

# ENDING LAMINATIONS AND CANNON-THURSTON MAPS

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WITH AN APPENDIX BY SHUBHABRATA DAS AND MAHAN MJ

ABSTRACT. In earlier work, we had shown that Cannon-Thurston maps exist for Kleinian surface groups without accidental parabolics. In this paper we prove that pre-images of points are precisely end-points of leaves of the ending lamination whenever the Cannon-Thurston map is not one-to-one.

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## 1. INTRODUCTION

1.1. **Statement of Results.** In earlier work we showed:

**Theorem 1.1.** [Mj14] *Let  $\rho : \pi_1(S) \rightarrow PSL_2(C)$  be a discrete faithful representation of a surface group with or without punctures, and without accidental parabolics. Let  $M = \mathbb{H}^3/\rho(\pi_1(S))$ . Let  $i$  be an embedding of  $S$  in  $M$  that induces a homotopy equivalence. Then the embedding  $\tilde{i} : \tilde{S} \rightarrow \tilde{M} = \mathbb{H}^3$  extends continuously to a map  $\hat{i} : \mathbb{D}^2 \rightarrow \mathbb{D}^3$ . Further, the limit set of  $\rho(\pi_1(S))$  is locally connected.*

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2010 *Mathematics Subject Classification.* 57M50, 20F67 (Primary); 20F65, 22E40 (Secondary).

This generalizes the first part of the next theorem due to Cannon and Thurston [CT85] (for 3 manifolds fibering over the circle) and Minsky [Min94] (for bounded geometry closed surface Kleinian groups):

**Theorem 1.2.** [CT85, CT07, Min94] *Suppose a closed surface group  $\pi_1(S)$  of bounded geometry acts freely and properly discontinuously on  $\mathbb{H}^3$  by hyperbolic isometries. Then the inclusion  $\tilde{i} : \tilde{S} \rightarrow \mathbb{H}^3$  extends continuously to the boundary. Further, pre-images of points on the boundary are precisely ideal boundary points of a leaf of the ending lamination, or ideal boundary points of a complementary ideal polygon whenever the Cannon-Thurston map is not one-to-one.*

In the main body of this paper, we generalize the second part of the above theorem to arbitrary Kleinian closed surface groups without accidental parabolics.

**Theorem 1.3.** *Suppose a closed surface group  $\pi_1(S)$  acts freely and properly discontinuously on  $\mathbb{H}^3$  by hyperbolic isometries. Then the inclusion  $\tilde{i} : \tilde{S} \rightarrow \mathbb{H}^3$  extends continuously to the boundary. Further, pre-images of points on the boundary are precisely ideal boundary points of a leaf of the ending lamination, or ideal boundary points of a complementary ideal polygon whenever the Cannon-Thurston map is not one-to-one.*

In passing from Theorem 1.2 to Theorem 1.3 we have removed the hypothesis of *bounded geometry*. In an appendix to the paper we extend Theorem 1.3 to the case of surfaces with cusps (Theorem A.4).

**1.2. Outline and Applications.** We first outline the main steps involved in the proof of the main Theorem 1.3. To fix notions, we let  $M$  be the convex core of a simply or doubly degenerate hyperbolic 3-manifold homotopy equivalent to a surface  $S$ . We also assume that an inclusion  $i : S \rightarrow M$  inducing the homotopy equivalence is fixed. Let  $\tilde{i} : \tilde{S} \rightarrow \tilde{M}$  denote the lift of  $i$  to universal covers.

**Recapitulation of Theorem 1.1 from [Mj14]:**

To show that a Cannon-Thurston map exists we have to show that  $\tilde{i}$  extends continuously to the boundary giving  $\hat{i} : \mathbb{D}^2 \rightarrow \mathbb{D}^3$ . The proof of the main Theorem 1.1 of [Mj14] proceeds (cf. Lemma 2.2 below) by showing that given a geodesic segment  $\lambda$  in (the intrinsic metric on)  $\tilde{S}$  lying outside a large ball about a fixed reference point  $o$  in  $\tilde{S}$ , the hyperbolic geodesic in  $\tilde{M}$  joining its end-points lies outside a large ball about  $\tilde{i}(o)$  in  $\tilde{M}$ . Towards this a *hyperbolic ladder*  $\mathcal{L}_\lambda$  is constructed in  $\tilde{M}$  containing  $\lambda$  satisfying the following:

- a) a (weak) quasiconvexity property,
- b) If  $\lambda$  lies outside a large ball about  $o$  in the intrinsic metric on  $\tilde{S}$ , then  $\mathcal{L}_\lambda$  lies outside a large ball about  $\tilde{i}(o)$  in  $\tilde{M}$ .

The quasiconvexity property of  $\mathcal{L}_\lambda$  ensures control over the hyperbolic geodesic in  $\tilde{M}$  joining the end-points of  $\tilde{i}(\lambda)$ . In particular, if  $\mathcal{L}_\lambda$  lies outside a large ball about  $\tilde{i}(o)$  in  $\tilde{M}$  then so does the geodesic in  $\tilde{M}$  joining the end-points of  $\tilde{i}(\lambda)$ . This guarantees the existence of the Cannon-Thurston map  $\hat{i}$  in Theorem 1.1.

**Scheme of proof of Theorem 1.3:**

Theorem 1.3 builds on Theorem 1.1 by describing the structure of the Cannon-Thurston map obtained in [Mj14]. The crux of the proof of Theorem 1.3 involves

an analysis of the structure of certain specific ladders  $\mathcal{L}_\lambda$ . The existence and weak quasiconvexity of these ladders was shown in [Mj14], but the analysis (see Steps 2, 3 below) was missing. In fact, even for punctured torus groups, where the existence of Cannon-Thurston maps was shown by McMullen [McM01], Theorem 1.3 is new. We now proceed with a step-by-step outline of the proof of Theorem 1.3.

Step 1) The easy part (Section 3.1) of Theorem 1.3 consists in showing that the end-points of leaves of ending laminations are identified by the Cannon-Thurston map  $\hat{i}$ . The essential point is that a leaf of the ending lamination in  $\tilde{S}$  can be approximated by the lifts to  $\tilde{S}$  of a sequence of closed curves in  $S$  whose geodesic realizations exit the relevant end of  $M$ . We shall refer to this step as the **forward direction** of Theorem 1.3.

Step 2) The hard part of the proof (Section 4 and Appendix A) consists in showing that if the Cannon-Thurston map  $\hat{i}$  identifies a pair of points, then they are the ideal end-points of a leaf of the ending lamination or an ideal complementary polygon. We shall refer to this step as the **reverse direction** of Theorem 1.3. Bi-infinite geodesics whose end-points are identified by  $\hat{i}$  are referred to as CT leaves (cf. Section 3).

The proof proceeds by analyzing the structure of the ladder  $\mathcal{L}_\lambda$  for  $\lambda$  a CT leaf. The heart of the proof lies in Proposition 4.7 (Asymptotic Quasigeodesic Rays) which essentially says that "vertical" quasigeodesic rays lying on such a ladder  $\mathcal{L}_\lambda$  are all asymptotic to some point  $z_\lambda \in \partial\mathbb{H}^3$ . Further the end-points of  $\lambda$  are identified with  $z_\lambda$  under  $\hat{i}$ .

Step 3) Given Proposition 4.7 there are two ways to complete the proof of Theorem 1.3:

- a) Look at the action of  $\pi_1(S)$  on the  $\mathbb{R}$ -tree dual to the ending lamination. If there is a CT leaf that is not a leaf of the ending lamination, then we construct a CT leaf (Section 4) whose ideal end-points consist of the attracting and repelling fixed points  $g^{-\infty}, g^\infty$  for some  $g \in \pi_1(S)$ . This is a contradiction as  $g$  is a hyperbolic (loxodromic) element. This is the approach taken in Section 4.
- b) Alternately use Proposition 4.7 and a Lemma of Bowditch (Lemma 9.2 of [Bow07]) to show that the collection of CT-leaves forms a lamination. Since the easy direction shows that the ending lamination is contained in the collection of CT leaves, this forces the collection of CT-leaves to exactly equal the ending lamination. This is the approach taken in Appendix A.

### Applications:

1) We prove the following strengthening of a rigidity Theorem due to Brock-Canary-Minsky [BCM12]:

**Theorem 4.10:** *Let  $G$  be a closed surface group. Let  $\rho(G) = \Gamma$  and  $\rho_1(G) = \Gamma_1$  be two simply or doubly degenerate representations of  $G$  into  $PSL_2(\mathbb{C})$  with limit sets  $\Lambda, \Lambda_1$ . Suppose that the  $G$ -actions on  $\Lambda, \Lambda_1$  are topologically conjugate. Then  $\rho$  and  $\rho_1$  are quasiconformally conjugate.*

2) Theorem 1.3 and its generalization Theorem A.4 are used to prove discreteness of commensurators of finitely generated infinite covolume Kleinian groups in [LLR11] and [Mj11].

3) Theorems 1.3 and A.4 are extended to arbitrary finitely generated Kleinian groups in [Mj10].

### Organization of the paper:

Section 2 of the paper deals with preliminary concepts and material from [Mj14]. Section 3.1 proves the easy direction of Theorems 1.3 and A.4: End-points of leaves of the ending lamination are identified by the Cannon-Thurston map. The arguments in Sections 2 and 3 give a unified treatment for surfaces with or without cusps. Section 4 proves the harder direction of Theorem 1.3 for surfaces without cusps. A slight modification of a fact proven for closed surfaces (Remark 4.6) will be used for cusped surfaces. We indicate this in Section 4 itself. Appendix A deals with surfaces with cusps.

**Acknowledgments:** The author would like to thank the referee(s) for pointing out errors and omissions and for suggesting corrections. The research for the case without parabolics is supported in part by a DST research grant DyNo. 100/IFD/8347/2008-2009. The research for the case with parabolics is supported in part by a CEFIPRA project grant 4301-1.

## 2. PRELIMINARIES

**2.1. Hyperbolic Metric Spaces.** Let  $(X, d_X)$  be a hyperbolic metric space and  $Y$  be a subspace that is hyperbolic with the inherited path metric  $d_Y$ . By adjoining the Gromov boundaries  $\partial X$  and  $\partial Y$  to  $X$  and  $Y$ , one obtains their compactifications  $\widehat{X}$  and  $\widehat{Y}$  respectively.

Let  $i : Y \rightarrow X$  denote inclusion.

**Definition 2.1.** *Let  $X$  and  $Y$  be hyperbolic metric spaces and  $i : Y \rightarrow X$  be an embedding. A Cannon-Thurston map  $\hat{i}$  from  $\widehat{Y}$  to  $\widehat{X}$  is a continuous extension of  $i$ .*

The following lemma (Lemma 2.1 of [Mit98]) says that a Cannon-Thurston map exists if and only if for all  $M > 0$  and  $y \in Y$ , there exists  $N > 0$  such that if a geodesic  $\lambda$  in  $Y$  lies outside an  $N$  ball around  $y$  in  $Y$ , then any geodesic in  $X$  joining the end-points of  $\lambda$  lies outside the  $M$  ball around  $i(y)$  in  $X$ . An equivalent statement is that the Cannon-Thurston map exists if and only if sets of small visual diameter go to sets of small visual diameter.

**Lemma 2.2.** *(Lemma 2.1 of [Mit98]) Let  $i : Y \rightarrow X$  be an inclusion of hyperbolic metric spaces. A Cannon-Thurston map from  $\widehat{Y}$  to  $\widehat{X}$  exists if and only if the following condition is satisfied:*

*Given  $y_0 \in Y$ , there exists a non-negative function  $M(N)$ , such that  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$ , and such that for all geodesic segments  $\lambda$  lying outside an  $N$ -ball around  $y_0 \in Y$ , any geodesic segment in  $X$  joining the end-points of  $i(\lambda)$  lies outside the  $M(N)$ -ball around  $i(y_0) \in X$ .*

**Relative Hyperbolicity and Electric Geometry** We refer the reader to [Far98] for terminology and details on relative hyperbolicity and electric geometry.

Let  $X$  be a  $\delta$ -hyperbolic metric space, and  $\mathcal{H}$  a family of  $C$ -quasiconvex,  $D$ -separated, collection of subsets. Recall [Mj14] that **electrocution** of the collection  $\mathcal{H}$  in  $X$  means constructing an auxiliary space  $X_{el} = X \bigcup_{H \in \mathcal{H}} (H \times I)$  with  $H \times \{0\}$  identified to  $H \subset X$  and  $H \times \{1\}$  equipped with the zero metric. This is a geometric ‘coning’ construction.

Then by work of Farb [Far98],  $X_{el}$  obtained by electrocuting the subsets in  $\mathcal{H}$  is a  $\Delta = \Delta(\delta, C, D)$  -hyperbolic metric space.

Now, let  $\alpha = [a, b]$  be a hyperbolic geodesic in  $X$  and  $\beta$  be an electric  $P$ -quasigeodesic without backtracking joining  $a, b$ . Starting from the left of  $\beta$ , replace each maximal subsegment, (with end-points  $p, q$ , say) lying within some  $H \times \{1\}$  ( $H \in \mathcal{H}$ ) by a hyperbolic geodesic  $[p, q]$ . The resulting **connected** path  $\beta_q$  is called an *electro-ambient representative* in  $X$ . Electro-ambient representatives are useful in light of the following.

**Lemma 2.3.** [Mj14] *Given  $\delta, C \geq 0, D > 0$ , there exists  $D_0$  such that the following holds:*

*Let  $(X, d)$  be a  $\delta$ -hyperbolic metric space, and  $\mathcal{H}$  a family of  $C$ -quasiconvex,  $D$ -separated, collection of subsets. Let  $(X_{el}, d_{el})$  denote the electric space obtained from  $X$  by electrocuting the family  $\mathcal{H}$ . Let  $\alpha = [a, b]$  be a geodesic in  $X$  and  $\beta_q$  be an electro-ambient representative of an electric geodesic joining  $a, b$ . Then  $\alpha, \beta_q$  lies within a bounded distance  $D_0$  of each other in  $(X_{el}, d_{el})$ . Further,  $\alpha$  lies within a bounded distance  $D_0$  of  $\beta_q$  in  $(X, d)$ .*

### Partial Electrocutation

Let  $Y$  be the convex core of a simply (resp. doubly) degenerate 3-manifold with cusps. After removing an open neighborhood of cusps we get a manifold with boundary of the form  $S \times J$ , where  $J = [0, \infty)$  (resp.  $\mathbb{R}$ ), where  $S$  is a surface with boundary. Surfaces minus open neighborhoods of cusps shall sometimes be referred to as *truncated surfaces*. Let  $\mathcal{B}$  denote the equivariant collection of horoballs in  $\tilde{Y}$  covering the cusps of  $Y$ . Let  $X$  denote  $\tilde{Y}$  minus the interior of the horoballs in  $\mathcal{B}$ . Let  $\mathcal{H}$  denote the collection of boundary horospheres. Then each  $H \in \mathcal{H}$  with the induced metric is isometric to a Euclidean product  $\mathbb{R} \times J$ . We shall need to equip each  $H \in \mathcal{H}$  is with a new pseudo-metric called the *partially electrocuted metric* by giving it the product of the zero metric (in the  $\mathbb{R}$ -direction) with the Euclidean metric (in the  $J$ -direction). The resulting space is quasi-isometric to what one would get by gluing to each  $H$  the mapping cylinder of the projection of  $H$  onto the  $J$ -factor. Let  $\mathcal{J}$  denote the collection of copies of  $J$  obtained in this construction and let  $(\mathcal{PEY}, d_{pel})$  denote the resulting partially electrocuted space. (See [MP11] for a more general discussion.) We have the following basic Lemma.

**Lemma 2.4.** [MP11]  *$(\mathcal{PEY}, d_{pel})$  is a hyperbolic metric space and the sets  $J_\alpha \in \mathcal{J}$  are uniformly quasiconvex.*

**2.2. Split Geometry.** We shall briefly recall the essential aspects of split geometry from [Min10, Mj14]. We shall also need the construction of certain quasiconvex ladder-like sets  $\mathcal{L}_\lambda$ . Since we shall deal with surfaces with cusps (or punctures) in the Appendix, we give a unified exposition for surfaces with or without punctures. If a finite area hyperbolic surface has cusps, we shall remove an open neighborhood of the cusp and denote the resulting *truncated surface* by  $S$ . In this subsection therefore  $S$  will denote a compact surface, possibly with boundary.

### Split level Surfaces

A **pants decomposition** of a compact surface  $S$ , possibly with boundary, is a disjoint collection of 3-holed spheres  $P_1, \dots, P_n$  embedded in  $S$  such that  $S \setminus \bigcup_i P_i$  is a disjoint collection of non-peripheral annuli in  $S$ , no two of which are homotopic.

Let  $N$  be the convex core of a hyperbolic 3-manifold minus an open neighborhood of the cusp(s). Then any end  $E$  of  $N$  is simply degenerate [Ago04, CG06, Can93] and homeomorphic to  $S \times [0, \infty)$ , where  $S$  is a compact surface, possibly with boundary. A closed geodesic in an end  $E$  homeomorphic to  $S \times [0, \infty)$  is **unknotted** if it is

isotopic in  $E$  to a simple closed curve in  $S \times \{0\}$  via the homeomorphism. A **tube** in an end  $E \subset N$  is a regular  $R$ -neighborhood  $N(\gamma, R)$  of an unknotted geodesic  $\gamma$  in  $E$ .

Let  $\mathcal{T}$  denote a collection of disjoint, uniformly separated tubes in ends of  $N$  such that

- a) all Margulis tubes in  $E$  belong to  $\mathcal{T}$  for all ends  $E$  of  $N$ .
- b) there exists  $\epsilon_0 > 0$  such that the injectivity radius  $injr_{d_x}(E) > \epsilon_0$  for all  $x \in E \setminus \bigcup_{T \in \mathcal{T}} \text{Int}(T)$  and all ends  $E$  of  $N$ .

Let  $F : N \rightarrow M$  be a bi-Lipschitz homeomorphism and let  $M(0)$  be the image of  $N \setminus \bigcup_{T \in \mathcal{T}} \text{Int}(T)$  in  $M$  under the bi-Lipschitz homeomorphism  $F$ . Let  $\partial M(0)$  (resp.  $\partial M$ ) denote the boundary of  $M(0)$  (resp.  $M$ ). Following [Mj14],  $M$  will be called the model manifold. The metrics on  $M$  and  $\widetilde{M}$  will be denoted by  $d_M$ . In [Min10, BCM12], the model manifold refers to  $M$  with considerable additional structure. In particular it involves the decomposition of  $M(0)$  into pieces of the form  $S_{0,4} \times I$  and  $S_{1,1} \times I$  where  $S_{0,4}$  and  $S_{1,1}$  refer to a sphere with 4 holes and a torus with one hole respectively. To distinguish between the model manifold in [Min10, BCM12] and that in this paper we shall refer to the former as the Minsky model. It should be pointed out that Minsky model was constructed by Minsky in [Min10] and proven to be bi-Lipschitz homeomorphic to the hyperbolic manifold  $N$  by Brock-Canary-Minsky in [BCM12].

Let  $(Q, \partial Q)$  be the unique hyperbolic pair of pants such that each component of  $\partial Q$  has length one.  $Q$  will be called the *standard* pair of pants. An isometrically embedded copy of  $(Q, \partial Q)$  in  $(M(0), \partial M(0))$  will be said to be *flat*.

**Definition 2.5.** A **split level surface** associated to a pants decomposition  $\{Q_1, \dots, Q_n\}$  of  $S$  in  $M(0) \subset M$  is an embedding  $f : \bigcup_i (Q_i, \partial Q_i) \rightarrow (M(0), \partial M(0))$  such that

- 1) Each  $f(Q_i, \partial Q_i)$  is flat
- 2)  $f$  extends to an embedding (also denoted  $f$ ) of  $S$  into  $M$  such that the interior of each annulus component of  $f(S \setminus \bigcup_i Q_i)$  lies entirely in  $F(\bigcup_{T \in \mathcal{T}} \text{Int}(T))$ .

Let  $S_i^s$  denote the union of the collection of flat pairs of pants in the image of the embedding  $f_i$ .

The class of *all* topological embeddings from  $S$  to  $M$  that agree with a split level surface  $f$  associated to a pants decomposition  $\{Q_1, \dots, Q_n\}$  on  $Q_1 \cup \dots \cup Q_n$  will be denoted by  $[f]$ .

We define a partial order  $\leq_E$  on the collection of split level surfaces in an end  $E$  of  $M$  as follows:

$f_1 \leq_E f_2$  if there exist  $g_i \in [f_i]$ ,  $i = 1, 2$ , such that  $g_2(S)$  lies in the unbounded component of  $E \setminus g_1(S)$ .

A sequence  $f_i$  of split level surfaces is said to exit an end  $E$  if  $i < j$  implies  $f_i \leq_E f_j$  and further for all compact subsets  $B \subset E$ , there exists  $L > 0$  such that  $f_i(S) \cap B = \emptyset$  for all  $i \geq L$ .

**Definition 2.6.** A curve  $v$  in  $S \subset E$  is  **$l$ -thin** if the core curve of the Margulis tube  $T_v(\subset E \subset N)$  has length less than or equal to  $l$ . A tube  $T \in \mathcal{T}$  is  $l$ -thin if its core curve is  $l$ -thin. A tube  $T \in \mathcal{T}$  is  $l$ -thick if it is not  $l$ -thin. A curve  $v$  is said to split a pair of split level surfaces  $S_i$  and  $S_j$  ( $i < j$ ) if  $v$  occurs

as a boundary curve of both  $S_i$  and  $S_j$ .

The collection of all  $l$ -thin tubes is denoted as  $\mathcal{T}_l$ . The union of all  $l$ -thick tubes along with  $M(0)$  is denoted as  $M(l)$ .

**Definition 2.7.** A pair of split level surfaces  $S_i$  and  $S_j$  ( $i < j$ ) is said to be  $k$ -separated if

- a) for all  $x \in S_i^s$ ,  $d_M(x, S_j^s) \geq k$
- b) Similarly, for all  $x \in S_j^s$ ,  $d_M(x, S_i^s) \geq k$ .

**Definition 2.8.** An  $L$ -bi-Lipschitz **split surface** in  $M(l)$  associated to a pants decomposition  $\{Q_1, \dots, Q_n\}$  of  $S$  and a collection  $\{A_1, \dots, A_m\}$  of complementary annuli (not necessarily all of them) in  $S$  is an embedding  $f : \cup_i Q_i \cup \cup_i A_i \rightarrow M(l)$  such that

- 1) the restriction  $f : \cup_i (Q_i, \partial Q_i) \rightarrow (M(0), \partial M(0))$  is a split level surface
- 2) the restriction  $f : A_i \rightarrow M(l)$  is an  $L$ -bi-Lipschitz embedding.
- 3)  $f$  extends to an embedding (also denoted  $f$ ) of  $S$  into  $M$  such that the interior of each annulus component of  $f(S \setminus (\cup_i Q_i \cup \cup_i A_i))$  lies entirely in  $F(\cup_{T \in \mathcal{T}_l} \text{Int}(T))$ .

**Note:** The difference between a split level surface and a split surface is that the latter may contain bi-Lipschitz annuli in addition to flat pairs of pants.

We denote split surfaces by  $\Sigma_i$  to distinguish them from split level surfaces  $S_i$ . Let  $\Sigma_i^s$  denote the union of the collection of flat pairs of pants and bi-Lipschitz annuli in the image of the split surface (embedding)  $\Sigma_i$ .

**Theorem 2.9.** [Mj14, Theorem 4.8] *Let  $N, M, M(0), S, F$  be as above and  $E$  an end of  $M$ . For any  $l$  less than the Margulis constant, let  $M(l) = \{F(x) : \text{injrad}_x(N) \geq l\}$ . Fix a hyperbolic metric on  $S$  such that each component of  $\partial S$  is totally geodesic of length one. There exist  $L_1 \geq 1$ ,  $\epsilon_1 > 0$ ,  $n \in \mathbb{N}$ , and a sequence  $\Sigma_i$  of  $L_1$ -bi-Lipschitz,  $\epsilon_1$ -separated split surfaces exiting the end  $E$  of  $M$  such that for all  $i$ , one of the following occurs:*

- (1) An  $l$ -thin curve  $v$  splits the pair  $(\Sigma_i, \Sigma_{i+1})$ , i.e.  $v$  splits the associated split level surfaces  $(S_i, S_{i+1})$ , which in turn form an  $l$ -thin pair.
- (2) there exists an  $L_1$ -bi-Lipschitz embedding

$$G_i : (S \times [0, 1], (\partial S) \times [0, 1]) \rightarrow (M, \partial M)$$

$$\text{such that } \Sigma_i^s = G_i(S \times \{0\}) \text{ and } \Sigma_{i+1}^s = G_i(S \times \{1\})$$

Finally, each  $l$ -thin curve in  $S$  splits at most  $n$  split level surfaces in the sequence  $\{\Sigma_i\}$ .

A model manifold  $M$  all of whose ends are equipped with a collection of exiting split surfaces satisfying the conclusions of Theorem 2.9 is said to be equipped with a **weak split geometry** structure.

Pairs of split surfaces satisfying Alternative (1) of Theorem 2.9 will be called an  $l$ -thin pair of split surfaces (or simply a thin pair if  $l$  is understood). Similarly, pairs of split surfaces satisfying Alternative (2) of Theorem 2.9 will be called an  $l$ -thick pair (or simply a thick pair) of split surfaces.

**Definition 2.10.** Let  $(\Sigma_i^s, \Sigma_{i+1}^s)$  be a thick pair of split surfaces in  $M$ . The closure of the bounded component of  $M \setminus (\Sigma_i^s \cup \Sigma_{i+1}^s)$  between  $\Sigma_i^s, \Sigma_{i+1}^s$  will be called a **thick block**.

Note that a thick block is uniformly bi-Lipschitz to the product  $S \times [0, 1]$  and that its boundary components are  $\Sigma_i^s, \Sigma_{i+1}^s$ .

**Definition 2.11.** *Let  $(\Sigma_i^s, \Sigma_{i+1}^s)$  be an  $l$ -thin pair of split surfaces in  $M$  and  $F(\mathcal{T}_i)$  be the collection of  $l$ -thin Margulis tubes that split both  $\Sigma_i^s, \Sigma_{i+1}^s$ . The closure of the union of the bounded components of  $M \setminus ((\Sigma_i^s \cup \Sigma_{i+1}^s) \cup_{F(T) \in F(\mathcal{T}_i)} F(T))$  between  $\Sigma_i^s, \Sigma_{i+1}^s$  will be called a split block. Equivalently, the closure of the union of the bounded components of  $M(l) \setminus (\Sigma_i^s \cup \Sigma_{i+1}^s)$  between  $\Sigma_i^s, \Sigma_{i+1}^s$  is a split block. Each connected component of a split block is a split component.*

**Remark 2.12.** [Mj14, Remark 4.12] For each lift  $\tilde{K} \subset \tilde{M}$  of a split component  $K$  of a split block of  $M(l) \subset M$ , there are lifts of  $l$ -thin Margulis tubes that share the boundary of  $\tilde{K}$  in  $\tilde{M}$ . Adjoining these lifts to  $\tilde{K}$  we obtain **extended split components**. Let  $\mathcal{K}'$  denote the collection of extended split components in  $\tilde{M}$ . Denote the collection of split components in  $\tilde{M}(l) \subset \tilde{M}$  by  $\mathcal{K}$ . Let  $\tilde{M}(l)$  denote the lift of  $M(l)$  to  $\tilde{M}$ . Then the inclusion of  $\tilde{M}(l)$  into  $\tilde{M}$  gives a quasi-isometry between  $\mathcal{E}(\tilde{M}(l), \mathcal{K})$  and  $\mathcal{E}(\tilde{M}, \mathcal{K}')$  equipped with the respective electric metrics. This follows from the last assertion of Theorem 2.9.

The electric metric on  $\mathcal{E}(\tilde{M}, \mathcal{K}')$  is called the **graph-metric** [Mj14, Section 4.3] and is denoted by  $d_G$ . The electric space will be denoted as  $(\tilde{M}, d_G)$ .

The electric metric on  $\mathcal{E}(\tilde{M}, \mathcal{K} \cup \mathcal{T}_l)$  is quasi-isometric to the electric metric on  $\mathcal{E}(\tilde{M}, \mathcal{K}')$ , again by the last assertion of Theorem 2.9. The electric space will be denoted as  $(\tilde{M}, d_G^1)$ .

**Definition 2.13.** *Let  $Y \subset \tilde{N}$  and  $X = F(Y)$ .  $X \subset \tilde{M}$  is said to be  $\Delta$ -graph quasiconvex if for any hyperbolic geodesic  $\mu$  joining  $a, b \in Y$ ,  $F(\mu)$  lies inside  $N_\Delta(X, d_G) \subset \mathcal{E}(\tilde{M}, \mathcal{K}')$ .*

For  $X$  a split component in a manifold, define  $CH(X) = F(CH(Y))$ , where  $CH(Y)$  is the convex hull of  $Y$  in  $\tilde{N}$ , provided the ends of  $N$  have no cusps, i.e.  $N = N^h$ . Else define  $CH(X)$  to be the image under  $F$  of  $CH(Y)$  minus cusps. Further, in order to ensure hyperbolicity of the universal cover, we partially electrocute the cusps of  $M$  (cf. Theorem 2.4).

Then  $\Delta$ -graph quasiconvexity of  $X$  is equivalent to the condition that  $dia_G(CH(X))$  is bounded by  $\Delta' = \Delta'(\Delta)$  as any split component has diameter one in  $(\tilde{M}, d_G)$ . We recall the following from [Mj14].

**Lemma 2.14.** [Mj14, Lemma 4.16] *Let  $E$  be a simply degenerate end of a simply or doubly degenerate hyperbolic 3-manifold  $N$  homotopy equivalent to a surface  $S$  and equipped with a weak split geometry model  $M$ . For  $K$  a split component contained in  $E$ , let  $\tilde{K}$  be a lift to  $\tilde{N}$ . Then there exists  $C_0 = C_0(K)$  such that the convex hull of  $\tilde{K}$  minus cusps lies in a  $C_0$ -neighborhood of  $\tilde{K}$  in  $\tilde{N}$ .*

**Proposition 2.15.** [Mj14, Proposition 4.23] *For  $K$  a split component,  $\tilde{K}$  is uniformly graph-quasiconvex in  $\tilde{M}$ , i.e. there exists  $\Delta'$  such that  $dia_G(CH(\tilde{K})) \leq \Delta'$  for all incompressible split components  $\tilde{K}$ .*

We summarize the conclusions of the above propositions below.

**Definition 2.16.** *A model manifold of weak split geometry is said to be of **split geometry** if*

- (1) Each split component  $\widetilde{K}$  is quasiconvex (not necessarily uniformly) in the hyperbolic metric on  $\widetilde{N}$ .
- (2) Equip  $\widetilde{M}$  with the graph-metric  $d_G$  obtained by electrocuting (extended) split components  $\widetilde{K}$ . Then the convex hull  $CH(\widetilde{K})$  of any split component  $\widetilde{K}$  has uniformly bounded diameter in the metric  $d_G$ .

Hence by Lemma 2.14 and Proposition 2.15 we have the following technical Theorem of [Mj14].

**Theorem 2.17.** [Min10, BCM12] [Mj14, Theorem 4.32] *Any simply or doubly degenerate hyperbolic 3-manifold homotopy equivalent to a surface is bi-Lipschitz homeomorphic to a Minsky model and hence to a model of split geometry.*

2.2.1. *Ladders:* For details on the construction of ladders, see [Mj14, Section 5]. Note that after welding the boundary components of  $S_i^s$  together in a split block, we obtain a bounded geometry surface  $S_i$  in  $M_{wel}$ . Thus  $S_i$  is the *connected* bounded geometry surface obtained from  $S_i^s$  by equipping it with the quotient topology dictated by welding. For convenience of notation, we redesignate this surface  $S_i$ . In the welded model manifold  $M_{wel}$ , we thus obtain a sequence of bounded geometry surfaces  $\{S_i\}$  exiting the end(s). The region between  $S_i$  and  $S_{i+1}$  is either a thick block or a split block.

From a geodesic  $\lambda = \lambda_0 \subset \widetilde{S} \times \{0\} \subset \widetilde{B}_0$  we constructed in [Mj14] a ‘hyperbolic ladder’  $\mathcal{L}_\lambda \subset \widetilde{M}_{wel}$  such that  $\lambda_i = \mathcal{L}_\lambda \cap \widetilde{S}_i$  is an electro-ambient quasigeodesic in the (path) electric metric on  $\widetilde{S}_i$  induced by the graph metric  $d_G$  on  $\widetilde{M}$ .  $\lambda_{i+1}$  is constructed inductively from  $\lambda_i$  (in [Mj14] or [Mj05]) by ‘flowing  $\lambda_i$  up’ in the block  $\widetilde{B}_i$ . More precisely,  $\widetilde{B}_i$  has a natural product structure and is bounded by  $\widetilde{S}_i$  and  $\widetilde{S}_{i+1}$ . Given  $\lambda_i$  joining  $p_i, q_i \in \widetilde{S}_i$ , there exist points  $p_{i+1}, q_{i+1} \in \widetilde{S}_{i+1}$  lying vertically above  $p_i, q_i$  respectively.  $\lambda_{i+1}$  is the electro-ambient geodesic in  $\widetilde{S}_{i+1}$  (equipped with the electric metric) joining  $p_{i+1}, q_{i+1}$ .

We also constructed a large-scale retract  $\Pi_\lambda : \widetilde{M} \rightarrow \mathcal{L}_\lambda$  such that the restriction  $\pi_i$  of  $\Pi_\lambda$  to  $\widetilde{S} \times \{i\}$  is, roughly speaking, a nearest-point retract of  $\widetilde{S} \times \{i\}$  onto  $\lambda_i$  in the (path) electric metric on  $\widetilde{S}_i$ .

We have the following basic theorem from [Mj14]

**Theorem 2.18.** [Mj14, Theorem 5.7] *There exists  $C > 0$  such that for any geodesic  $\lambda = \lambda_0 \subset \widetilde{S} \times \{0\} \subset \widetilde{B}_0$ , the retraction  $\Pi_\lambda : \widetilde{M} \rightarrow \mathcal{L}_\lambda$  satisfies:*

$$\text{Then } d_G(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_G(x, y) + C.$$

2.2.2. *qi Rays:* We also have the following from [Mj14].

**Lemma 2.19.** [Mj14, Lemma 5.9] *There exists  $C \geq 0$  such that for  $x_i \in \lambda_i$  there exists  $x_{i-1} \in \lambda_{i-1}$  with  $d_G(x_i, x_{i-1}) \leq C$ . Similarly there exists  $x_{i+1} \in \lambda_{i+1}$  with  $d_G(x_i, x_{i+1}) \leq C$ . Hence, for all  $n$  and  $x \in \lambda_n$ , there exists a  $C$ -quasigeodesic ray  $r$  such that  $r(i) \in \lambda_i \subset \mathcal{L}_\lambda$  for all  $i$  and  $r(n) = x$ .*

Further, by construction of split blocks,  $d_G(x_i, S_{i-1}) = 1$ . Therefore inductively,  $d_G(x_i, S_j) = |i - j|$ . Hence  $d_G(x_i, x_j) \geq |i - j|$ . By construction,  $d_G(x_i, x_j) \leq C|i - j|$ .

Hence, given  $p \in \lambda_i$  the sequence of points  $x_n, n \in \mathbb{N} \cup \{0\}$  (for simply degenerate groups) or  $n \in \mathbb{Z}$  (for totally degenerate groups) with  $x_i = p$  gives by Lemma 2.19

above, a quasigeodesic in the  $d_G$ -metric. Such quasigeodesics shall be referred to as  $d_G$ -quasigeodesic rays.

### 3. LAMINATIONS

**3.1. Ideal points are identified by Cannon-Thurston Maps.** We would like to know exactly which points are identified by the Cannon-Thurston map, whose existence is ensured by Theorem 1.1. Let  $i : \tilde{S} \rightarrow \tilde{M}$  denote inclusion. Let  $\hat{i}$  be the continuous extension of  $i$  to the disk  $\mathbb{D} = (\mathbb{H}^2 \cup S_\infty^1)$  in Theorem 1.1. Let  $\partial i$  denote the restriction of  $\hat{i}$  to the boundary  $S_\infty^1$ .

As mentioned in the introductory Section 1.2, we shall first prove the **forward direction** of Theorem 1.3. Proposition 3.1 below shows that the existence of a Cannon-Thurston map automatically guarantees that end-points of leaves of the ending lamination are identified by the Cannon-Thurston map.

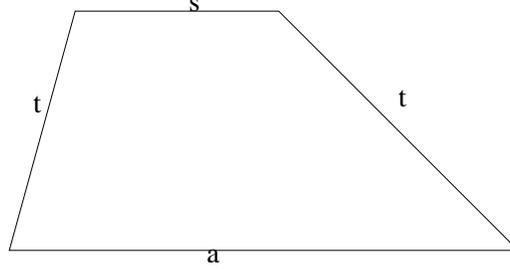
**Proposition 3.1.** *Let  $u, v$  be either ideal end-points of a leaf of an ending lamination, or ideal boundary points of a complementary ideal polygon. Then  $\partial i(u) = \partial i(v)$ .*

*Proof.* ( cf. Lemma 3.5 of [Mit97]. See also [Mj14].) We shall use two facts in the proof:

- 1) the fact (due to Bonahon [Bon86] and Thurston [Thu80]) that surface groups are tame and that for  $M$  there exist simple closed curves  $a_i$  on  $S$  whose geodesic realizations exit the end.
- 2) the fact (due to Thurston [Thu80] Ch. 9) that the sequence of simple closed curves  $a_i$  converges to the ending lamination in the space of measured laminations. It follows that after lifting to the universal cover, any leaf of the ending lamination is a Chabauty topology limit of bi-infinite geodesics  $\tilde{a}_i$  (lifts of  $a_i$ ).

Since any end  $E$  of  $M$  is geometrically tame [Thu80], there exists  $C_0$  such that there exists a sequence of closed geodesics  $s_i$  with length at most  $C_0$  exiting the end. We shall refer to such geodesics as ‘bounded geodesics’. Let  $a_i$  be geodesics in the intrinsic metric on the base surface  $S$  ( $= S_0 \subset M$ ) freely homotopic to  $s_i$ . We can assume further [Bon86] that  $a_i$ ’s are simple. Join  $a_i$  to  $s_i$  by the shortest geodesic  $t_i$  in  $M$  connecting the two curves.

For any leaf  $l$  of the ending lamination, we have a subsequence of the  $a_i$ ’s whose Hausdorff limit in  $S$  contains  $l$ . Abusing notation slightly let us denote the subsequence as  $\{a_i\}$ . In the universal cover, we obtain segments  $a_{fi} \subset \tilde{S}$  which are finite segments whose end-points are identified by the covering map  $P : \tilde{M} \rightarrow M$ . We also assume that  $P$  is injective restricted to the interior of  $a_{fi}$ ’s mapping to  $a_i$ . Similarly there exist segments  $s_{fi} \subset \tilde{M}$  which are finite segments whose end-points are identified by the covering map  $P : \tilde{M} \rightarrow M$ . We also assume that  $P$  is injective restricted to the interior of  $a_{fi}$ ’s. The finite segments  $s_{fi}$  and  $a_{fi}$  are chosen in such a way that there exist lifts  $t_{1i}, t_{2i}$ , joining end-points of  $a_{fi}$  to corresponding end-points of  $s_{fi}$ . The union of these four pieces looks like a trapezium (See figure below, where we omit subscripts for convenience).

Figure 2: *Trapezium*

Next, given any lift  $\lambda$  of the leaf  $l$  to  $\tilde{S}$ , we may choose translates of the finite segments  $a_{f_i}$  (under the action of  $\pi_1(S)$ ) appropriately, such that  $a_{f_i}$  converge to  $\lambda$  in (the Hausdorff/Chabauty topology on closed subsets of)  $\mathbf{H}^2$ . For each  $a_{f_i}$ , let

$$\beta_{f_i} = t_{1i} \circ s_{f_i} \circ \overline{t_{2i}}$$

where  $\overline{t_{2i}}$  denotes  $t_{2i}$  with orientation reversed. Then  $\beta_{f_i}$ 's are uniform hyperbolic quasigeodesics in  $\tilde{M}$  (since  $s_{f_i}$  is short). If the translates of  $a_{f_i}$  we are considering have end-points lying outside large balls around a fixed reference point  $p \in \tilde{S}$ , it is easy to check that  $\beta_{f_i}$ 's lie outside large balls about  $p$  in  $\tilde{M}$ .

At this stage we invoke the existence theorem for Cannon-Thurston maps, Theorem 1.1. Since  $a_{f_i}$ 's converge to  $\lambda$  and there exist uniform hyperbolic quasigeodesics  $\beta_{f_i}$ , joining the end-points of  $a_{f_i}$  and exiting all compact sets, it follows that  $\partial i(u) = \partial i(v)$ , where  $u, v$  denote the boundary points of  $\lambda$ .

Hence if we define  $u, v$  to be equivalent if they are the end-points of a leaf of the ending lamination, then the transitive closure of this relation has as elements of an equivalence class

- a) either ideal end-points of a leaf of a lamination,
- b) or ideal boundary points of a complementary ideal polygon,
- c) or a single point in  $S_\infty^1$  which is not an end-point of a leaf of a lamination.  $\square$

**Definition 3.2.** Let  $H$  be a finitely presented group acting on a hyperbolic space  $X$  with quotient  $M$ . Let  $X_H$  be a 2-complex with fundamental group  $H$ , and  $i : X_H \rightarrow M$  be a map inducing an isomorphism of fundamental groups. Then  $i$  lifts to  $\tilde{i} : \tilde{X}_H \rightarrow X$ . A bi-infinite geodesic  $\lambda$  in  $\tilde{X}_H \subset X$  will be called a **leaf of the abstract ending lamination** for  $i : X_H \rightarrow M$ , if

- 1) there exists a set of geodesics  $\sigma_i$  in  $M$  exiting every compact set
- 2) there exists a set of geodesics  $\alpha_i$  in  $X_H$  with  $i(\alpha_i)$  freely homotopic to  $\sigma_i$
- 3) there exist finite lifts  $\tilde{\alpha}_i$  of  $\alpha_i$  in  $(\tilde{X}_H)$  such that the natural covering map  $\Pi : \tilde{X}_H \rightarrow X_H$  is injective away from end-points of  $\tilde{\alpha}_i$
- 4)  $\tilde{\alpha}_i$  converges to  $\lambda$  in the Chabauty topology

Proposition 3.1 and its proof readily generalize to

**Proposition 3.3.** Suppose  $H$  is hyperbolic and  $\tilde{i} : \tilde{X}_H \rightarrow X$  extends to a Cannon-Thurston map on boundaries. Let  $u, v$  be end-points of a leaf of an abstract ending lamination. Then  $\partial i(u) = \partial i(v)$ .

To distinguish between the ending lamination and bi-infinite geodesics whose end-points are identified by  $\partial i$ , we make the following definition.

**Definition 3.4.** A **CT leaf**  $\lambda_{CT}$  is a bi-infinite geodesic whose end-points are identified by  $\partial i$ .

An **EL leaf**  $\lambda_{EL}$  is a bi-infinite geodesic whose end-points are ideal boundary points of either a leaf of the ending lamination, or a complementary ideal polygon.

Then to prove the main theorem 1.3 it remains to show that

- **A CT leaf is an EL leaf.**

**3.2. Leaves of Laminations.** Our first observation is that any semi-infinite geodesic (in the *hyperbolic metric* on  $\tilde{S}$ ) contained in a *CT leaf* in the base surface  $\tilde{S} = \tilde{S} \times \{0\} \subset \tilde{B} = \tilde{B} \times \{0\}$  has infinite diameter in the *graph metric*  $d_G$  restricted to  $\tilde{S} \times \{0\}$ , i.e. the induced path metric on  $\tilde{S} \times \{0\}$ . This follows from the following somewhat stronger assertion.

**Lemma 3.5.** *Given  $k \geq 0$ , there exists  $C \geq 0$  such that if  $B = \cup_{0 \leq i \leq k} B_i$  and  $\lambda \subset \tilde{B}$  is a bi-infinite geodesic in the intrinsic metric on  $\tilde{B}$ , whose end-points are identified by the Cannon-Thurston map, then for any split component  $\tilde{K}$ ,  $dia_{hyp}(\lambda \cap \tilde{K}) \leq C$*

*Proof.* Suppose not.

Then there exist split components  $\tilde{K}(i) \subset \tilde{B}$ , such that  $dia_{hyp}(\lambda \cap \tilde{K}(i)) \geq 3i$ , where  $dia_{hyp}$  denotes diameter in the hyperbolic metric on  $\tilde{M}$ . Acting on  $\tilde{B}$  by elements  $h_i$  of the surface group  $\pi_1(S)$ , we may assume that there exists a sequence of segments  $\lambda^i \subset h_i \cdot \lambda$  such that

- $\lambda^i$  is approximately centered about a fixed origin 0 in a fixed lift  $\tilde{K}$  of a fixed split component  $K$ , i.e.  $\lambda^i$  pass uniformly close to 0 and end-points of  $\lambda^i$  are at distance  $\geq i$  from 0. This is possible since  $B$  contains finitely many split blocks.

Since  $\tilde{K}$  is quasiconvex, it follows that the  $\lambda^i$ 's are uniform quasigeodesics in  $\tilde{M}$ . Hence, the sequence  $\{h_i \cdot \lambda\}$  converges to a bi-infinite quasigeodesic  $\lambda^\infty$  in the Chabauty topology. Since the set of *CT leaves* are closed in the Chabauty topology, it follows that  $\lambda^\infty$  is a *CT leaf*.

But, this is a contradiction, as we have noted already that  $\lambda^\infty$  is a quasigeodesic.  $\square$

**Corollary 3.6. CT leaves have infinite diameter**

Let  $\lambda_+(\subset \lambda \subset \tilde{S} \times \{0\} = \tilde{S})$  be a semi-infinite geodesic (in the hyperbolic metric on  $\tilde{S}$ ) contained in a *CT leaf*  $\lambda$ . Then  $dia_G(\lambda_+)$  is infinite, where  $dia_G$  denotes diameter in the graph metric restricted to  $\tilde{S}$ .

*Proof.* Put  $k = 1$  in Lemma 3.5.  $\square$

Using Lemma 3.5, we shall now show:

**Proposition 3.7.** *There exists a function  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$  such that the following holds:*

Let  $\lambda$  be a *CT leaf*. Also for  $p, q \in \tilde{M}$ , let  $\overline{pq}$  denote a geodesic in  $(\tilde{M}, d_G)$  joining  $p, q$ . If  $a_i, b_i \in \lambda$  be such that  $d(a_i, 0) \geq N$ ,  $d(b_i, 0) \geq N$ , then  $d_G(\overline{a_i b_i}, 0) \geq M(N)$ , where  $d$  denotes the hyperbolic metric on  $\tilde{M}$  and  $d_G$  the graph metric.

*Proof.* Suppose not. Let  $\lambda_+$  and  $\lambda_-$  denote the ideal end points of  $\lambda$ . Then there exists  $C \geq 0$ ,  $a_i \rightarrow \lambda_-$ ,  $b_i \rightarrow \lambda_+$  such that  $d_G(\overline{a_i b_i}, 0) \leq C$ . That is, there exist  $p_i \in \overline{a_i b_i}$  such that  $d_G(0, p_i) \leq C$ . Due to the existence of a Cannon-Thurston map



of  $\tilde{I}$ . Then the limit set of any connected component of the transitive closure of this relation contains a pair of poles  $g^\infty$  and  $g^{-\infty}$  for some element  $g \in \pi_1(S)$ .

*Proof.* Let  $\mathcal{T}$  be the  $\mathbb{R}$ -tree dual to  $\tilde{\Lambda}$ . Let  $I_0$  be a fixed lift of  $I$  to  $\tilde{S}$ . Then  $I_0 \subset \tilde{S}$  projects to an embedded non-trivial interval (also called  $I_0$ ) in  $\mathcal{T}$  under the quotient map  $q$  that identifies leaves of  $\tilde{\Lambda}$  to points. The orbit of  $I_0$  under  $\pi_1(S)$  acting on  $\mathcal{T}$  is a forest, in fact a sub-forest  $\mathcal{F}$  of  $\mathcal{T}$ .

Let  $\mathcal{T}_1$  be the connected component of  $\mathcal{F}$  containing  $I_0$ . If  $g\mathcal{T}_1 \cap \mathcal{T}_1 \neq \emptyset$ , then  $g\mathcal{T}_1 \subset \mathcal{T}_1$ . Hence  $g^{-1}\mathcal{T}_1 \cap \mathcal{T}_1 \neq \emptyset$ , and we finally have that  $\mathcal{T}_1$  is invariant under  $g^n$  for all integers  $n$ . This shows that  $q^{-1}(\mathcal{T}_1) \subset \tilde{S}$  contains the pole corresponding to the infinite order element  $g$ .

Thus we need finally the existence of a  $g$  as in the previous paragraph. It suffices to show that for any non-trivial  $I_0$ , there exists  $g \in \pi_1(S)$  such that  $gI_0 \cap I_0 \neq \emptyset$ . But this follows from minimality of  $\Lambda$ , using the fact that each leaf is dense in  $\Lambda$ , and hence that there exists  $g \in \pi_1(S)$  such that  $gI_0$  and  $I_0$  are transverse to a common leaf  $\lambda$  of  $\tilde{\Lambda}$ .  $\square$

#### 4.2. Rays Contained in Ladders.

**Definition 4.2.** Let  $X, Y, Z$  be geodesically complete metric spaces such that  $X \subset Y \subset Z$ .  $X$  is said to **coarsely separate**  $Y$  into  $Y_1$  and  $Y_2$  if

- (1)  $Y_1 \cup Y_2 = Y$
- (2)  $Y_1 \cap Y_2 = X$
- (3) For all  $M \geq 0$ , there exist  $y_1 \in Y_1$  and  $y_2 \in Y_2$  such that  $d(y_1, Y_2) \geq M$  and  $d(y_2, Y_1) \geq M$
- (4) There exists  $C \geq 0$  such that for all  $y_1 \in Y_1$  and  $y_2 \in Y_2$  any geodesic in  $Z$  joining  $y_1, y_2$  passes through a  $C$ -neighborhood of  $X$ .

Let  $\lambda = \lambda_0$  be any bi-infinite geodesic in  $\tilde{S}$ . Let  $\mathcal{L}_\lambda$  be the ladder corresponding to  $\lambda$  as in Theorem 2.18.

We now fix a quasigeodesic ray  $r_0$  as in Lemma 2.19, and consider a translate  $r' = h \cdot r_0$  passing through  $z \in \lambda_m \subset \mathcal{L}_\lambda$ , i.e.  $r'(m) = z$ . Let  $\Pi_\lambda \cdot r' = r \subset \mathcal{L}_\lambda$ . Each  $r(i)$  cuts  $\lambda_i$  into two pieces  $\lambda_i^-$  and  $\lambda_i^+$  with ideal boundary points  $\lambda_{i,-\infty}, \lambda_{i,\infty}$  respectively.

We shall show that  $r$  coarsely separates  $\mathcal{L}_\lambda$  into  $\mathcal{L}_\lambda^+$  and  $\mathcal{L}_\lambda^-$ , where

$$\begin{aligned} \mathcal{L}_\lambda^+ &= \bigcup_i \lambda_i^+ \\ \mathcal{L}_\lambda^- &= \bigcup_i \lambda_i^- \end{aligned}$$

and  $\lambda_i^+$  (resp.  $\lambda_i^-$ ) is the segment of  $\lambda$  joining  $r(i)$  to the ideal end-point  $\lambda_{i,-\infty}$  (resp.  $\lambda_{i,\infty}$ ).

We need to repeatedly apply Theorem 2.18 to prove the above assertion.

Given  $r'$ , we construct two hyperbolic ladders  $\mathcal{L}_\lambda^{+'}$  and  $\mathcal{L}_\lambda^{-'}$ , obtained by joining the points  $r'(i)$  to the ideal end-points  $\lambda_{i,-\infty}, \lambda_{i,\infty}$  respectively, of  $\lambda_i \subset \tilde{S} \times \{i\}$ . Then  $\mathcal{L}_\lambda^{+'}$  and  $\mathcal{L}_\lambda^{-'}$  are  $C$ -quasiconvex (in the graph metric  $d_G$ ) by Theorem 2.18. Further,  $\Pi_\lambda \cdot r'(i) = r(i)$  by definition of  $r$ . Hence,

$$\begin{aligned} \Pi_\lambda(\mathcal{L}_\lambda^{-'}) &= \mathcal{L}_\lambda^- \\ \Pi_\lambda(\mathcal{L}_\lambda^{+'}) &= \mathcal{L}_\lambda^+ \end{aligned}$$

Further,

$$\begin{aligned}\mathcal{L}_\lambda^- \cup \mathcal{L}_\lambda^+ &= \mathcal{L}_\lambda \\ \mathcal{L}_\lambda^- \cap \mathcal{L}_\lambda^+ &= r\end{aligned}$$

and there exists a  $K_0$  (independent of  $r_0, h, \lambda$ ) such that  $\mathcal{L}_\lambda^-, \mathcal{L}_\lambda^+, \mathcal{L}_\lambda, r$  are all  $K_0$ -quasiconvex.

Criterion (3) of Definition 4.2 in this context is given by Lemma 3.6 : *CT leaves have infinite diameter.*

To prove that  $r$  separates  $\mathcal{L}_\lambda$  into  $\mathcal{L}_\lambda^-, \mathcal{L}_\lambda^+$ , we need to show first:

**Lemma 4.3.** *For all  $K_0 \geq 0$ , there exists  $K_1 \geq 0$  such that if  $p \in \mathcal{L}_\lambda^-, q \in \mathcal{L}_\lambda^+$  with  $d_G(p, q) \leq K_0$ , then there exists  $z \in r$  such that  $d_G(p, z) \leq K_1$  and  $d_G(q, z) \leq K_1$ .*

*Proof.* Let  $\Pi_\lambda^+$  denote the sheetwise retract of Theorem 2.18 onto  $\mathcal{L}_\lambda^+$ . Then

$$\Pi_\lambda^+(\lambda_i^-) = r(i)$$

and

$$\Pi_\lambda^+(x) = x$$

for all  $x \in \mathcal{L}_\lambda^+$ .

Hence

$$\begin{aligned}\Pi_\lambda^+(q) &= q \\ \Pi_\lambda^+(p) &= z = r(i)\end{aligned}$$

for some  $z \in r$  and some  $i$ .

Therefore, by Theorem 2.18 again,

$$d_G(q, z) \leq C d_G(p, q) = C K_0.$$

Choosing  $K_1 = C K_0 + K_0$  (and using the triangle inequality for  $p, q, z$ ) the Lemma follows.  $\square$

We are now in a position to prove:

**Theorem 4.4.**  *$r$  coarsely separates  $\mathcal{L}_\lambda$  into  $\mathcal{L}_\lambda^-, \mathcal{L}_\lambda^+$ .*

*Proof.* We have already shown

$$\begin{aligned}\mathcal{L}_\lambda^- \cup \mathcal{L}_\lambda^+ &= \mathcal{L}_\lambda \\ \mathcal{L}_\lambda^- \cap \mathcal{L}_\lambda^+ &= r\end{aligned}$$

and there exists a  $K_0$  (independent of  $r_0, h, \lambda$ ) such that  $\mathcal{L}_\lambda^-, \mathcal{L}_\lambda^+, \mathcal{L}_\lambda, r$  are all  $K_0$ -quasiconvex.

Criterion (3) of Definition 4.2 is given by Lemma 3.6.

Finally given  $u \in \mathcal{L}_\lambda^-$  and  $v \in \mathcal{L}_\lambda^+$ , let  $\overline{uv}$  be the geodesic in  $(\widetilde{M}, d_G)$  joining  $u, v$ . Then  $\Pi_\lambda(\overline{uv})$  is a "dotted quasigeodesic", i.e. there is a sequence of points  $u = p_0, p_1, \dots, p_n = v$ , where  $d_G(p_i, p_{i+1}) \leq C$  (and the constant  $C$  is obtained from Theorem 2.18). Further,  $p_0 \in \mathcal{L}_\lambda^-, p_n \in \mathcal{L}_\lambda^+$  and  $p_i \in \mathcal{L}_\lambda$  for all  $i$ . Therefore there exists  $m$  such that  $p_m \in \mathcal{L}_\lambda^-, p_{m+1} \in \mathcal{L}_\lambda^+$ , with  $d_G(p_m, p_{m+1}) \leq C$ . Hence, by Lemma 4.3, there exists  $K_1 \geq 0$  such that there exists  $z \in r$  with  $d_G(p_m, z) \leq K_1$  and  $d_G(p_{m+1}, z) \leq K_1$ .

Finally, by Theorem 2.17,  $(\widetilde{M}, d_G)$  is hyperbolic, and therefore the "dotted quasigeodesic"  $u = p_0, p_1, \dots, p_n = v$  lies in a uniformly bounded neighborhood of the geodesic  $\overline{uv}$ . That is, there exists  $C_1 \geq 0$  such that for all  $u \in \mathcal{L}_\lambda^-$  and  $v \in \mathcal{L}_\lambda^+$ ,

the geodesic  $\overline{uv}$  in  $(\widetilde{M}, d_G)$  joining  $u, v$  passes through a  $C_1$ -neighborhood of  $r$ . This proves (4) in Definition 4.2 and hence we conclude that  $r$  coarsely  $\mathcal{L}_\lambda$  into  $\mathcal{L}_\lambda^-, \mathcal{L}_\lambda^+$ .  $\square$

We shall have need for the following Proposition, whose proof is exactly along the lines of Theorem 4.4 above.

**Proposition 4.5.** *Let  $\mu, \lambda$  be two bi-infinite geodesics on  $\widetilde{S}$  such that  $\mu \cap \lambda \neq \emptyset$ . Then  $\mathcal{L}_\lambda \cap \mathcal{L}_\mu$  contains a quasigeodesic ray  $r$  coarsely separating both  $\mathcal{L}_\lambda$  and  $\mathcal{L}_\mu$ .*

**Remark 4.6.** Proposition 4.5 generalizes readily to cusped surfaces  $S^h$  to show that if  $\mu, \lambda$  are two bi-infinite geodesics on  $\widetilde{S}^h$  such that  $\mu \cap \lambda \neq \emptyset$ , then  $\mathcal{L}_\lambda \cap \mathcal{L}_\mu$  contains a quasigeodesic ray  $r$ .

To see this it suffices to note that if  $\mu \cap \lambda \neq \emptyset$ , then  $\mu_i \cap \lambda_i \neq \emptyset$  for all  $i$ . Hence we may construct a quasigeodesic ray  $r$  contained in both  $\mathcal{L}_\lambda$  and  $\mathcal{L}_\mu$ .

One last Proposition to be used in the proof of Theorem 1.3 is the following which says in particular that any two quasigeodesic rays lying on  $\mathcal{L}_\lambda$  are asymptotic with respect to the graph metric  $d_G$ .

**Proposition 4.7. Asymptotic Quasigeodesic Rays**

*Given  $K \geq 1$  there exists  $\alpha$  such that if  $\lambda$  is a CT-leaf then there exists  $z \in \partial\widetilde{M}$  satisfying the following:*

*If  $r_1$  and  $r_2$  are  $K$ -quasi-geodesic rays contained in  $\mathcal{L}_\lambda$  then there exists  $N \in \mathbb{N}$  such that*

- 1)  $r_j(n) \rightarrow z = \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$  as  $n \rightarrow \infty$ , for  $j = 1, 2$ .
- 2)  $d_G(r_1(n), r_2(n)) \leq \alpha$  for all  $n \geq N$

*Proof.* By Proposition 3.7, we find that if  $a_i, b_i \in \lambda = \lambda_0$  such that  $a_i, b_i$  converge to ideal points  $\lambda_{0,-\infty}, \lambda_{0,\infty}$  (denoted  $\lambda_{-\infty}, \lambda_\infty$  for convenience), then  $\Pi_\lambda(\overline{a_i b_i})$  leaves large balls about 0. More precisely there exists  $L_i \rightarrow \infty$  as  $i \rightarrow \infty$  such that  $\Pi_\lambda(\overline{a_i b_i})$  lies outside the  $L_i$ -ball about 0.

Also, by Theorem 4.4 above, each  $r_j$  coarsely separates  $\mathcal{L}_\lambda$ . Hence  $\Pi_\lambda(\overline{a_i b_i})$  passes close to  $r_j(n_{i(j)})$  for some  $n_{i(j)} \in \mathbb{N}$ , where  $n_i \rightarrow \infty$  as  $i \rightarrow \infty$ . We conclude that any such  $r_j$  converges on  $\partial\widetilde{M}$  to the same point as  $\partial(\lambda_{-\infty}) = \partial(\lambda_\infty)$ . This proves (1).

In particular any two quasigeodesic rays lying on  $\mathcal{L}_\lambda$  are asymptotic with respect to the graph metric  $d_G$ . This proves (2).  $\square$

**4.3. Main Theorem for Simply Degenerate Groups.** We are now in a position to prove the main Theorem 1.3 of this paper for closed surfaces. For ease of exposition we shall deal with the simply degenerate case first and then indicate the additional niceties for doubly degenerate groups. Recall that for a simply degenerate manifold  $M = S \times J$ , where  $J = [0, \infty)$ . For a totally degenerate manifold  $J = (-\infty, \infty)$  and it is the presence of two ends, positive and negative, that necessitates further care. The split level surfaces are indexed by  $0, 1, \dots, \infty$  for a simply degenerate manifold and by  $\mathbb{Z}$  for a totally (doubly) degenerate manifold. For a doubly degenerate group, there will be *two* ending laminations, one for each end and a slight modification of the proof below will be necessary to identify and distinguish these. The constructions of ladders and blocks are otherwise identical in both cases.

**Theorem 4.8.** *Let  $\partial i(a) = \partial i(b)$  for  $a, b \in S_\infty^1$  be two distinct points that are identified by the Cannon-Thurston map corresponding to a simply degenerate closed surface group (without accidental parabolics). Then  $a, b$  are either ideal end-points of a leaf of the ending lamination (in the sense of Thurston), or ideal boundary points of a complementary ideal polygon. Further, if  $a, b$  are either ideal end-points of a leaf of a lamination, or ideal boundary points of a complementary ideal polygon, then  $\partial i(a) = \partial i(b)$ .*

*Proof.* The second statement has been shown in Proposition 3.1.

To prove the first statement, let  $\partial i(a) = \partial i(b)$  for  $a, b \in S_\infty^1$ . Then  $(a, b) = \lambda \subset \widetilde{S}_0 \subset \widetilde{M}$  is a CT-leaf.

Suppose  $\lambda$  and  $\mu$  are intersecting CT leaves, i.e.  $\partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$  and  $\partial i(\mu_{-\infty}) = \partial i(\mu_\infty)$ .

As before, let  $\lambda_i$  and  $\mu_i$  be intersections of the ladders  $\mathcal{L}_\lambda$  and  $\mathcal{L}_\mu$  with the horizontal sheets. Then  $r(i) = \lambda_i \cap \mu_i$  is a quasigeodesic ray by Proposition 4.5. By Proposition 4.7,  $r(i)$  converges to a point  $z$  on  $\partial \widetilde{M}$  as  $i \rightarrow \infty$  such that  $z = \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty) = \partial i(\mu_{-\infty}) = \partial i(\mu_\infty)$ . Hence the Cannon-Thurston map identifies the endpoints of any two intersecting CT leaves  $\lambda$  and  $\mu$ .

If possible, suppose that the CT leaf  $\lambda$  is not an EL-leaf. Then  $\lambda$  intersects the ending lamination transversely (since the ending lamination is a filling lamination without any closed leaves) and there exist EL-leaves  $\mu$  for which  $\lambda \cap \mu \neq \emptyset$ . By Proposition 3.1, each such  $\mu$  is a CT-leaf. Hence, by the previous paragraph, the Cannon-Thurston map  $\partial i$  identifies the end points of  $\lambda$  with the endpoints of each such EL-leaf  $\mu$  for which  $\lambda \cap \mu \neq \emptyset$ . Let  $z(\in S_\infty^2)$  denote this common image under  $\partial i$ .

Since  $\lambda$  is not an EL-leaf, it contains a non-trivial geodesic subsegment  $I$  transverse to the ending lamination. Then the common image (under  $\partial i$ ) of end-points of all EL leaves  $\mu$  intersecting  $I$  transversely is  $z$ .

By Lemma 4.1  $(\partial i)^{-1}(z)$  contains a pair of poles  $g^{-\infty}, g^\infty$  for some  $g \in \pi_1(S)$ . This is because the equivalence class defined by  $I$  as in Lemma 4.1 consists of pairs of points all of which are identified (under  $\partial i$ ) with  $z$ .

This is a contradiction as a pair of poles forms the end-points of a quasigeodesic in  $\widetilde{M}$ . We conclude that  $\lambda$  must be an EL-leaf.  $\square$

**4.4. Modifications for Totally Degenerate Groups.** We elaborate on the modifications indicated in the first paragraph of Section 4.3, to pass from the simply degenerate case to the totally degenerate case. The construction of the ‘hyperbolic ladder’  $\mathcal{L}_\lambda$  as in the discussion preceding Theorem 2.18 is done with indexing set  $\mathbb{Z}$  in place of  $\mathbb{N}$ . In particular the quasigeodesic ray of Lemma 2.19 is replaced by a bi-infinite quasigeodesic  $r$ . However, as a hyperbolic metric space  $\mathbb{Z}$  has two boundary points  $+\infty$  and  $-\infty$ . Correspondingly we have two ending laminations  $\Lambda_+$  and  $\Lambda_-$ . The easy direction of Theorem 1.3 given by Proposition 3.1 then goes through verbatim to show that  $\Lambda_+ \cup \Lambda_- \subset \Lambda_{CT}$ .

We need to find a way of distinguishing the  $+$  and  $-$  directions in  $\Lambda_{CT}$ . To implement this, note that the discussion preceding Proposition 4.7 shows that if  $\lambda \in \Lambda_{CT}$ , i.e.  $\partial i(\lambda_\infty) = \partial i(\lambda_{-\infty})$ , then we have a bi-infinite quasigeodesic  $r : \mathbb{Z} \rightarrow \mathcal{L}_\lambda$  such that  $\partial i(\lambda_\infty) = \partial i(\lambda_{-\infty}) = r(\alpha)$ , where  $\alpha$  is either  $+\infty$  or  $-\infty$ . Define  $\Lambda_{CT}^+ \subset \Lambda_{CT}$  (resp.  $\Lambda_{CT}^- \subset \Lambda_{CT}$ ) to be the collection of CT-leaves whose endpoints

are identified in the  $+\infty$  (resp.  $-\infty$ ) direction, i.e.  $\alpha = +\infty$  (resp.  $-\infty$ ). Then the forward direction of Theorem 1.3 given by Proposition 3.1 shows that  $\Lambda_+ \subset \Lambda_{CT}^+$  and  $\Lambda_- \subset \Lambda_{CT}^-$ . Since both ending laminations  $\Lambda_+$  and  $\Lambda_-$  are individually filling arational minimal laminations, the proof of Theorem 4.8 (the reverse direction for simply degenerate groups) now shows that in fact  $\Lambda_+ = \Lambda_{CT}^+$  and  $\Lambda_- = \Lambda_{CT}^-$ .

**4.5. Application: Rigidity.** In [BCM12], Brock-Canary-Minsky prove the following Rigidity Theorem.

**Theorem 4.9.** *Let  $G$  be a closed surface group. If  $\rho$  and  $\rho'$  are two discrete faithful representations of  $G$  into  $PSL_2(\mathbb{C})$  that are conjugate by an orientation-preserving homeomorphism of  $\widehat{\mathbb{C}}$ , then  $\rho$  and  $\rho'$  are quasiconformally conjugate.*

We strengthen this by weakening the hypothesis of Theorem 4.9 to a topological conjugacy only on limit sets (rather than all of  $\widehat{\mathbb{C}}$ ).

**Theorem 4.10.** *Let  $G$  be a closed surface group. Let  $\rho(G) = \Gamma$  and  $\rho_1(G) = \Gamma_1$  be two simply or doubly degenerate representations of  $G$  into  $PSL_2(\mathbb{C})$  with limit sets  $\Lambda, \Lambda_1$ . Suppose that the  $G$ -actions on  $\Lambda, \Lambda_1$  are topologically conjugate. Then  $\rho$  and  $\rho_1$  are quasiconformally conjugate.*

*Proof.* We first deal with the simply degenerate case. By Theorem 1.3, the pre-images of the Cannon-Thurston maps  $\partial i$  and  $\partial i_1$  from  $\partial G (= S^1)$  to  $\Lambda$  or  $\Lambda_1$  are given by end-points of leaves of the ending lamination (or ideal points of complementary polygons) whenever  $\partial i$  and  $\partial i_1$  are non-injective. Thus the  $G$ -action on  $\Lambda, \Lambda_1$  pulls back to a  $G$ -equivariant homeomorphism  $\phi : \partial G \rightarrow \partial G$  taking the ending lamination of  $\rho$  to that of  $\rho_1$ . Re-marking by an isomorphism of  $G$  if necessary, we can ensure that the homeomorphism be the identity on  $\partial G$ . Hence the ending laminations of  $\rho$  and  $\rho_1$  are the same.

In the doubly degenerate case, the same argument shows that the pairs of ending laminations for  $\rho$  and  $\rho_1$  are the same. Hence if  $\rho$  and  $\rho_1$  are doubly degenerate, they have the same end-invariants. By the Ending Lamination Theorem [BCM12],  $\rho$  and  $\rho_1$  are **conformally conjugate**.

In the simply degenerate case, the conformal structures corresponding to the geometrically finite ends for  $\rho$  and  $\rho_1$  are quasiconformal deformations of each other (since the quotient of the domain of discontinuity is a connected finite volume Riemann surface). Since the ending laminations of  $\rho$  and  $\rho_1$  are the same, it follows therefore that the quotient manifolds are bi-Lipschitz homeomorphic by the Ending Lamination Theorem [BCM12]. Hence  $\rho$  and  $\rho_1$  are **quasiconformally conjugate**.  $\square$

## Appendix A (by Shubhabrata Das and Mahan Mj) Surfaces with Cusps <sup>1</sup>

We now deal with surfaces with cusps.  $S^h$  will denote a finite volume hyperbolic surface with cusps.  $S$  will denote a truncated surface, i.e.  $S^h$  minus an open neighborhood of the cusps. The arguments in this Section can be easily adapted to

<sup>1</sup>The work in this Appendix forms part of SD's PhD thesis written under the supervision of MM. The proof given here was discovered jointly considerably after the work on the earlier Sections was completed. Hence we have retained both approaches.

furnish a slightly different proof of Theorem 1.3 for surfaces without cusps. As in the previous Section we deal with the case of simply degenerate groups first.

**Equivalence Relations on  $S^1$ :**

Suppose that a group  $G$  acts on  $S^1$  preserving a closed equivalence relation  $\mathcal{R}$ . An example  $\mathcal{R}_{\mathcal{L}}$  of such a relation comes from a lamination  $\mathcal{L}$ , where two points on  $S^1$  are declared equivalent if they are end-points of a leaf of  $\mathcal{L}$ . The equivalence relation  $\mathcal{R}_{\mathcal{L}}$  is obtained as the transitive closure of this relation.

**Definition A.1.** [Bow07] *Two disjoint subsets,  $P, Q \subset S^1$  are linked if there exist linked pairs,  $\{x, y\} \subset P$  and  $\{z, w\} \subset Q$ .  $\mathcal{R}$  is unlinked if distinct equivalence classes are unlinked.*

The following Lemma due to Bowditch give us a way of recognizing relations coming from laminations.

**Lemma A.2.** (Lemma 9.2 of [Bow07]) *Let  $\mathcal{R}$  be a non-empty closed unlinked  $G$ -invariant equivalence relation on  $S^1$ . Suppose that no pair of fixed points of any loxodromic are identified under  $\mathcal{R}$ . Then there is a unique complete perfect lamination,  $\mathcal{L}$ , on  $S$  such that  $\mathcal{R} = \mathcal{R}_{\mathcal{L}}$ .*

Let  $\mathcal{R}_{CT}$  denote the equivalence relation on  $S^1$  induced by the Cannon-Thurston map for a simply degenerate punctured surface group (cf. Theorem 1.1). Let  $\Lambda$  denote the ending lamination. By Proposition 3.1 pairs of end-points of leaves of  $\Lambda$  are contained in  $\mathcal{R}_{CT}$ . Hence, for simply degenerate groups, it suffices to show that  $\mathcal{R}_{CT}$  is induced by a lamination since no other lamination can properly contain  $\Lambda$ . By Lemma A.2 it suffices to show that  $\mathcal{R}_{CT}$  is unlinked. The next Proposition is the analogue of Proposition 4.7 for cusped surfaces.

**Proposition A.3.** *Let  $i : \widetilde{S}^h \rightarrow \widetilde{M}^h$  be an inclusion of the universal cover of a punctured surface into the universal cover of the convex core  $M^h$  of a simply degenerate 3-manifold. Let  $\partial i$  be the associated Cannon-Thurston map. If  $\lambda$  is a CT-leaf in  $\widetilde{S}^h$ ,  $\mathcal{L}_\lambda$  the corresponding ladder, and  $r = r(n) \subset \mathcal{L}_\lambda$  a qi ray, then there exists  $z \in \partial \widetilde{M}^h$  such that  $r(n) \rightarrow z = \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$  as  $n \rightarrow \infty$ .*

*Proof.* We first observe that both end-points  $\lambda_{-\infty}, \lambda_\infty$  of the CT leaf  $\lambda$  cannot be parabolics. For then they would have to be base points of *different* horoballs in  $\widetilde{M}$  as they correspond to different lifts of the cusp(s) of  $M$ .

**Case 1:** Both  $\lambda_{-\infty}, \lambda_\infty$  are non-parabolic.

The proof of Proposition 4.7 goes through in this context mutatis mutandis.

**Case 2:** Exactly one of  $\lambda_{-\infty}, \lambda_\infty$  is a parabolic.

Without loss of generality assume that  $\lambda_{-\infty}$  is a parabolic. Let  $B$  be the horoball in  $\widetilde{M}^h$  based at  $w = \partial i(\lambda_{-\infty})$  and let  $H$  be the horosphere boundary of  $B$ . Let  $o$  be the point of intersection of  $\lambda$  with  $H$ . For  $p, q \in \widetilde{M}^h$ ,  $(p, q)_h$  and  $\overline{pq}$  will denote respectively geodesics in  $(\widetilde{M}^h, d)$  and  $(\widetilde{M}^h, d_G)$ .

Choose a sequence of points  $a_n, b_n \in \lambda$  such that  $a_n \rightarrow \lambda_{-\infty}$  and  $b_n \rightarrow \lambda_\infty$ . Then by the existence of Cannon-Thurston maps for  $i : \widetilde{S}^h \rightarrow \widetilde{M}^h$  (Theorem 1.1) it follows that there exists a function  $M(n) \rightarrow \infty$  as  $n \rightarrow \infty$  such that  $(a_n, b_n)_h$  lies outside  $B_{M(n)}(o) \subset \widetilde{M}^h$ . Hence, if  $q_n = (a_n, b_n)_h \cap H$  then  $d(q_n, o) \geq M(n)$  and the geodesic subsegment  $(q_n, b_n)_h$  lies outside  $B_{M(n)}(o) \subset \widetilde{M}^h$ .

Let  $N = \widetilde{M^h} \setminus \bigcup_\alpha B_\alpha$  be the complement of open horoballs and  $d_G$  be the graph metric on  $N$  obtained after first partially electrocuting horospheres (cf. Section 2.1). By Lemma 2.3  $(q_n, b_n)_h$  and  $\overline{q_n b_n}$  lie in a bounded neighborhood of each other in  $(N, d_G)$ .

The  $d_G$ -distance  $d_G(o, q_n)$  is equal to the number of vertical blocks between  $o$  and  $q_n$ . But  $a_n \rightarrow \lambda_{-\infty}$  implies  $q_n \rightarrow \infty$  in  $\widetilde{M^h}$ . Hence  $d_G(o, q_n) \rightarrow \infty$  as  $a_n \rightarrow \lambda_{-\infty}$ .

By Corollary 3.6,  $d_G(o, b_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Hence by Proposition 3.7 there exists a function  $M_1(n) \rightarrow \infty$  as  $n \rightarrow \infty$  such that  $\overline{q_n b_n}$  lies outside  $B_{M_1(n)}(o) \subset (N, d_G)$ .

Now recall that  $\Pi_\lambda : N \rightarrow \mathcal{L}_\lambda$  is a coarse Lipschitz retract by Theorem 2.18. Hence  $\Pi_\lambda[\overline{q_n b_n}] \subset \mathcal{L}_\lambda$  is a uniform quasigeodesic in  $(N, d_G)$ .

Further, since  $q_n$  belongs to  $H$  and since  $\Pi_\lambda$  essentially fixes the horosphere  $H$ , it follows that  $d_G(\Pi_\lambda(q_n), q_n) \leq 1$ . Also  $\Pi_\lambda(b_n) = b_n$ . Therefore there exists a function  $M_2(n) \rightarrow \infty$  as  $n \rightarrow \infty$  such that  $\Pi_\lambda[\overline{q_n b_n}]$  lies outside  $B_{M_2(n)}(o) \subset (N, d_G)$ .

Next, since  $H \cap \mathcal{L}_\lambda$  and  $b_n$  lie on different sides of the qi ray  $r = r(n) \subset \mathcal{L}_\lambda$  it follows that there exists  $z_n \in \overline{q_n b_n}$  such that  $d_G(z_n, r)$  is uniformly bounded.

Also there exists  $t_n \in (q_n, b_n)_h$  such that  $d_G(z_n, t_n)$  and hence  $d_G(t_n, r)$  is uniformly bounded.

Since  $t_n \in (q_n, b_n)_h$  it follows that  $t_n \rightarrow \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$ . Since  $d_G(t_n, r)$  is uniformly bounded, there exists  $s_n \in r$  such that  $d_G(t_n, s_n)$  is uniformly bounded and therefore  $t_n, s_n$  are separated by a uniformly bounded number of split components. By uniform graph quasiconvexity of split components (Theorem 2.17) it follows that  $s_n \rightarrow \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$ .

Finally if  $r_{s_n}$  denotes the part of the ray  $r$  ‘above’  $s_n$ , (i.e.  $[s_n, \infty)$ ) then joining points of  $r_{s_n}$  in successive blocks by hyperbolic geodesics we obtain an electroambient quasigeodesic  $\sigma_n$ . By Lemma 2.3 there exist hyperbolic geodesics  $\tau_{m,n}$  joining  $r(m), r(n)$  for  $m > n$  and contained in a bounded neighborhood of  $\sigma_n \cup B$  in  $\widetilde{M^h}$ . Hence  $r(n) \rightarrow \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$  as  $n \rightarrow \infty$ .  $\square$

We are now in a position to prove the analogue of Theorem 1.3 for surfaces with cusps.

**Theorem A.4.** *Let  $\partial i(a) = \partial i(b)$  for  $a, b \in S_\infty^1$  be two distinct points that are identified by the Cannon-Thurston map corresponding to a simply degenerate surface group (without accidental parabolics). Then  $a, b$  are either ideal end-points of a leaf of the ending lamination, or ideal boundary points of a complementary ideal polygon. Further, if  $a, b$  are either ideal end-points of a leaf of a lamination, or ideal boundary points of a complementary ideal polygon, then  $\partial i(a) = \partial i(b)$ .*

*Proof.* The second statement has been shown in Proposition 3.1.

To prove the first statement, it suffices to show that  $\mathcal{R}_{CT}$  is unlinked. Suppose now that  $\lambda$  and  $\mu$  are intersecting CT leaves, i.e.  $\partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty)$  and  $\partial i(\mu_{-\infty}) = \partial i(\mu_\infty)$ .

Consider the ladders  $\mathcal{L}_\lambda$  and  $\mathcal{L}_\mu$ . Let  $r(i) \subset \lambda_i \cap \mu_i$  be a quasigeodesic ray as per Remark 4.6. By Proposition A.3,  $r$  converges to a point  $z$  on  $\partial \widetilde{M}$  such that  $z = \partial i(\lambda_{-\infty}) = \partial i(\lambda_\infty) = \partial i(\mu_{-\infty}) = \partial i(\mu_\infty)$ . Hence if  $\{a, b\}, \{c, d\} \in \mathcal{R}_{CT}$ , then either  $\{a, b, c, d\}$  are all mutually related in  $\mathcal{R}_{CT}$ , or  $\{a, b\}, \{c, d\}$  are unlinked. By Lemma A.2,  $\mathcal{R}_{CT}$  is induced by a lamination  $\Lambda_{CT}$ . By Proposition 3.1, the ending

lamination  $\Lambda_{EL}$  is contained in  $\Lambda_{CT}$ . Since  $\Lambda_{EL}$  is filling and arational, it follows that  $\Lambda_{EL} = \Lambda_{CT}$ .  $\square$

The modifications necessary to pass from the simply degenerate case to the totally degenerate case are exactly as in the case of surfaces without cusps.

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