Consider the matrix: \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \omega
\end{bmatrix}
\]

where \(\omega\) is a root of unity. Such a linear transformation \(T\) fixes the hyperplane. Let \(f\) be a polynomial map from \(\mathbb{C}^n \to \mathbb{C}\). Consider the function \(f \circ T\). Let \(x\) be a point on the fixed hyperplane. Then:

\[(f \circ T)(x) = f(T(x)) = f(x) - f(x) = 0.\]

\(\Rightarrow f \circ T\) vanishes on the hyperplane, thus \(f \circ T \in \mathfrak{h}\), where \((\mathfrak{h}) \triangleq \mathbb{C}[x_1, \ldots, x_n]\) is the ideal generated by \(\mathfrak{h}\).

Thus we have a general definition:

Let \(S = \bigoplus_{n=0}^{\infty} S_n\) be an \(\mathbb{C}\)-graded algebra over \(S_0 = \mathbb{C}\) and let \(G\) be a finite group acting on \(S\) by degree preserving \(\mathbb{C}\)-automorphisms.

Def: \(g \in G\) is a pseudo-reflection if there is a homogeneous element \(s_g \in S\) of degree, not a zero divisor such that

\[\forall s \in S, \ g(s) = \sigma \in s \sigma S.\]

Theorem: Let \(S = \bigoplus_{n=0}^{\infty} S_n\), \(G\) a finite group generated by pseudo-reflections and \(R = S_{Q}\). Then \(S\) is a finitely generated free module over \(R\).

(a) \(S\) is a finitely generated module over \(R\) because \(S\) is integral over \(R\). Hence \(S/S_{Q+}\) is a finitely generated module over \(R/R_{+}\). But \(R/R_{+} = \mathbb{C}\) so \(S/R_{+}S\) is a finite dimensional vector space over \(\mathbb{C}\).
(b) Let \( f_1, \ldots, f_k \) generate \( S/\mathfrak{m}S \) over \( R/\mathfrak{m} \), then we claim that \( f_1, \ldots, f_k \) generate \( S \) over \( R \).

Proof: we proceed by induction on the degree of \( f \); the element to be expressed at an element of \( \Sigma f_i \).

(i) Let \( \deg(f) = 0 \), then for first note that given a homogenous element \( f \) of \( S \), there exist \( \mathfrak{m} \)-primary ideals \( \mathfrak{a}_i \) such that \( f = \sum f_i = \sum s_i r_i \), where \( \deg(s_i) > 0 \) and \( \deg(s_i) < \deg(f) \).

(ii) Let \( \deg(f) = 0 \) then \( \deg(s_i) < 0 \Rightarrow s_i = 0 \Rightarrow f = \sum f_i \).

(ii) Assume the result for \( \deg(f) < n \). Let \( f \) be a homogenous element with \( \deg(f) = n \).

Then \( f = \sum s_i r_i + \sum c_i f_i \), where \( \deg(s_i) < n \).

\( \Rightarrow s_i \in \Sigma R f_i \); this proves the claim.

(c) We now have a candidate for the free basis of \( S \) over \( R \). To show that the above \( f_i \) indeed generate \( S \) over \( R \) as a free module we make the following observation:

Claim: If \( f_1, \ldots, f_n \) are homogenous elements and \( \sum f_i = 0 \), then \( f_i \in \mathfrak{m} \).

We assume this and prove:

(d) If \( f_1, \ldots, f_n \) are forms linearly independent over \( S/\mathfrak{m}S \) then they are linearly independent over \( S \) over \( R \).

We show: Not linearly independent in \( S \) over \( R \) \( \Rightarrow \) not linearly independent in \( S/\mathfrak{m}S \)
we proceed by induction on the number of forms k.

(i) if \( f_1 \cdot x_1 = 0 \) and \( x_1 \neq 0 \) then by the claim \( f_1 \in SR_+ \Rightarrow f_1 = 0 \)

(ii) Let's assume the result for \( k' < k \) and \( k > 1 \)

If \( \sum f_i x_i = 0 \) then by the eta \( x_1 \in (x_2, \ldots, x_k) \) or \( x_1 \in (x_2, \ldots, x_k) \)

If the second case is true then \( f_1 \in SR_+ \Rightarrow f_1 = 0 \) hence the set \( f_1, \ldots, f_k \) is linearly dependent.

therefore \( x_1 \in (x_2, \ldots, x_k) \) \( \Rightarrow x_1 = \sum x_i x_i' \ x_i' \in \mathbb{R} \).

\[ \Rightarrow f_1 x_1 + f_2 x_2 + \cdots + f_k x_k = 0 \]

\[ \Rightarrow (f_2 + f_1 x_2') x_2 + \cdots + (f_k + f_1 x_k') x_k = 0. \]

This by induction \( \frac{f_2}{f_1}, \frac{f_3}{f_1}, \ldots, \frac{f_k}{f_1} \) are linearly dependent \( \Rightarrow f_1, \ldots, f_k \) are linearly dependent.

we now prove

(c) Proof of claim:-

Let \( f_1, \ldots, f_k \) be forms in \( S \) such that \( \sum f_i x_i = 0 \) for \( x_i \in \mathbb{R} \)

If \( x_1 \notin (x_2, \ldots, x_k) \) then \( f_1 \in SR_+ \). We proceed by induction on the degree of \( f_1 \). If \( \deg(f_1) = 0 \) then

\[ x_1 = \frac{1}{f_1} \sum_{i=2}^{k} f_i x_i \] applying the reynolds's operator

\[ x_1 = \frac{1}{f_1} \sum P(f_i) x_i \]

\[ \Rightarrow x_1 \in (x_2, \ldots, x_k) \rightarrow \Leftarrow \]
let \( \text{deg}(f_i) > 0 \) and \( \Sigma f_i x_i = 0 \). If \( g \in G \) is a pseudoreflection then \( \Sigma g_i (f_i) x_i = 0 \).

\[ \Rightarrow \Sigma (f_i - g(f_i)) x_i = 0, \exists s g \in S \text{ of degree such that } f_i - g(f_i) = e_i s g, \text{ here } \text{deg}(e_i) < \text{deg}(f_i) \]

This \( \Sigma e_i x_i = 0 \) and \( \text{deg}(e_i) < \text{deg}(f_i) \Rightarrow e_i \in \text{CSR} \), so if \( g \) is a pseudoreflection then \( f_i - g(f_i) \in \text{CSR} \).

But \( G \) is generated by pseudoreflections, this means that \( f_i - g(f_i) \in \text{CSR} \), \( \forall g \in G \). Hence \( f_i - g(f_i) \in \text{CSR} \).

But \( f(f_i) \in \text{CSR} \) if \( \text{deg}(f_i) > 0 \) \( \Rightarrow f_i \in \text{CSR} \).

The main result allows us to prove the following corollary:

If \( G \) acts on \( \mathbb{C}[x_1, \ldots, x_n] \) such that \( \text{deg}(x_i) = \text{deg}(g(x_i)) \), \( \forall g \in G \), then the ring of invariants is a polynomial ring if \( G \) is generated by pseudo-reflections.

Proof: \( S = \mathbb{C}[x_1, \ldots, x_n] \) is regular, \( S^G \) is the ring of invariants. \( S \) is finitely generated free over \( S^G \), then since \( S \) is regular \( S^G \) is regular and a regular graded ring is a polynomial algebra.