Homotopical Height

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Joint with Indranil Biswas and Dishant Pancholi
Question

Given f.p. group $G$ and a class $\mathcal{C}$ of smooth manifolds (e.g. symplectic, contact, Kähler etc), what is the obstruction to constructing a $K(G, 1)$ manifold within the class $\mathcal{C}$?

Definition

$G$ - finitely presented group;
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$ht_{\mathcal{C}}(G)$ is $-\infty$ if $G$ is not the $\pi_1$ of some manifold in $\mathcal{C}$.
If $\pi_2(M) \neq 0$ for all $M \in \mathcal{C}$ with $\pi_1(M) = G$, then $ht_{\mathcal{C}}(G)$ is 2.
$ht_{\mathcal{C}}(G)$ is greater than or equal to $n$ if there exists a manifold $M \in \mathcal{C}$ such that $\pi_1(M) = G$ and $\pi_i(M) = 0$ for every $1 < i < n$.
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$G$ is of type $FP$ if it admits a finite $K(G, 1)$ space.

Hardness or softness (à la Gromov): Class $C$ is of

Type 1: $ht_c(G) = \infty$ for all groups of type $FP$;
Type 2: if $ht_c(\{1\}) = \infty$ for the trivial group;
Type 3: if $ht_c(G) \geq 0$ for all groups of type $FP$;
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Type 1 – softest; Type 4 – hardest
Type 2 and Type 3 are classes that exhibit intermediate behavior of different (and not quite comparable) kinds.
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Type 2 and Type 3 are classes that exhibit intermediate behavior of different (and not quite comparable) kinds.
Examples relevant to this talk:

Type 1: Closed Almost Complex Manifolds: $\mathcal{A}_C$, $ht_{\mathcal{A}_C}(G) = \infty$ for all groups of type $FP$.

Type 2: Closed Complex Analytic Manifolds: $ht_{\mathcal{A}_A}({\{1\}}) = \infty$ for the trivial group (Calabi-Eckmann).

Type 3: Closed Complex Analytic Manifolds: $ht_{\mathcal{A}_A}(G) \geq 0$ for all groups of type $FP$ (Gompf-Taubes).

Type 4: Smooth Complex Projective Manifolds: $ht_{\mathcal{P}}(G) = -\infty$ for some group of type $FP$. 
Examples relevant to this talk:

**Type 1**: Closed Almost Complex Manifolds: $\mathcal{AC}$, $ht_{AC}(G) = \infty$ for all groups of type $FP$

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First, we show that the class $\mathcal{AC}$ of Closed Almost Complex Manifolds is soft of Type I.

Proof Idea:
Step 1: $X$ is a finite $K(G,1)$. Embed $X$ in $\mathbb{R}^N$ for large $N$. Take regular neighborhood of $X$ and let $M$ be its boundary. Then $M$ has higher homotopy groups vanishing till as far as one likes (taking $N$ larger and larger). Also $M$ is of codimension one in $\mathbb{R}^N$. Hence $TM$ is stably trivial. Therefore $M \times \mathbb{R}^m$ has trivial tangent bundle for all large enough $m$. Hence $M \times \mathbb{R}^m$ is almost complex for all large enough $m$ whenever $m + \text{dim}(M)$ is even.

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First, we show that the class $\mathcal{AC}$ of Closed Almost Complex Manifolds is soft of Type I.

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Step 1: $X$ is a finite $K(G,1)$. Embed $X$ in $\mathbb{R}^N$ for large $N$. Take regular neighborhood of $X$ and let $M$ be its boundary. Then $M$ has higher homotopy groups vanishing till as far as one likes (taking $N$ larger and larger). Also $M$ is of codimension one in $\mathbb{R}^N$. Hence $TM$ is stably trivial. Therefore $M \times \mathbb{R}^m$ has trivial tangent bundle for all large enough $m$. Hence $M \times \mathbb{R}^m$ is almost complex for all large enough $m$ whenever $m + \text{dim}(M)$ is even.

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Complex Manifolds

Calabi–Eckmann manifolds are complex with underlying real manifold $S^{2m+1} \times S^{2n+1}$. $C_n$ is the Calabi–Eckmann manifold with underlying manifold $S^{2n+1} \times S^{2n+1}$. $G$ – finite group. $H$ – subgroup of $G$. $M$ a complex manifold on which $H$ acts freely by holomorphic automorphisms. 

$[G : H] = N$. $M^N$ is the space of maps from $G/H$ to $M$. Then $M^N$ can be naturally identified with the space of all $H$–equivariant maps from $G$ to $M$. Diagonal action of $H$ on $M^N$ naturally extends to a $G$–action on $M^N$ using the left–translation action of $G$ on $G/H$. 
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$$i_g = [G : \langle g \rangle] = \text{index of } \langle g \rangle \text{ in } G.$$ There is an action of $G$ on the Cartesian product $C_n^{ig}$ by holomorphic automorphisms, such that the action of the subgroup $\langle g \rangle$ is free. Hence diagonal action of $G$ on the product

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is free and by holomorphic automorphisms. Let $V_n = W_n/G$ denote the quotient manifold. Then $\pi_1(V_n) = G$, and $\pi_i(V_n) = 0$ for $1 < i \leq 2n$. Hence $ht_{CA}(G) = \infty$ for any finite group $G$. 
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Complex Projective Manifolds

Question (Kollar)

Is a projective group $G_1$ commensurable to a group $G$, admitting a $K(G,1)$ space which is a smooth quasi-projective variety?

Dimca, Papadima and Suciu have furnished examples of finitely presented groups giving a negative answer to this Question.
Motivation
Almost Complex Manifolds
Complex Manifolds
Complex Projective Manifolds

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(Bieri) Let $1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$ be a short exact sequence of groups, with both $G$, $Q$ PD groups. Further suppose that $N$ is not a PD group of finite cohomology dimension $cd(G) − cd(Q)$. Then $N$ is not of type FP. Hence, $N$ cannot have a $K(G,1)$ space homotopy equivalent to a finite CW complex. In particular, $N$ cannot have a quasiprojective $K(G,1)$ space.
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Suppose

a) $M$ is a closed orientable $2n$–dimensional manifold,

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(e.g. any smooth complex projective variety $M$ realizing $ht_S(G)$ or a hyperplane section of $M$)

Then

- $H^p(G, \mathbb{Z}G) = 0$ for $0 < p < n$,
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Ingredient 3: Topological Lefschetz fibration:

**Definition**

A topological Lefschetz fibration on a smooth, closed, oriented $2n$–manifold $M$ consists of the following data:

1. a closed orientable 2-manifold $S$,
2. a finite set of points $K = \{b_i\} \subset M$ called the critical set,
3. a smooth map $f : M \longrightarrow S$ whose differential $df$ is surjective outside $K$,
4. for each critical point $x$ of $f$, there are orientation preserving coordinate charts about $x$ and $f(x)$ (into $\mathbb{C}^n$ and $\mathbb{C}$, respectively) in which $f$ is given by $f(z_1, \cdots, z_n) = \sum_{i=1}^{n} z_i^2$, and
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Ingredient 4: (Complex) Morse Theory gives the following topological generalization of a Theorem of Dimca-Papadima-Suciu:

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Let $f : M \to S$ be an irrational topological Lefschetz fibration that is not a Kodaira fibration, with $\dim M = 2n + 2$, $n \geq 2$. Let $K$ be the finite critical set of $f$. Further suppose that $\tilde{M}$ is contractible. Let $F$ denote the regular fiber and $N = \pi_1(F)$. Then

a) $\pi_k(F) = 0$ for $1 < k < n$,

b) $\pi_n(F)$ is a free $\mathbb{Z}N$-module, with generators in one-to-one correspondence with $K \times \pi_1(S)$,

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Let $f : M \to S$ be an irrational topological Lefschetz fibration that is not a Kodaira fibration, with $\dim M = 2n + 2$, $n \geq 2$. Let $K$ be the finite critical set of $f$. Further suppose that $\tilde{M}$ is contractible. Let $F$ denote the regular fiber and $N = \pi_1(F)$. Then

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Theorem

d) $N$ cannot be of type $FP$; in particular, there does not exist a quasiprojective $K(N, 1)$ space.

Sketch of Proof:
Note that $\pi_1(M)$ and $\pi_1(S)$ are PD groups of dimension $(2n + 2)$ and 2 respectively. To show that $N$ cannot be of type $FP$, it suffices (by Theorem 4) to show that $N$ cannot be a $PD(2n)$ group. Spectral Sequence Proposition gives

\[
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If $N$ is PD(2n) group, then $H^n(N, \mathbb{Z}N) = H^{n+1}(N, \mathbb{Z}N) = 0$ because $n \geq 2$. Hence $H^n(F, \mathbb{Z}N) = H^n(\tilde{F}, \mathbb{Z}N)^N$.

Now, by Poincaré Duality and the Hurewicz’ Theorem, we have,

$$H^n(F, \mathbb{Z}N) = H^n_c(\tilde{F}) = H_n(\tilde{F}) = \pi_n(\tilde{F}) = \bigoplus_I \mathbb{Z},$$

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Homotopical Height
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$$H^n(\tilde{F}, \mathbb{Z}N)^N = (\text{Hom}_{\mathbb{Z}}(H_n(\tilde{F}), \mathbb{Z}N))^N = (\text{Hom}_{\mathbb{Z}}(\pi_n(\tilde{F}), \mathbb{Z}N))^N = \prod_{I} \mathbb{Z}N,$$

where $\prod_{I} \mathbb{Z}N$ denotes the direct product of a collection of copies of $\mathbb{Z}N$ indexed by $I$. 
Hence

\[ \bigoplus_I \mathbb{Z} = \prod_I \mathbb{Z}^N. \]

Since \( f : M \to S \) is irrational, \( I \) is countably infinite. Therefore, \( \bigoplus_I \mathbb{Z} \) is countable and \( \prod_I \mathbb{Z}^N \) is uncountable and the two cannot be equal. A contradiction.
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