

$$c_1(E) := c_1(\det E),$$

where $\det E = \Lambda^r E$ is the determinant line bundle of $E$. Given a Hermitian metric $h$ on $E$, it is represented as

$$c_1(E) = \left[ \frac{\sqrt{-1}}{2\pi} \text{tr}(F_h) \right]$$

since for the curvature form of the induced metric $\det h$ on $\det E$, we have

$$\frac{1}{2\pi\sqrt{-1}} \partial \bar{\partial} \log \det h = \frac{\sqrt{-1}}{2\pi} \text{tr}(F_h).$$

The Chern class of a coherent sheaf $\mathcal{E}$ can be defined in the same way once we have defined a determinant line bundle $\det \mathcal{E}$. This can be done as follows. If

$$0 \rightarrow E_m \rightarrow \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow 0$$

is an exact sequence of vector bundles, one can show that

$$\bigotimes_{i=0}^m (\det E_i)^{\otimes (-1)^i}$$

is isomorphic to the trivial line bundle. Given a coherent sheaf $\mathcal{E}$, there are local resolutions

$$0 \rightarrow \mathcal{E}_n \rightarrow \cdots \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}|_U \rightarrow 0$$

is isomorphic to the trivial line bundle. Given a coherent sheaf $\mathcal{E}$, there are local resolutions

$$0 \rightarrow \mathcal{E}_n \rightarrow \cdots \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}|_U \rightarrow 0$$

is isomorphic to the trivial line bundle.
of $\mathcal{E}$ on open sets $U \subset X$ by locally free coherent sheaves $\mathcal{E}_0, \ldots, \mathcal{E}_n$. Let $E_i$ be the vector bundle corresponding to the sheaf $\mathcal{E}_i$. It turns out that

$$(\det \mathcal{E})|_U := \bigotimes_{i=0}^n (\det E_i)^{\otimes (-1)^i}$$

is independent of the choice of local resolution and defines a line bundle $\det \mathcal{E}$ on $X$, the \textit{determinant line bundle} of $\mathcal{E}$. Then the Chern class of $\mathcal{E}$ is again defined to be

$$c_1(\mathcal{E}) := c_1(\det \mathcal{E}).$$

\textbf{Definition 3.1.} The \textit{g-degree} of a coherent sheaf $\mathcal{E}$ is defined as

$$\deg_g(\mathcal{E}) := (c_1(\mathcal{E}) \cup [\omega_g]^{n-1}) \cap [X]$$

$$= \int_X c_1(\mathcal{E}) \wedge \omega_g^{n-1} \in \mathbb{R}.$$ 

\textbf{Remark 3.2.} This also works in the more general setting of a \textit{semi-Kähler manifold}, i.e. with a metric $g$ satisfying $d(\omega_g^{n-1}) = 0$. By representing $c_1(\mathcal{E})$ using the curvature form of a Hermitian metric on $\det \mathcal{E}$, it can even be generalized to \textit{Gauduchon manifolds}, i.e. where $g$ satisfies $\partial \bar{\partial}(\omega_g^{n-1}) = 0$. The advantage of the latter is that Gauduchon metrics always exist on compact complex manifolds [Ga77]. In this case, however, the degree does not only depend on $c_1(\mathcal{E})$ and is, in fact, only a holomorphic instead of a topological invariant.

Let us consider some special cases:

- \textbf{dim $X = 1$:} In this case, the degree is obviously independent of the Kähler metric $g$.
- \textbf{$X$ is projective:} Choose an ample line bundle $H$ on $X$ and let $g$ be the Kähler metric induced by the polarization $H$. Then we have $[\omega_g] = c_1(H)$. In this case, the $g$-degree of $\mathcal{E}$, which is then called the $H$-degree, is an integer and can be seen as an intersection product of line bundles:

$$\deg_H(\mathcal{E}) = \det(\mathcal{E}) \cdot H^{n-1} \in \mathbb{Z}.$$

\section{Stability}

Suppose we are given a holomorphic vector bundle $E$ on $X$ together with a Hermitian-Einstein metric $h$, i.e. satisfying

$$\sqrt{-1} \Lambda_g F_h = \lambda \text{id}_E$$

for some $\lambda \in \mathbb{R}$. Taking the trace and integrating over $(X, g)$, we obtain

$$\int_X \sqrt{-1} \text{tr}(\Lambda_g F_h) \frac{\omega_g^{n-1}}{n!} = \lambda \text{rk}(E) \text{vol}_g(X),$$

where $\text{rk}(E)$ is the rank of $E$ and $\text{vol}_g(X)$ denotes the volume of $X$ with respect to $g$. The left-hand side equals

$$2\pi \int_X \frac{\sqrt{-1}}{2\pi} \text{tr}(F_h) \wedge \frac{\omega_g^{n-1}}{(n-1)!} = \frac{2\pi}{(n-1)!} \deg_g(E).$$

This means that we must have

$$\lambda = \frac{2\pi \deg_g(E)}{(n-1)! \text{rk}(E) \text{vol}_g(X)}.$$

(2)
In particular, since this expression is independent of the Hermitian-Einstein metric $h$, there is only one possible value for $\lambda$, which is a topological invariant of $E$.

Now suppose we are given a non-trivial coherent subsheaf $F$ of the sheaf $E = \mathcal{O}_X(E)$ of holomorphic sections of $E$. The $g$-degree of $F$ can be computed using any Hermitian metric $h$ on $E$ (not necessarily Hermitian-Einstein) as follows. Since $F$ is torsion-free (see above), there is a closed analytic subset $S \subset X$ of codimension $\geq 2$ such that $F$ is locally free on $X \setminus S$. This means that there is a holomorphic subbundle $F'$ of $E|_{X\setminus S}$ such that

$$F'|_{X\setminus S} = \mathcal{O}(F).$$

Let $\pi$ be the orthogonal projection of $E$ onto $F$ over $X \setminus S$ with respect to $h$. Then we have the following formula.

**Proposition 4.1** (Chern-Weil formula, e. g. [Si88, Lemma 3.2]).

$$deg_g(F) = \frac{\sqrt{-1}}{2\pi n} \int_{X\setminus S} tr(\pi \Lambda_g F_h) \omega^n_g - \frac{1}{2\pi n} \int_{X\setminus S} |\bar{\partial} \pi|^2 \omega^n_g.$$ 

Now if $h$ is a Hermitian-Einstein metric as above, the Chern-Weil formula yields

$$deg_g(F) \leq \frac{\lambda \text{rk}(F)(n-1)! \text{vol}_g(X)}{2\pi},$$

where $\text{rk}(F) := \text{rk}(F)$ is defined to be the rank of the vector bundle corresponding to $F$ on $X \setminus S$. Using (2), it follows that

$$\frac{deg_g(F)}{\text{rk}(F)} \leq \frac{deg_g(E)}{\text{rk}(E)}.$$

For a non-trivial torsion-free coherent sheaf $E$, we call

$$\mu_g(E) := \frac{deg_g(E)}{\text{rk}(E)}$$

the $g$-slope of $E$. We have the following definition.

**Definition 4.2.** Let $E$ be a torsion-free coherent sheaf.

1. $E$ is called semistable (with respect to $g$) if $\mu_g(F) \leq \mu_g(E)$ holds for every coherent subsheaf $F$ of $E$ with $0 < \text{rk}(F)$.
2. If the strict inequality holds for every coherent subsheaf $F$ of $E$ with $0 < \text{rk}(F) < \text{rk}(E)$, then $E$ is called stable (with respect to $g$).

A holomorphic vector bundle $E$ is called (semi)stable if its sheaf $E$ of holomorphic sections is (semi)stable. This notion is commonly referred to as stability in the sense of Mumford-Takemoto (see also [Ta72]) or slope-stability. Making use of this definition, we can summarize the above discussion as follows.

**Lemma 4.3.** Every holomorphic vector bundle admitting a Hermitian-Einstein metric must be semistable.

Corresponding to Corollary 2.10, we have the following trivial remark.

**Remark 4.4.**

1. Every line bundle is stable.
2. If $\dim X = 1$, the stability of $E$ does not depend on the choice of Kähler metric. (This is because the degree of sheaves is independent of the Kähler metric.)
Now the question arises whether the existence of a Hermitian-Einstein metric already implies the stability of the bundle. This is easily seen to be false by using the following observations.

**Lemma 4.5** ([Ko87, Corollary V.7.14]). *Every stable holomorphic vector bundle is simple (see Definition 2.7).*

This implies that every stable holomorphic vector bundle is irreducible, i.e. it cannot be decomposed as a non-trivial direct sum of holomorphic subbundles. On the other hand, given non-trivial holomorphic vector bundles $E_1, \ldots, E_m$ with $m \geq 2$ of the same $g$-slope, each admitting a Hermitian-Einstein metric, their direct sum also admits a Hermitian-Einstein metric while not being stable. Thus the existence of a Hermitian-Einstein metric cannot imply stability in general. However, such direct sum decompositions turn out to be the only obstructions to stability of Hermitian-Einstein bundles, a fact which can be made precise as follows.

**Definition 4.6.** A torsion-free coherent sheaf $E$ is said to be polystable (with respect to $g$) if it decomposes as a direct sum

$$E = E_1 \oplus \cdots \oplus E_m$$

of stable coherent subsheaves of the same $g$-slope $\mu_g(E_1) = \cdots = \mu_g(E_m)$.

Again, a holomorphic vector bundle $E$ is called polystable if the sheaf $E = O_X(E)$ is polystable.

**Remark 4.7.** Given $E = E_1 \oplus \cdots \oplus E_m$ as above, it follows that $E$ is semistable and its $g$-slope equals

$$\mu_g(E) = \mu_g(E_1) = \cdots = \mu_g(E_m).$$

Now there is a stronger version of Lemma 4.3.

**Proposition 4.8** ([Ko82, Theorem 2.4], [Lue83, p. 245]). *Every holomorphic vector bundle $E$ admitting a Hermitian-Einstein metric must be polystable. In particular, if $E$ is irreducible, then it is stable.*

**Proof.** By Lemma 4.3, $E$ is semistable. Suppose that $E$ is not stable. Then there is a coherent subsheaf $F$ of $E = O_X(E)$ with $0 < \text{rk}(F) < \text{rk}(E)$ such that $\mu_g(F) = \mu_g(E)$. From the Chern-Weil formula (Proposition 4.1), it follows that

$$\int_{X \setminus S} |\bar{\partial} \pi|^2 h_\omega n_g = 0,$$

where the notation is as above. This means that the projection $\pi$ is holomorphic, i.e. we have a decomposition

$$E = F \oplus F^\perp$$

on $X \setminus S$, where $F$ is the holomorphic vector bundle on $X \setminus S$ corresponding to $F$ and $F^\perp$ is its orthogonal complement in $E$ with respect to the Hermitian-Einstein metric $h$. A theorem of Lübke (see [Lue83]) guarantees that this decomposition extends to $X$. Restricting $h$ to $F$ resp. $F^\perp$, we obtain a Hermitian-Einstein metric on $F$ resp. $F^\perp$ with the same Einstein factor $\lambda$. This implies that

$$\mu_g(E) = \mu_g(F) = \mu_g(F^\perp)$$

and if one of the bundles $F$ and $F^\perp$ is not stable, we iterate the argument, which eventually yields the desired decomposition of $E$. \qed
Examples 4.9. According to [Ko87], Examples IV.6.3 (see also V.8.7), the following irreducible holomorphic vector bundles on compact Kähler manifolds admit Hermitian-Einstein metrics and are therefore stable.

(1) The tangent and cotangent bundles of a compact irreducible Hermitian symmetric space.

(2) The symmetric tensor power $S^kT^nP^n$ of the tangent bundle of the complex projective space $P^n$.

(3) The exterior power $\Lambda^kT^nP^n$ of the tangent bundle of the complex projective space $P^n$.

Now we can formulate the Kobayashi-Hitchin correspondence, which clarifies the relationship between stability and the existence of Hermitian-Einstein metrics by giving a converse to Proposition 4.8.

Theorem 4.10. Let $E$ be a holomorphic vector bundle on a compact Kähler manifold $(X, g)$. If $E$ is stable, then $E$ admits a Hermitian-Einstein metric, which is unique up to a constant multiple.

By the above considerations, one can deduce the following corollary.

Corollary 4.11 (Kobayashi-Hitchin correspondence). Let $E$ be a holomorphic vector bundle on a compact Kähler manifold $(X, g)$. Then $E$ admits a Hermitian-Einstein metric if and only if $E$ is polystable.

The proof of Theorem 4.10 was initiated by Donaldson, who treated the case of compact Riemann surfaces [Do83] by giving a new proof of the famous theorem of Narasimhan and Seshadri [NS65] on the relation between polystable holomorphic vector bundles of degree 0 and finite-dimensional unitary representations of the fundamental group. It was Donaldson who attributed the problem to Kobayashi and Hitchin. Later he gave proofs for projective-algebraic surfaces [Do83] and projective-algebraic manifolds of arbitrary dimension [Do87]. The proof for the general case of compact Kähler manifolds is due to Uhlenbeck and Yau [UY86], [UY89].

We give a very brief overview of a method of proof developed by Simpson [Si88], which is a combination of the methods of Donaldson and Uhlenbeck-Yau. Let $E$ be a stable holomorphic vector bundle on $X$. Choose some Hermitian metric $h_0$ on $E$, the background metric, and consider the evolution equation

$$\frac{dh_t}{dt} h_t^{-1} = - (\sqrt{-1} A_g F h_t - \lambda \text{id}_E)$$

for smooth families $(h_t)$ of Hermitian metrics on $E$ over a real interval, where $\lambda$ is as above. Using the theory of parabolic partial differential equations and a-priori estimates, one can show that there is a solution defined on the interval $[0, \infty)$, see [Do85].

Up to this point, the stability of $E$ is not needed. If the solution converges to a Hermitian metric $h_\infty$ in an appropriate sense as $t \to \infty$, the limit can be seen to be a Hermitian-Einstein metric. Otherwise, one has to produce a contradiction to the stability of $E$, i.e., one has to find a coherent subsheaf $F$ of $\mathcal{E} = \mathcal{O}_X(E)$ with

$$0 < \text{rk}(F) < \text{rk}(\mathcal{E}) \quad \text{and} \quad \mu_g(F) \geq \mu_g(\mathcal{E}).$$

(3)

This is called a destabilizing subsheaf. In the course of the proof (see [Si88]), it is first produced in the following weak sense.

Definition 4.12. A weakly holomorphic subbundle of $E$ is an $L^2$ section $\pi$ of $\text{End} E$ with $L^2$ first-order weak derivatives satisfying

$$\pi = \pi^* = \pi^2 \quad \text{and} \quad (\text{id}_E - \pi) \circ \bar{\partial} \pi = 0,$$

where $\pi^*$ is the adjoint of $\pi$ with respect to $h_0$. 


The conditions (3) can be explained for such objects. It then remains to show that this produces a coherent subsheaf of $\mathcal{E}$ in the ordinary sense, which is due to a regularity theorem of Uhlenbeck-Yau [UY86], an alternative proof of which can be found in [Po05].

References


