A theorem abo on finite reflection groups.

① Consider the matrix: $\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & \omega \end{bmatrix}$

where ω is a root of unity. Such a linear transformation T fixes the hyperplane. Let f be a polynomial map from $\mathbb{C}^n \longrightarrow \mathbb{C}$. Consider the function $f \circ T$. Let x be a point on the fixed hyperplane. Then:

$(f - f\circ T)x = f(x) - f\circ T(x) = f(x) - f(x) = 0.$

$\Rightarrow f - f\circ T$ vanishes on the hyperplane, Thus $f - f\circ T \in (H)$. where $(H) \subseteq \mathbb{C}[x_1, \dots, x_n]$ is the ideal generated by H.

Thus we have a general definition:

Let $S = \sum_{n=0}^{\infty} S_n$ be a noetherian graded algebra over $S_0 = \mathbb{C}$ and let G be a finite group acting on S by degree preserving $\mathbb{C}$-automorphisms.

Def^n: $g \in G$ is a pseudo-reflection if there is a homogenous element s_g of +ve degree, not a zero divisor such that $\forall\, s \in S$, $g(s) - s \in s_g S$.

Theorem:- Let $S = \sum_{n=0}^{\infty} S_n$, G a finite group generated by pseudo-reflections and $R = S^G$. Then S is a finitely generated free module over R.

(a) S is a finitely generated module over R because S is integral over R. Hence S/SR_+ is a finitely generated module over R/R_+. But $R/R_+ = \mathbb{C}$ so S/R_+S is a finite dimensional vector space over $\mathbb{C}$.

(b) Let $\bar{f}_1, \cdots, \bar{f}_k$ generate S/R_+S over R/R_+, then we claim that $f_1, \cdots, f_k$ generate S over R.

Proof: we proceed by induction on the degree of f, the element to be expressed ~~at~~ an an element of $\sum Rf_i$.

~~(i) let $\deg(f) = 0$, then for~~

first note that given a homogenous element $f \in S$. $\exists\, c_i$ such that $f - \sum c_i f_i = \sum s_i r_i$, where $\deg(r_i) > 0$ and $\deg(s_i) < \deg(f)$.

(i) let $\deg(f) = 0$ then $\deg(s_i) < 0 \Rightarrow s_i = 0 \ \forall i$

$\Rightarrow f = \sum c_i f_i$

(ii). Assume the result for $\deg(f) < n$. ~~The co~~. Let f be a homogenous element with $\deg(f) = n$.

Then $f = \sum s_i r_i + \sum c_i f_i$ where $\deg(s_i) < n$

$\Rightarrow s_i \in \sum R_i f_i$ this proves the claim.

(c). we now have a candidate for the free basis of S over R. To show that the above f_i indeed generate S over R as a free module we make the following observation:

Claim: If $f_1, \cdots f_n \in S$ are homogenous elements and $\sum f_i r_i = 0$, $r_1 \notin (r_2, \cdots r_k)$ then $f_1 \in R_+S$.

we assume this and prove:

(d). If $\bar{f}_1, \cdots, \bar{f}_n$ are forms linearly independent ~~over~~ in S/R_+S then they are linearly independent over in S over R.

we show: Not linearly independent in S over $R \Rightarrow$ not linearly independent in S/R_+S.

we proceed by induction on the number of forms k.

(i) if $f_1 r_1 = 0$ and $r_1 \neq 0$ then by the claim $f_1 \in SR_+ \Rightarrow \bar{f}_1 = 0$

(ii) Let's assume the result for $k' < k$ and $k > 1$

If $\sum f_i r_i = 0$ Then ~~by the cla~~ $r_1 \in (r_2, \cdots r_k)$ or $r_1 \notin (r_2, \cdots r_k)$

If the second case is true then $f_1 \in SR_+ \Rightarrow \bar{f}_1 = 0$ hence the set $\bar{f}_1, \cdots, \bar{f}_k$ is linearly dependent.

therefore $r_1 \in (r_2, \cdots, r_k) \Rightarrow r_1 = \sum r_i r_i'$ $r_i' \in R$.

$\Rightarrow f_1 r_1 + f_2 r_2 + \cdots + f_k r_k = 0$

$\Rightarrow (f_2 + f_1 r_2') r_2 + \cdots + (f_k + f_1 r_k') r_k = 0.$

Thus by induction $\overline{f_2 + f_1 r_2'}, \cdots \overline{f_k + f_1 r_k'}$ are linearly dependent $\Rightarrow \bar{f}_1 \cdots \bar{f}_k$ are linearly dependent.

we now prove

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(e) Proof of claim:-

Let $f_1, \cdots, f_k$ be forms in S such that $\sum f_i r_i = 0$ for $r_i \in R$

If $r_1 \notin (r_2, \cdots, r_k)$ then $f_1 \in SR_+$. we proceed by induction on the degree of f_1. If $\deg(f_1) = 0$ then

$r_1 = \frac{1}{f_1} \sum_{i=2}^{k} f_i r_i$ applying the reynold's operator

$r_1 = \frac{1}{f_1} \sum \rho(f_i) r_i$

$\Rightarrow r_1 \in (r_2 \cdots r_k) \rightarrow\leftarrow$

let $\deg(f_i) > 0$ and $\sum f_i r_i = 0$, if $g \in G$ is a pseudoreflection then $\sum g(f_i) r_i = 0$.

$\Rightarrow \sum (f_i - g(f_i)) r_i = 0$, $\exists$ $s_g \in S$ of +ve degree such that

$f_i - g(f_i) = e_i s_g$, here $\deg(e_i) < \deg(f_i)$

Thus $\sum e_i r_i = 0$ and $\deg(e_1) < \deg(f) \Rightarrow e_1 \in SR_+$

so if g is a pseudoreflection then $f_1 - g(f_1) \in SR_+$

But G is generated by pseudoreflections, this means that $f_1 - g(f_1) \in SR_+$ $\forall g \in G$. Hence $f_1 - \rho(f_1) \in SR_+$

But $\rho(f_1) \in R_+$ if $\deg(f_1) > 0 \Rightarrow f_1 \in SR_+$

The main result allows us to prove the following corollary:

If G acts on $\mathbb{C}[X_1, \cdots, X_n]$ such that $\deg(x) = \deg(g(x))$ $\forall g \in G$, then the ring ~~on~~ of invariants is a polynomial ring if G is generated by pseudo-reflections.

Proof:- $S = \mathbb{C}[X_1, \cdots X_n]$ is regular, S^G is the ring of invariants. S is finitely generated free over S^G, then since S is regular S^G is regular and a regular graded ring is a polynomial algebra.