Rotation spectra and exotic group actions on the circle

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ABSTRACT. We develop a general framework for producing uncountable families of exotic actions of certain classically studied groups on the circle. We show that if G is a closed surface group then there are connected components of the nonlinear representation variety $\mathrm{Hom}(G, \mathrm{Homeo}^+(S^1))$ which contain uncountably many semi-conjugacy classes of faithful G–actions, thus complementing a result of K. Mann. We also exhibit non-semi-conjugate actions of mapping class groups of surfaces with boundary on the circle. In the process of establishing these results, we prove general combination theorems for indiscrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ which apply to all finitely generated Fuchsian groups and all limit groups, a topological Baumslag Lemma, and general combination theorems for representations into Baire topological groups. As a corollary, we show that for many of the groups, there exist uncountably many integer-valued subadditive quasi-morphisms which are defect–one and which are linearly independent in $H^2_b(G;\mathbb{Z})$. We also give a mostly self-contained reconciliation of the various notions of semi-conjugacy in the extant literature by showing that they are all equivalent.

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Date: November 1, 2016.

1991 *Mathematics Subject Classification.* 57M05, 37E45 (Primary); 20F65 57S05 (Secondary).

Key words and phrases. Rotation spectrum, limit groups, surface group representation, Baumslag Lemma, Combination Theorem.

1. Introduction

In this paper, we study finitely generated groups which are classically known to act faithfully on the circle. The purpose of this article is to systematically construct uncountable families of actions of these groups which have essentially different dynamics. The tools we develop allow us to construct many *exotic* actions of classically studied groups, i.e. actions which are not semi-conjugate to the "usual" or "standard" actions of these groups.

In the course of developing tools to build exotic group actions, we prove several *combination theorems*, which construct a framework for building dense subgroups of $\mathrm{PSL}_2(\mathbb{R})$,viewed as a subgroup of the group $\mathrm{Homeo}^+(S^1)$ of orientation–preserving homeomorphisms of the circle. Dense subgroups of $\mathrm{PSL}_2(\mathbb{R})$ fall outside the purview of classically studied objects, and to study them we employ methods from representation varieties, dynamics in one dimension, and hyperbolic geometry in two and three dimensions.

Before stating the results of this paper, we adopt in this section the standing assumption that *all actions, unless otherwise stated, are faithful*. Let G be a group and let

$$\rho_1, \rho_2 \colon G \to \mathrm{Homeo}^+(S^1)$$

be two actions. Roughly speaking, we say that ρ_1 and ρ_2 are *semi-conjugate* if there exists a monotone degree one map $h\colon S^1 \to S^1$ such that

$$h \circ \rho_1 = \rho_2 \circ h$$

after lifting the actions to $\mathbb{R}$. The reader is directed to Section 2 for a comprehensive discussion of this definition. We say that ρ_1 and ρ_2 are *equivalent* if there exists an $\alpha \in \mathrm{Aut}(G)$ such that $\rho_1 \circ \alpha$ is semi-conjugate to ρ_2. Equivalence is indeed an equivalence relation. We again direct the reader to Section 2 for more detail. We will say that a group action $\rho\colon G \to \mathrm{Homeo}^+(S^1)$ is *projective* if $\rho(G)$ is conjugate into $\mathrm{PSL}_2(\mathbb{R})$. Note that projective actions are always analytic, but the converse is false in general; see Subsection 1.5.7.

We now summarize the principal results of this paper.

1.1. Combination theorems and indiscrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$.

Let G be a countable group and let

$$\rho\colon G \to \mathrm{PSL}_2(\mathbb{R}) \leqslant \mathrm{Homeo}^+(S^1)$$

be an action of G on the circle. Before stating the main combination theorem, we need to introduce some terminology.

For a group L, we let $T(L)$ denote the set of torsion elements. Let $R \subseteq \mathbb{R}$. We say a representation

$$\lambda\colon L \to \mathrm{PSL}_2(\mathbb{R})$$

is *R–free* if

$$\operatorname{tr}\circ\lambda(L\backslash 1)\cap R=\varnothing.$$

We say that λ is *almost R–free* if

$$\operatorname{tr}\circ\lambda(L\backslash T(L))\cap R=\varnothing.$$

Most of the time, we will assume R contains $\{\pm 2\}$ by enlarging R if necessary.

The concept of R–freeness of a representation is a generalization of the notion of a parabolic-free representation, which in turn is a generalization of a hyperbolic structure without cusps. A projective action $\lambda\colon L\to \mathrm{PSL}(2,\mathbb{R})$ is *parabolic-free* if $\lambda(L\backslash 1)$ does not contain parabolic elements. A nontrivial torsion element in $\mathrm{PSL}_2(\mathbb{R})$ is never parabolic. So if R contains $\{\pm 2\}$, then a faithful, almost R–free representation is parabolic–free.

For

$$\Lambda\subseteq \mathrm{Hom}(L,\mathrm{PSL}_2(\mathbb{R}))$$

and $g\in L$, we denote

$$\Lambda(g)=\{\lambda(g)\mid \lambda\in\Lambda\}.$$

Let

$$p\colon \widetilde{\mathrm{PSL}_2(\mathbb{R})}\to \mathrm{PSL}_2(\mathbb{R})$$

denote the universal covering map. We will say that a representation

$$\lambda\colon L\to \mathrm{PSL}(2,\mathbb{R})$$

is *liftable* if λ factors as

$$L\to \widetilde{\mathrm{PSL}_2(\mathbb{R})}\to \mathrm{PSL}_2(\mathbb{R}).$$

We say a subgroup $L\leqslant \mathrm{PSL}_2(\mathbb{R})$ is liftable if so is the embedding $L\to \mathrm{PSL}_2(\mathbb{R})$.

For a group A and its subgroup C, we denote by $A{*_C}$ the HNN extension of A amalgamated along C with the identity map on C. In other words, we have

$$A{*_C}=A*_C(C\times\langle s\rangle),$$

where here we denote a generator of $\mathbb{Z}$ by s. Recall C is *malnormal* in A if $C\cap C^g=\{1\}$ for all $g\in A\backslash C$. A group is *2-torsion-free* if no elements have order two.

A finitely generated group L is *flexible* (*liftable-flexible*, respectively) if there exists $g_0\in L\backslash T(L)$ such that for each countable subset $R\subseteq\mathbb{R}$ we have a faithful almost R–free (and liftable, respectively) representation $\lambda\colon L\to \mathrm{PSL}_2(\mathbb{R})$ with the property that $\lambda(g_0)$ is elliptic. Let $\mathscr{F}$ and $\mathscr{F}^\sim$ denote the classes of flexible and liftable-flexible groups, respectively.

The following result is the principal combination theorem which we establish in this paper:

Theorem 1.1. *Let $\mathscr{K}=\mathscr{F}$ or $\mathscr{K}=\mathscr{F}^\sim$.*

(1) Infinite cyclic groups are in $\mathscr{K}$.

(2) *If* $H \leqslant G \in \mathscr{K}$ *and* $[G : H] < \infty$*, then* $H \in \mathscr{K}$*.*
(3) *If* A *and* B *are in* $\mathscr{K}$*, then so is* $A * B$*.*
(4) *If* A *is in* $\mathscr{K}$ *and* B *is a finite cyclic group, then* $A * B \in \mathscr{K}$*.*
(5) *Let* $L \in \mathscr{K}$ *and* $A, B, C \leqslant L$ *such that* $C \leqslant A \cap B$*. If* C *is malnormal and maximal abelian in* A *and in* B*, then* $A *_C B \in \mathscr{K}$*.*
(6) *If* $A \in \mathscr{K}$ *and* C *is a malnormal maximal abelian subgroup of* A*, then* $A*_C \in \mathscr{K}$*.*

A relatively straightforward application of Theorem 1.1 above is for a *double* of a flexible group. For a group A and a subgroup C, we denote by $A *_C \bar{A}$ the amalgamation of two copies of A along Id_C.

Corollary 1.2. *Let* A *be a group and let* C *be a malnormal maximal abelian subgroup of* A*. If* A *is flexible or liftable-flexible, then so is* $A *_C \bar{A}$*.*

With some more work, we obtain the following very general result about circle actions of *limit groups*. Recall that a limit group is a finitely generated group which is fully residually free.

Corollary 1.3. *Nontrivial limit groups are liftable-flexible.*

Within the context of groups which "classically" act on the circle through discrete projective actions, we have the following:

Theorem 1.4. (1) *Every nonuniform lattice in* $\mathrm{PSL}_2(\mathbb{R})$ *is flexible.*
(2) *Every torsion-free lattice in* $\mathrm{PSL}_2(\mathbb{R})$ *is liftable-flexible.*

Flexibility is important to us because it allows us to construct many "dynamically distinct" actions of groups on the circle. See the discussion below.

In the course of establishing Theorem 1.1, we prove several other results which are of independent interest. Probably the most important of these are several general versions of Baumslag's Lemma.

Classically, Baumslag's Lemma [5] is a criterion for certifying that elements of free groups do not reduce to the identity, and is fundamental in the study of residually free (i.e. limit) groups. We prove several generalizations of Baumslag's Lemma, which we will not state here for the sake of brevity. These include a topological Baumslag Lemma for group actions on any Hausdorff topological space (Lemma 3.18), a continuous Baumslag Lemma which applies to subgroups of $\mathrm{PSL}_2(\mathbb{C})$ (Lemma 3.19), and a hyperbolic Baumslag Lemma which applies to word–hyperbolic groups and mapping class groups of surfaces (Corollary 3.20). Baumslag's Lemma coupled with a Baire category argument for representation varieties allows us to prove a number of combination theorems for faithful (but possibly indiscrete) representations. The reader is directed to Section 3 for more detail.

Relaxing the hypothesis of a projective action of a group on S^1, we also prove the following combination theorem for smooth actions:

Theorem 1.5. *Let $G, H \leqslant \mathrm{Diff}^\infty(S^1)$ be countable groups consisting of fully supported orientation–preserving C^∞ diffeomorphisms of S^1. Then for a generic choice of $\psi \in \mathrm{Diff}^\infty(S^1)$, we have that*

$$G * H \cong \langle G, H^\psi \rangle \leqslant \mathrm{Diff}^\infty(S^1).$$

Here, a homeomorphism (or diffeomorphism) ψ is *fully supported* if the fixed point set $\mathrm{Fix}\,\psi$ has empty interior. The reader is directed to Theorem 3.30 for a more detailed statement of the result above.

1.2. **Uncountable families of exotic group actions on the circle.** The combination theorems discussed above are the technical tools we use to construct large families of actions of "classical" groups on the circle with essentially different dynamical behaviors, which is to say that the actions are not semi–conjugate to each other.

We distinguish between various semi-conjugacy classes of actions by estimating their *rotation spectra*. Recall that if

$$f \in \mathrm{Homeo}^+(S^1)$$

then f has a well-defined *rotation number* $\mathrm{rot}(f)$, which is given by lifting f to

$$\widetilde{f} \in \mathrm{Homeo}^+(\mathbb{R})$$

and computing the limit

$$\lim_{n\to\infty} \frac{\widetilde{f}^n(0)}{n} \pmod{\mathbb{Z}}.$$

The *rotation spectrum* of an action $\phi \colon G \to \mathrm{Homeo}^+(S^1)$ is given by

$$\Sigma(\phi) := \mathrm{rot}\circ\phi(G) = \{\mathrm{rot}\circ\phi(g) \mid g \in G\} \subseteq \mathbb{R}.$$

One can also define the *marked rotation spectrum* of ϕ by

$$\Sigma^M(\phi) := \mathrm{rot}\circ\phi = \{(g, \mathrm{rot}\circ\phi(g)) \mid g \in G\} \in \mathbb{R}^G.$$

If

$$\Lambda \subseteq \mathrm{Hom}(G, \mathrm{PSL}_2(\mathbb{R}))$$

and $g \in G$, we write

$$\mathrm{rot}\,\Lambda(g) := \{\mathrm{rot}\,\lambda(g) \mid \lambda \in \Lambda\}.$$

Because the rotation number is a semi-conjugacy invariant of a homeomorphism [57, 53], it is clear that the rotation spectrum of an action is an equivalence invariant of an action, and that the marked rotation spectrum is a semi-conjugacy invariant of an action. Note that the authors do not claim to have invented the notion of rotation spectrum; see [17, 35] for instance, as well as the discussion in Subsection 1.5.5 below.

Proposition 1.6. *If G is a countable flexible group, then there exists $g_0 \in G$ and a uncountable set*

$$\Lambda \subseteq \mathrm{Hom}(G, \mathrm{PSL}_2(\mathbb{R}))$$

of faithful parabolic–free representations such that all of the following hold:

(i) for each $\lambda \in \Lambda$, each element of $\Sigma(\lambda)$ is either rational or transcendental;
(ii) the set $\mathrm{rot}\,\Lambda(g_0)$ is $\mathbb{Z}$–linearly independent.
(iii) for all distinct $\lambda, \lambda' \in \Lambda$, we have

$$\Sigma(\lambda) \cap \Sigma(\lambda') \subseteq \mathbb{Q};$$

(iv) for all $g \in G$, and for all distinct $\lambda, \lambda' \in \Lambda$, the elements $\lambda(g_0)$ and $\lambda'(g)$ are not conjugate in $\mathrm{Homeo}^+(S^1)$;
(v) for all elements

$$g, h \in G \backslash T(G)$$

and for all distinct

$$\lambda, \lambda' \in \Lambda,$$

the elements $\lambda(g)$ and $\lambda'(h)$ are not conjugate in $\mathrm{PSL}_2(\mathbb{R})$;

Moreover, if G is liftable-flexible, then we can further require that each representation in Λ is liftable.

In [54], K. Mann shows that if G is a closed surface group and if Z is a component of the representation space

$$X = \mathrm{Hom}(G, \mathrm{Homeo}^+(S^1))$$

which contains a geometric representation, then Z consists of one semi-conjugacy class. It is an open question whether or not X has infinitely many connected components [54]. The representations in

$$\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbb{R}))$$

with Euler number 0 form a connected component [39], and we show that this component contains uncountably many equivalence classes, as follows easily from Proposition 1.6. We thus obtain the following.

Corollary 1.7. *Let G be a closed surface group. Then some component of*

$$\mathrm{Hom}(G, \mathrm{Homeo}^+(S^1))$$

contains uncountably many distinct equivalence classes of faithful actions.

Corollary 1.7 is complemented by work of M. Wolff. See Subsection 1.5.3 below.

An immediate cohomological consequence of Proposition 1.6 is the following:

Corollary 1.8. *If G is liftable-flexible, then there exists a set Λ of integer-valued subadditive defect–one quasimorphisms such that*

$$\{[\lambda] \in \mathrm{HQM}(G;\mathbb{Z}) \mid \lambda \in \Lambda\}$$

is uncountable and linearly independent.

Here,

$$\mathrm{HQM}(G;\mathbb{Z}) = \mathrm{QM}(G;\mathbb{Z})/(B(G;\mathbb{Z}) \oplus H^1(G;\mathbb{Z}))$$

denotes the kernel of the natural map

$$H_b^2(G;\mathbb{Z}) \to H^2(G;\mathbb{Z}).$$

See Section 2 for a more detailed discussion of quasimorphisms, circle actions, and bounded cohomology.

Shifting the focus to some more complicated groups, we have the following complement to the results in this subsection:

Theorem 1.9 (cf. Proposition 7.3)**.** *Let S be a surface of genus $g \geqslant 2$ with one boundary component, and let $x \in S$ be a marked point in the interior of S. Then the mapping class group* $\mathrm{Mod}(S,x)$ *has at least two inequivalent actions on S^1.*

1.3. **An axiomatic approach to combination theorems for representations.** In Section 5 below, we develop an axiomatic framework for combination theorems for group representations where the target group is a general Baire topological group $\mathfrak{G}$. It would have been possible to start with the general axiomatic framework of Section 5 and prove some of the facts proven in Section 4. However, Section 4 deals with a far more concrete setup, and the axioms in Section 5 might seem technical and (overly) abstract to the reader; this is why we have chosen the present expository mode of providing concrete examples in Section 4 motivating the more abstract axiomatic nature of Section 5 that follows.

The notion of a trace is replaced by a more general notion of a *tracial structure*. The definition of a tracial structure (Definition 5.1 below) is somewhat technical, but the most important concepts encoded in a tracial structure are a generalized trace (a continuous function on conjugacy classes) and a generalized notion of parabolic elements. Thus, the generalized trace is simply a continuous map

$$\tau \colon \mathfrak{G} \to \mathscr{S},$$

where $\mathscr{S}$ is a topological space, and where τ is constant on conjugacy classes. The notion of generalized parabolic elements in this situation is defined accordingly:

$$\mathscr{P} = \tau^{-1}(\tau(1)),$$

i.e. the generalized parabolic elements are precisely those whose traces agree with that of the identity with respect to the tracial structure given by τ. Part of the definition of a tracial structure is that parabolic elements are not allowed to be torsion.

To state the axiomatic combination theorem precisely, we need several notions (cf. Definitions 5.4 below). Let $R \subseteq \mathscr{S}$ be countable. We say $L \leqslant \mathfrak{G}$ is *$\mathscr{P}$–free* if $L \cap \mathscr{P} = \varnothing$. A subgroup L of $\mathfrak{G}$ is *almost R–free* if

$$\tau(L \backslash T(L)) \cap R = \varnothing,$$

where here $T(L)$ denotes the set of torsion elements of L. We say a representation is *$\mathscr{P}$–free* or *almost R–free* if its image has the corresponding property. We say a finitely generated group L is *in the class $\mathscr{F}(\mathfrak{G})$* if for each countable subset $R \subseteq \mathscr{S}$ there exists a faithful almost R–free representation $L \to \mathfrak{G}$ with some "elliptic" element in the image; see Definition 5.6 for a precise statement.

Theorem 1.10 (cf. Theorem 5.7 below)**.** *(1) Infinite cyclic groups are in $\mathscr{F}(\mathfrak{G})$.*

(2) If $A \leqslant B \in \mathscr{F}(\mathfrak{G})$, then $A \in \mathscr{F}(\mathfrak{G})$.
(3) If $A, B \in \mathscr{F}(\mathfrak{G})$, then $A * B \in \mathscr{F}(\mathfrak{G})$.
(4) Let A and B be finitely generated nontrivial subgroups of a group L and let C be a malnormal maximal abelian subgroup of both A and B. If $L \in \mathscr{F}(\mathfrak{G})$, then $A *_C B \in \mathscr{F}(\mathfrak{G})$.
(5) Let A be a finitely generated group and let C be a malnormal maximal abelian subgroup of A. If $A \in \mathscr{F}(\mathfrak{G})$, then $A*_C \in \mathscr{F}(\mathfrak{G})$.

The consequence of Theorem 1.10 which is of primary interest to us is the following rather general fact:

Corollary 1.11. *Every nontrivial limit group lies in the class $\mathscr{F}(\mathrm{PSL}_2(\mathbb{R}))$.*

1.4. **Flexibility and rigidity.** Among group actions on the circle with a given rotation spectrum, there appears to be a mixture of rigidity and flexibility, as we illustrate with the following results.

On the one hand, projective representations impose a high degree of rigidity. The following fact is relatively well–known (see for instance Katok's book [45], Theorem 2.5.4):

Proposition 1.12 (cf. Theorem 6.5 and Theorem 6.12)**.** *Let G be a finitely generated group and let ϕ be a faithful projective action of G with $\Sigma(\phi) = \{0\}$. Then $\phi(G)$ is conjugate to a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. In particular, either:*

(1) The group G is the fundamental group of a closed surface S_g of genus $g \geqslant 2$ and ϕ corresponds to a complete hyperbolic structure on S_g; in particular, up to an automorphism of G, there are only two conjugacy classes of such actions;
(2) The group G is free, and there are finitely many equivalence classes of such projective G–actions on S^1, two for each homeomorphism type of a surface S with $\pi_1(S) \cong G$.

On the other hand, even marked rotation spectra do not form a complete equivalence invariant of closed surface group actions:

Theorem 1.13 (cf. Theorem 6.11)**.** *There exists a faithful action ϕ of a torsion-free Fuchsian group G on S^1 with (marked) rotation spectrum $\{0\}$ and such that ϕ is not semi-conjugate to a projective action of G.*

The proof of Theorem 1.13 is remarkable in that it uses several nontrivial facts about hyperbolic 3–manifolds which fiber over the circle. We remark furthermore that Theorem 1.13 considers flexibility of actions within a class of actions with a fixed rotation spectrum. Thus, this result is one of the several in this paper which are completely different from (as opposed to complementary to) the other results in literature which study flexibility and rigidity of actions via rotation numbers (cf. [17, 21, 35, 53], for instance).

K. Mann has pointed out an alternative argument for Theorem 1.13 which appeared in her thesis [51]. Her argument shares certain features with the proof given in this paper, though the crux of her argument (which relies on Thurston norm and maximal Euler classes) and motivations for considering the problem in the first place are different from those of the authors.

Result similar to Theorem 1.13 have also been produced by Barbot–Fenley [3], using pseudo-Anosov flows. See Subsection 1.5.8 below.

The following result exhibits exotic actions of $\mathrm{PSL}_2(\mathbb{Z})$ on the circle, among other exotic group actions:

Theorem 1.14. *For all pairs of distinct integers $n, m > 1$, there exists a faithful C^∞ action of $\mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$ on S^1 which is not semi-conjugate to a discrete projective action of* $\mathrm{PSL}_2(\mathbb{R})$.

If one drops the assumption of faithfulness, even projective representations with equal and nonzero rotation spectra can be pairwise inequivalent.

Proposition 1.15 (cf. Proposition 6.6)**.** *Let $G = \pi_1(S_g)$, where $g \geqslant 2$. Then there exist infinitely many pairwise inequivalent actions of G on S^1 with a fixed rotation spectrum that factor through a fixed faithful projective action of $\pi_1(S_{g-1})$.*

Non–faithful projective actions of surface groups have been studied for their geometric purposes in their own right; see [34].

1.5. **Notes and references.**

1.5.1. *Circle actions and quasi-morphisms.* Among the original motivations for this paper was a question posed to the authors by M. Bestvina and K. Fujiwara, namely whether one can explicitly produce distinct, subadditive, integer–valued, defect–one quasimorphisms of a free group or a surface group. This paper answers their question by producing uncountable families of examples which come from quasimorphisms of the corresponding groups.

It was pointed out to the authors by D. Calegari that producing $\{-1,0,1\}$–coboundary defect–one quasimorphisms from a given quasimorphism is relatively straightforward, so that producing uncountably many linearly independent ones is not such a huge generalization. In this paper however, the quasi-morphisms we produce are subadditive and $\{0,1\}$–coboundary, these two properties being not so trivial to simultaneously guarantee, since such a quasimorphism (as a second bounded cohomology class) corresponds to a semi-conjugacy class of a circle action that lifts to the real line.

1.5.2. *Generalizations to other semi-simple algebraic groups.* Many of the results about $\mathrm{PSL}_2(\mathbb{R})$ which we discuss in this paper generalize suitably to other semi-simple algebraic groups. When discussing algebraic groups, we concentrate on $\mathrm{PSL}_2(\mathbb{R})$ since we are primarily interested in the dynamical picture. Indeed, lattices in non-split higher rank semi-simple linear algebraic groups do not admit any interesting actions on the circle [15, 71, 37]. Thus, while the continuous version of Baumslag's Lemma (see Lemma 3.19 below) admits a generalization to other algebraic groups, for instance, we avoid making such generalizations for the sake of brevity and clarity, especially since the dynamical consequences would be rather limited for us. We reiterate, however, that this paper develops a representation–theoretic perspective for studying dynamics in low dimensions which fits into a larger context of indiscrete subgroups of semi-simple algebraic groups.

1.5.3. *Towards a Teichmüller Theory for indiscrete representations.* Most classical constructions of free group actions on various spaces, and more generally of free product actions, rely on some version of the ping–pong lemma [26, 68, 48]. The novelty of the approach given here is in the construction of dense subgroups of $\mathrm{PSL}_2(\mathbb{R})$ with various exotic properties, which are situations to which ping–pong is unadaptable. We note that, even though we are concentrating on dense subgroups of $\mathrm{PSL}_2(\mathbb{R})$, our constructions are generally informed by and often rely on classical discrete subgroup constructions.

The rotation spectrum of a representation is an equivalence invariant of the representation, which plays a role similar to the length spectrum for a discrete representation of a group G into $\mathrm{PSL}_2(\mathbb{R})$. Classical Teichmüller Theory gives conditions under which length spectra determine a surface group representation up to conjugacy (see [62], for instance). In the situation of indiscrete actions, the length spectrum for hyperbolic elements in the image will be dense inside of $\mathbb{R}^+$, and it seems likely that rotation spectrum alone will not determine the semi-conjugacy class of a faithful action:

Question 1.16. *Let G be a closed surface group and let ρ_1, ρ_2 be projective actions of G on S^1. If $\Sigma(\rho_1) = \Sigma(\rho_2)$, under what conditions is ρ_1 is equivalent to ρ_2?*

We make three further remarks about rigidity of actions. First, there is a sense in which the marked rotation spectrum (with some extra data) does in fact determine a circle action up to semi-conjugacy, which is given by Matsumoto's Theorem [57]. We will give an exact statement of this result in Theorem 2.15.

Secondly, we briefly consider actions of $\mathrm{PSL}_2(\mathbb{Z})$ on S^1 which "look like" the standard one. Of course any finite order element in $\mathrm{PSL}_2(\mathbb{Z})$ will have nonzero rotation number under any action on S^1, so that the rotation spectrum of any such action is nonzero. However, one may consider actions of $\mathrm{PSL}_2(\mathbb{Z})$ where all infinite order elements have rotation number zero. The following is a corollary of the work of Matsuda in [56]:

Theorem 1.17. *Up to an automorphism of* $\mathrm{PSL}_2(\mathbb{Z})$*, there is only one conjugacy class of projective actions of* $\mathrm{PSL}_2(\mathbb{Z})$ *on S^1 such that all infinite order elements have zero rotation number.*

Thirdly, the work of M. Wolff gives an answer to Question 1.16 in the case of marked rotation spectra:

Theorem 1.18 (M. Wolff [72], see also [73])**.** *Let G be a group and let*

$$\rho_1, \rho_2 \in \mathrm{Hom}(G, \mathrm{PSL}_2(\mathbb{R})).$$

Suppose that ρ_1 is non–elementary and indiscrete, and suppose that for all $g \in G$ we have

$$\mathrm{rot}(\rho_1(g)) = \mathrm{rot}(\rho_2(g)).$$

Then ρ_1 and ρ_2 are conjugate in $\mathrm{PSL}_2(\mathbb{R})$.

In the case of unmarked rotation spectra, it seems that relatively few properties of a projective group action are determined by the unmarked rotation spectrum. See [55].

1.5.4. *Dense limit subgroups of algebraic groups.* The work in Section 3 is closely related to, and partially inspired by, the work of Breuillard–Gelander–Souto–Storm [12] and of Barlev–Gelander [4] in which those authors studied the relationship between dense free subgroups and dense limit subgroups of semi-simple algebraic groups. In the case of $\mathrm{PSL}_2(\mathbb{R})$ actions, we show not only the existence of dense non–free limit subgroups, (thus recovering the main results of [12] and [4] in this case) but in fact an enormous abundance of such subgroups, as witnessed by our constructions of uncountable families of representations which are not only pairwise non–conjugate in $\mathrm{PSL}_2(\mathbb{R})$, but which give rise to pairwise inequivalent actions on the circle.

Perhaps the most significant conceptual difference between the present work and that of [12] and of [4] is the form in which Baumslag lemmas are stated, proven and used. In [12] and [4], the fundamental combinatorial group theory tool is the Baumslag lemma for free groups. In the present work, we use continuous Baumslag lemmas adapted to the ambient Lie group, and topological Baumslag lemmas adapted to more general setups in which north–south dynamics is present. It is thus that we are able to apply our machinery to a larger array of groups in order to deduce the dynamical consequences outlined above.

1.5.5. *Relationship to the work of Calegari and Calegari–Walker.* Various rigidity and flexibility phenomena concerning rotation numbers for groups acting on the circle have been studied by many authors as we have indicated above, and will indicate further below. As for the ideas of this paper which concern rotation numbers, the most closely related work is probably the work of Calegari in [17] and Calegari–Walker in [21].

In [21], Calegari and Walker study actions of the rank two free group F_2 on the circle, and the various rigidity and rationality phenomena that can be described therein. For instance, if two free generators act with rational rotation numbers r and s then the supremum $R(w, r, s)$ of the possible rotation numbers achieved by each fixed positive word $w \in F_2$ in those generators is again rational. Moreover, the denominator of $R(w, r, s)$ is well–controlled and computable. The value $R(w, r, s)$ is locally constant from the right for a positive w and $r, s \in \mathbb{Q}$.

In [17], Calegari considers (among other things) the following problem: let $\alpha \in \mathbb{R}/\mathbb{Z}$. Is there a finitely presented group G acting on S^1 and a $g \in G$ such that $\operatorname{rot}\rho(g) = \{\pm\alpha, 0\}$ as ρ varies over all possible actions of G on the circle? If the answer is yes then α is called a *forceable number*. Calegari shows that the set of forceable numbers is very large: lifting the set of forceable numbers to $\mathbb{R}$, one obtains an infinite dimensional rational vector subspace of $\mathbb{R}$. Moreover, the set of forceable numbers contains infinitely many algebraically independent transcendental numbers.

Perhaps the most significant difference between this work and that of Calegari [17] is the fact that in the present work we study possible rotation numbers within the context of actions of a single group, i.e. we do not pretend to consider all actions of a given finitely presented group on the circle. Our purpose is the opposite, in fact: we use large diversity in rotation numbers to deduce the existence of many actions.

Moreover, as noted above, we expend a considerable effort to produce inequivalent actions of groups on the circle which are not distinguished by rotation numbers, which is to say flexibility of actions with a fixed rotation spectrum.

1.5.6. *Dense sets of faithful projective surface group actions.* In [27], Deblois and Kent show that for a closed surface S_g, the set of faithful representations inside the representation variety

$$\operatorname{Hom}(\pi_1(S_g), \mathrm{PSL}_2(\mathbb{K}))$$

is dense in the classical analytic topology, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$. In fact, they show that the set of faithful representations is a dense G_δ in the classical topology. In their proof, they even use a fact which may be regarded as a transcendental/number theoretic Baumslag Lemma in order to establish the properness of vanishing loci in the representation variety, though the authors do not call it by that name. The work of Deblois–Kent in the case of $\mathbb{K} = \mathbb{R}$ can be used to construct many non–conjugate actions on the circle (see [52]), though questions of dynamical independence of these actions are not addressed.

The purpose of Deblois and Kent was to prove a conjecture of W. Goldman, namely that the set of faithful representations is dense in the representation variety. As such, dynamical considerations did not enter into their discussion. On the one hand, the present work builds a more general framework for studying exotic group actions on the circle. As a consequence of developing such a framework, we deduce the existence of an abundance of exotic circle actions, not just for surface groups but also for free groups, limit groups, and many other groups beyond the scope of the methods used in [27]. On the other hand, this paper does not recover the Deblois–Kent result, since their considerations apply to all components of the representation variety, whereas the topology of the representation variety is of secondary importance to us.

Deblois–Kent's methods are geometric in the sense that they rely essentially on the geometrization of surfaces. Our approach is a combination of a representation–theoretic and an algebro–geometric approach which is informed by ideas from geometry. Thus, we are able to deduce dynamical consequences for groups beyond those appearing in two–dimensional hyperbolic geometry. The particular features of our work which do not fall under the purview of [27] include but are not limited to combination theorems for indiscrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$, general Baumslag Lemmas, and applicability to limit groups. Moreover, our discussion of dynamical independence via flexibility (cf. Section 4 below) does not follow formally from Deblois–Kent. Finally, while the present article focuses on projective actions of groups, a large part of the paper concerns general smooth and even topological actions of groups on the circle. The following question attempts a natural generalization Deblois–Kent result in our setting:

Question 1.19. *Let G be a nonabelian limit group. Is the set of faithful parabolic-free projective representations of G very general in* $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbb{R}))$*?*

In Question 1.19, the terminology *very general* is a certain Baire category condition. See Subsection 3.4 for a precise discussion.

1.5.7. *Projective actions versus analytic actions.* Questions similar to those addressed in this paper are studied in [1] in the context of analytic group actions on the circle. A group action on the circle is called *analytic* if every element acts by a homeomorphism which is locally given by a convergent power series in the local parameter. The group $\mathrm{PSL}_2(\mathbb{R})$ acts on the circle by analytic diffeomorphisms, though not every analytic action of a group corresponds to a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. In [1], the authors of that paper define a combinatorial invariant called a *topological skeleton*, which plays the role of a Markov partition, and which allows them to classify virtually free group actions on the circle which are analytic.

Among the consequences of the main results of [1] are certain rigidity results for semi-conjugacy of actions with a minimal exceptional set. They also obtain results which are related to Theorem 1.13. Namely, they produce certain *locally discrete* finitely generated free groups of analytic diffeomorphisms of S^1 which have exotic dynamics. For example, they produce a two–generated subgroup of analytic diffeomorphisms which acts minimally, which is free of rank two, where one generator acts like a hyperbolic element of $\mathrm{PSL}_2(\mathbb{R})$, and where one generator has two parabolic fixed points. In particular, such an action cannot be semi-conjugate to a projective action. The authors show that the rotation spectrum of the actions they consider are finite.

Thus, the authors of [1] produce examples of a similar ilk to those furnished in Theorem 1.13, in the case of free groups. Their methods are very different from those employed in this paper, and they do not extend to analytic closed surface group actions.

1.5.8. *Non–Fuchsian exotic actions.* Groups which act on the circle in ways that resemble Fuchsian group actions but which are not projective actions (as in Theorem 1.13) have been studied by several authors, as has been pointed out by M. Triestino. The ideas find their origin in Thurston [67], which were then clarified and further developed by Brooks in his appendix to Bott's article [8], and then further studied by Tsuboi [69].

The work in [1] suggests that locally discrete analytic virtually free group actions should be semi-conjugate into a certain overgroup studied by Tsuboi in [69]. As discussed above, other examples of non–Fuchsian exotic actions appear in [51] and [3].

1.5.9. *Groups without exotic actions.* There are many groups which naturally act on the circle with various degrees of regularity, but which do not admit any (or at least very few) "exotic" actions. Precisely, there are finitely generated groups

$$G < \mathrm{Diff}^k(S^1) < \mathrm{Homeo}^+(S^1)$$

such that if $k \gg 0$ (generally $k \geqslant 2$ suffices), then any two faithful C^k actions on G on $\mathrm{Homeo}^+(S^1)$ have periodic points and are therefore semi-conjugate to actions of finite cyclic groups. Natural classes of groups with such properties lie in the class of right-angled Artin groups. For an explicit example, one can consider groups of the form $G = A \times B$, where A and B are both nontrivial free products of free abelian groups. Standard applications of Denjoy–Koppell type arguments show that any faithful C^2 action of G has a periodic point. We remark that in general, right-angled Artin groups admit no faithful actions on the circle of C^2 or higher regularity (see [2], also [30, 43]).

It seems that the degree of regularity one requires of group actions can have a significant effect on the existence or nonexistence of exotic actions. For instance, note that every right-angled Artin group occurs as a subgroup of the mapping class group of some surface with boundary (see [49] and the references therein), and these mapping class groups do admit exotic actions (see Section 7 below), but no faithful mapping class group action is conjugate to a C^2 action by [2].

1.6. **Outline of the paper.** We have striven to make the present article as self–contained as possible. In the preliminaries and the appendix (Sections 2 and A below) we gather various cohomological facts about circle actions, where an object of central importance is the bounded Euler class of an action. We single out Theorem 2.17 for the attention of the reader, since it is difficult to find a definitive statement of the equivalence of all the various definitions of semi-conjugacy in the literature.

Section 3 contains the main technical tools of the paper, including the various incarnations of Baumslag's Lemma, a discussion of flexibility and liftability, and the combination theorems.

Section 4 produces uncountable families of inequivalent Fuchsian and nontrivial limit group actions on the circle, all arising from indiscrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$, and relying on the work in Section 3.

Section 5 develops a more axiomatic setup for the discussion in Section 3 and in Section 4, and establishes indiscrete combination theorems for general limit groups and in the setup where the target group is a general Baire topological group.

Section 6 addresses the Question 1.16, especially in the case of zero rotation spectrum. We produce torsion-free Fuchsian group actions on the circle which are not semi-conjugate to any projective action of these groups.

Section 7 briefly addresses mapping class group actions and exhibits inequivalent actions of mapping classes on the circle.

The logical dependence of the sections is as follows. Section 2 is basic but can be skipped by readers familiar with circle actions and bounded cohomology. Section A complements Section 2 by giving details and proofs. Section 4 relies on Section 3. Section 5 is logically independent from the other sections, but is conceptually informed by Sections 3 and 4. Section 6 relies on some terminology and ideas from Section 4, but is largely independent. Section 7 is independent.

2. Preliminaries on semi-conjugacy

Ghys employed cohomological methods to “study the dynamics” [36] of discrete group actions on the circle. More precisely, he defined a relation between group actions he called *semi-conjugacy* and exhibited a correspondence between cohomology classes and semi-conjugacy classes; see Theorem 2.13 and 2.19.

Recently, several other closely related notions of semi-conjugacy were suggested in the literature [19, 14, 53]. These modifications were aimed at converting semi-conjugacy into an equivalence relation and further, to have a powerful characterization of circle actions via cohomology classes. For a complete reference on bounded cohomology of discrete groups, the reader is directed to Frigerio’s recent book [33], which includes a discussion of second bounded cohomology and circle actions.

Since we will be needing these different notions of semi-conjugacy in different contexts, we summarize the relevant notions here and refer the reader to Section A for proofs of equivalence and references. A reconciliation of these notions is summarized in Theorem 2.17 below. A different and more detailed exposition of many (but not all) of these equivalent notions has been given in [14]. The ideas go back to Ghys’ original article [36], and to [19, 14, 53]. The reader is also directed to [16].

Throughout this section, *we let G be a countable group*. For $r \in \mathbb{R}$, we let $T(r)$ denote the translation by r on $\mathbb{R}$ or on S^1. For brevity, we denote

$$T = T(1)\colon \mathbb{R} \to \mathbb{R}.$$

Let us write $\mathscr{H} = \mathrm{Homeo}^+(S^1)$ and

$$\tilde{\mathscr{H}} = \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) = \{f \in \mathrm{Homeo}_+(\mathbb{R})\colon fT = Tf\}.$$

There exists a universal central extension

$$1 \to \langle T \rangle \to \tilde{\mathscr{H}} \to \mathscr{H} \to 1.$$

Circle homeomorphisms

Definition 2.1. For $f \in \tilde{\mathscr{H}}$, we define the *translation number*

$$\mathrm{trans}(f) = \lim_{n\to\infty} \frac{f^n(x)}{n},$$

which is independent of the choice of $x \in \mathbb{R}$. The Poincaré’s *rotation number* of $f \in \mathscr{H}$ is defined as

$$\mathrm{rot}(f) = \mathrm{tran}(\tilde{f}) \pmod{\mathbb{Z}}$$

where $\tilde{f} \in \tilde{\mathscr{H}}$ is an arbitrary lift of f.

Then $\mathrm{rot}\colon \mathscr{H} \to S^1$ is continuous with respect to the usual (uniform) topology on $\mathscr{H}$. Moreover, we have

$$\mathrm{rot}(f^n) = n\,\mathrm{rot}(f)$$

and

$$\mathrm{rot}(f^g) = \mathrm{rot}(f)$$

for $f, g \in \mathscr{H}$ and $n \in \mathbb{Z}$. The readers may refer to [61] for more details.

Definition 2.2. A map $h\colon \mathbb{R} \to \mathbb{R}$ is *monotone degree one (on $\mathbb{R}$)* if h is monotone increasing and if

$$h \circ T = T \circ h.$$

We say $h\colon S^1 \to S^1$ is *monotone degree one (on S^1)* if h is the projection of some monotone degree one map on $\mathbb{R}$.

Remark 2.3. The constant map $c\colon x \mapsto 0$ on S^1 is monotone degree one (projection of $x \mapsto \lfloor x \rfloor$), and continuous. However, c does not have a continuous monotone degree one lift. A *surjective* monotone degree one map on S^1 is continuous and has a continuous lift to $\mathbb{R}$.

Let $M = S^1$ or $M = \mathbb{R}$ and

$$\rho_1, \rho_2 \in \mathrm{Hom}(G, \mathrm{Homeo}_+(M)).$$

We say ρ_1 *dominates* ρ_2 and write $\rho_1 \succcurlyeq \rho_2$ if there exists a (not necessarily continuous) monotone degree one map $h\colon M \to M$ such that the following diagram commutes for each $g \in G$:

$$\begin{array}{ccc} M & \xrightarrow{\rho_1(g)} & M \\ \big\downarrow h & & \big\downarrow h \\ M & \xrightarrow{\rho_2(g)} & M. \end{array}$$

To emphasize the *dominating map h*, we also say ρ_1 *dominates* ρ_2 *by h* and write $\rho_1 \succcurlyeq_h \rho_2$.

Semi-conjugacy

The following definition is due to Takamura [66] and also to Bucher–Frigerio–Hartnick [14].

Definition 2.4. Let

$$\rho_1, \rho_2 \in \mathrm{Hom}(G, \mathscr{H}).$$

We say ρ_1 and ρ_2 are *symmetrically semi-conjugate*, and write

$$\rho_1 \sim_{\mathrm{sym}} \rho_2,$$

if $\rho_1 \succcurlyeq \rho_2$ and $\rho_2 \succcurlyeq \rho_1$.

It is immediate that $\sim_{\mathrm{sym}}$ is an equivalence relation on $\mathrm{Hom}(G, \mathscr{H})$. Calegari–Dunfield defined another equivalence relation on $\mathrm{Hom}(G, \mathscr{H})$:

Definition 2.5 ([20, 19] and [61, Remark 2.1.4])**.** Let

$$\rho_1, \rho_2 \in \mathrm{Hom}(G, \mathscr{H}).$$

We say ρ_1 and ρ_2 are *monotone equivalent*, and write

$$\rho_1 \sim_{\mathrm{mono}} \rho_2,$$

if there exists $\rho_0 \in \mathrm{Hom}(G, \mathscr{H})$ and *surjective* monotone degree one maps

$$h_1, h_2 \colon S^1 \to S^1$$

such that $\rho_0 \succcurlyeq_{h_i} \rho_i$ for $i = 1, 2$.

In Theorem 2.17, we will see that the monotone equivalence indeed defines an equivalence relation. Mann [53] gave yet another definition of semi-conjugacy, based on [38].

Definition 2.6. We say that

$$\rho_1, \rho_2 \in \mathrm{Hom}(G, \mathscr{H})$$

are $\mathbb{R}$*–semi-conjugate*, and write $\rho_1 \sim_{\mathbb{R}} \rho_2$, if there exists a central extension

$$1 \to \mathbb{Z} \to \tilde{G} \to G \to 1$$

and two representations

$$\tilde{\rho}_i \colon \tilde{G} \to \tilde{\mathscr{H}}$$

such that $\tilde{\rho}_1 \succcurlyeq \tilde{\rho}_2$ and moreover, the following diagram commutes for $i = 1$ and 2:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & \searrow & \big\downarrow{\scriptstyle 1 \mapsto T} & & \big\downarrow{\scriptstyle \tilde{\rho}_i} & & \big\downarrow{\scriptstyle \rho_i} & \nearrow & \\
 & & \langle T \rangle & \longrightarrow & \tilde{\mathscr{H}} & \longrightarrow & \mathscr{H} & &
\end{array}$$

Common minimalization à la Ghys

There is yet another different definition of an equivalence relation using the notion of *minimal actions*. This relation is based on the ideas in [38, Theorem 6.6] and [53, Definition 5.8]. Recall an action

$$\rho \in \mathrm{Hom}(G, \mathscr{H})$$

is called *minimal* if all the orbits of $\rho(G)$ are dense. Every action

$$\rho \in \mathrm{Hom}(G, \mathscr{H})$$

is either minimal, has a finite orbit, or leaves invariant a Cantor set; see [61, Theorem 2.1.1].

Definition 2.7. We say $\bar{\rho} \in \mathrm{Hom}(G, \mathscr{H})$ is a *minimalization* of $\rho \in \mathrm{Hom}(G, \mathscr{H})$ if one of the following holds:

(i) $\rho(G)$ has a finite orbit and $\bar{\rho} = T \circ \mathrm{rot} \circ \rho$;
(ii) there exists a surjective monotone degree one map $h\colon S^1 \to S^1$ such that $\rho \geqslant_h \bar{\rho}$ for some minimal representation $\bar{\rho} \in \mathrm{Hom}(G, \mathscr{H})$.

Remark 2.8. In (i), we see $\bar{\rho}(G)$ is a finite cyclic subgroup of $\mathrm{SO}(2, \mathbb{Q})$. In particular, $\bar{\rho}$ in this case is not minimal. So (i) and (ii) are mutually exclusive.

Definition 2.9. We say ρ_1 and ρ_2 are *minimalization equivalent*, and write $\rho_1 \sim_{\min} \rho_2$, if ρ_1 and ρ_2 have a common minimalization.

Lemma 2.10. *Let*
$$\rho_1, \rho_2\colon G \to \mathrm{Homeo}^+(S^1)$$
be minimal actions of G*. Then* ρ_1 *and* ρ_2 *are* $\mathbb{R}$*-semi-conjugate if and only if they are conjugate.*

This lemma is well-known; see [53] for instance. We also give a proof of this lemma in Appendix (Lemma A.4).

Bounded Euler classes

Let $p \in \mathbb{R}$. There exists a unique section $s^p\colon \mathscr{H} \to \tilde{\mathscr{H}}$ satisfying the condition that
$$s^p(f)(p) \in [p, p+1)$$
for all $f \in \mathscr{H}$. We can define the *Euler cocycle based at* p, denoted as
$$\mathrm{eu}^p\colon \mathscr{H} \times \mathscr{H} \to \{0, 1\},$$
by the following condition:
$$s^p(f)s^p(g) = T(\mathrm{eu}^p(f, g))s^p(fg).$$

It is a routine exercise to see that eu^p is a bounded two–cocycle on $\mathscr{H}$, taking values in $\{0, 1\}$. Moreover,
$$\mathrm{eu} = [\mathrm{eu}^p] \in H^2(\mathscr{H}; \mathbb{Z})$$
and
$$\mathrm{eu}_b = [\mathrm{eu}^p] \in H_b^2(\mathscr{H}; \mathbb{Z})$$
are independent of the choice of $p \in \mathbb{R}$. The classes eu and eu_b are called the *Euler class* and the *bounded Euler class*, respectively. We often write $s = s^0$ and $\mathrm{eu} = \mathrm{eu}^0$. The class
$$\mathrm{eu} \in H^2(\mathscr{H}; \mathbb{Z})$$
corresponds to the universal central extension of $\mathscr{H}$. Also the natural map
$$H_b^2(\mathscr{H}; \mathbb{Z}) \to H^2(\mathscr{H}; \mathbb{Z})$$
sends eu_b to eu.

We note rotation numbers can be computed from Euler cocycles:

Lemma 2.11. *For $f \in \mathscr{H}$, we have*

$$\operatorname{rot}(f) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \operatorname{eu}(f, f^k) \pmod{\mathbb{Z}}.$$

Proof. For $f, g \in \mathscr{H}$, observe that $\operatorname{eu}(f, g) = 1$ if and only if

$$s(f) \circ s(g)(0) \geqslant 1.$$

Hence we have the following:

$$\operatorname{trans}(s(f)) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \left|(s(f)^k(0), s(f)^{k+1}(0)] \cap \mathbb{Z}\right| = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \operatorname{eu}(f, f^k). \quad \square$$

Euler classes of representations

For each representation $\rho \colon G \to \mathscr{H}$, we define a two-cocyle

$$\rho^* \operatorname{eu} \colon \mathscr{H} \times \mathscr{H} \to \{0, 1\}$$

by the relation

$$\rho^* \operatorname{eu}(a, b) = \operatorname{eu}(\rho(a), \rho(b)).$$

The following is essentially due to Poincaré.

Lemma 2.12 (Poincaré)**.** *Let $\rho_1, \rho_2 \in \operatorname{Hom}(G, \mathscr{H})$. If*

$$\rho_1^* \operatorname{eu}_b = \rho_2^* \operatorname{eu}_b,$$

then

$$\operatorname{rot}(\rho_1(g)) = \operatorname{rot}(\rho_2(g))$$

for every $g \in G$.

Proof. We have a bounded map $\beta \colon G \to \mathbb{Z}$ such that

$$\rho_1^* \operatorname{eu} = \rho_2^* \operatorname{eu} + \partial \beta$$

as group cocycles. From Lemma 2.11, we have for each $g \in G$:

$$\begin{aligned}
\operatorname{rot}(\rho_1 g) &= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \operatorname{eu}(\rho_1 g, \rho_1 g^k) = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \rho_1^* \operatorname{eu}(g, g^k) \\
&= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \left(\rho_2^* \operatorname{eu}(g, g^k) + \beta(g) + \beta(g^k) - \beta(g^{k+1})\right) \\
&= \lim_n \left(\frac{1}{n} \sum_{k=0}^{n-1} \rho_2^* \operatorname{eu}(g, g^k) + \frac{\beta(1) - \beta(g^n)}{n} \right) + \beta(g) \\
&= \operatorname{rot}(\rho_2(g)) \pmod{\mathbb{Z}}. \qquad \square
\end{aligned}$$

Recall the *exponent* of g in a group is the smallest positive integer N such that $g^N = 1$. We note the following from the appendix; see Lemma A.5.

Lemma 2.13. *Let $\rho \in \mathrm{Hom}(G, \mathscr{H})$. Then ρ has a finite orbit of cardinality $N < \infty$ if and only if the exponent of $\rho^*\mathrm{eu}_b$ is N in $H^2_b(G;\mathbb{Z})$.*

The canonical Euler cocycle

Definition 2.14 ([57])**.** For $f, g \in \mathscr{H}$ and their arbitrary lifts $\tilde{f}, \tilde{g} \in \tilde{\mathscr{H}}$, we write the *canonical Euler cocycle*

$$\tau(f, g) = \mathrm{tran}(\tilde{f}\tilde{g}) - \mathrm{tran}(\tilde{f}) - \mathrm{tran}(\tilde{g}).$$

The canonical Euler cocycle is independent of the choice of the lifts. Moreover τ is a representative cocycle for the image $\mathrm{eu}_b^{\mathbb{R}}$ of eu_b under the map

$$H^2_b(\mathscr{H};\mathbb{Z}) \to H^2_b(\mathscr{H};\mathbb{R}).$$

Theorem 2.15. [57] *Let G be a group generated by a set S and let ρ_1, ρ_2 be actions of G on S^1. Then $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b$ if and only if:*

(i) we have $\mathrm{rot}(\rho_1(s)) = \mathrm{rot}(\rho_2(s))$ for all $s \in S$, and

(ii) for each pair of elements $g, h \in G$, we have $\tau(\rho_1(g), \rho_1(h)) = \tau(\rho_2(g), \rho_2(h))$.

The following is the converse of Lemma 2.12.

Lemma 2.16. *Let $\rho_1, \rho_2 \in \mathrm{Hom}(G, \mathscr{H})$, where $G = \langle f \rangle \cong \mathbb{Z}$. If*

$$\mathrm{rot} \circ \rho_1(f) = \mathrm{rot} \circ \rho_2(f),$$

then $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b$.

Proof. For $f \in \mathscr{H}$ and $n \in \mathbb{Z}$, we have

$$s(f^n) = T\left(-\lfloor s(f)^n(0) \rfloor\right) \circ s(f)^n.$$

So we see the following:

$$\mathrm{trans}(s(f^n)) = \mathrm{trans}(s(f)^n) - \lfloor s(f)^n(0) \rfloor = n \cdot \mathrm{trans}(s(f)) - \lfloor s(f)^n(0) \rfloor,$$
$$\mathrm{trans}(s(f^m)s(f^n)) = (m+n) \cdot \mathrm{trans}(s(f)) - \lfloor s(f)^m(0) \rfloor - \lfloor s(f)^n(0) \rfloor.$$

It follows immediately that $\tau(f^m, f^n) = 0$ for all $f \in \mathscr{H}$ and $m, n \in \mathbb{Z}$. Hence

$$\rho_1^*\tau = \rho_2^*\tau = 0,$$

so that we can apply Theorem 2.15. □

The equivalence

The following Theorem reconciles the various notions of semi-conjugacy. See Section A for a proof.

Theorem 2.17. *Let G be a countable group. For two representations*

$$\rho_1, \rho_2 \in \mathrm{Hom}(G; \mathscr{H}),$$

the following are all equivalent.

(1) *ρ_1 and ρ_2 are symmetrically semi-conjugate.*
(2) *ρ_1 and ρ_2 are monotone equivalent.*
(3) *ρ_1 and ρ_2 are $\mathbb{R}$-semi-conjugate.*
(4) *ρ_1 and ρ_2 are minimalization equivalent.*
(5) *ρ_1 and ρ_2 have the same bounded Euler classes.*
(6) *ρ_1 and ρ_2 have the same canonical Euler cocycles, and for some generating set S of G, we have $\mathrm{rot}(\rho_1(s)) = \mathrm{rot}(\rho_2(s))$ for every $s \in S$.*

We will thus say that ρ_1 and ρ_2 are *semi-conjugate* if one of the equivalent conditions in Theorem 2.17 holds. We then write $\rho_1 \sim_s \rho_2$.

Corollary 2.18. *If $\rho_1 \sim_s \rho_2$, then $\mathrm{rot} \circ \rho_1 = \mathrm{rot} \circ \rho_2$ as maps.*

Bounded cohomology and quasi-morphisms

Let us define $H^2_{\{0,1\}}(G; \mathbb{Z})$ as the set of classes in $H^2(G; \mathbb{Z})$ which can be represented by $\{0, 1\}$–valued cocycles. For

$$\rho \in \mathrm{Hom}(G, \mathrm{Homeo}_+(S^1)),$$

we have

$$\rho^* \mathrm{eu}_b \in H^2_{\{0,1\}}(G; \mathbb{Z}).$$

Theorem 2.19 ([37, 38])**.** *There exists a one-to-one correspondence:*

$$\mathrm{Hom}(G, \mathscr{H})/\sim_s \to H^2_{\{0,1\}}(G; \mathbb{Z}),$$

given by

$$[\rho] \mapsto \rho^* \mathrm{eu}_b.$$

The fact that the map in Theorem 2.19 is well-defined follows from Theorem 2.17.

For a group G, let $\mathrm{QM}(G; \mathbb{Z})$ denote the abelian group of integer–valued quasi-morphisms on G. Define

$$\mathrm{HQM}(G; \mathbb{Z}) = \mathrm{QM}(G; \mathbb{Z}) / \left(B(G; \mathbb{Z}) \oplus H^1(G; \mathbb{Z})\right)$$

where $B(G; \mathbb{Z})$ consists of bounded maps $G \to \mathbb{Z}$.

The *defect* $D(\alpha)$ of $\alpha \in \mathrm{QM}(G; \mathbb{Z})$ is defined by

$$D(\alpha) = \sup |\partial \alpha| = \sup\{|\alpha(g) + \alpha(h) - \alpha(gh)| : g, h \in G\}.$$

So if $\alpha \in \mathrm{QM}(G; \mathbb{Z})$ has defect one, then α has the smallest possible defect among nontrivial integral quasimorphisms. Recall a map $f : G \to \mathbb{Z}$ is *subadditive* if

$$f(xy) \leqslant f(x) + f(y)$$

for all $x, y, \in G$. For example, the map $n \mapsto \lceil \pi n \rceil$ is a defect–one subadditive quasimorphism $\mathbb{Z} \to \mathbb{Z}$.

We say a representation

$$\rho \in \mathrm{Hom}(G, \mathscr{H})$$

is *liftable* if ρ factors as

$$G \to \tilde{\mathscr{H}} \to \mathscr{H}.$$

This is equivalent to the condition

$$\rho^* \mathrm{eu}_b \in \ker(H^2_b(G;\mathbb{Z}) \to H^2(G;\mathbb{Z})).$$

If ρ is liftable, then we see that $\rho^*\mathrm{eu} = \partial\alpha$ as cocycles for some map $\alpha\colon G \to \mathbb{Z}$. Since $\rho^*\mathrm{eu}$ is $\{0, 1\}$-valued, we see that

$$\partial\alpha(x, y) = \alpha(x) + \alpha(y) - \alpha(xy) \in \{0, 1\}.$$

In other words, α is an integral defect–one subadditive quasimorphism. Note

$$[\alpha] \in \mathrm{HQM}(G;\mathbb{Z})$$

maps to $\rho^*\mathrm{eu}_b$ by the well-known exact sequence [19]:

$$0 \to \mathrm{HQM}(G;\mathbb{Z}) \to H^2_b(G;\mathbb{Z}) \to H^2(G;\mathbb{Z}).$$

By Theorem 2.19, we have the following.

Lemma 2.20. *If two non-semi-conjugate representations*

$$\rho_1, \rho_2 \in \mathrm{Hom}(G, \mathbb{Z})$$

are liftable, then they correspond to distinct subadditive defect-one quasimorphisms in $\mathrm{HQM}(G;\mathbb{Z})$.

3. Combination theorems for faithful indiscrete representations

This section deals with one of the principal technical tools of the paper, namely combination theorems for faithful representations. We shall be concerned with building faithful representations of free products with amalgamation $G = A *_C B$, or HNN extensions $A*_C$ from faithful representations of A, B. The representations we build are typically indiscrete. The following are some of the main tools in this section:

(1) A genericity or *Baire category argument* in Section 3.4, which makes precise the notion of a property satisfied by a generic point in a real or complex algebraic variety. This is applied later to representation varieties.

(2) A topological Baumslag Lemma in Section 3.6, which gives sufficient conditions to guarantee nontriviality of a word

$$w(t_1, \cdots, t_k) = g_1\mu_1(t_1)\cdots g_k\mu_k(t_k)$$

for large values of the parameters t_i. Here the $\mu_j(t_j)$'s are one-parameter subgroups of a continuous (possibly analytic) group. The proof is quite general and reminiscent of Tits' proof [68] of the Tits' alternative for discrete linear groups in that the underlying idea consists of a ping-pong argument.

(3) The one-parameter subgroups occurring in Section 3.6 are often (generalizations of) parabolic and hyperbolic one-parameter subgroups of $\mathrm{PSL}_2(\mathbb{R})$. To handle elliptic subgroups we use a complexification and Zariski density trick by embedding $\mathrm{PSL}_2(\mathbb{R})$ in $\mathrm{PSL}_2(\mathbb{C})$ and reduce the elliptic case to the hyperbolic one (Lemma 3.19).

(4) The goal of this section is the set of combination theorems in Section 3.8. The three ingredients above are put together here. We use in an essential way, the algebro-geometric structure of representation varieties to establish that a generic point is faithful *provided* no word (varying according to suitably chosen parameters) is identically trivial. The topological Baumslag Lemma and the complexification trick now furnish this sufficient condition.

(5) The last combination theorem (Section 3.9) is one in the smooth category and its proof follows the lines of the usual Baire category argument for the real line.

In Section 5.7 we shall develop tools for relative versions of the combination theorems where elements of the edge subgroup are constrained by an algebraic condition, for instance by demanding that the image of C lies in a particular subgroup. We focus primarily on *projective* group actions on the circle, i.e. representations into $\mathrm{PSL}_2(\mathbb{R})$; but many of the proofs go through in the more general context of representations into linear algebraic semisimple groups.

3.1. **Terminology.** The reader may recall some of the following definitions from the introduction, which we restate here for the convenience of the reader.

We set

$$x \sim -x$$

for each $x \in \mathbb{R}$ and

$$\mathbb{R}' = \mathbb{R}/\!\sim .$$

Note that

$$\mathrm{tr}\colon\ \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{R}'.$$

For convenience of exposition, we will often blur the distinction between $\mathbb{R}'$ and $\mathbb{R}$ unless there is a danger of confusion.

For a group L, we let $T(L)$ denote the set of torsion elements. Let $R \subseteq \mathbb{R}$. We say a representation

$$\lambda \colon L \to \mathrm{PSL}_2(\mathbb{R})$$

is *R–free* if

$$\mathrm{tr} \circ \lambda(L \backslash 1) \cap R = \varnothing.$$

We say that λ is *almost R–free* if

$$\mathrm{tr} \circ \lambda(L \backslash T(L)) \cap R = \varnothing.$$

Most of the time, we will assume R contains $\{\pm 2\}$ by enlarging R if necessary.

A projective action

$$\lambda \colon L \to \mathrm{PSL}(2, \mathbb{R})$$

is *parabolic-free* if $\lambda(L \backslash 1)$ does not contain parabolic elements. Parabolic-free actions are sometimes called *aparabolic*, but we elect to use the former terminology. A nontrivial torsion element in $\mathrm{PSL}_2(\mathbb{R})$ is never parabolic. So if R contains $\{\pm 2\}$, then a faithful, almost R–free representation is parabolic–free.

For

$$\Lambda \subseteq \mathrm{Hom}(L, \mathrm{PSL}_2(\mathbb{R}))$$

and $g \in L$, we denote

$$\Lambda(g) = \{\lambda(g) \mid \lambda \in \Lambda\}.$$

Let

$$p \colon \widetilde{\mathrm{PSL}_2(\mathbb{R})} \to \mathrm{PSL}_2(\mathbb{R})$$

denote the universal covering map. Recall that we say a representation

$$\lambda \colon L \to \mathrm{PSL}(2, \mathbb{R})$$

is *liftable* if λ factors as

$$L \to \widetilde{\mathrm{PSL}_2(\mathbb{R})} \to \mathrm{PSL}_2(\mathbb{R}).$$

We say a subgroup $L \leqslant \mathrm{PSL}_2(\mathbb{R})$ is liftable if so is the embedding $L \to \mathrm{PSL}_2(\mathbb{R})$.

We will often denote a generator of $\mathbb{Z}$ by s. For a group A and its subgroup C, we denote by $A*_C$ the HNN extension of A amalgamated along C with the identity map on C. In other words, we have

$$A*_C = A *_C (C \times \langle s \rangle).$$

A group is *2-torsion-free* if no elements have order two.

3.2. **Flexibility.**

Definition 3.1. A finitely generated group L is *flexible* (*liftable-flexible*, respectively) if there exists an element

$$g_0 \in L \backslash T(L)$$

such that for each countable subset $R \subseteq \mathbb{R}$ we have a faithful almost R–free (and liftable, respectively) representation $\lambda\colon L \to \mathrm{PSL}_2(\mathbb{R})$ with the property that $\lambda(g_0)$ is elliptic. We call g_0 an *anchor* of L.

Remark 3.2. (1) A liftable-flexible group is flexible.
(2) A torsion group can never be flexible, since $L \backslash T(L) = \varnothing$. A liftable-flexible group is torsion-free, since so is

$$\widetilde{\mathrm{PSL}_2(\mathbb{R})} \leqslant \mathrm{Homeo}_+(\mathbb{R}).$$

(3) The representation λ appearing in Definition 3.1 is indiscrete. This comes from the observation that λ maps an anchor to an irrational rotation. We will see later that λ is dense if L is non-virtually solvable (Lemma 6.1).
(4) The flexibility of a group implies that the group admits many "dynamically distinct" projective actions on the circle. See Proposition 4.1 for a more precise description.
(5) Flexibility is closely related to the existence of certain quasi-morphisms; see Corollary 4.4.

Example 3.3. Infinite cyclic groups are liftable-flexible. To see this, let us write $\mathbb{Z} = \langle s \rangle$. Let $R \subseteq \mathbb{R}$ be a given countable set containing ± 2. Consider

$$\mathrm{SO}(2) \subseteq \mathrm{PSL}_2(\mathbb{R}),$$

and let

$$S = \{v \in \mathrm{SO}(2) \mid \mathrm{tr}(v^k) \in R \text{ for some } k \in \mathbb{Z} \backslash 0\}.$$

Since S is countable, we can choose some

$$v_0 \in \mathrm{SO}(2) \backslash S.$$

Note that v_0 is an irrational rotation. We define $\lambda\colon \mathbb{Z} \to \mathrm{PSL}_2(\mathbb{R})$ by $\lambda(s) = v_0$, and see that

$$\lambda(\mathbb{Z} \backslash 1) \cap R = \varnothing.$$

In particular, $\mathbb{Z}$ is flexible. By choosing an arbitrary preimage

$$\tilde{v}_0 \in \widetilde{\mathrm{PSL}_2(\mathbb{R})}$$

of v_0 and defining $\tilde{\lambda}(s) = \tilde{v}_0$, we see that $\mathbb{Z}$ is liftable-flexible.

Example 3.4. The infinite dihedral group

$$L = \mathbb{Z}_2 * \mathbb{Z}_2$$

is *not* flexible. To see this, note that the image of an arbitrary faithful projective representation of L is generated by two rotations of angle π. By writing those two rotations as ab and bc for some reflections a, b, c, we see that

$$(ab) \cdot (bc) = ac$$

has an infinite order. This means that $\mathrm{Fix}(a)$ and $\mathrm{Fix}(c)$ are parallel in $\mathbb{H}^2$ and hence each element in the image of L is either an involution or a hyperbolic isometry. In particular, L cannot have an anchor.

Let us summarize the main result of this section. Let $\mathscr{F}$ and $\mathscr{F}^{\sim}$ denote the classes of flexible and liftable-flexible groups, respectively. Since $\mathrm{Homeo}_+(\mathbb{R})$ is torsion-free, so are the groups in $\mathscr{F}^{\sim}$. We have already seen the part (1) of the following theorem in Example 3.3.

Theorem 3.5. *Let $\mathscr{H} = \mathscr{F}$ or $\mathscr{H} = \mathscr{F}^{\sim}$.*

(1) Infinite cyclic groups are in $\mathscr{H}$.
(2) If $H \leqslant G \in \mathscr{H}$ and $[G : H] < \infty$, then $H \in \mathscr{H}$.
(3) If A and B are in $\mathscr{H}$, then so is $A * B$.
(4) If A is in $\mathscr{H}$ and B is a finite cyclic group, then so is $A * B$.
(5) Let $L \in \mathscr{H}$ and $A, B, C \leqslant L$ such that $C \leqslant A \cap B$. If C is malnormal maximal abelian in A and in B, then $A *_C B \in \mathscr{H}$.
(6) If $A \in \mathscr{H}$ and C is a malnormal maximal abelian subgroup of A, then $A*_C \in \mathscr{H}$.

Remark 3.6. In parts (5) and (6), the hypotheses on the normalizer can be replaced by the assumption that A and B do not have 2–torsion. This is because if μ is a maximal abelian subgroup of $\mathrm{PSL}_2(\mathbb{R})$ then $N(\mu)\backslash\mu$ consists of elements of order 2. See Lemma 5.8 (3) for other characterizations of $N(\mu)$ in a more general setting.

A particularly simple application of the combination theorem is for a *double* of a flexible group. For a group A and a subgroup C, we denote by $A *_C \bar{A}$ the amalgamation of two copies of A along Id_C.

Corollary 3.7. *Let A be a group and let C be a malnormal maximal abelian subgroup of A. If A is flexible or liftable-flexible, then so is $A *_C \bar{A}$.*

Proof. Apply Theorem 3.5 (5) for $L = A = B$. □

3.3. **Warm-up: Genus-two Surface.** For an overview of the techniques to come in this section, let us exhibit uncountably many distinct faithful parabolic-free representations of the fundamental group L of a closed oriented surface S with genus two. This result will be strengthened to Theorem 4.9 below.

We will assume that we have a faithful parabolic-free representation

$$\lambda\colon F_2 = \langle a, b\rangle \to \mathrm{PSL}_2(\mathbb{R}).$$

We can choose a discrete λ corresponding to a hyperbolic structure of a torus with one funnel. Or, we can find an indiscrete λ such that $\lambda([a, b])$ is an irrational rotation, as in Example 4.18.

Let us denote by μ the one-parameter group containing $\lambda([a, b])$. Again, we allow μ to be either hyperbolic or elliptic. For each $\nu \in \mu$, we define $\rho_\nu\colon L \to \mathrm{PSL}_2(\mathbb{R})$ by

$$(a, b, c, d) \mapsto (\lambda(a), \lambda(b), \lambda(a)^\nu, \lambda(b)^\nu).$$

Since

$$[\lambda(a)^\nu, \lambda(b)^\nu] = \lambda([a, b])^\nu = \lambda([a, b]),$$

we see ρ_ν is well-defined. Note that $\rho_1 = \lambda$. The goal of this example is to show the following:

Claim. *There exists a dense subset $\mathscr{D} \subseteq \mu$ such that for each $\nu \in \mathscr{D}$, the map ρ_ν is faithful and parabolic-free.*

Let us enumerate

$$L \setminus 1 = \{g_1, g_2, \ldots\}.$$

For each $n \in \mathbb{N}$, we define

$$X_n = \{\nu \in \mu \mid \mathrm{tr} \circ \rho_\nu(g_i) = \pm 2 \text{ for some } i \leqslant n\}.$$

Define a map $f_n\colon L \to F_2$ by

$$f_n\colon (a, b, c, d) \to (a, b, a^{[a,b]^n}, b^{[a,b]^n}).$$

Then a classical lemma of Baumslag [5] in combinatorial group theory (which is stated as Lemma 3.13) implies that if $n \in \mathbb{N}$, then there exists $M = M(n)$ such that $f_M(g_i) \neq 1$ for all $1 \leqslant i \leqslant n$.

For each n, we put

$$\nu_n = \lambda([a, b])^{M(n)}.$$

Then we see $\rho_{\nu_n}(g_i) \neq 1$ for each $1 \leqslant i \leqslant n$. Since we are assuming λ is parabolic-free and $\rho_{\nu_n}(L) \leqslant \lambda(F_2)$, we see that ρ_{ν_n} is parabolic-free. In particular, $\rho_{\nu_n} \in \mu \setminus X_n$.

Since μ is a circle or a real line and X_n is a proper algebraic subset of ν for $n \geqslant 1$, we see that each X_n is finite. Put

$$\mathscr{D} = \mu \setminus \bigcup_n X_n$$

so that $\mathscr{D}$ is dense in X. Then for each $\nu \in \mathscr{D}$ the map ρ_ν is faithful and parabolic-free.

3.4. **Very General Points, Abundance and Stable Injectivity.** Our techniques involve both the topological category as well as that of (real or complex) algebraic varieties. Thus a choice of terminology needs to be made, which we set out to do here.

First we set up the terminology in the topological category. Recall that a G_δ set in a topological space X is the countable intersection of open subsets. We shall refer to a property as being satisfied by a *very general* point in X if it is satisfied by a point in a (suitably chosen) dense G_δ subset of X.

The corresponding notion in algebraic geometry is more restrictive. We will regard linear algebraic groups as affine algebraic sets and equip them with the Zariski topology. We shall refer to a property as being satisfied by a *very general* point in an algebraic set X if it is satisfied by a point in the complement of a countable union of subsets

$$\{X_1, X_2, \ldots\}$$

of X such that each X_i is a proper algebraic subset of an irreducible component of X. We shall say that a subset Y of an algebraic set X is *abundant* in X if Y contains very general points of an irreducible component of X.

We gather together some facts for use later. All algebraic sets will be defined over $\mathbb{R}$ or $\mathbb{C}$. The following is a standard fact. For example, one can see Lemma 2.4 of [11] for instance.

Lemma 3.8. *Let X be an algebraic set. If*

$$X_1 \subseteq X_2 \subseteq \cdots$$

is a countable chain of proper algebraic subsets of X, then

$$X \setminus \bigcup_{i=1}^{\infty} X_i$$

is abundant in X. If, furthermore, X has a positive dimension, then

$$X \setminus \bigcup_{i=1}^{\infty} X_i$$

is uncountable.

Proof. Each X_i is closed in the Zariski topology, and hence is either nowhere dense in X or contains an irreducible component of X. Since X has finitely many components and each X_i is a proper subset, there exists an irreducible component V of X

such that $V \cap X_i$ is proper in V for each i. By definition,

$$Y = V \setminus \bigcup_i X_i$$

is very general in V. For the second part, we note that Y is uncountable since it contains a dense G_δ set. □

We indicate here a general method of extracting a large (uncountable) family of faithful representations by applying Lemma 3.8. The main idea in this comes from translating a concept from the (discrete) context of limit groups into the (algebraic geometry) context of representations into algebraic groups.

Definition 3.9 ([64])**.** Let G be a group. We say a sequence of group homomorphisms

$$(f_n)_{n\geqslant 1} \subseteq \mathrm{Hom}(G, H_n)$$

is *stably injective* if for each $g \in G\setminus\{1\}$ there exists $n_0 > 0$ such that $f_n(g) \neq 1$ for all $n \geqslant n_0$.

For terminological convenience, we will also say a sequence of maps is *stably injective* if the sequence has a stably injective subsequence. Stable injectivity appears naturally in the theory of limit groups. Recall [64] that a group G is a *limit group* if there exists a stably injective sequence

$$\{\rho_n : G \to F\}_{n\in\mathbb{N}}$$

for some fixed nonabelian free group F.

Limit groups are torsion-free and finitely presented [7]. Nonabelian free groups and hyperbolic orientable surface groups are limit groups. To see the latter fact, one can consider the sequence of Dehn twist automorphisms along a separating curve; see [5] or [47, Lemma 3.6].

As a warm-up, we now give a simple illustration of the Baire category argument in connection with stably injective maps.

Lemma 3.10. *Let G be a finitely generated group, and $\mathfrak{G}$ be an algebraic group. If there exists a stably injective sequence of representations*

$$\{f_n : G \to \mathfrak{G}\}_{n\in\mathbb{N}}$$

which contains infinitely many distinct representations, then there exist uncountably many faithful representations $G \to \mathfrak{G}$. In fact the set of faithful representations is abundant.

Proof. Write $G = \langle S \rangle$. We first observe that

$$X = \mathrm{Hom}(G, \mathfrak{G}) \subseteq \mathfrak{G}^S$$

is an algebraic set. This is obvious when G is finitely presented. If not, the Hilbert Basis Theorem still guarantees that finitely many relations of G would suffice to define the set X.

Note that X has a positive dimension since

$$\{f_n\}_{n\in\mathbb{N}} \subseteq X.$$

Let us enumerate

$$G\backslash 1 = \{g_1, g_2, \ldots\}.$$

For each $n \in \mathbb{N}$, we define an algebraic set

$$X_n = \{\rho \in X \colon \rho(g_i) = 1 \text{ for some } i \leqslant n\},$$

The stable injectivity implies that X_n is proper for all $n \in \mathbb{N}$. Then Lemma 3.8 implies that

$$\mathscr{D} = X \backslash \bigcup_{n\geqslant 1} X_n$$

is uncountable. Note that $\mathscr{D}$ is precisely the set of faithful representations. □

Let L be an arbitrary limit group. An obvious consequence of Lemma 3.10 is that if an algebraic group $\mathfrak{G}$ contains a nonabelian free group F, then $\mathfrak{G}$ contains an isomorphic copy of L. Actually, more can be said. In [7, Exercises 8 and 9] and [70], it is shown that for an arbitrary finite subset $A \subseteq L$, one can find a parabolic-free embedding

$$\rho\colon L \to \mathrm{PSL}_2(\mathbb{R})$$

in such a way that $\rho(A)$ consists of hyperbolics. This latter fact will be strengthened in this paper; see Corollaries 4.7 and 5.21. Moreover, we will see that $\rho(L)$ can be chosen to be purely hyperbolic for a nonabelian L only if the representation is discrete; see Lemma 6.1.

The essence of our combination theorem is to construct an embedding (with a desired property) of a group, where the embedding is built upon known embeddings of "smaller" groups. This technique can be described as follows.

Lemma 3.11. *Let G be a finitely generated subgroup of an algebraic group $\mathfrak{G}$. Consider a group presentation*

$$K = \langle G \cup A \mid R \rangle$$

where A is a finite set and R consists of words on $G \cup A$. We enumerate $K = \{1, g_1, g_2, \ldots\}$. Define

$$X = \mathrm{Hom}_G(K, \mathfrak{G})$$

as the set of representations $K \to \mathfrak{G}$ which extend the inclusion $G \leqslant \mathfrak{G}$. For $n \in \mathbb{N}$, we let

$$X_n = \{\rho \in X \colon \rho(g_i) = 1 \text{ for some } i \leqslant n\}.$$

(1) If X_n is proper for each $n \in \mathbb{N}$, then abundantly many representations $\rho \in X$ are faithful.
(2) If X_n never contains an irreducible component of X for every $n \in \mathbb{N}$, then a very general representation $\rho \in X$ is faithful.

Remark 3.12. If the natural map $G \to K$ is not injective, then X is empty.

Proof. We may assume X is nonempty. Define an algebraic set

$$X = \mathrm{Hom}_G(K, \mathfrak{G}).$$

The hypotheses imply that each X_n is a proper algebraic subset of X. Note that

$$\mathscr{D} = X \setminus \bigcup_n X_n$$

precisely consists of faithful representations. Then by Lemma 3.8 any point of

$$\mathscr{D} = X \setminus \bigcup_n X_n$$

is a very general point in an irreducible component of X. □

3.5. **One-parameter subgroups.** Let us regard $\mathrm{PSL}_2(\mathbb{R})$ as the set of Möbius transformations on the upper half-plane of $\mathbb{C}$. In $\mathrm{PSL}_2(\mathbb{R})$, we consider

$$\mathrm{Rot}(t) = \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix}, \quad \exp(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad p(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

These isometries are respectively an elliptic rotation about $i \in \mathbb{H}^2$ by the angle t, the hyperbolic translation by distance t along the geodesic $[0, \infty]$, and a parabolic isometry fixing ∞.

By regarding

$$\mathrm{PSL}_2(\mathbb{R}) \subseteq \mathrm{Homeo}^+(\partial\mathbb{H}^2),$$

we have a restriction

$$\mathrm{rot}\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{R}/\mathbb{Z}.$$

Let $\ell\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathbb{R}$ be given by

$$\ell(g) = \inf\{d_{\mathrm{hyp}}(x, g \cdot x) \colon x \in \mathbb{H}^2\}.$$

Then $\mathrm{rot}(g) = 0$ for a non-elliptic g, and $\ell(g) = 0$ for a non-hyperbolic g. Moreover, we have

$$|\operatorname{tr} g| = \begin{cases} |2\cos(\pi \,\mathrm{rot}(g))|, & \text{if } g \text{ is elliptic;} \\ 2\cosh(\ell(g)/2), & \text{if } g \text{ is hyperbolic.} \end{cases}$$

We always regard the identity as an elliptic, parabolic and hyperbolic isometry simultaneously.

A subgroup of $\mathrm{PSL}_2(\mathbb{R})$ which is a conjugate of the group

$$\mathrm{SO}(2) = \mathrm{Rot}(2\pi\mathbb{R}/\mathbb{Z}),$$

or the group $\exp(\mathbb{R})$, or the group $p(\mathbb{R})$ is called a *one-parameter subgroup*. Each one-parameter subgroup μ is an algebraic subgroup and equipped with a homomorphism

$$\mu\colon \mathbb{R} \to \mathrm{PSL}_2(\mathbb{R}).$$

A group $H \leqslant \mathrm{PSL}_2(\mathbb{R})$ is *elementary* if H admits a finite orbit on $\mathbb{H}^2 \cup \partial\mathbb{H}^2$. An elementary subgroup is conjugate into the group $\mathrm{Rot}(\mathbb{R})$, the group

$$N(\exp(\mathbb{R})) = \exp(\mathbb{R}) \rtimes \mathrm{Rot}(\pi),$$

or the group $p(\mathbb{R})$, where $N(G)$ denotes the normalizer of G.

We also consider one-parameter subgroups $\mu\colon \mathbb{R} \to \mathrm{PSL}_2(\mathbb{C})$. In short, they are conjugate to one of the following.

$$\mathrm{Rot}(t) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix}, \quad \exp(t\lambda) = \begin{pmatrix} e^{\lambda t/2} & 0 \\ 0 & e^{-\lambda t/2} \end{pmatrix}, \quad p(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

where $\lambda \neq 0$ is not purely imaginary. Note that for

$$1 \neq g \in \mathrm{PSL}_2(\mathbb{R})$$

(or in $\mathrm{PSL}_2(\mathbb{C})$ as the case be), the set $\mathrm{Fix}\, g$ consists of at most two points in $\partial\mathbb{H}^d$ where $d = 2, 3$.

Still more generally, we will call a group homomorphism μ from $\mathbb{Z}$ or $\mathbb{R}$ into a given group $\mathfrak{G}$ as an (integral or real) *one-parameter subgroup*. An integral one-parameter subgroup is simply a cyclic subgroup of $\mathfrak{G}$. The parametrization of a real one-parameter subgroup in a topological group is usually required to be continuous.

3.6. **A topological Baumslag Lemma.** We will provide combination theorems where edge subgroups are constrained by an algebro-geometric condition. The main idea is an extension of the following, called Baumslag Lemma in the literature. We denote by F_n a free group of rank n.

Lemma 3.13 ([5]). *Suppose we have*

$$u, g_1, g_2, \ldots, g_k \in F_n$$

such that $[u, g_i] \neq 1$ *for each i. Then there exists* $m > 0$ *such that*

$$g_1 u^{m_1} g_2 u^{m_2} \cdots g_k u^{m_k} \neq 1$$

whenever $|m_i| \geqslant m$ *for each* $i = 1, \ldots, k$.

This lemma generalizes for multiple "twisting words" [47, 4] or for torsion-free word-hyperbolic groups [41]. Our continuous version of Baumslag's lemma below will imply both of these generalizations. See Remark 3.21.

Motivated by Baumslag Lemma, we make the following definition.

Definition 3.14. Let X be a topological space and μ be an (integral or real) one-parameter subgroup of $\mathrm{Homeo}(X)$. We say μ is *attracting* if the following hold:

(i) $\mathrm{Fix}\,\mu$ is a nonempty compact set;
(ii) for each open neighborhood U of $\mathrm{Fix}\,\mu$ we have
$$\mu(\pm t)(X\backslash U) \subseteq U$$
for all sufficiently large $t > 0$.

Definition 3.15. Let X be a topological space and (C^+, C^-) be a pair of disjoint compact subsets of X. Suppose f is a map from $\mathbb{Z}^k$ or $\mathbb{R}^k$ to $\mathrm{Homeo}(X)$, which is not necessarily a homomorphism. We say f is *topologically Baumslag* (*relative to* (C^+, C^-)) if for each pair of disjoint open neighborhoods (U^+, U^-) of (C^+, C^-), we have
$$f(t_1, \ldots, t_k)^{\pm 1}(X\backslash U^{\mp}) \subseteq U^{\pm}$$
whenever all $|t_i|$'s are sufficiently large.

Attracting one-parameter subgroups generalize non-elliptic one-parameter subgroups of
$$\mathrm{PSL}_2(\mathbb{R}) \leqslant \mathrm{Homeo}^+(S^1).$$
If a map f is topologically Baumslag into $\mathrm{PSL}_2(\mathbb{R})$, then $f(t_1, \ldots, t_k)$ is hyperbolic whenever all $|t_i|$'s are sufficiently large. Let us illustrate a key example, which is actually a warm-up for the full Topological Baumslag Lemma (Lemma 3.18).

Example 3.16. Let $a = \mathrm{Rot}(\pi/2)$ and $b = \exp(1)$ in $\mathrm{PSL}_2(\mathbb{R})$, and set $f(t) = ab^t$ for $t \in \mathbb{Z}$. Put
$$C^- = \{0, \infty\}$$
and
$$C^+ = a.\{0, \infty\}$$
in $S^1 = \partial\mathbb{H}^2$. Consider a small open neighborhood $U^- \supseteq C^-$ in S^1, and put $U^+ = a.U^-$. Then for all sufficiently large $|t|$, we see that
$$f(t).(X\backslash U^-) = ab^t.(X\backslash U^-) \subseteq a.U^- = U^+,$$
$$f(t)^{-1}.(X\backslash U^+) = b^{-t}a^{-1}.(X\backslash U^+) = b^{-t}.(X\backslash U^-) \subseteq U^-.$$

Therefore the map $f\colon \mathbb{R} \to \mathrm{PSL}_2(\mathbb{R})$ is topologically Baumslag.

Remark 3.17. (1) If
$$f(t)^{\pm 1}(X \backslash U^{\mp}) \subseteq U^{\pm}$$
for some $t > 0$, then
$$f(t)^{\pm n}(X \backslash U^{\mp}) \subseteq U^{\pm}$$
for every positive integer n. This is because $U^{\pm} \subseteq X \backslash U^{\mp}$.

(2) Note that a topologically Baumslag map could be constantly the identity if
$$X = C^{+} \coprod C^{-}.$$

(3) Let X be a connected and Hausdorff space and let f be a topologically Baumslag map into $\mathrm{Homeo}(X)$. We pick disjoint open neighborhoods $U^{\pm}$ of $C^{\pm}$ and
$$x_0 \notin U^{+} \cup U^{-}.$$
If $|t_i|$'s are all sufficiently large, then
$$f(t_1, \ldots, t_k)^n . x_0 \in U^{+} \cup U^{-}$$
for all $n \neq 0$, and so
$$f(t_1, \ldots, t_k)^n \neq \mathrm{Id}\,.$$
That is, the cyclic group $\langle f(t_1, \ldots, t_k) \rangle$ is isomorphic to $\mathbb{Z}$. We also note that f cannot be a homomorphism, even when $k = 1$. This is because
$$f(t)^{-1} . x_0 \in U^{-}$$
and
$$f(-t) . x_0 \in U^{+}$$
and hence, $f(-t) \neq f(t)^{-1}$ for all large $|t|$.

(4) Suppose μ is a one-parameter subgroup of
$$\mathrm{Isom}^{+}(\mathbb{H}^d) \subseteq \mathrm{Homeo}_{+}(S^{d-1}).$$
Then μ is attracting if and only if μ is a nontrivial parabolic or a nontrivial hyperbolic.

(5) If X is normal, then we can relax the compactness hypothesis of $\mathrm{Fix}\,\mu$ to closedness, with similar implications. Such variations on our theme are beyond the scope of this paper.

We will call the following a *Topological Baumslag Lemma*. Some of the ideas go back to the original ping-pong argument of Tits [68]. In the statement, f is only a function (not a homomorphism).

Lemma 3.18. *Let X be a Hausdorff space. Define*

$$f(t_1, t_2, \ldots, t_k) = g_1\mu_1(t_1)g_2\mu_2(t_2)\cdots g_k\mu_k(t_k),$$

where $g_i \in \mathrm{Homeo}(X)$ and each μ_i is an (integral or real) attracting one-parameter subgroup of $\mathrm{Homeo}(X)$. Assume

$$\mathrm{Fix}\,\mu_i \cap g_{i+1}\,\mathrm{Fix}\,\mu_{i+1} = \varnothing$$

for each $i = 1, \ldots, k$ where indices are taken cyclically modulo k. Then the map f is topologically Baumslag relative to $(g_1\,\mathrm{Fix}\,\mu_1, \mathrm{Fix}\,\mu_k)$.

Proof. Choose an open neighborhood U_i of $\mathrm{Fix}\,\mu_i$ such that

$$U_i \cap g_{i+1}(U_{i+1}) = \varnothing$$

for each $1 \leqslant i \leqslant k$. There exists $N > 0$ such that

$$\mu_i(\pm t)(X \backslash U_i) \subseteq U_i$$

for all $t \geqslant N$.

Fix $(t_1, t_2, \ldots, t_k)$ such that $|t_i| \geqslant N$ for each i, and let

$$w = f(t_1, t_2, \ldots, t_k).$$

It suffices to verify the ping-pong condition of Definition 3.15 for w with respect to the pair of disjoint open sets $(g_1(U_1), U_k)$. For each i, we first note

$$g_i\,\mu_i(t_i)(X \backslash U_i) \subseteq g_i(U_i) \subseteq X \backslash U_{i-1},$$

$$\mu_i(-t_i)g_i^{-1}(X \backslash g_i(U_i)) = \mu_i(-t_i)(X \backslash U_i) \subseteq U_i \subseteq X \backslash g_{i+1}(U_{i+1}).$$

So we have that

$$\begin{aligned} w(X \backslash U_k) &= \prod_{i=1}^{k} g_i\,\mu_i(t_i)(X \backslash U_k) \subseteq \prod_{i=1}^{k-1} g_i\,\mu_i(t_i)(X \backslash U_{k-1}) \\ &\subseteq \cdots \subseteq g_1\mu_1(t_1)(X \backslash U_1) \subseteq g_1(U_1), \end{aligned}$$

$$\begin{aligned} w^{-1}(X \backslash g_1(U_1)) &= \prod_{i=k}^{1} \mu_i(-t_i)g_i^{-1}(X \backslash g_1(U_1)) \subseteq \prod_{i=k}^{2} \mu_i(-t_i)g_i^{-1}(X \backslash g_2(U_2)) \\ &\subseteq \cdots \subseteq \mu_k(-t_k)g_k^{-1}(X \backslash g_k(U_k)) = \mu_k(-t_k)(X \backslash U_k) \subseteq U_k. \end{aligned}$$

Thus we establish the lemma. ◻

We will mainly employ Lemma 3.18 for the projective setting.

Lemma 3.19. *Define* $f\colon \mathrm{PSL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ *by*

$$f(u) = g_1 u^{m_1} g_2 u^{m_2} \cdots g_k u^{m_k}$$

where $g_i \in \mathrm{PSL}_2(\mathbb{C})$ *and where* $m_i \in \mathbb{Z}\backslash 0$. *Suppose* $\mu\colon \mathbb{R} \to \mathrm{PSL}_2(\mathbb{C})$ *is a one-parameter subgroup such that* $[\mu, \mu^{g_i}]$ *contains some nontrivial non-parabolic element for each i.*

(1) *If* μ *is hyperbolic or parabolic, then for all* $M > 0$ *there exists* $N > 0$ *such that whenever* $|t| \geqslant N$ *we have*

$$\ell \circ f \circ \mu(t) \geqslant M.$$

(2) *If* μ *is elliptic of period* 2π, *then for each countable subset* $R \subseteq \mathbb{R}$, *we have*

$$\mathrm{tr} \circ f \circ \mu(t) \notin R$$

for all but countably many $t \in 2\pi\mathbb{R}/\mathbb{Z}$. *In particular, we can require that* $f \circ \mu(t)$ *is an irrational rotation or a nontrivial hyperbolic for such a t.*

Proof. Let $\mathrm{Fix}\,\mu$ denote the fixed point set of the action of μ on $\partial\mathbb{H}^3$. The condition

$$\mathrm{tr}[\mu, \mu^g] \neq \{\pm 2\}$$

is equivalent to the condition that

$$\mathrm{Fix}\,\mu \cap g\,\mathrm{Fix}\,\mu = \varnothing.$$

(1) By Lemma 3.18, we have a pair of two disjoint compact sets

$$(g_1\,\mathrm{Fix}\,\mu, \mathrm{Fix}\,\mu)$$

relative to which f is topologically Baumslag.

Choose a sequence of a pair of open neighborhoods $\{(U_n^+, U_n^-)\}$ of

$$(g_1\,\mathrm{Fix}\,\mu, \mathrm{Fix}\,\mu)$$

such that

$$d_{\mathrm{hyp}}(U_n^+, U_n^-) \to \infty.$$

Since

$$f \circ \mu(t)^{\pm 1}(X \backslash U_n^{\mp}) \subseteq U_n^{\pm}$$

for all sufficiently large $|t|$, we see that

$$\ell \circ f \circ \mu(t) \to \infty$$

as $|t| \to \infty$.

(2) We use a complexification trick, embedding $\mathrm{PSL}_2(\mathbb{R})$ into $\mathrm{PSL}_2(\mathbb{C})$ and using Zariski density of $\mathrm{SO}(2)$ in $\mathbb{C}^\times$.

Regard $\mathbb{H}^2$ as a unit disk properly embedded in $\mathbb{H}^3$ with the Poincaré ball model. We may assume μ is the rotation group about the center of $\mathbb{H}^3$ preserving $\mathbb{H}^2$. Then we can write

$$f \circ \mu(2t) = \phi(e^{it}),$$

where here we write
$$\phi(\zeta) = \prod_{j=1}^{k} g_j \begin{pmatrix} \zeta^{m_j} & 0 \\ 0 & \zeta^{-m_j} \end{pmatrix}.$$

Let
$$R = \{r_1, r_2, \ldots\},$$
where $r_1 = \pm 2$ and $\mathbb{Q} \subseteq R$. We define an algebraic set
$$Y_n = \{\zeta \in \mathbb{C}^\times : \operatorname{tr} \phi(\zeta) = r_j \text{ for some } 1 \leqslant j \leqslant n\}.$$
If some Y_n is infinite, then $Y_n = \mathbb{C}^\times$ since every infinite set is Zariski dense in $\mathbb{C}^\times$. For a hyperbolic one-parameter group $\mu' \colon \mathbb{R} \to \mathrm{PSL}_2(\mathbb{C})$ whose axis is perpendicular to $\mathbb{H}^2$ at $\operatorname{Fix} \mu$, we can write
$$\phi(e^{t/2}) = \prod_{j=1}^{k} g_j \mu'(m_j t).$$
Since
$$\operatorname{Fix} \mu \cap g_i \operatorname{Fix} \mu = \varnothing,$$
we see $\operatorname{Axis}(\mu')$ and $g_i \operatorname{Axis}(\mu')$ have disjoint closures. By the part (1), we see Y_n cannot contain the unit circle. Hence, we have that
$$\bigcup_{n \geqslant 1} Y_n \cap S^1$$
is countable. □

We can apply the Topological Baumslag Lemma to other groups with north-south dynamics.

Corollary 3.20. *Let G be a group and $k \geqslant 1$. Suppose that*
$$g_1, \ldots, g_k, u_1, \ldots, u_k \in G$$
satisfy
$$[u_i, g_{i+1} u_{i+1} g_{i+1}^{-1}] \neq 1$$
for each
$$i \in \{1, 2, \ldots, k\},$$
where the indices are taken modulo k. Define $f \colon \mathbb{Z}^k \to G$ by
$$f(t_1, \ldots, t_k) = g_1 u_1^{t_1} g_2 u_2^{t_2} \cdots g_k u_k^{t_k}.$$

(1) If G is a torsion-free word-hyperbolic group, then $f(t_1, \ldots, t_k)$ has an infinite order whenever all $|t_i|$'s are sufficiently large.
(2) If G is a mapping class group of a closed oriented hyperbolic surface and if each u_i is pseudo-Anosov, then $f(t_1, \ldots, t_k)$ has an infinite order whenever all $|t_i|$'s are sufficiently large.

We emphasize again that f is merely a function and not a homomorphism.

Proof. Let $G = H$ for some torsion-free nonelementary word-hyperbolic group H, or $G = \mathrm{Mod}(S)$ for some closed oriented hyperbolic surface S. We accordingly let $X = \partial H$ or $X = \mathrm{PML}(S)$, so that $G \hookrightarrow \mathrm{Homeo}(X)$. Note also that ∂H is metrizable and so, Hausdorff. Two elements in H or two pseudo-Anosovs in $\mathrm{Mod}(S)$ either commute or have disjoint fixed point sets [58] . Hence we have

$$\mathrm{Fix}\, u_i \cap g_{i+1} \mathrm{Fix}\, u_{i+1} = \mathrm{Fix}\, u_i \cap \mathrm{Fix}\, g_{i+1} u_{i+1} g_{i+1}^{-1} = \varnothing,$$

and Lemma 3.18 applies. The infinite order claim follows from that H is torsion-free and that $\mathrm{PML}(S)$ is connected. □

Remark 3.21. (1) The hypothesis

$$[u_i, g_{i+1} u_{i+1} g_{i+1}^{-1}] \neq 1$$

in Corollary 3.20 is equivalent to the hypothesis that the elements u_i and

$$g_{i+1} u_{i+1} g_{i+1}^{-1}$$

never belong to a common cyclic subgroup of G. In the special case

$$u_1 = \cdots = u_k = u,$$

this hypothesis is equivalent to

$$[u, g_i] \neq 1$$

for each i. This is because torsion-free word-hyperbolic groups and mapping class groups do not admit Baumslag–Solitar relations that are not commutators [46, Theorem 8.2]. So we note that the corollary is a direct generalization of the original Baumslag's Lemma.

(2) In [41, Lemma 5.4], Corollary 3.20 was proved for the case when G is torsion-free word-hyperbolic and

$$u_1 = u_2 = \cdots = u_k.$$

In [47, Lemma 3.6], the corollary was proved for the special case that G is free; see also [4, Lemma 3.5].

In the case $G = \mathrm{Mod}(S)$, the previous corollary does not conclude that f is pseudo-Anosov.

Question 3.22. *If G is a mapping class group, is $f(t_1, \ldots, t_k)$ in Corollary 3.20 necessarily pseudo-Anosov for all sufficiently large $|t_i|$'s?*

3.7. **Liftability from combinations.** Let us note a lemma concerning general liftability. The example to keep in mind is the case when

$$p\colon \widetilde{\mathrm{PSL}_2(\mathbb{R})} \to \mathrm{PSL}_2(\mathbb{R})$$

and μ is a one-parameter subgroup of $\mathrm{PSL}_2(\mathbb{R})$. We will denote the inner automorphism γ_ν defined by a group element ν as

$$\gamma_\nu(g) = g^\nu.$$

For $\nu \in \mathrm{PSL}_2(\mathbb{R})$, we let

$$\delta_\nu\colon \mathbb{Z} = \langle s \rangle \to \mathrm{PSL}_2(\mathbb{R})$$

be defined by $\delta_\nu(s) = \nu$.

Lemma 3.23. *Let $p\colon \tilde{\mathfrak{G}} \to \mathfrak{G}$ be a surjective homomorphism between groups. Suppose we have a commutative diagram of groups as follows.*

$$\begin{array}{ccc} & & \tilde{\mathfrak{G}} \\ & \overset{\tilde{\lambda}}{\nearrow} & \downarrow p \\ L' & \xrightarrow{\lambda} & \mathfrak{G}. \end{array}$$

Let $\mu \leqslant \mathfrak{G}$ be such that $p^{-1}(\mu)$ is abelian, and let $\nu \in \mu$.

(1) Suppose there exist subgroups A, B, C of L' such that $C \leqslant A \cap B$ and $\lambda(C) \leqslant \mu$. Let us define

$$\lambda' = \gamma_\nu \circ \lambda\restriction_B\colon B \to \mathfrak{G}.$$

If we put $L = A *_C B$, and define $\rho\colon L \to \mathfrak{G}$ as an extension of $\lambda\restriction_A$ and λ', then there exists $\tilde{\rho}\colon L \to \tilde{\mathfrak{G}}$ such that $p\tilde{\rho} = \rho$.

(2) Suppose there exists a subgroup C of L' such that $\lambda(C) \leqslant \mu$. Let us define

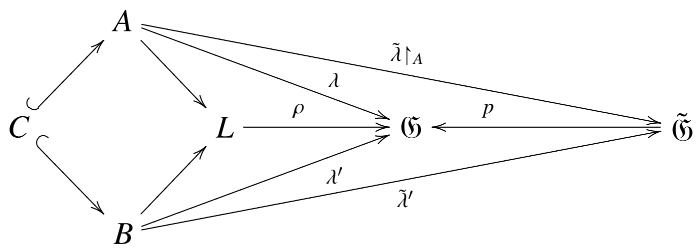
$$\lambda' = \lambda\restriction_C \times \delta_\nu\colon C \times \mathbb{Z} \to \mathfrak{G}.$$

If we put $L = L' *_C$, and define $\rho\colon L \to \mathfrak{G}$ as an extension of $\lambda\restriction_{L'}$ and λ', then there exists $\tilde{\rho}\colon L \to \tilde{\mathfrak{G}}$ such that $p\tilde{\rho} = \rho$.

For the relationship between groups in the lemma, see the commutative diagram below.

Proof of Lemma 3.23. We fix $\tilde{\nu} \in p^{-1}(\nu)$. First note that ρ is uniquely determined by the given conditions.

(1) Let $\tilde{\lambda}' = \gamma_{\tilde{\nu}} \circ \tilde{\lambda}\restriction_B$. Using the assumption that $p^{-1}(\mu)$ is abelian, we obtain the following commutative diagram.



So we obtain a lift $\tilde{\rho}\colon L \to \tilde{\mathfrak{G}}$ from the universality of L. We have $p\tilde{\rho} = \rho$ from the uniqueness of ρ.

(2) We define $A = L'$ and $B = C \times \langle s \rangle$, so that L satisfies $L = A *_C B$. Define $\tilde{\lambda}'\colon B \to \mathfrak{G}$ by

$$\tilde{\lambda}'(cs^k) = \tilde{\lambda}(c)\tilde{\nu}^k, \quad \text{for } c \in C, \text{ and } k \in \mathbb{Z}.$$

The rest of the proof is identical to (1) in verbatim. □

To prove Theorem 3.5, it now suffices to show that each group is flexible. Then the liftability part will be a trivial consequence of Lemma 3.23.

3.8. **Proof of combination theorems.** We are now in a position to prove our first combination theorem. The *word length* of an element w in a free (or amalgamated free) product $G * H$ (or $G *_A H$) is the smallest k such that

$$w = a_1 a_2 \cdots a_k,$$

where $a_i \in G \cup H$ for each i. Note that $\mathrm{PSL}_2(\mathbb{R})$ is an irreducible variety since it is connected. In particular, the notion of being *very general* and that of being *abundant* coincide in $\mathrm{PSL}_2(\mathbb{R})$.

Proof of Theorem 3.5 (1) and (2). The part (1) is in Example 3.3.

For the part (2), we let g_0 be an anchor of L and $R \subseteq \mathbb{R}$ be a countable set. We can find a faithful, almost R–free (and liftable if $\mathscr{H} = \mathscr{F}^\sim$) representation

$$\lambda \in \mathrm{Hom}(L, \mathrm{PSL}_2(\mathbb{R}))$$

such that $\lambda(g_0)$ is an irrational rotation. There is $N > 0$ such that

$$h_0 = g_0^N \in H.$$

Then $\lambda\restriction_H$ is a faithful, almost R–free (and liftable if $\mathscr{H} = \mathscr{F}^\sim$) such that $\lambda(h_0)$ is an irrational rotation. □

The parts (3) and (4) of Theorem 3.5 is an immediate consequence of a stronger result, that is, Lemma 3.25. After the lemma, then we have only to note that the liftability follows from the universality of free products, and also that the existence of an anchor is guaranteed by an anchor of the group A in the statement.

We first make the following observation as a warm-up, which is interesting in its own.

Lemma 3.24. *Suppose $\alpha\colon A \to \mathrm{PSL}_2(\mathbb{R})$ is a faithful representation of a nontrivial finitely generated group A. For each $v \in \mathrm{PSL}_2(\mathbb{R})$, we let $L = A * \mathbb{Z}$ and*

$$\rho_v = \alpha * \delta_v\colon L \to \mathrm{PSL}_2(\mathbb{R}).$$

Then for each countable subset R of $\mathbb{R}$, the set

$$\mathscr{D} = \{v \in \mathrm{PSL}_2(\mathbb{R}) \mid \rho_v \text{ is faithful and } \operatorname{tr} \rho_v(L \backslash A^L) \cap R = \varnothing\}$$

is very general in $\mathrm{PSL}_2(\mathbb{R})$. If, furthermore, α is almost R–free, then so is ρ_v for each $v \in \mathscr{D}$.

Proof of Lemma 3.24. The crucial step of the proof is the following claim.

Claim. *If*

$$g \in L \backslash A^L,$$

then there exists a hyperbolic one–parameter subgroup μ such that for each $x \in \mathbb{R}$ the set

$$U_x = \{v \in \mu \mid \operatorname{tr} \circ \rho_v(g) = x\}$$

is closed and nowhere dense in μ.

We may assume $g \notin \mathbb{Z}^L$. Let us write

$$\rho_v(g) = \prod_{i=1}^{k} g_i v^{m_i}$$

for some $k > 0$, for elements $g_i \in A \backslash 1$, and for exponents $m_i \in \mathbb{Z} \backslash 0$, possibly after conjugation. Choose a point

$$P \in S^1 \backslash \cup_i \operatorname{Fix} g_i.$$

Then for some open neighborhood U of P, we have

$$U \cap (\cup_i g_i U) = \varnothing.$$

There exists a hyperbolic one-parameter group $\mu(\mathbb{R})$ such that $\operatorname{Fix} \mu \in U$, and so,

$$\operatorname{Fix} \mu \cap (\cup_i g_i \operatorname{Fix} \mu) = \varnothing.$$

By Lemma 3.19, we see that

$$\ell \circ \rho_{\mu(t)}(g) \to \infty$$

for $|t| \to \infty$. In particular, U_x is proper and hence, closed nowhere dense.

From Lemma 3.8, we see

$$\mathrm{PSL}_2(\mathbb{R})\backslash\mathscr{D} = \bigcup_{x\in R}\bigcup_{g\in L\backslash A^L}\{\nu\in\mathrm{PSL}_2(\mathbb{R}) \mid \mathrm{tr}\circ\rho_\nu(g) = x\}$$

is an F_σ set with empty interior. In particular, $\mathscr{D}$ is very general.

If $g \in L$ is conjugate to $h \in A$, then $\rho_\nu(g)$ is conjugate to $\alpha(h)$. It follows that if α is almost R–free, then so is ρ_ν for each $\nu \in \mathscr{D}$. ☐

Lemma 3.25. *Suppose $\alpha\colon A \to \mathrm{PSL}_2(\mathbb{R})$ and $\beta\colon B \to \mathrm{PSL}_2(\mathbb{R})$ are faithful representations of nontrivial finitely generated groups A and B. For each $\nu \in \mathfrak{G}$, we let $L = A * B$ and*

$$\rho_\nu = \alpha * (\gamma_\nu\circ\beta)\colon L \to \mathrm{PSL}_2(\mathbb{R}).$$

Then for each countable subset R of $\mathbb{R}$, the set

$$\mathscr{D} = \{\nu\in\mathrm{PSL}_2(\mathbb{R}) \mid \rho_\nu \text{ is faithful and } \mathrm{tr}\,\rho_\nu(L\backslash(A\cup B)^L)\cap R = \varnothing\}$$

is very general in $\mathrm{PSL}_2(\mathbb{R})$. *If, furthermore, α and β are almost R–free, then so is ρ_ν for each $\nu \in \mathscr{D}$.*

Proof. We first prove the following.

Claim. *If*

$$g \in L\backslash(A\cup B)^L,$$

then there exists a hyperbolic one–parameter subgroup μ such that for each $x \in \mathbb{R}$ the set

$$U_x = \{\nu\in\mu \mid \mathrm{tr}\circ\rho_\nu(g) = x\}$$

is closed and nowhere dense in μ.

Let us write

$$\rho_\nu(g) = a_1\nu^{-1}b_1\nu a_2\nu^{-1}b_2\nu\cdots\nu^{-1}b_k\nu$$

for some $k \geqslant 1$, for some $a_i \in A\backslash 1$ and for some $b_i \in B\backslash 1$, possibly after conjugating suitably. As in Lemma 3.24, there exists a hyperbolic one-parameter group $\mu(\mathbb{R})$ such that

$$\mathrm{Fix}\,\mu \cap (\cup_i(a_i\,\mathrm{Fix}\,\mu \cup b_i\,\mathrm{Fix}\,\mu)) = \varnothing.$$

We can apply Lemma 3.19 to see that U_x is proper and hence closed nowhere dense.

From Lemma 3.8, we see

$$\mathrm{PSL}_2(\mathbb{R})\backslash\mathscr{D} = \bigcup_{x\in R}\bigcup_{g\in L\backslash(A\cup B)^L}\{\nu\in\mathrm{PSL}_2(\mathbb{R}) \mid \mathrm{tr}\circ\rho_\nu(g) = x\}$$

is an F_σ set with empty interior.

The almost R–free part of the lemma is obvious as in the proof of Lemma 3.24. ☐

We let $N_A(B)$ denote the normalizer of B in A. We denote by $Z_A(c)$ the centralizer of $c \in A$.

Lemma 3.26. *Let A be a group and C be a maximal abelian subgroup of A. If there exists an embedding $\lambda\colon A \to \mathrm{PSL}_2(\mathbb{R})$, then the following hold.*

(1) For each $c \in C\setminus 1$, we have $C = Z_A(c)$.
(2) The following two conditions are equivalent:
  (i) $N_A(C) = C$.
  (ii) C is malnormal in A.

Proof. (1) There uniquely exists a one-parameter subgroup μ containing $\lambda(C)$. So we have

$$\lambda(Z_A(c)) = Z_{\lambda(A)}(\lambda(c)) = \lambda(A) \cap \mu \leqslant \lambda(C)$$

by the maximality of C. The opposite inclusion $C \leqslant Z_A(c)$ is obvious from that C is abelian.

(2) Let us first prove (i) $\Rightarrow$(ii). Suppose C is not malnormal in A. Then for some $g \in A\setminus C$ and for some $c, c' \in C\setminus 1$, we have $c^g = c'$. We have

$$C = Z_A(c') = Z_A(c^g) = Z_A(c)^g = C^g.$$

It follows that $g \in N_A(C)$ and hence, $N_A(C) \neq C$.

The opposite implication is immediate from the definition of malnormality. ◻

The following two lemmas strengthen the parts (5) and (6) of Theorem 3.5. The liftability in Theorem 3.5 is an immediate consequence of Lemma 3.23. We postpone the proof of the existence of an anchor here. We will give such a proof in detail for a more general setting later; see the proof of Theorem 5.7.

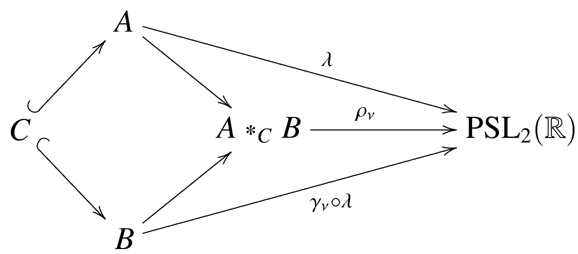
Lemma 3.27. *We let L' be a group and let A, B be finitely generated nontrivial subgroups of L' satisfying the following:*

(i) There is a subgroup $C \leqslant A \cap B$ such that C is malnormal maximal abelian in A and in B;
(ii) There is a representation $\lambda\colon L' \to \mathrm{PSL}_2(\mathbb{R})$ such that each of the restrictions $\lambda\restriction_A$ and $\lambda\restriction_B$ is faithful and parabolic-free.

Let μ be the one-parameter group containing $\lambda(C)$, and for each $\nu \in \mu$, we define

$$\rho_\nu\colon L = A *_C B \to \mathrm{PSL}_2(\mathbb{R})$$

by the following commutative diagram:



Then for each countable subset R of $\mathbb{R}$*, the set*

$$\mathscr{D} = \{\nu \in \mu \mid \rho_\nu \text{ is faithful and } \operatorname{tr}\rho_\nu(L\backslash(A \cup B)^L) \cap R = \varnothing\}$$

is very general in μ*. If, furthermore,* α *and* β *are almost R–free, then so is* ρ_ν *for each* $\nu \in \mathscr{D}$*.*

Lemma 3.28. *We let A be a finitely generated group such that the following hold:*

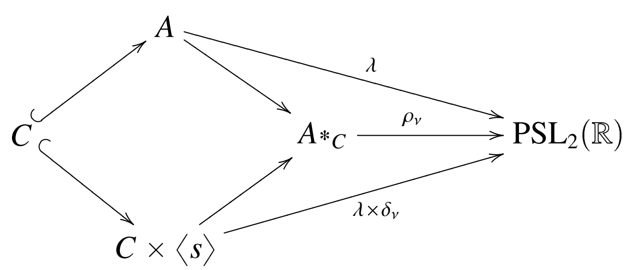
(i) there is $C \leqslant A$ *which is malnormal maximal abelian in A;*

(ii) there is a faithful and parabolic-free representation $\lambda\colon A \to \mathrm{PSL}_2(\mathbb{R})$*.*

Let μ *be the one-parameter group containing* $\lambda(C)$*, and for each* $\nu \in \mu$*, we define*

$$\rho_\nu\colon L = A*_C \to \mathrm{PSL}_2(\mathbb{R})$$

by the following commutative diagram:



Then for each countable subset R of $\mathbb{R}$*, the set*

$$\mathscr{D} = \{\nu \in \mu \mid \rho_\nu \text{ is faithful and } \operatorname{tr}\rho_\nu(L\backslash A^L) \cap R = \varnothing\}$$

is very general in μ*. If, furthermore,* λ *is almost R–free, then so is* ρ_ν *for each* $\nu \in \mathscr{D}$*.*

Proof of Lemma 3.27. Denote by $\hat{\mu}$ the normalizer of μ. As in Lemma 3.25, we first observe the following.

Claim. *If* $g \in L\backslash(A \cup B)^L$*, then for each* $x \in S$ *the set*

$$U_x = \{\nu \in \mu \mid \operatorname{tr} \circ \rho_\nu(g) = x\}$$

is closed and nowhere dense in μ*.*

In order to prove the claim, let us write

$$g = \prod_{i=1}^{k} a_i b_i$$

for some $k \geqslant 1$, for some elements $a_i \in A\backslash C$, and for some elements $b_i \in B\backslash C$ possibly after a conjugating suitably. By Lemma 3.26, we have $N_A(C) = C = N_B(C)$ as subgroups of L'. We see that

$$\lambda((A \cup B)\backslash C) \cap \hat{\mu} = \varnothing.$$

Let us fix $c \in C\backslash 1$. Assume that for some

$$h \in (A \cup B)\backslash C$$

we have that

$$\operatorname{tr} \lambda[c^h, c] = \pm 2.$$

Since parabolics are of infinite order, so is $\lambda[c^h, c]$. Since $\lambda\restriction_A$ and $\lambda\restriction_B$ are faithful and parabolic-free, it follows that $[c^h, c] = 1$. This contradicts that $\lambda(h) \notin \hat{\mu}$. We thus see that

$$\operatorname{tr} \lambda[c^h, c] \neq \pm 2$$

for all $h \in (A \cup B)\backslash C$. The claim follows from Lemma 3.19.

We see

$$\mathrm{PSL}_2(\mathbb{R})\backslash\mathscr{D} = \bigcup_{x\in R} \bigcup_{g\in L\backslash(A\cup B)^L} \{\nu \in \mu \mid \operatorname{tr} \circ\rho_\nu(g) = x\}.$$

is an F_σ set with empty interior. □

Proof of Lemma 3.28. Put $L = A*_C$. Denote by $\hat{\mu}$ the normalizer of μ.have As in Lemma 3.27, we first observe the following.

Claim. *If $g \in L\backslash A^L$, then for each $x \in S$ the set*

$$U_x = \{\nu \in \mu \mid \operatorname{tr} \circ\rho_\nu(g) = x\}$$

is closed and nowhere dense in μ.

Let us first consider the case when g is conjugate into $C \times \langle s\rangle$. There exist some $c \in C$ and some $k \in \mathbb{Z}\backslash 0$ such that $g = cs^k$, possibly after a conjugating suitably. We have

$$\operatorname{tr} \circ\rho_\nu(g) = \operatorname{tr}(c\nu^k) \in \operatorname{tr}(\mu)$$

for $\nu \in \mu$. The conclusion is obvious in this case.

As the second case of the claim, let us assume g is not conjugate into $C \times \langle s\rangle$ so that

$$g = \prod_{i=1}^{k} a_i \nu^{m_i}$$

for some $k \geqslant 1$, for some elements $a_i \in A\backslash C$, and for some elements $m_i \in \mathbb{Z}\backslash 0$ possibly after a suitable conjugation. By Lemma 3.26 again, we see that

$$\lambda(A\backslash C) \cap \hat{\mu} = \varnothing.$$

Let us fix $c \in C\backslash 1$. Assume that for some $h \in A\backslash C$ we have

$$\operatorname{tr} \lambda[c^h, c] \pm 2.$$

We know $\lambda[c^h, c]$ is not torsion. Since λ is faithful and parabolic-free, it follows that $[c^h, c] = 1$. This contradicts the fact that $\lambda(h) \notin \hat{\mu}$. So, we obtain

$$\operatorname{tr} \lambda[c^h, c] \neq \pm 2$$

for all $h \in A\backslash C$. The claim follows from Lemma 3.19.

The rest of the proof follows the same line of reasoning. Namely, we decompose $\mathscr{D}$ into a countable union of closed nowhere dense subsets, by separately considering the elements of L that is conjugate into A and also the elements that is not. □

Example 3.29. Let

$$\rho\colon F_2 = \langle a, b\rangle \to \mathrm{PSL}_2(\mathbb{R})$$

be a faithful parabolic-free representation (for example, a Schottky group), and let $\mu\colon \mathbb{R} \to \mathrm{PSL}_2(\mathbb{R})$ be the one-parameter subgroup containing $\rho([a, b])$. Recall we have a presentation

$$\pi_1(S_2) = \langle a, b, c, d | [a, b] = [c, d]\rangle.$$

By Lemma 3.27, the following representation

$$\rho_\nu\colon \pi_1(S_2) \to \mathrm{PSL}_2(\mathbb{R})$$

is faithful, parabolic-free and liftable for a very general $\nu \in \mu(\mathbb{R})$:

$$(a, b, c, d) \mapsto (\rho(a), \rho(b), \rho(a)^\nu, \rho(b)^\nu).$$

This illustrates a general strategy behind Corollary 3.7. Note that discrete faithful representations of closed surface groups are not liftable [39].

3.9. **A combination theorem for smooth actions.** Let $\mathrm{Diff}^\infty(M)$ denote the group of C^∞ orientation preserving diffeomorphisms of a one-manifold M. Recall that $G \leqslant \mathrm{Diff}^\infty(S^1)$ is *fully supported* if the fixed point set of each element $g \in G\backslash 1$ has empty interior. A group

$$G \subseteq \mathrm{Diff}^\infty(S^1)$$

has a free orbit if G acts freely on some G–orbit. Following the strategy of [40, Corollary 2.6] and [38], we will use the Baire category argument in the theorem below.

Theorem 3.30. *Suppose G and H are countable, fully supported subgroups of the group $\mathrm{Diff}^\infty(S^1)$. Then for a very general choice of*

$$(\psi, x) \in \mathrm{Diff}^\infty(S^1) \times S^1,$$

the group $\langle G, H^\psi\rangle$ is isomorphic to $G * H$ and has $\langle G, H^\psi\rangle \cdot x$ as a free orbit. In particular, $G * H$ admits a faithful C^∞ action on S^1.

Theorem 3.30 seems to be a folklore theorem, the details of whose proof we record here for the convenience of the reader. In the proof below, for $w \in G * H$, we let $\|w\|$ denote the word–length of w.

Proof. For $\psi \in \mathrm{Diff}^\infty(S^1)$, define $\rho_\psi : G * H \to \langle G, H^\psi\rangle$ by the following:

$$\rho_\psi \restriction_G = \mathrm{Id}_G, \quad \rho_\psi(h) = h^\psi \text{ for } h \in H.$$

Let

$$X = \mathrm{Diff}^\infty(S^1) \times S^1.$$

Since $\mathrm{Diff}^\infty(S^1)$ is a Frechét space [42], we see X is Baire. For each

$$w \in G * H \backslash 1$$

we define a closed set

$$X_w = \{(w, x) \in X \mid \rho_\psi(w)(x) = x\}.$$

By the Baire Category Theorem, it suffices to show the following.

Claim. *X_w is nowhere dense for $w \neq 1$.*

Let w be a shortest word such that X_w has a nonempty interior. Pick a nonempty open set $V \subseteq X_w$. If

$$w = g \in G \backslash 1,$$

then we have

$$X_g = \mathrm{Diff}^\infty(S^1) \times \mathrm{Fix}\, g$$

is nowhere dense. If

$$w = h \in H \backslash 1,$$

then we obtain

$$X_h = \{(\psi, x) \in X \mid \psi^{-1} h \psi(x) = x\} = \bigcup \{\psi \times \psi^{-1} \mathrm{Fix}\, h \mid \psi \in \mathrm{Diff}^\infty(S^1)\}$$

is nowhere dense. So, we conclude that $\|w\| \geqslant 2$.

For $u, v \in G * H$, we let

$$Y_{u,v} = \{(\psi, x) \in X \mid \rho_\psi(u)\rho_\psi(v)(x) = \rho_\psi(v)(x)\}.$$

There exists a continuous map $\Phi_v : X_u \to Y_{u,v}$ defined by

$$(\psi, x) \mapsto (\psi, \rho_\psi(v)^{-1}(x)).$$

As $\Phi_{v^{-1}} \circ \Phi_v$ is the identity, we have a homeomorphism $X_u \to Y_{u,v}$. By minimality, the set

$$Y = \bigcup \{Y_{u,v} \mid \|u\|, \|v\| < \|w\|, \text{ and } u \neq 1\}$$

is a countable union of closed nowhere dense sets, and hence has empty interior. So $V_0 = V \backslash Y$ is nonempty.

For each $u \in G * H$, we have

$$X_{uwu^{-1}} = Y_{w,u^{-1}} \approx X_w.$$

So, we may assume

$$\rho_\psi(w) = \psi^{-1} h_k \psi g_k \cdots \psi^{-1} h_1 \psi g_1$$

possibly after a conjugation. In particular, $\|w\| = 2k$. Let us pick an arbitrary $(\psi, x_0) \in V_0$. Define

$$x_1, \ldots, x_{2k}$$

by

$$x_{2i+1} = g_i(x_{2i})$$

and by

$$x_{2i+2} = \psi^{-1} h_i \psi(x_{2i+1}).$$

By assumption, $x_{2k} = \rho_\psi(w) = x_0$. Moreover, the points

$$x_0, x_1, \ldots, x_{2k-1}$$

are all distinct since $V_0 \cap Y_{u,v} = \varnothing$ for $\|u\|, \|v\| < \|w\|$.

Note that

$$\psi(x_0) = \psi(x_{2k}) = h_k \psi(x_{2k-1}).$$

Choose a small closed interval J containing x_0 such that $x_i \notin J$ for each $0 < i < 2k$. By perturbing ψ, we can pick $(\phi, x_0) \in V$ such that

$$\phi\restriction_{S^1 \backslash J} = \psi\restriction_{S^1 \backslash J}$$

and

$$\phi(x_0) \neq h_k \psi(x_{2k-1}).$$

If $i \leqslant k$, then

$$\phi^{-1} h_i \phi g_i(x_{2i-2}) = \phi^{-1} h_i \phi(x_{2i-1}) = \phi^{-1} h_i \psi(x_{2i-1}) = \psi^{-1} h_i \psi(x_{2i-1}) = x_{2i}$$

since $h_i \psi(x_{2i-1}) \notin \psi(J)$ and since

$$\phi^{-1}\restriction_{S^1 \backslash \psi(J)} = \psi^{-1}\restriction_{S^1 \backslash \psi(J)}.$$

Hence,

$$\begin{aligned} \rho_\phi(w)(x_0) &= \prod_{i=k}^{1} \phi^{-1} h_i \phi g_i(x_0) = \prod_{i=k}^{2} \phi^{-1} h_i \phi g_i(x_2) = \cdots \\ &= \phi^{-1} h_k \phi g_k(x_{2k-2}) = \phi^{-1} h_k \psi(x_{2k-1}) \neq x_0. \end{aligned}$$

So $\phi \notin X_w$. We have a contradiction since we have assumed $(\phi, x_0) \in V$. □

Theorem 3.30 implies that if $G \leqslant \mathrm{Diff}^\infty(S^1)$ is fully supported then $G * \mathbb{Z}$ embeds into $\mathrm{Diff}^\infty(S^1)$ as a subgroup admitting almost every point as a free orbit. One actually has a genericity statement as below.

Corollary 3.31. *Let $G \leqslant \mathrm{Diff}^\infty(S^1)$ be countable and fully supported. Then for a very general choice of*

$$(\psi, x) \in \mathrm{Diff}^\infty(S^1) \times S^1,$$

the group $\langle G, \psi\rangle$ is isomorphic to $G * \mathbb{Z}$ and has $\langle G, \psi\rangle \cdot x$ as a free orbit.

Proof. Recall our notation $\mathbb{Z} = \langle s\rangle$. If $w \in G * \mathbb{Z}$, we let $\|w\|$ denote the word length in the following sense

$$\|w\| = \min\{\ell \mid w = s_1 s_2 \cdots s_\ell, \quad s_i \in G \cup \{s\}^{\pm 1}\}.$$

For $\psi \in \mathrm{Diff}^\infty(S^1)$, define

$$\rho_\psi : G * \langle s\rangle \to \mathrm{Diff}^\infty(S^1)$$

by $\rho_\psi \restriction_G = \mathrm{Id}_G$ and $\rho_\psi(s) = \psi$. Then we can proceed very similarly to the proof of Theorem 3.30, so we omit the details. □

4. Exotic circle actions from flexibility

In this section we provide uncountably many distinct exotic circle actions of flexible groups, and make a connection to the existence of quasi-morphisms on those groups. We will prove that most Fuchsian groups and their iterated generalized doubles are flexible. Limit groups will be shown to be liftable-flexible.

4.1. Flexibility and Exotic Circle Actions.

For a group L, an element $g \in L$ and a set $\Lambda \subseteq \mathrm{Hom}(L, \mathrm{PSL}_2(\mathbb{R}))$, we let

$$\mathrm{rot}\,\Lambda(g) = \{\mathrm{rot} \circ \lambda(g) \mid \lambda \in \Lambda\}.$$

Each flexible group admits uncountably many distinct faithful representations satisfying the following strong form of non-conjugacy. Recall the statement of Proposition 1.6 from the introduction:

Proposition 4.1. *If L is a flexible group, then there exists $g_0 \in L$ and a uncountable set*

$$\Lambda \subseteq \mathrm{Hom}(L, \mathrm{PSL}_2(\mathbb{R}))$$

of faithful parabolic–free representations such that all of the following hold:

(i) for each $\lambda \in \Lambda$, each element of $\Sigma(\lambda)$ is either rational or transcendental;
(ii) the set $\mathrm{rot}\,\Lambda(g_0)$ is $\mathbb{Z}$–linearly independent.

(iii) for all distinct $\lambda, \lambda' \in \Lambda$*, we have*

$$\Sigma(\lambda) \cap \Sigma(\lambda') \subseteq \mathbb{Q};$$

(iv) for all $g \in L$*, and for all distinct* $\lambda, \lambda' \in \Lambda$*, the elements* $\lambda(g_0)$ *and* $\lambda'(g)$ *are not conjugate in* $\mathrm{Homeo}^+(S^1)$*;*

(v) for all elements

$$g, h \in L \backslash T(L)$$

and for all distinct

$$\lambda, \lambda' \in \Lambda,$$

the elements $\lambda(g)$ *and* $\lambda'(h)$ *are not conjugate in* $\mathrm{PSL}_2(\mathbb{R})$*;*

Moreover, if L is liftable-flexible, then we can further require that each representation in Λ *is liftable.*

Proof. Let g_0 be an anchor of L. We define

$$\Lambda \subseteq \mathrm{Hom}(L, \mathrm{PSL}_2(\mathbb{R}))$$

be a maximal set of faithful parabolic–free representations such that all of the following conditions hold:

(A) for each λ, the element $\lambda(g_0)$ is an irrational rotation;
(B) for each $\lambda \in \Lambda_0$, the set $\mathrm{rot} \circ \lambda(L \backslash T(L))$ contains transcendental numbers only;
(C) the set $\mathrm{rot}\, \Lambda(g_0)$ is $\mathbb{Z}$–linearly independent;
(D) for all distinct $\lambda, \lambda' \in \Lambda$, we have

$$\mathrm{tr} \circ \lambda(L \backslash T(L)) \cap \mathrm{tr} \circ \lambda'(L \backslash T(L)) = \varnothing;$$

(E) We further require that each map in Λ is liftable if L is liftable-flexible.

Let R be the set of all algebraic numbers, and

$$R' = \{\mathrm{tr} \circ \mathrm{Rot}(2\pi u) \mid u \in R\}.$$

By applying Definition 3.1 to R', we have some

$$\lambda \in \mathrm{Hom}(L, \mathrm{PSL}_2(\mathbb{R}))$$

such that $\lambda(g_0)$ is an irrational rotation and

$$\mathrm{rot} \circ \lambda(L \backslash T(L))$$

contains no algebraic numbers. It follows that Λ is not empty.

Suppose that one can enumerate the elements of Λ as

$$\Lambda = \{\lambda_1, \lambda_2, \ldots\}.$$

We then set

$$S = \{\mathrm{tr} \circ \lambda_i(g) \mid i \in \mathbb{N} \text{ and } g \in L\}.$$

Let S' be the $\mathbb{Z}$–span of

$$\{\mathrm{rot}\circ\lambda_i(g) \mid i \in \mathbb{N} \text{ and } g \in L\}.$$

We define

$$S'' = \{\mathrm{tr}\circ\mathrm{Rot}(2\pi u) \mid u \in S'\}.$$

By flexibility, we have a faithful almost $(S \cup S'')$–free representation

$$\lambda \in \mathrm{Hom}(L, \mathrm{PSL}_2(\mathbb{R})).$$

This contradicts the maximality of Λ. So we have that Λ is uncountable.

The conclusion (i) and (ii) follow from (B) and (C). For (iii), suppose

$$\mathrm{rot}\circ\lambda(g) = \mathrm{rot}\circ\lambda'(h) \notin \mathbb{Q}$$

for some $g, h \in L$. Then

$$g, h \notin T(L)$$

since the rotation numbers of torsion elements are rational. We have that $\lambda(g)$ and $\lambda'(h)$ are elliptics with the same trace. This contradicts (D).

For (iv), suppose

$$\mathrm{rot}\circ\lambda(g_0) = \mathrm{rot}\circ\lambda'(g).$$

Note that $\lambda(g_0)$ is an irrational rotation. So $g \notin T(L)$. However, (D) implies that

$$\mathrm{tr}\circ\lambda(g_0) \neq \mathrm{tr}\circ\lambda'(g)$$

whence we have a contradiction.

For (v), we simply note

$$\mathrm{tr}\circ\lambda(g) \neq \mathrm{tr}\circ\lambda'(h).$$

□

Remark 4.2. In the above proof, if L is non–virtually solvable, then the existence of an anchor and Lemma 6.1 together imply that each $\lambda \in \Lambda$ is indiscrete.

4.2. **Flexibility and Quasi-morphisms.** Let us note a cohomological observation.

Lemma 4.3. *Let L be a countable group and $g_0 \in L$. Suppose*

$$\Lambda \subseteq \mathrm{Hom}(L, \mathrm{PSL}_2(\mathbb{R}))$$

satisfies that

$$\{\mathrm{rot}\circ\lambda(g_0) \mid \lambda \in \Lambda\}$$

is $\mathbb{Z}$–linearly independent. Then

$$\{\rho^*\mathrm{eu}_b \colon \rho \in \Lambda\}$$

is linearly independent in $H^2_b(L;\mathbb{Z})$.

Proof. Let us suppose that

$$k_0\rho_0^*\mathrm{eu}_b = \sum_{i=1}^{n} k_i\rho_i^*\mathrm{eu}_b$$

for some $\rho_i \in \Lambda$ and nonzero integers k_i. Then there exists a bounded map $\beta\colon G \to \mathbb{Z}$ such that

$$k_0\rho_0^*\mathrm{eu} = \sum_{i=1}^{n} k_i\rho_i^*\mathrm{eu} + \partial\beta$$

as two–cocycles. By Lemma 2.11, we have

$$\begin{aligned}
\operatorname{rot}\rho_0(g_0) &= \lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\rho_0^*\mathrm{eu}(g_0, g_0^{j-1})\\
&= \lim_{N\to\infty}\frac{1}{k_0N}\sum_{j=1}^{N}\left(\sum_{i=1}^{n}k_i\rho_i^*\mathrm{eu}(g_0,g_0^{j-1}) + \beta(g_0) + \beta(g_0^{j-1}) - \beta(g_0^{j})\right)\\
&= \frac{1}{k_0}\beta(g_0) + \sum_{i=1}^{n}\frac{k_i}{k_0}\operatorname{rot}\rho_i(g_0).
\end{aligned}$$

This contradicts the $\mathbb{Z}$-linear independence hypothesis. □

It is known that $\mathrm{HQM}(L;\mathbb{Z})$ is infinite dimensional if L is a nontrivial free group, a nonelementary word-hyperbolic group or a mapping class group [13, 29, 6]. Our construction shows that for a liftable-flexible L, the abelian group $\mathrm{HQM}(L;\mathbb{Z})$ has uncountable dimension even when we count only defect–one subadditive quasimorphisms:

Corollary 4.4. *If L is liftable-flexible, then there exists a set Λ of integer-valued subadditive defect–one quasimorphisms such that*

$$\{[\lambda] \in \mathrm{HQM}(L;\mathbb{Z}) \mid \lambda \in \Lambda\}$$

is uncountable and linearly independent.

Proof. This is immediate from Proposition 4.1 (ii), along with Lemmas 2.20 and 4.3. □

4.3. **Limit groups.**

Definition 4.5 (cf. [24]). (1) An epimorphism $\phi\colon L \to L'$ is called a *generalized double over L'* if $L = A *_C B$ for some finitely generated nontrivial groups A and B such that C is maximal abelian both in A and in B, and such that $\phi\restriction_A$ and $\phi\restriction_B$ are both injective.

(2) An epimorphism $\phi\colon L \to L'$ is called a *centralizer extension* if $L = A*_C$ for some finitely generated nontrivial group A such that C is maximal abelian in A and such that $\phi\restriction_A$ is injective.
(3) If there is a sequence of epimorphisms

$$L = L_n \xrightarrow{f_n}\!\!\!\!\twoheadrightarrow L_{n-1} \xrightarrow{f_{n-1}}\!\!\!\!\twoheadrightarrow \cdots \longrightarrow\!\!\!\!\twoheadrightarrow L_1 \xrightarrow{f_1}\!\!\!\!\twoheadrightarrow L_0,$$

such that each f_i is a generalized double or a centralizer extension, then we say L is an *iterated generalized double over* L_0 and write $L \in \mathrm{IGD}(L_0)$.
(4) In the part (3), we say a group L is an *iterated malnormal generalized double over* L_0 if the amalgamating subgroup of L_i in each step f_i is malnormal in the vertex groups of L_i. We write $L \in \mathrm{IMGD}(L_0)$ in this case.

The following characterizes limit groups as iterated generalized doubles of free groups.

Theorem 4.6 ([64, 24])**.** *Every limit group is in* $\mathrm{IGD}(F_r)$ *for some* r*.*

If μ is a one-parameter subgroup of $\mathrm{PSL}_2(\mathbb{R})$, then $N(\mu)\backslash\mu$ is either empty or consisting entirely of two-torsion. Suppose $L = A *_C B$ (or $L = A*_C$) where C is maximal abelian in A and in B. If L is without 2-torsion, then for each faithful projective representation λ of L, we see that the normalizer of $\lambda(C)$ in $\lambda(A)$ is $\lambda(C)$. Then by Lemma 3.26, the malnormality hypothesis of Theorem 3.5 is already guaranteed. Recall a liftable-flexible group is torsion-free. So we have the following.

Corollary 4.7. *(1) If* L_0 *is flexible, then every group in* $\mathrm{IMGD}(L_0)$ *is flexible.*
(2) If L_0 *is a flexible group without 2-torsion, then every group in* $\mathrm{IGD}(L_0)$ *is flexible.*
(3) If L_0 *is liftable-flexible, then every group in* $\mathrm{IGD}(L_0)$ *is liftable-flexible. In particular, every nontrivial limit group is liftable-flexible.*

In the next section, we will provide more delicate control on the rotation numbers of an anchor in a limit group; see Corollary 5.16.

As a consequence of flexibility, we find uncountably many distinct subadditive quasi-morphisms.

Corollary 4.8. *Every nontrivial limit group admits a set* Λ *of integer-valued subadditive defect–one quasimorphisms such that*

$$\{[\lambda] \in \mathrm{HQM}(L;\mathbb{Z}) \mid \lambda \in \Lambda\}$$

is uncountable and linearly independent.

4.4. **Lattices in** $\mathrm{PSL}_2(\mathbb{R})$**.** A lattice L in $\mathrm{PSL}_2(\mathbb{R})$ can be realized as the fundamental group of a complete hyperbolic 2–orbifold S of finite type.

Theorem 4.9. *(1) Every nonuniform lattice in* $\mathrm{PSL}_2(\mathbb{R})$ *is flexible.*
(2) Every torsion-free lattice in $\mathrm{PSL}_2(\mathbb{R})$ *is liftable-flexible.*

We state a result on uniform lattices with torsion in Theorem 4.14. Let us first note that some uniform lattices in $\mathrm{PSL}_2(\mathbb{R})$ are *not* flexible. For $p, q, r > 0$, we define

$$\Delta(p,q,r) = \langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle.$$

If $p, q, r > 0$ and

$$1/p + 1/q + 1/r < 1,$$

then $\Delta(p,q,r)$ is called a (p,q,r)*-hyperbolic triangle group*. A hyperbolic triangle group can be realized as a uniform lattice in $\mathrm{PSL}_2(\mathbb{R})$.

Proposition 4.10. *Hyperbolic triangle groups are not flexible.*

Proof. Let $p, q, r > 0$ and

$$1/p + 1/q + 1/r < 1.$$

We will show that $\Delta(p,q,r)$ admits only finitely many conjugacy classes of faithful projective representations.

Let

$$\rho\colon \Delta(p,q,r) \to \mathrm{PSL}_2(\mathbb{R})$$

be a faithful representation, and a, b, c denote the standard generators as above. Possibly after applying a suitable conjugation, we may assume that

$$\rho(a) = \mathrm{Rot}(2u\pi/p)$$

and that

$$\rho(b) = \exp(-t)\,\mathrm{Rot}(2v\pi/q)\exp(t)$$

for some

$$u, v \in (0, \max(p,q)) \cap \mathbb{Z}$$

and $t \in \mathbb{R}$. Choose reflections

$$\sigma_0, \sigma_1, \sigma_2 \in \mathrm{PGL}_2(\mathbb{R})$$

such that $\mathrm{Fix}\,\sigma_1$ is the geodesic joining 0 and ∞, such that

$$\rho(a) = \sigma_0\sigma_1,$$

and such that

$$\rho(b) = \sigma_1\sigma_2.$$

Then we obtain

$$1 = \rho(ab)^r = (\sigma_0\sigma_2)^r.$$

It follows that $\operatorname{Fix}\sigma_0$ and $\operatorname{Fix}\sigma_2$ intersects with the angle $2\pi w/r$ for some $w \in (0, r) \cap \mathbb{Z}$. The angles

$$(2\pi u/p, 2\pi v/q, 2\pi w/r)$$

determine the conjugacy class of ρ. ◻

The following group–theoretic lemma will come handy when finding stably injective sequences.

Lemma 4.11. *For a residually finite group G and a finite subset $P \subseteq G$, the natural surjections*

$$\rho_n \colon G \to G/\langle\langle \{g^n \mid g \in P\}\rangle\rangle$$

are stably injective.

Proof. Let $1 \neq h \in G$. There exists a finite index normal subgroup N of G such that $h \notin N$. Then for all $n \geqslant [G : N]$ we have $h \notin \langle\langle\{g^{n!} \mid g \in P\}\rangle\rangle \leqslant N$. ◻

For brevity, we let

$$\zeta_n = \operatorname{Rot}(2\pi/n).$$

Proof of Theorem 4.9. Let us fix a countable subset

$$R = \{r_1, r_2, \ldots\} \subseteq \mathbb{R}$$

containing ± 2. Let $L = \pi_1^{\mathrm{orb}}(S)$ where S is a hyperbolic 2–orbifold.

Case 1. L is non-uniform.

The orbifold S has at least one cusp in this case. As we are only concerned with the isomorphism type of L, we may assume S is planar. That is, we can write

$$L = \langle a_1, \ldots, a_k, b_1, \ldots, b_r \mid a_i^{m_i} = 1 \text{ for each } 1 \leqslant i \leqslant k\rangle \cong \mathop{\ast}_{i=1}^{k} \mathbb{Z}_{m_i} * F_r$$

for some integers $k \geqslant 0$ and $r \geqslant 0$, and for exponents

$$2 \leqslant m_1 \leqslant m_2 \leqslant \ldots, m_k$$

such that $k + r > 0$. Note that

$$\chi(S) = 2 - \sum_{i=1}^{k}(1 - 1/m_i) - (r + 1) < 0.$$

Recall $\mathbb{Z}$ is flexible by Theorem 3.5. So the same theorem implies that L is flexible for $r \geqslant 1$. We now assume $r = 0$. By Theorem 3.5 (4) and an induction on $k \geqslant 2$, it suffices to deal with the case $k = 2$.

Let

$$L = \langle a, b \mid a^m = b^n = 1\rangle.$$

For each $\nu \in \mathrm{PSL}_2(\mathbb{R})$ we define $\rho_\nu(a) = \zeta_m$ and $\rho_\nu(b) = \zeta_n^\nu$. As in Lemma 3.25, let us consider a very general set

$$\mathscr{D} = \{\nu \in \mathrm{PSL}_2(\mathbb{R}) \mid \rho_\nu \text{ is faithful and } \operatorname{tr}\rho_\nu(L\backslash T(L)) \cap R = \varnothing\}.$$

Note that $T(L) = (\langle a\rangle \cup \langle b\rangle)^L$.

Since L is a lattice, we have

$$1/m + 1/n < 1.$$

So for a sufficiently large $p > 0$, we have

$$f_p \colon L \to \mathrm{PSL}_2(\mathbb{R})$$

defined as the composition

$$L \to L/\langle\!\langle (ab)^p \rangle\!\rangle \cong \Delta(m, n, p) \hookrightarrow \mathrm{PSL}_2(\mathbb{R}).$$

We can further require that $f_p(a) = \zeta_m$. Let us write $f_p(b) = \zeta_n^x$ for some $x \in \mathrm{PSL}_2(\mathbb{R})$ so that

$$\operatorname{rot}(\zeta_m \cdot \zeta_n^x) = 1/p.$$

Since the set $\mathscr{D}$ is very general and the map rot is continuous, we can find $y \in \mathscr{D}$ that is sufficiently close to x so that

$$0 < \operatorname{rot}(\zeta_m \cdot \zeta_n^y) < 1.$$

Then ρ_y is a desired almost R–free representation and $g_0 = ab$ is an anchor of L.

We note that if L is torsion-free, then L is a free group in this case, and hence liftable-flexible.

Case 2. L is uniform and torsion-free. We have $L = \pi_1(S)$ for some closed hyperbolic surface S. By Theorem 3.5 (2), it suffices to consider the case that S has genus 2. We have a decomposition

$$L = F_2 *_{\mathbb{Z}} F_2 = \langle a, b\rangle *_{[a,b]=[c,d]} \langle c, d\rangle.$$

So Corollary 3.7 implies that L is liftable-flexible. This strategy was also exhibited in Example 3.29. One can also use Corollary 4.7 for this case. □

Remark 4.12. A connected component of

$$\operatorname{Hom}(\pi_1(S_g), \mathrm{PSL}_2(\mathbb{R}))$$

is precisely determined by its Euler number [39]. Theorem 4.9 shows that the component where the Euler number is zero contains uncountably many distinct semi-conjugacy classes of faithful parabolic-free actions. On the other hand, the components with the maximal Euler numbers consist of single semi-conjugacy classes [54].

Let us now consider uniform lattices with torsion.

Definition 4.13. Let S be a hyperbolic 2–orbifold. We say S is *sporadic* if S has genus g, and no cusps, and $k \geqslant 0$ cone points of orders

$$p_1 \leqslant \cdots \leqslant p_k$$

such that one of the following holds:

(i) $g = 0$ and $k = 3$;
(ii) $g = 0, k = 4$ and $p_1 = p_2 = p_3 = 2$;
(iii) $g = 0, k = 5$ and $p_1 = p_2 = p_3 = p_4 = p_5 = 2$;
(iv) $g = 1$ and $k = 1$;
(v) $g = 1, k = 2$ and $p_1 = p_2 = 2$.

A lattice in $\mathrm{PSL}_2(\mathbb{R})$ is *non-sporadic* if the corresponding orbifold is not sporadic. For example, a torsion-free or a non-uniform lattice in $\mathrm{PSL}_2(\mathbb{R})$ is non-sporadic. There are only finitely many possible homeomorphism types of the underlying surfaces of sporadic hyperbolic 2–orbifolds.

A uniform non-sporadic lattice with torsion exhibits a slightly weaker form of flexibility, which still guarantees uncountably many inequivalent actions:

Theorem 4.14. *Let L be a uniform non-sporadic lattice with torsion. Then there exists a nontrivial decomposition $L = A *_C B$ satisfying the following: for each countable set $R \subseteq \mathbb{R}$ we have a faithful parabolic-free representation*

$$\lambda \colon L \to \mathrm{PSL}_2(\mathbb{R})$$

such that $\lambda(C)$ is nontrivial elliptic and such that

$$\mathrm{tr} \circ \lambda(L \backslash (A \cup B)^L) \cap R = \varnothing.$$

In particular, L admits uncountably many distinct inequivalent indiscrete projective representations.

Proof of Theorem 4.14. We use the notations L, S and R in the proof of Theorem 4.9. Since we are assuming S is non-sporadic, we can decompose $S = S_1 \cup_c S_2$ for some simple closed geodesic c such that each $S_i \backslash c$ is homeomorphic to a complete single-cusped hyperbolic 2–orbifold.

We can write $L = A *_C B$, where here $A \cong \pi_1^{\mathrm{orb}}(S_1)$, where $B \cong \pi_1^{\mathrm{orb}}(S_2)$, and where $C = \langle c \rangle$. The infinite cyclic group C is maximal abelian in A and in B. By considering A as a single-cusped hyperbolic orbifold group and C as its peripheral (and parabolic) subgroup, we deduce that $N_A(C) = C$. Similarly we have $N_B(C) = C$. By Lemma 3.26, we see that C is malnormal in A and in B.

Define an algebraic set

$$X = \{\lambda \in \mathrm{Hom}(L, \mathrm{PSL}_2(\mathbb{R})) \mid \lambda(c) \in \mathrm{SO}(2)\}.$$

For each finite subset

$$Q \subseteq (A \cup B) \backslash 1,$$

we define

$$X(Q) = \{\lambda \in X \mid \operatorname{tr} \circ \lambda(g) = \pm 2 \text{ for some } g \in Q\}.$$

For $p \gg 0$, we let

$$A_p = A/\langle\langle c^p \rangle\rangle$$

and we let

$$B_p = B/\langle\langle c^p \rangle\rangle.$$

There exist discrete faithful cocompact (hence parabolic-free) representations

$$\alpha_p \colon A_p \to \mathrm{PSL}_2(\mathbb{R}), \quad \beta_p \colon B_p \to \mathrm{PSL}_2(\mathbb{R})$$

such that

$$\alpha_p(c) = \beta_p(c) = \zeta_p.$$

Define $f_p \colon L \to \mathrm{PSL}_2(\mathbb{R})$ as the composition

$$L = A *_C B \longrightarrow L/\langle\langle c^p \rangle\rangle = A_p *_{\mathbb{Z}_p} B_p \xrightarrow{\alpha_p * \beta_p} \mathrm{PSL}_2(\mathbb{R}).$$

Then $\{f_p \restriction_A\}_{p \in \mathbb{N}}$ and $\{f_p \restriction_B\}_{p \in \mathbb{N}}$ are both stably injective by Lemma 4.11. Also the images of $f_p \restriction_A$ and $f_p \restriction_B$ do not contain nontrivial parabolics.

If

$$Q \subseteq (A \cup B) \backslash 1$$

is finite, then there exists $p \gg 0$ such that $1 \notin f_p(Q)$. In other words, $f_p \in X \backslash X(Q)$. Note that X has a positive dimension since $f_p \neq f_q$ for $p \neq q$.

We let

$$\mathscr{D} = \{\lambda \in X \mid \operatorname{tr} \circ \lambda(g) \neq \pm 2 \text{ for all } g \in (A \cup B) \backslash 1\}.$$

By Lemma 3.8, we conclude that $\mathscr{D}$ is abundant in X.

By applying Lemma 3.27 to $\lambda \in \mathscr{D}$, we obtain some $\rho \colon L \to \mathrm{PSL}_2(\mathbb{R})$ such that

$$\operatorname{tr} \circ \rho(L \backslash (A \cup B)^L) \cap R = \varnothing$$

and such that

$$\pm 2 \notin \operatorname{tr} \circ \rho((A \cup B)^L \backslash 1).$$

By the definition of X, the element $\rho(c)$ is an irrational rotation.

Let

$$\Lambda \subseteq \mathrm{Hom}(L, \mathrm{PSL}_2(\mathbb{R}))$$

be a maximal subset with all distinct rotation spectra. Since $\operatorname{rot} \lambda(c)$ can avoid an arbitrary countable subset of $\mathbb{R}$, we see that Λ is uncountable. It follows that L admits uncountably many distinct inequivalent indiscrete projective representations. □

4.5. **Simultaneous control of rotation numbers.** So far we have developed techniques of finding uncountably many distinct faithful indiscrete actions of a given group. We will now describe an approach to simultaneous control the rotation numbers of finitely many group elements.

Let

$$\mathscr{E} = \mathrm{tr}^{-1}(-2,2) \subseteq \mathrm{PSL}_2(\mathbb{R})$$

be the set of elliptics. We have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{SO}(2) \times \mathrm{PSL}_2(\mathbb{R}) & \xrightarrow{\ \xi\ } & \mathscr{E} & \hookrightarrow & \mathrm{PSL}_2(\mathbb{R}) \\ & \searrow^{\eta} & \downarrow{\scriptstyle \bar{\eta}} & & \\ & & \mathrm{SO}(2) & & \end{array}$$

where $\xi(x, y) = x^y, \eta(x, y) = x$ and

$$\bar{\eta}(x) = \mathrm{Rot}(2\pi \,\mathrm{rot}(x)).$$

Note that $\mathscr{E}$ and $\bar{\eta}$ are not algebraic.

Theorem 4.15. *Let G be a finitely generated group and $A \subseteq G$ be a nonempty finite set. Suppose there is a stably injective sequence of parabolic-free representations*

$$\{\rho_n : G \to \mathrm{PSL}_2(\mathbb{R})\}_{n \geqslant 1}$$

such that for each $n \geqslant 1$, the set $\rho_n(A)$ consists of elliptics with the same rotation number which belongs to $(0, 1/n]$. Then there exists a uncountable set Λ of faithful parabolic-free representations $G \to \mathrm{PSL}_2(\mathbb{R})$ such that:

(i) *$\lambda(A)$ consists of elliptics with the same rotation number for each $\lambda \in \Lambda$;*
(ii) $\mathrm{rot} \circ \lambda(A) \neq \mathrm{rot} \circ \mu(A)$ *for $\lambda \neq \mu$ in Λ;*
(iii) $\Sigma(\lambda) \neq \Sigma(\mu)$ *for $\lambda \neq \mu$ in Λ;*

Remark 4.16. A flexible group G satisfies the above three conclusions where A is the set consisting of one anchor. See Proposition 4.1.

Proof of Theorem 4.15. We have defined the maps $\xi, \eta, \bar{\eta}$ above. Fix a finite set $B \subseteq G \backslash A$ so that $G = \langle A \cup B \rangle$. Define an algebraic set

$$W = \{(\alpha, \beta) : \alpha \in (\mathrm{SO}(2) \times \mathrm{PSL}_2(\mathbb{R}))^A \text{ and } \beta \in \mathrm{PSL}_2(\mathbb{R})^B$$
$$\text{such that } \eta \circ \alpha(a) = \eta \circ \alpha(a') \text{ for } a, a' \in A\}.$$

Then we can define another algebraic subset $X \subseteq W$ as the set of

$$\chi = (\alpha, \beta) \in W$$

such that there exists

$$\phi_\chi \in \mathrm{Hom}(G, \mathrm{PSL}_2(\mathbb{R}))$$

satisfying the following:

$$\phi_\chi(x) = \begin{cases} \xi \circ \alpha(x), & \text{if } x \in A \\ \beta(x), & \text{if } x \in B. \end{cases}$$

Moreover, each ρ_m can be written as $\rho_m = \phi_{\chi_m}$ for some $\chi_m \in X$. For each finite subset $Q \subseteq G \setminus 1$, we let

$$X(Q) = \{\chi \in X \colon \phi_\chi(Q) \text{ contains a parabolic element}\}.$$

Let us write

$$G \setminus 1 = \bigcup_n Q_n$$

for some increasing sequence of finite subsets $\{Q_n\}$. Since $\{\rho_m\}$ is stably injective and parabolic-free, each $X(Q_n)$ is proper in X. Also, X is not finite by the condition on $\rho_n(A)$. So,

$$X_0 = X \setminus \bigcup_n X(Q_n)$$

is uncountable by Lemma 3.8. Then each ϕ_χ is faithful and parabolic-free for each $\chi \in X_0$.

Let us define

$$\Lambda = \{\rho_\chi \mid \chi \in X_0\},$$

and fix $a \in A$.

Claim. *The set* rot $\Lambda(a)$ *is uncountable.*

Suppose the contrary so that

$$\bar{\eta}\{\lambda(a) \colon \lambda \in \Lambda\} \subseteq \{\zeta_1, \zeta_2, \ldots\} \subseteq \mathrm{SO}(2).$$

Consider an algebraic set

$$Y_n = \{(\alpha, \beta) \in X \colon \eta \circ \alpha(a) = \zeta_j \text{ for some } j \leqslant n\}$$

so that

$$X_0 \subseteq \bigcup_{n \geqslant 1} Y_n.$$

Let $\chi_m = (\alpha_m, \beta_m)$ so that

$$\rho_m(a) = \phi_{\chi_m}(a) = \xi \circ \alpha_m(a).$$

By the hypothesis, the set

$$\{\eta \circ \alpha_m(a) = \bar{\eta} \circ \rho_m(a) \colon m \geqslant 1\}$$

is infinite and so, each Y_n is proper in X. We have

$$X = X_0 \cup \left(\bigcup_{n \geqslant 1} X(Q_n) \right) \subseteq \bigcup_{n \geqslant 1} Y_n \cup \left(\bigcup_{n \geqslant 1} X(Q_n) \right) \subseteq X.$$

This contradicts Lemma 3.8, and the claim is proved.

Since $X \subseteq W$, we see that the conclusion (i) holds for (every subset of) Λ. Let Ω be a maximal subset of Λ such that the following hold for all distinct $\lambda, \mu \in \Omega$:

(a) $\operatorname{rot} \circ \lambda(a) \neq \operatorname{rot} \circ \mu(a)$;
(b) $\Sigma(\lambda) \neq \Sigma(\mu)$.

Suppose Ω is countable. Then

$$U = \bigcup \{\Sigma(\mu) \mid \mu \in \Omega\} \subseteq \mathbb{R}/\mathbb{Z}$$

is countable. Since $\operatorname{rot} \Lambda(a)$ is uncountable, there exists $\lambda \in \Lambda \backslash \Omega$ such that

$$\operatorname{rot} \circ \lambda(a) \notin U.$$

By maximality, the set $\Omega \cup \{\lambda\}$ violates the condition (a) or (b) above. Since

$$\operatorname{rot} \lambda(a) \neq \operatorname{rot} \mu(g)$$

for all $g \in G$ and $\mu \in \Omega$, we see $\Sigma(\lambda) = \Sigma(\mu)$ for some $\mu \in \Omega$. This is a contradiction since

$$\operatorname{rot} \circ \lambda(a) \in \Sigma(\lambda).$$

It follows that Ω is uncountable. □

Remark 4.17. In Theorem 4.15, we have the condition that "$\rho(A)$ consists of elliptics with the same rotation number" for $\rho = \rho_n$ or $\rho = \lambda \in \Lambda$. More generally, one can modify this condition to the following:

> For some $k > 0$ and a $k \times A$ integer matrix B, we have $B \cdot \operatorname{rot} \circ \rho(A) = 0 \in \mathbb{R}^k$.

Here we regard $\operatorname{rot} \circ \rho(A)$ as an $|A|$–dimensional vector. For a proof, we have only to modify W as

$$W = \{(\alpha, \beta) \colon \alpha \in V \text{ and } \beta \in \mathrm{PSL}_2(\mathbb{R})^B\}.$$

for some algebraic set

$$V \subseteq (\mathrm{SO}(2) \times \mathrm{PSL}_2(\mathbb{R}))^A.$$

Theorem 4.15 is a special case when we have the following matrix:

$$B(x_1, \ldots, x_{|A|})^t = (x_1 - x_2, x_2 - x_3, \ldots, x_{|A|-1} - x_{|A|})^t,$$

for which we consider the algebraic set

$$V = \{\alpha \in (\mathrm{SO}(2) \times \mathrm{PSL}_2(\mathbb{R}))^A \mid \eta \circ \alpha(a) = \eta \circ \alpha(a') \text{ for } a, a' \in A\}.$$

The rest of the modified proof is almost identical. See Corollary 5.21 for a generalization along this line.

Example 4.18. Let G be the fundamental group of a closed orientable surface with genus two. Recall we have actually shown that G is flexible in Theorem 4.9. For an illustrative purpose, let us prove that G satisfies the hypothesis of Theorem 4.15 rather than resorting to a combination theorem.

We have a sequence of maps as follows.

$$\begin{array}{ccc} G = \langle a,b,c,d \mid [a,b] = [c,d]\rangle & \overset{\iota}{\hookrightarrow} & H = \langle a,b,t \mid [[a,b],t] = 1\rangle \\ & & \downarrow f_p \\ \iota(c) = a^t,\ \iota(d) = b^t & & F_2 = \langle a,b\rangle \\ & & \downarrow g_q \\ f_p(t) = [a,b]^p & & \pi_1^{\mathrm{orb}}(T_q) = \langle a,b \mid [a,b]^q = 1\rangle \overset{\rho_q}{\hookrightarrow} \mathrm{PSL}_2(\mathbb{R}). \end{array}$$

Here, g_q is the natural quotient map, and T_q is a compact genus–one hyperbolic orbifold with exactly one cone point, whose order is q. The group $\pi_1^{\mathrm{orb}}(T_q)$ admits a discrete faithful cocompact projective representation ρ_q such that

$$\rho_q[a,b] = \zeta_q.$$

The map f_p is stably injective by Baumslag's Lemma for free groups, and so is g_q by Lemma 4.11.

For each $p \geqslant 1$ we can choose a sufficiently large

$$q = q(p) \geqslant p$$

such that the sequence

$$\{\phi_p = g_{q(p)} \circ f_p \colon H \to \pi_1^{\mathrm{orb}}(T_{q(p)})\}$$

is stably injective. Note that

$$\mathrm{rot} \circ \phi_p[a,b] = 1/q.$$

So Theorem 4.15 applies, with $A = \{[a,b]\}$. In particular, we have a uncountable set

$$\Lambda \subseteq \mathrm{Hom}(H, \mathrm{PSL}_2(\mathbb{R}))$$

of faithful parabolic-free representations such that

$$\mathrm{rot}\,\Lambda([a,b]) = \{\mathrm{rot} \circ \lambda([a,b]) \colon \lambda \in \Lambda\}$$

is uncountable.

5. Axiomatics

In this section, we abstract out key ingredients of our combination theorems targeting more general Baire topological groups. Since we do not assume local compactness for the ambient topological group (in contrast to [4] for instance), we cannot use the structure theorem of Montgomery–Zippin.

5.1. Tracial structure of a topological group.

Definition 5.1. Let $\mathfrak{G}$ be a topological group which is Hausdorff and Baire. If $\mathfrak{G}$ satisfies all of the following conditions, then we say that $\mathfrak{G}$ is equipped with a *tracial structure*.

(A) **Trace.** There is a topological space $\mathscr{S}$ and a continuous map $\tau\colon \mathfrak{G} \to \mathscr{S}$, such that τ is constant on each conjugacy class.
(B) **Parabolic.** Let us write $\mathscr{P} = \tau^{-1}(\tau(1))$. Then the following hold:
  (i) $\mathscr{P} \neq \mathfrak{G}$.
  (ii) If $g \in \mathscr{P}$, then $\langle g \rangle \subseteq \mathscr{P}$.
  (iii) $\mathscr{P}$ does not contain nontrivial torsion.
  We say $H \leqslant \mathfrak{G}$ is *generic* if $H \nsubseteq \mathscr{P}$.
(C) **Elliptic.** We have a distinguished uncountable open set $\mathscr{E}_0 \subseteq \mathscr{S} \backslash \tau(1)$ and let $\mathscr{E} = \tau^{-1}(\mathscr{E}_0)$, which is possibly equal to $\mathfrak{G} \backslash \mathscr{P}$.
(D) **Abelian.** Every generic, maximal abelian subgroup μ is perfect (i.e. without isolated points), Baire, and contains the centralizer of every $g \in \mu \backslash 1$. Moreover, the set

$$\{\nu \in \mu \mid x = \tau(\nu^m) \text{ for some } m \in \mathbb{Z} \backslash 0\}$$

is countable for each $x \in \mathscr{S}$.
(E) **Algebraic Baumslag.** Suppose that $k \geqslant 1$, that

$$g_1, \ldots, g_k \in \mathfrak{G} \backslash 1,$$

that $m_i \in \mathbb{Z} \backslash 0$, and that $x \in \mathscr{S}$, and let

$$T_x = \{\nu \in \mathfrak{G} \mid \tau (g_1 \nu^{m_1} g_2 \nu^{m_2} \cdots g_k \nu^{m_k}) = x\}.$$

Let μ be a generic and maximal abelian subgroup. Then each of the following hold:
  (i) T_x is nowhere dense in $\mathfrak{G}$;
  (ii) Either $\mu \subseteq T_x$ or $T_x \cap \mu$ is nowhere dense in μ;
  (iii) If $\mu \subseteq T_x$, then $[\mu^{g_i}, \mu] \subseteq \mathscr{P}$ for some i.

Remark 5.2. (1) The key example to keep in mind is when τ is the usual trace map on a matrix group. See Example 5.3 and later parts of this section for further examples.

(2) A subgroup $\mu \leqslant \mathfrak{G}$ is generic and maximal abelian if and only if $\mu = Z(c)$ for some $c \in \mathfrak{G} \backslash \mathscr{P}$. In this case, the normalizer

$$\hat{\mu} = N(\mu) = \{g \in \mathfrak{G} \mid \mu^g \subseteq \mu\}$$

is called *maximal elementary*. See Lemma 5.8 for a different characterization of $\hat{\mu}$.

(3) The group $\mathfrak{G}$ is nontrivial since $\mathfrak{G} \neq \mathscr{P}$. By the condition **Algebraic Baumslag** (i), the set $\mathscr{P}$ is closed nowhere dense. In fact, Lemma 5.8 implies that $\mathfrak{G}$ and $\tau(\mathfrak{G})$ is uncountable.

(4) The condition **Elliptic** is optional: if $\mathscr{E}$ is not specified, we simply let

$$\mathscr{E} = \tau^{-1}(\mathscr{S} \backslash \tau(1)).$$

Example 5.3. Let $\mathfrak{G} = \mathrm{PSL}_2(K)$ for $K = \mathbb{R}$ or $K = \mathbb{C}$, and let τ be the trace map. Then we let $\mathscr{P}$ be the set of parabolics and $\mathscr{E}$ be the set of elliptics. Since $\mathfrak{G}$ is connected in this example, it is irreducible. The condition **Algebraic Baumslag** follows from Baumslag Lemma for $\mathrm{PSL}_2(K)$, that is Lemma 3.19. It is routine to verify the rest of the conditions.

Throughout this section, we assume $\mathfrak{G}$ is a topological group equipped with a tracial structure as in Definition 5.1.

Definition 5.4. Let $R \subseteq \mathscr{S}$ be countable.

(1) We say $L \leqslant \mathfrak{G}$ is *$\mathscr{P}$–free* if $L \cap \mathscr{P} = \{1\}$.

(2) A subgroup L of $\mathfrak{G}$ is *almost R–free* if

$$\tau(L \backslash T(L)) \cap R = \varnothing.$$

(3) We say a representation (or a set of representations) is *$\mathscr{P}$–free* or *almost R–free* if its image has the corresponding property.

Remark 5.5. If L is a torsion group, then every representation of L is $\mathscr{P}$–free and almost R–free for every $R \subseteq \mathscr{S}$, by definition.

Definition 5.6. A finitely generated group L is said to be *in the class $\mathscr{F}(\mathfrak{G})$* if there exists an element

$$g_0 \in L \backslash T(L)$$

such that for each countable subset $R \subseteq \mathscr{S}$ we have a faithful almost R–free representation $\lambda\colon L \to \mathfrak{G}$ with the property that $\lambda(g_0) \in \mathscr{E}$.

We will mostly abbreviate $\mathscr{F}(\mathfrak{G})$ by $\mathscr{F}$ when the ambient group $\mathfrak{G}$ is clear from the context. Propositions 5.15 and 4.1 exhibit the significance of the class $\mathscr{F}$. The main result of this section is the following combination theorem.

Theorem 5.7. *(1) Infinite cyclic groups are in $\mathscr{F}$.*

(2) If $A, B \in \mathscr{F}$, then $A * B \in \mathscr{F}$.

(3) *Let A and B be finitely generated nontrivial subgroups of a group L' and C is a malnormal maximal abelian subgroup of both A and B. If $\langle A, B\rangle \in \mathscr{F}$, then $A *_C B \in \mathscr{F}$.*

(4) *Let A be a finitely generated group and let C be a malnormal maximal abelian subgroup of A. If $A \in \mathscr{F}$, then $A*_C \in \mathscr{F}$.*

5.2. **Preliminary observations.**

Lemma 5.8. *Let $\mu \leqslant \mathfrak{G}$ be maximal abelian and generic.*

(1) *For each $g \in \mu\backslash 1$, the set*

$$\{h \in \mathfrak{G} \mid g \in \langle h\rangle\}$$

is countable.

(2) *μ is closed and nowhere dense in $\mathfrak{G}$.*

(3) *The normalizer $\hat{\mu}$ of μ satisfies*

$$\begin{aligned}\hat{\mu} &= \{g \in \mathfrak{G} \mid [\mu^g, \mu] = 1\} = \{g \in \mathfrak{G} \mid [\nu^g, \nu] = 1 \text{ for some } \nu \in \mu\backslash 1\} \\ &= \{g \in \mathfrak{G} \mid [\nu^g, \nu'] = 1 \text{ for some } \nu, \nu' \in \mu\backslash 1\} = \{g \in \mathfrak{G} \mid \mu \cap \mu^g \neq 1\}.\end{aligned}$$

(4) *The set*

$$\tau(\mu) = \{\tau(\nu) \mid \nu \in \mu\}$$

is an uncountable set.

(5) *For each $x \in \mathscr{S}$, $k \in \mathbb{Z}\backslash 0$ and $c \in \mu$, the set*

$$\{\nu \in \mu \mid \tau(c\nu^k) = x\}$$

is closed and nowhere dense in μ.

Proof. (1) Let A be the given set and $h \in A$. Then $h \in Z(g) = \mu$ and so,

$$A = \{h \in \mu \mid g \in \langle h\rangle\} \subseteq \{h \in \mu \mid \tau(g) \in \tau\langle h\rangle\}.$$

By the condition **Abelian**, the set A is countable.

(2) Choose $c \in \mu\backslash\mathscr{P}$. Then $\mu = Z(c)$ is defined by the polynomial $[c, \nu] = 1$. It follows that μ is closed.

By the condition **Algebraic Baumslag** (i), there exists $\nu \in \mathfrak{G}\backslash 1$ such that

$$\tau[\nu^c, \nu] \neq \tau(1).$$

Since

$$[\nu^c, \nu] = [c, \nu]^c \cdot [c, \nu]^{c\nu^{-1}},$$

we see that $[c, \nu] \neq 1$. In particular, $\mu = Z(c)$ is proper and so nowhere dense.

(3) Let us write the given statement as

$$\hat{\mu} = \hat{\mu}_1 = \hat{\mu}_2 = \hat{\mu}_3 = \hat{\mu}_4.$$

Then the inclusions

$$\hat{\mu} \subseteq \hat{\mu}_1 \subseteq \hat{\mu}_2 \subseteq \hat{\mu}_3 \subseteq \hat{\mu}_4$$

are almost immediate to verify. In order to prove $\mu_4 \subseteq \hat{\mu}$, let us assume $g \in \hat{\mu}_4$. That is, we have $\nu, \nu' \in \mu\backslash 1$ such that $\nu^g = \nu'$. Then

$$\mu = Z(\nu') = Z(\nu^g) = Z(\nu)^g = \mu^g.$$

So we have $g \in \hat{\mu}$.

(4) Since $\mathfrak{G}$ is Hausdorff and μ is perfect, each singleton in μ is closed nowhere dense. It follows that μ is uncountable. Since

$$\mu \cap \tau^{-1}(x)$$

is countable for each $x \in \mathscr{S}$, we see that $\tau(\mu)$ is uncountable.

(5) By **Algebraic Baumslag** again, it suffices to show that the set

$$U_x = \{\nu \in \mu \mid \tau(c\nu^k) = x\}$$

is properly contained in μ. If not, then $\tau(c\nu^k) = x$ for all $\nu \in \mu$, and **Abelian** implies

$$\{c\nu^k \mid \nu \in \mu\} = \{a_1, a_2, \ldots\}.$$

It follows that

$$\tau(\mu) = \bigcup_i \{\tau(\nu) \mid \nu \in \mu \text{ and } \nu^k = c^{-1}a_i\}$$

is countable again by **Abelian**. This contradicts part (4). □

Lemma 5.9. *Let A be a group and C be a maximal abelian subgroup of A. If there exists an embedding $\lambda\colon A \to \mathfrak{G}$, then the following two conditions are equivalent:*

(i) $N_A(C) = C$.
(ii) C is malnormal in A.

Proof. After employing the condition **Abelian**, the proof is identical to that of Lemma 3.26. □

We can augment the condition **Algebraic Baumslag** as follows.

Lemma 5.10. *If $R \subseteq \mathscr{S}$ is a countable subset and if*

$$g_1, g_2, \ldots, g_k \in \mathfrak{G}\backslash 1,$$

then for a very general $\nu \in \mathfrak{G}$, we have $\tau(\nu) \notin R$ and

$$\tau[\nu^{g_i}, \nu] \notin R.$$

In particular, for such a very general ν_0 and $\mu = Z(\nu_0)$, the set

$$\{\nu \in \mu \mid \tau\left(g_1\nu^{m_1}g_2\nu^{m_2}\cdots g_k\nu^{m_k}\right) = x\}.$$

is nowhere dense in μ for each $x \in \mathscr{S}$.

Proof. Since the property of being very general is closed under taking a finite intersection, it suffices to show for the case $k = 1$. We let $g_1 = g$. By enlarging R if necessary, we may assume $\tau(1) \in R$. Let

$$A = \bigcup_{x \in R} \{v \in \mathfrak{G} \mid \tau[v^g, v] = x \quad \text{or} \quad \tau(v) = x\}.$$

By the conditions **Algebraic Baumslag** (i), we see that A is an F_σ set with empty interior. An arbitrary element $v_0 \in \mathfrak{G} \backslash A$ satisfies the conclusion. ◻

5.3. **Free product.** Let us denote the inner automorphism defined by For each $v \in \mathfrak{G}$, we define $\gamma_v \colon \mathfrak{G} \to \mathfrak{G}$ and $\delta_v \colon \mathbb{Z} \to \mathfrak{G}$ by

$$\gamma_v(g) = g^v \text{ and } \delta_v(s) = v.$$

Lemma 5.11. *Suppose $\alpha \colon A \to \mathfrak{G}$ is a faithful representation from a nontrivial finitely generated group A. For each $v \in \mathfrak{G}$, we let $L = A * \mathbb{Z}$ and*

$$\rho_v = \alpha * \delta_v \colon L \to \mathrm{PSL}_2(\mathbb{R}).$$

Then for each countable subset R of $\mathscr{S}$, the set

$$\mathscr{D} = \{v \in \mathfrak{G} \mid \rho_v \textit{ is faithful and } \operatorname{tr} \rho_v(L \backslash A^L) \cap R = \varnothing\}$$

is very general in $\mathfrak{G}$. If, furthermore, α is almost R–free, then so is ρ_v for each $v \in \mathscr{D}$.

Proof. The proof is almost identical to that of Lemma 3.24. The key claim in this case is the following.

Claim. *If*

$$g \in L \backslash A^L,$$

then there exists a generic maximal abelian μ such that for each $x \in S$ the set

$$U_x = \{v \in \mu \mid \tau \circ \rho_v(g) = x\}$$

is properly contained in μ.

Let us write

$$\rho_v(g) = \prod_{i=1}^{k} g_i v^{m_i}$$

for some $k > 0$, for some elements $g_i \in A \backslash 1$, and for some exponents $m_i \in \mathbb{Z} \backslash 0$, possibly after a suitable conjugation. Put

$$\mathscr{C} = \{v \in \mathfrak{G} \mid [v^h, v] \in \mathscr{P} \text{ for some } h \in A \backslash 1\}.$$

By Lemma 5.10, we can pick (even very generally)

$$v_0 \in \mathfrak{G} \backslash (\mathscr{C} \cup \mathscr{P}).$$

If we set $\mu = Z(\nu_0)$, then the condition **Algebraic Baumslag** (iii) implies that U_x is closed and nowhere dense. The claim is thus established.

From Lemma 3.8, we see

$$\mathfrak{G}\backslash\mathscr{D} = \bigcup_{x\in R}\bigcup_{g\in L\backslash A^L}\{\nu \in \mathfrak{G} \mid \tau \circ \rho_\nu(g) = x\}$$

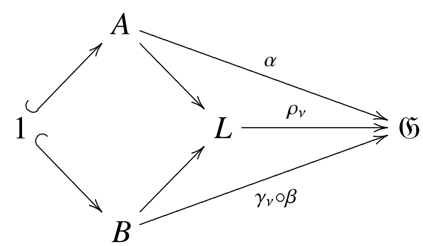
is an F_σ set with empty interior. In particular, $\mathscr{D}$ is very general.

If $g \in L$ is conjugate to $h \in A$, then $\rho_\nu(g)$ is conjugate to $\alpha(h)$. It follows that if α is almost R–free, then so is ρ_ν for each $\nu \in \mathscr{D}$. ◻

Lemma 5.12. *Suppose $\alpha\colon A \to \mathfrak{G}$ and $\beta\colon B \to \mathfrak{G}$ are faithful representations from nontrivial finitely generated groups A and B. For each $\nu \in \mathfrak{G}$, we define*

$$\rho_\nu\colon L = A * B \to \mathfrak{G}$$

by the following commutative diagram:



Then for each countable subset R of $\mathscr{S}$, the set

$$\mathscr{D} = \{\nu \in \mathfrak{G} \mid \rho_\nu \text{ is faithful and } \operatorname{tr}\rho_\nu(L\backslash(A \cup B)^L) \cap R = \varnothing\}$$

is very general in $\mathfrak{G}$. If, furthermore, α and β are almost R–free, then so is ρ_ν for each $\nu \in \mathscr{D}$.

Proof. We start with the following claim.

Claim. *If*

$$g \in L\backslash(A \cup B)^L,$$

then there exists a generic maximal abelian μ such that for each $x \in S$ the set

$$U_x = \{\nu \in \mu \mid \tau \circ \rho_\nu(g) = x\}$$

is properly contained in μ.

Writing

$$g = a_1b_1a_2b_2 \cdots a_kb_k$$

for some $k \geqslant 1$, for some $a_i \in A\backslash 1$ and for some $b_i \in B\backslash 1$, possibly after conjugating suitably. Put

$$\mathscr{C} = \{\nu \in \mathfrak{G} \mid [\nu^h, \nu] \in \mathscr{P} \text{ for some } h \in (A \cup B)\backslash 1\}.$$

By Lemma 5.10, we can pick (even very generally)

$$\nu_0 \in \mathfrak{G} \backslash (\mathscr{C} \cup \mathscr{P}).$$

If we set $\mu = Z(\nu_0)$, then the condition **Algebraic Baumslag** (iii) implies that U_x is closed and nowhere dense. This proves the claim.

We see

$$\mathfrak{G} \backslash \mathscr{D} = \bigcup_{x \in R} \bigcup_{g \in L \backslash (A \cup B)^L} \{\nu \in \mathfrak{G} \mid \tau \circ \rho_\nu(g) = x\}$$

is an F_σ set with empty interior.

For the almost R-free part of the conclusion, note that

$$T(L) \subseteq (T(A) \cup T(B))^L$$

and also that τ is a conjugacy class invariant. □

5.4. **Free product with amalgamation.**

Lemma 5.13. *We let L' be a group and let A, B be finitely generated nontrivial subgroups of L' satisfying the following:*

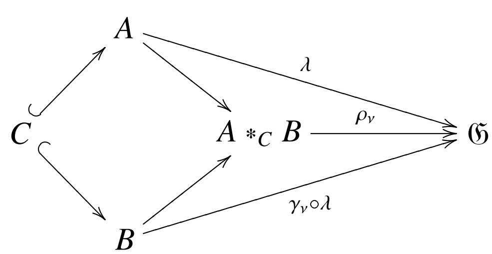
(i) There is a $C \leqslant A \cap B$ such that C is malnormal maximal abelian in A and in B;

(ii) There is a representation $\lambda\colon L' \to \mathfrak{G}$ such that the restrictions $\lambda\restriction_A$ and $\lambda\restriction_B$ are faithful and $\mathscr{P}$–free.

Let μ be the generic maximal abelian group containing $\lambda(C)$, and for each $\nu \in \mu$, we define

$$\rho_\nu\colon A *_C B \to \mathfrak{G}$$

by the following commutative diagram:



Then for each countable subset R of $\mathscr{S}$, the set

$$\mathscr{D} = \{\nu \in \mu \mid \rho_\nu \textit{ is faithful and } \operatorname{tr} \rho_\nu(L \backslash (A \cup B)^L) \cap R = \varnothing\}$$

is very general in μ. If, furthermore, α and β are almost R–free, then so is ρ_ν for each $\nu \in \mathscr{D}$.

Proof. Put $L = A *_C B$. Denote by $\hat{\mu}$ the normalizer of μ. As in Lemma 5.12, we first observe the following.

Claim. *If $g \in L \backslash (A \cup B)^L$, then for each $x \in S$ the set*

$$U_x = \{\nu \in \mu \mid \tau \circ \rho_\nu(g) = x\}$$

is closed and nowhere dense in μ.

In order to prove the claim, let us write

$$g = \prod_{i=1}^{k} a_i b_i$$

for some $k \geqslant 1$, for some elements $a_i \in A \backslash C$, and for some elements $b_i \in B \backslash C$ possibly after conjugation. Lemma 5.9 implies that

$$\lambda((A \cup B) \backslash C) \cap \hat{\mu} = \varnothing.$$

Let us fix $c \in C \backslash 1$. Assume that for some

$$h \in (A \cup B) \backslash C$$

we have that $\lambda[c^h, c] \in \mathscr{P}$. Since $\mathscr{P}$ is torsion-free, we know $\lambda[c^h, c]$ is not torsion. Since λ is almost R–free and $\lambda \restriction_A$ and $\lambda \restriction_B$ are faithful, it follows that $[c^h, c] = 1$. This is a contradiction, since Lemma 5.8 (3) implies that $[c^h, c] \neq 1$. We thus see that

$$\lambda[c^h, c] \notin \mathscr{P}$$

for all $h \in (A \cup B) \backslash C$. The claim follows from the condition **Algebraic Baumslag**.

We see

$$\mathfrak{G} \backslash \mathscr{D} = \bigcup_{x \in R} \bigcup_{g \in L \backslash (A \cup B)^L} \{\nu \in \mu \mid \tau \circ \rho_\nu(g) = x\}.$$

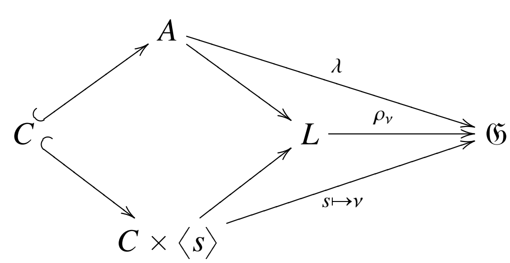
is an F_σ set with empty interior. □

5.5. **Centralizer extensions.**

Lemma 5.14. *We let A be a finitely generated group such that the following hold:*

(i) there is $C \leqslant A$ which is maximal abelian in A;
(ii) for each $g \in A \backslash C$ we have $[C^g, C] \neq \{1\}$;
(iii) there is a faithful and $\mathscr{P}$–free representation $\lambda\colon A \to \mathfrak{G}$.

Let μ be the generic maximal abelian group containing $\lambda(C)$, and for each $\nu \in \mu$, we define $\rho_\nu\colon L = A *_C \to \mathfrak{G}$ by the following commutative diagram:



Then for each countable subset R of $\mathscr{S}$, the set

$$\mathscr{D} = \{\nu \in \mu \mid \rho_\nu \text{ is faithful and } \operatorname{tr}\rho_\nu(L\backslash A^L) \cap R = \varnothing\}$$

is very general in μ. If, furthermore, α and β are almost R–free, then so is ρ_ν for each $\nu \in \mathscr{D}$.

Proof. Denote by $\hat{\mu}$ the normalizer of μ. As in Lemma 5.13, we first observe the following.

Claim. *If $g \in L\backslash A^L$, then for each $x \in S$ the set*

$$U_x = \{\nu \in \mu \mid \tau \circ \rho_\nu(g) = x\}$$

is closed and nowhere dense in μ.

Let us first consider the case when g is conjugate into $C \times \langle s\rangle$. There exist some $c \in C$ and some $k \in \mathbb{Z}\backslash 0$ such that $g = cs^k$, possibly after a conjugating suitably. We have

$$\tau \circ \rho_\nu(g) = \tau(c\nu^k) \in \tau(\mu)$$

for $\nu \in \mu$. Lemma 5.8 (5) implies the claim in this case.

As the second case of the claim, let us assume g is not conjugate into $C \times \langle s\rangle$ so that

$$g = \prod_{i=1}^{k} a_i \nu^{m_i}$$

for some $k \geqslant 1$, for some elements $a_i \in A\backslash C$, and for some elements $m_i \in \mathbb{Z}\backslash 0$ possibly after a suitable conjugation. By Lemma 5.9 again, we have

$$\lambda(A\backslash C) \cap \hat{\mu} = \varnothing.$$

Let us fix $c \in C\backslash 1$. We have seen in the proof of Lemma 5.13 that

$$\lambda[c^h, c] \notin \mathscr{P}$$

for all $h \in A\backslash C$. The claim follows from the condition **Algebraic Baumslag** (iii).

The rest of the proof follows the same line of reasoning as in Lemmas 3.28 and 5.13. Namely, we decompose $\mathscr{D}$ into a countable union of closed nowhere dense subsets, by separately considering the elements of L that is conjugate into A and also the elements that is not. □

5.6. **The class $\mathscr{F}$.**

Proof of Theorem 5.7. For the proof, let us fix a countable subset $R \subseteq \mathscr{S}$ containing $\tau(1)$.

(1) Since $\mathscr{E}$ is uncountable, we can pick

$$c \in \mathscr{E}\backslash\tau^{-1}(R).$$

We write $\mu = Z(c)$ and $\hat{\mu} = N(\mu)$. Let

$$\mathscr{C} = \{\nu \in \mu \mid \tau(\nu^k) \in R \text{ for some } k \in \mathbb{Z}\backslash 0\}.$$

Then $\mathscr{C}$ is an F_σ set with an empty interior by Lemma 5.8 (5). In particular, $\mu\backslash\mathscr{C}$ is very general in μ.

Since $c \in \mu \cap \mathscr{E}$ and $\mathscr{E}$ is open, we can find

$$\nu \in (\mu \cap \mathscr{E})\backslash\mathscr{C}.$$

Then $\rho_\nu \colon \mathbb{Z} \to \mathfrak{G}$ defined by $s \mapsto \nu$ satisfies that ρ_ν is faithful, almost R–free and that $\rho_\nu(s) \in \mathscr{E}$.

(2) Let $\alpha \colon A \to \mathfrak{G}$ and $\beta \colon B \to \mathfrak{G}$ be faithful and almost R–free. Assume also that we have

$$g_0 \in A\backslash T(A)$$

such that $\alpha(g_0) \in \mathscr{E}$. Then the set $\mathscr{D}$ defined in Lemma 5.12 is very general. Note that

$$\rho_\nu(g_0) = \alpha(g_0) \in \mathscr{E}$$

for each $\nu \in \mathscr{E}$, where ρ_ν is as defined in the same lemma. This proves the part *(2)*.

(3) Let us start with a faithful almost R–free representation

$$\lambda \colon L' = \langle A, B \rangle \to \mathfrak{G},$$

and

$$g_0 \in L'\backslash T(L')$$

such that $\lambda(g_0) \in \mathscr{E}$. Let $\mu \leqslant \mathfrak{G}$ be the maximal abelian subgroup containing $\lambda(C)$. Recall the definition of ρ_ν and $\mathscr{D}$ in Lemma 5.13, where $\mathscr{D}$ is shown to be very general in μ. Pick a shortest word

$$g_1 \in L = A *_C B$$

that maps to g_0. If g_1 is conjugate into $A \cup B$, then

$$\tau \circ \rho_\nu(g_1) = \tau \circ \lambda(g_0),$$

and therefore $\rho_\nu(g_1) \in \mathscr{E}$. Otherwise, let us write

$$g_1 = a_1 b_1 \cdots a_k b_k$$

for some elements $a_i \in A\backslash C$ and some elements $b_i \in B\backslash C$, possibly after a suitable conjugation. Since

$$\rho_1(g_1) = \lambda(g_0) \in \mathscr{E},$$

we see that there is an open neighborhood V in μ containing the identity such that $\rho_\nu(g_1) \in \mathscr{E}$ for each $\nu \in V$. Then ρ_ν is a desired representation for an arbitrary choice of $\nu \in V \cap \mathscr{D}$.

(4) Let $\lambda \colon A \to \mathfrak{G}$ be a faithful almost R–free representation and let

$$g_0 \in A\backslash T(A)$$

satisfy that $\lambda(g_0) \in \mathscr{E}$. We let $\mu \leqslant \mathfrak{G}$ be the maximal abelian subgroup containing $\lambda(C)$. With respect to the definition of ρ_ν and $\mathscr{D}$ in Lemma 5.14, we have seen $\mathscr{D}$ is very general in μ. Note that

$$\rho_\nu(g_0) = \lambda(g_0) \in \mathscr{E}.$$

□

The groups in $\mathscr{F}$ admit uncountably many faithful representations satisfying the following strong version of non-conjugacy.

Proposition 5.15. *Let R be a countable subset of $\mathscr{S}$, and let $L \in \mathscr{F}$. Then there exists a uncountable set*

$$\Lambda \subseteq \mathrm{Hom}(L, \mathfrak{G})$$

of faithful almost R–free representations such that for all

$$g, h \in L \backslash T(L)$$

and for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$, the element $\lambda(g)$ is not conjugate to $\mu(h)$ in $\mathfrak{G}$.

Proof. Let $\Lambda \subseteq \mathrm{Hom}(L, \mathfrak{G})$ be a maximal set of faithful almost R–free representations such that

$$\tau \circ \lambda(L \backslash T(L)) \cap \tau \circ \mu(L \backslash T(L)) = \varnothing$$

for all distinct λ, μ in Λ. Suppose that one can enumerate the elements of Λ as

$$\Lambda = \{\lambda_1, \lambda_2, \ldots\}.$$

We then set

$$R' = \{\tau \circ \lambda_i(g) \mid i \in \mathbb{N} \text{ and } g \in L\}.$$

By the definition of $\mathscr{F}$, we have a faithful almost R'–free representation λ, which contradicts the maximality. The conclusion follows. □

We illustrate an application of generalization pursued in this section, by controlling possible rotation numbers of an anchor:

Corollary 5.16. *Let L be a nontrivial limit group. Pick an arbitrary nonempty open interval $J \subseteq S^1$. Then there exists $g_0 \in L$ and a uncountable set Λ of faithful projective parabolic–free representations such that the following hold:*

(i) $\Sigma(\lambda) \cap \Sigma(\mu) \subseteq \mathbb{Q}$ *for all distinct λ, μ in Λ;*
(ii) $\Sigma(\lambda) \backslash \mathbb{Q}$ *is a nonempty set consisting of transcendental numbers for each $\lambda \in \Lambda$.*
(iii) $\mathrm{rot} \circ \lambda(g_0) \in J$.

Proof. The abelian case is easy by embedding L into $\mathrm{SO}(2)$. We assume L is nonabelian. Let

$$\mathscr{E} = \mathrm{rot}^{-1}(J) \cap \mathrm{tr}^{-1}(-2, 2) \subseteq \mathrm{PSL}_2(\mathbb{R})$$

and run the same argument as in Corollary 4.7 and Theorem 5.7. □

5.7. **UV Structures.** As another abstraction of our techniques, we generalize Theorem 4.15 to the context of an arbitrary algebraic group in order to control certain "spectra" of representations.

Definition 5.17. (1) Let $\mathfrak{G}$ be an algebraic group and $k > 0$. For each $i = 1, 2, \ldots, k$, we assume there is a commutative diagram

$$\begin{array}{ccccc} U_i & \xrightarrow{\xi_i} & \xi_i(U_i) & \hookrightarrow & \mathfrak{G}\ , \\ & \searrow^{\eta_i} & \downarrow{\bar{\eta}_i} & & \\ & & V_i & & \end{array}$$

such that
  (i) U_i and V_i are algebraic sets;
  (ii) ξ_i and η_i are polynomial maps;
  (iii) $\mathfrak{G} \setminus \bigcup_{i=1}^{k} \xi_i(U_i)$ is an algebraic set.

In this situation, we say $\mathfrak{G}$ is equipped with a *UV–structure*.

(2) Suppose G is a finitely generated group, and

$$A = \{A_1, \ldots, A_k\}$$

is a collection of finite subsets $A_i \subseteq G$. Let us fix algebraic subset $W_i \subseteq V_i^{A_i}$ for each

$$i \in \{1, 2, \ldots, k\},$$

and put

$$W = \{W_1, \ldots, W_k\}.$$

We say a representation $\rho\colon G \to \mathfrak{G}$ is *UV–compatible (with respect to A and W)* if the following hold.
  (i) $\rho(G) \subseteq \bigcup_{i=1}^{k} \xi_i(U_i)$.
  (ii) $\rho(A_i) \subseteq \xi_i(U_i)$ and $\bar{\eta}_i \circ \rho\restriction_{A_i} \in W_i$ for each i.

Remark 5.18. (1) We assume neither that $\xi_i(U_i)$ is algebraic nor that $\bar{\eta}_i$ is a polynomial map. The map $\bar{\eta}_i$ merely makes the diagram commute.

(2) If (i) and (ii) are satisfied, we can force (iii) by setting

$$U_{k+1} = \xi_{k+1}(U_{k+1}) = V_{k+1} = \mathfrak{G}.$$

(3) If V_i's are not specified, then we will assume $V_i = \{0\}$.

Example 5.19. Let us define a UV–structure on $\mathfrak{G} = \mathrm{PSL}_2(\mathbb{R})$ by:

$$U_1 = \mathrm{SO}(2) \times \mathrm{PSL}_2(\mathbb{R}), V_1 = \mathrm{SO}(2), U_2 = \exp(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R}), V_2 = \exp(\mathbb{R}),$$

and $\eta_i(x, y) = x$ and $\xi_i(x, y) = x^y$ for $i = 1, 2$. Note that $\xi_1(U_1)$ is the set of elliptics, and $\xi_2(U_2)$ is that of hyperbolics. Then $\bar{\eta}_1, \bar{\eta}_2$ are well-defined by

$$\bar{\eta}_1(x) = \mathrm{Rot}(2\pi \operatorname{rot}(x))$$

and

$$\bar{\eta}_2(x) = \exp \ell(x).$$

Here ℓ denotes the translation length function on $\mathrm{Isom}_+(\mathbb{H}^2)$. For a representation $\rho \in \mathrm{Hom}(G, \mathfrak{G})$, the set

$$\bar{\eta}_1(\rho(G) \cap \xi_1(U_1))$$

is the rotation spectrum of ρ. We may call

$$\bar{\eta}_2(\rho(G) \cap \xi_2(U_2))$$

as the *(hyperbolic translation) length spectrum* of ρ. The set of parabolics is written as

$$\mathrm{PSL}_2(\mathbb{R}) \setminus (\xi_1(U_1) \cup \xi_2(U_2)),$$

which is algebraic.

Then Theorem 4.15 generalizes as follows.

Theorem 5.20. *Let us fix an algebraic group $\mathfrak{G}$ equipped with a UV–structure, and a finitely generated group G. Let A and W be as in Definition 5.17. Suppose there is a stably injective sequence of UV–compatible (with respect to A and W) representations*

$$\{\rho_n \colon G \to \mathfrak{G}\}_{n \geqslant 1}$$

such that whenever $m \neq n$, we have

$$\bar{\eta}_i \circ \rho_m(a) \neq \bar{\eta}_i \circ \rho_n(a)$$

for each

$$i \in \{1, \ldots, k\}$$

and $a \in A_i$. Then there exists a uncountable collection of faithful UV–compatible (with respect to A and W) representations $G \to \mathfrak{G}$ such that

$$\{\bar{\eta}_i \circ \lambda(a) \colon \lambda \in \Lambda\}$$

is uncountable for each i and $a \in A_i$.

The proof is actually an easy variation from Theorem 4.15. From Theorem 5.20, we obtain a strengthening of Theorem 4.15 as follows.

Corollary 5.21. *In addition to the hypotheses of Theorem 4.15, let us assume that A' is a finite subset of G such that $\rho_n(A')$ consists of hyperbolics with the same translation length which belongs to $(0, 1/n]$. Then there exists a uncountable set Λ of faithful parabolic-free representations $G \to \mathrm{PSL}_2(\mathbb{R})$ such that the conditions of (i) through (iii) of Theorem 4.15 hold and moreover,*

(iv) $\lambda(A')$ consists of hyperbolics with the same translation length for each $\lambda \in \Lambda$;
(v) $\ell \circ \lambda(a) \neq \ell \circ \lambda'(a)$ for $a \in A'$ and $\lambda \neq \lambda'$ in Λ;
(vi) $\{\ell \circ \lambda(a) \colon \lambda \in \Lambda\}$ is $\mathbb{Z}$-linearly independent for $a \in A'$;

(vii) $\ell \circ \lambda(G) \neq \ell \circ \lambda'(G)$ *for* $\lambda \neq \lambda'$ *in* Λ.

The proof is immediate from Theorem 5.20 and closely follows the idea of Theorem 4.15.

Example 5.22. Let

$$H = \langle a, b, t \mid [[a, b], t] = 1 \rangle.$$

Recall the UV structure of $\mathrm{PSL}_2(\mathbb{R})$ in Example 5.19. In an alternative proof of Theorem 4.9, we exhibited a stably injective sequence

$$\phi_p \colon H \to \mathrm{PSL}_2(\mathbb{R})$$

such that

$$\mathrm{rot} \circ \phi_p(t) = 1/p$$

for $p \geqslant 2$. We can further require that $\phi_p(a)$ is a hyperbolic element of length at most $1/p$ by considering the moduli space of a torus with one cone point.

So, Corollary 5.21 implies that there exists a uncountable set

$$\Lambda \subseteq \mathrm{Hom}(H, \mathrm{PSL}_2(\mathbb{R}))$$

of faithful parabolic-free representations with all distinct rotation spectra and with all distinct length spectra.

6. Rotation spectrum, rigidity, and flexibility

In this section, we consider free group and surface group actions on the circle, and we develop conditions under which the equivalence class of an action is determined by the rotation spectrum, and when the semi-conjugacy class of the action is determined by the marked rotation spectrum.

In the case of indiscrete representations of groups into $\mathrm{PSL}_2(\mathbb{R})$, there is a lack of a geometric interpretation of such representations which is as well-developed as Teichmüller theory in the case of discrete representations. In this section, we consider the degree to which marked rotation spectrum can supplant marked length spectrum as a (sometimes nearly complete) semi-conjugacy invariant.

Sections 6.1 and 6.2 (see also Theorems 6.5 and 6.11) establish rigidity results. Here we conclude that any two representations of a group into $\mathrm{PSL}_2(\mathbb{R})$ with zero marked length spectrum are semi-conjugate (see also [54]). The remaining subsections focus more on flexibility results. Examples of flexibility are given in different contexts: indiscrete representations into $\mathrm{PSL}_2(\mathbb{R})$ or discrete representations into larger groups, especially $\mathrm{PSL}_2(\mathbb{C})$. Sections 6.3, 6.4 and 6.5 deal respectively with closed surface groups, free groups and free products of finite groups (e.g. $\mathrm{PSL}_2(\mathbb{Z})$).

6.1. **A characterization of indiscrete subgroups of** $\mathrm{PSL}_2(\mathbb{R})$**.** It will be useful for us to have a dynamical characterization of subgroups of $\mathrm{PSL}_2(\mathbb{R})$ which are not discrete. The following lemma is probably well-known to experts in algebraic groups and Lie groups (see [63, 45], for instance). We include a proof for the convenience of the reader. This lemma was mentioned in Remark 4.2.

Lemma 6.1. *Let G be a finitely generated non–virtually solvable subgroup of* $\mathrm{PSL}_2(\mathbb{R})$*. The following are equivalent:*

(1) The rotation spectrum of G is infinite;
(2) The group G is not discrete;
(3) The group G is dense.

Proof of Lemma 6.1. First, note that G is Zariski dense, since any subgroup of $\mathrm{PSL}_2(\mathbb{R})$ which is not virtually solvable cannot be contained in a Borel subgroup and hence is Zariski dense.

We first prove that G is dense if and only if it is Zariski dense and not discrete. If G is dense then its Zariski closure is a closed subgroup of $\mathrm{PSL}_2(\mathbb{R})$ containing G, and hence is all of $\mathrm{PSL}_2(\mathbb{R})$. Conversely, suppose G is Zariski dense and not discrete. Then the closure of G in the usual topology on $\mathrm{PSL}_2(\mathbb{R})$ contains a component of positive dimension which is a closed subgroup and is hence a Lie subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Since G is Zariski dense, this subgroup must be all of $\mathrm{PSL}_2(\mathbb{R})$. Thus, we establish the equivalence of (2) and (3).

Now let G be a dense subgroup of $\mathrm{PSL}_2(\mathbb{R})$. We claim that the rotation spectrum of G is infinite. Indeed, the conjugacy class of

$$\mathrm{SO}(2)\backslash\{1\} < \mathrm{PSL}_2(\mathbb{R})$$

is an open subset of $\mathrm{PSL}_2(\mathbb{R})$, since it is determined by the open *algebraic* condition $|\operatorname{tr}(A)| < 2$. In particular, the intersection of G with this conjugacy class is non-trivial. Note that if the rotation spectrum of G were finite, then the trace function would take on only finitely many values in $(-2, 2)$ when restricted to G. However, if

$$\varnothing \neq U \subset (-2, 2)$$

is open then

$$G \cap \operatorname{tr}^{-1}(U) \neq \varnothing,$$

so that the rotation spectrum of G must indeed be infinite. Thus, (3) implies (1).

For the final implication, suppose G is discrete but that it has infinite rotation spectrum. Then G cannot contain any infinite order elliptic elements, and so G must contain elliptic elements of arbitrarily high order. Moreover, being a finitely generated linear group, we have that G contains a finite index subgroup H which is torsion-free by Selberg's Lemma. If n is the index of H in G and $g \in G$ is arbitrary, then $g^{n!} \in H$. It follows that G must have a bounded exponent, a contradiction. Thus, the negation of (2) implies the negation of (1), so that (1) implies (2). □

We remark that Lemma 6.1 really is specific to $\mathrm{PSL}_2(\mathbb{R})$. It fails already in $\mathrm{PSL}_2(\mathbb{C})$, where "infinite rotation spectrum" is replaced by "infinite order elliptic element" (see [50]).

To illustrate Lemma 6.1 a little, we give an example of a dense subgroup of $\mathrm{PSL}_2(\mathbb{R})$ with infinite purely arithmetic rotation spectrum, and we explicitly find elliptic elements.

Consider the ring of integers $\mathbb{Z}[\sqrt{2}]$. For any $\epsilon > 0$, choose elements

$$a, b, c, d \in \mathbb{Z}[\sqrt{2}]$$

such that

(1) $0 < a + d < \epsilon$ (this is possible as $\mathbb{Z}[\sqrt{2}]$ is dense in $\mathbb{R}$),
(2) $b = 1, c = ad - 1$.

The matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$$

is easily checked to lie in the finitely generated dense subgroup

$$\mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}]) < \mathrm{PSL}_2(\mathbb{R}),$$

and to be an elliptic element of $\mathrm{PSL}_2(\mathbb{R})$. The rotation number θ is given by

$$2\cos\theta = (a + d),$$

which in turn lies in the interval

$$[\arccos(\epsilon/2), \pi/2],$$

thus providing infinitely many distinct arithmetic rotation numbers. More generally, such examples can be constructed from Hilbert modular groups $\mathrm{PSL}_2(\mathscr{O})$, where $\mathscr{O}$ is the ring of integers in a totally real number field.

The following observation is an immediate consequence of Lemma 6.1:

Corollary 6.2. *Let $G < \mathrm{PSL}_2(\mathbb{R})$ be a finitely generated group, and suppose that the marked rotation spectrum of G is identically zero. Then G is torsion-free and discrete.*

6.2. **Teichmüller Theory and circle actions.** In this subsection we gather some facts from Teichmüller Theory which allow us to analyze projective actions of free and surface groups with zero marked rotation spectrum.

Lemma 6.3. *(cf. Mann* [53]*) Let S be an orientable surface and let*

$$\phi_1, \phi_2 \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$$

be discrete and faithful representations of $\pi_1(S)$ *corresponding to complete finite volume hyperbolic structures on* S*, with a fixed orientation. Then the corresponding actions of* ϕ_1 *and* ϕ_2 *on* S^1 *are conjugate in* $\mathrm{Homeo}^+(S^1)$.

Proof. The two representations ϕ_1 and ϕ_2 of $\pi_1(S)$ (considered up to conjugacy in $\mathrm{PSL}_2(\mathbb{R})$) correspond to two points X_1 and X_2 in the Teichmüller space $\mathscr{T}(S)$ or the Teichmüller space $\mathscr{T}(\overline{S})$ with the opposite orientation. It suffices to show that if

$$X_1, X_2 \in \mathscr{T}(S)$$

then the corresponding representations are conjugate in $\mathrm{Homeo}^+(S^1)$. Between any two points in $\mathscr{T}(S)$, there is a quasi-conformal map taking one hyperbolic structure to the other. Writing h for such a map between X_1 and X_2, we lift h to the universal covers of X_1 and X_2 respectively. If S is a closed surface then h induces a quasi–isometry $\widetilde{h}$ of the universal covers of X_1 and X_2, which are both identified with $\mathbb{H}^2$. Thus, $\widetilde{h}$ induces a homeomorphism $\partial\widetilde{h}$ of S^1 which conjugates the actions of ϕ_1 and ϕ_2 on S^1.

If S is not closed then h need not induce a quasi–isometry because S is non-compact. In this case, we consider h as a self-homeomorphism of S and note that h induces a $\pi_1(S)$–equivariant homeomorphism between fundamental domains for X_1 and X_2 in $\mathbb{H}^2$. By cutting out small neighborhoods of the cusps of S, we obtain a sequence of nested compact surfaces $\{S^i\}_{i\in\mathbb{N}}$ whose union is S. Writing X_1^i for the restriction of X_1 to S^i and X_2^i for the image of X_1^i under h, we get a sequence of homeomorphisms

$$\{h^i \colon X_1^i \to X_2^i\}_{i\in\mathbb{N}}$$

which lift to $\pi_1(S)$–equivariant homeomorphisms between fundamental domains for X_1^i and X_2^i in $\mathbb{H}^2$. Moreover, these homeomorphisms are compatible with respect to inclusion, in the sense that if $i \leqslant j$ then $h^j = h^i$ when restricted to S^i.

The total preimages $\widetilde{X_1^i}$ and $\widetilde{X_2^i}$ of X_1^i and X_2^i in $\mathbb{H}^2$ (which are identified with the corresponding universal covers) have boundaries which are naturally identified with S^1, after including the limit sets. In the limit as i tends to infinity, both boundaries are identified with $\partial\mathbb{H}^2$, and

$$h = \lim_{i\to\infty} h^i$$

induces a homeomorphism of S^1 conjugating the two actions of $\pi_1(S)$. □

Remark 6.4. When S is closed, representations corresponding to points in the two different Teichmüller spaces $\mathscr{T}(S)$ and $\mathscr{T}(\overline{S})$ are not conjugate in $\mathrm{Homeo}^+(S^1)$ because they have different Euler numbers.

6.3. **Closed surface groups.** For closed surface groups, we consider rigidity and flexibility of projective actions, and we give examples of closed surface group actions on the circle whose marked rotation spectra are identically zero but which are not semi-conjugate to any projective action of a surface group.

6.3.1. *Projective actions.* The following characterizes projective closed surface group actions with rotation spectrum identically zero, up to conjugacy. The proof is a fairly easy combination of standard facts from Teichmüller theory:

Theorem 6.5. *Let S_g be a closed surface and let*

$$\phi\colon \pi_1(S_g) \to \mathrm{PSL}_2(\mathbb{R}) < \mathrm{Homeo}^+(S^1)$$

be a faithful representation with $\Sigma(\phi) = \{0\}$. Then $\phi(\pi_1(S_g))$ is a discrete cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$. *Moreover, there are exactly two conjugacy classes of such actions in* $\mathrm{Homeo}^+(S^1)$.

Proof. The fact that $\phi(\pi_1(S_g))$ is a discrete cocompact subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is an immediate consequence of Lemma 6.1. By Lemma 6.3, we have that two discrete faithful surface group representations into $\mathrm{PSL}_2(\mathbb{R})$ with equal Euler numbers give rise to conjugate actions on S^1. The Euler number of such a representation is a conjugacy invariant in $\mathrm{Homeo}^+(S^1)$, and it takes on exactly two values for discrete surface group representations. □

Question 1.16 from the introduction asks to what degree marked spectrum generally controls the equivalence class of a projective surface group action. In light of Theorem 6.5 it seems likely that there should be many inequivalent projective actions with a fixed nonzero marked rotation spectrum. If one drops the requirement of faithfulness then one can produce inequivalent projective actions with a fixed (unmarked) rotation spectrum:

Proposition 6.6. *Let $G = \pi_1(S_g)$, where $g \geqslant 2$. Then there exist infinitely many pairwise inequivalent actions of G on S^1 with a fixed rotation spectrum that factor through a fixed faithful projective action of $\pi_1(S_{g-1})$.*

Proof. Choose a separating simple closed curve on S_g and contract it to a point, so that the resulting space X_g is a wedge of S_{g-1} with S_1. We have

$$\pi_1(X_g) \cong \pi_1(S_{g-1}) * \mathbb{Z}^2.$$

Fix a faithful projective action ϕ of $\pi_1(S_{g-1})$ on S^1 with nonzero rotation spectrum. If $g \geqslant 3$ then we can find such actions using Theorem 4.9, for instance. If $g = 2$ then one can simply take any two independent irrational rotations in $\mathrm{SO}(2)$ to get a faithful projective representation of $\mathbb{Z}^2 = \langle (1,0), (0,1) \rangle$. Let a denote a simple closed curve on S_g that maps to $(1,0)$ in $\pi_1(X_g)$.

Now, choose a sequence of elements

$$\{\xi_i\}_{i\in\mathbb{N}} \subseteq \phi(\pi_1(S_{g-1}))$$

with pairwise distinct rotation numbers. Extend the identity map on $\pi_1(S_{g-1})$ to

$$\alpha_i \colon \pi_1(S_{g-1}) * \mathbb{Z}^2 \to \pi_1(S_{g-1})$$

by

$$\alpha_i((1,0)) = \xi_i$$

and by

$$\alpha_i((0,1)) = 1.$$

We let ϕ_i be the composition

$$\pi_1(S_g) \longrightarrow \pi_1(S_{g-1}) * \mathbb{Z}^2 \xrightarrow{\alpha_i} \pi_1(S_{g-1}) \xhookrightarrow{\phi} \mathrm{PSL}_2(\mathbb{R}).$$

Note that $\Sigma(\phi_i) = \Sigma(\phi)$ for all i. We may a priori require that

$$\mathrm{rot}(\xi_i^n) \neq \mathrm{rot}(\xi_j)$$

for all $n \geqslant 1$ and $i \neq j$. Let a be a simple closed curve on S_g that maps to the $(1,0)$–curve in

$$\pi_1(S_{g-1}) * \mathbb{Z}^2.$$

Suppose ϕ_i is semi-conjugate to $\phi_j \circ \psi$ for some $\psi \in \mathrm{Aut}(G)$ and $i \neq j$. Every automorphism of $G = \pi_1(S_g)$ is induced by an extended mapping class of S_g (by the Dehn–Nielsen–Baer Theorem [31]). Since ϕ_i and ϕ_j are minimal, we see

$$h\phi_i(g)h^{-1} = \phi_j \circ \psi(g)$$

for some fixed $h \in \mathrm{Homeo}_+(S^1)$. Since

$$\phi_j \circ \psi(a) = \xi_j^n$$

for some $n \neq 0$. we have a contradiction. □

6.3.2. *Nonlinear actions.* We now wish to construct an infinite family of faithful closed surface group actions on the circle which are not semi-conjugate to any projective action of a closed surface group on the circle. Moreover, these actions will have (marked) rotation spectrum identically zero, and hence will share some of the features of discrete projective surface group actions. To perform this construction, we will take a detour through closed hyperbolic 3–manifolds.

As we remarked in the introduction, K. Mann has a construction of such actions which appears in her thesis [51]. Her argument also uses 3–manifold topology, though whereas the main engine of her proof is Thurston norm and Jørgensen's inequality, our main engine is the work of Kahn–Markovic on the surface subgroup conjecture.

Let M be a closed hyperbolic 3–manifold which fibers over the circle. Then M is homeomorphic to the mapping torus of a closed surface S with pseudo-Anosov monodromy ψ. We have that the universal cover $\widetilde{M}$ of M is naturally homeomorphic to

$$\widetilde{S} \times \mathbb{R} \cong \mathbb{H}^2 \times \mathbb{R},$$

where the $\mathbb{R}$ factor gives rise to a natural foliation of $\widetilde{M}$ by lines. There is a representative pseudo-Anosov homeomorphism Ψ in the isotopy class of ψ which preserves a lamination $\lambda \subset S$, which gives a suspended codimension one foliation $\Lambda \subset M$ which is transverse to S. This lamination in turn lifts to a lamination $\widetilde{\Lambda} \subset \widetilde{M}$ which is transverse to $\widetilde{S}$.

The identification of $\widetilde{S}$ with $\mathbb{H}^2$ gives rise to a natural faithful action of $\pi_1(M)$ on $S^1 \cong \partial\mathbb{H}^2$. This action extends the natural action of $\pi_1(S) < \pi_1(M)$ on S^1 by a copy of $\mathbb{Z}$, which acts on S^1 with a finite even number of fixed points, and whose dynamics are alternately attracting and repelling.

The number of these fixed points of the $\mathbb{Z}$–action on the circle at infinity depends on on the pseudo-Anosov monodromy, as well as the particular choice of lift of $\mathbb{Z}$ to $\pi_1(M)$ under the surjection $\pi_1(M) \to \mathbb{Z}$.

Lemma 6.7. *Let Ψ be a homeomorphism representing a pseudo-Anosov mapping class which preserves a lamination $\lambda \subset S$ and let $\widetilde{\Psi}$ be a lift of Ψ to the universal cover $\widetilde{S}$ of S which fixes a complementary region of $\widetilde{S}\backslash\widetilde{\lambda}$. Then the action of $\widetilde{\Psi}$ on S^1 has, after passing to a finite nonzero power if necessary, at least four fixed points on S^1.*

Proof. This follows from [23]. □

Now, consider the natural action of $\pi_1(M)$ on S^1.

Lemma 6.8. *After passing to a finite cover of M if necessary, each element of $\pi_1(M)$ acts on S^1 with at least one fixed point. In particular, we have* $\mathrm{rot}(g) = 0$ *for every nonidentity g lying in such a subgroup.*

Proof. We use the fact that there is a $\pi_1(S)$–equivariant continuous surjection ϕ from $S^1(\cong \partial\mathbb{H}^2)$ to $S^2(\cong \partial\mathbb{H}^3)$, namely a Cannon–Thurston map [22, 60]. Furthermore, ϕ intertwines the action by the infinite cyclic group $\mathbb{Z}$ generated by the pseudo-Anosov monodromy ψ on S^1 with the action of a hyperbolic element on S^2. It follows that ϕ intertwines the actions of $\pi_1(M)$ on S^1 and on S^2.

Any element in $g \in \pi_1(M)$ acting on S^2 has a fixed point p as it is a hyperbolic transformation of $\mathbb{H}^3$, since we assume that M is closed. By the elementary properties of the Cannon–Thurston map [22, 60], we have that the action of g on S^1 preserves $\phi^{-1}(p)$ as a set. Moreover, since p is the fixed point of a hyperbolic element and ϕ is uniformly finite–to–one, it also follows that $\phi^{-1}(p)$ is finite and of bounded cardinality. Passing to a finite index subgroup of $\pi_1(M)$ if necessary, we may assume that for every $g \in \pi_1(M)$, g fixes every point of $\phi^{-1}(p)$. □

Lemma 6.9 (See [44])**.** *Let $t \in \pi_1(M)$ be an element. There exists an $N \gg 0$ and a closed surface subgroup $G < \pi_1(M)$ such that $t^N \in G$.*

We are grateful to J. Kahn for explaining the relevant part of [44] to us in the argument below.

Proof of Lemma 6.9. The construction of Kahn and Markovic in [44] proceeds by gluing pairs of pants together to construct closed surface subgroups $G < \pi_1(M)$. This is arranged in such a way that for all $R > 0$ sufficiently large, and $\epsilon > 0$ sufficiently small, there exists a G containing conjugates of all elements of $\pi_1(M)$ that have translation length $\epsilon-$close to $2R$.

Now, let $H = \langle t \rangle$ be any cyclic subgroup of $\pi_1(M)$. For sufficiently large $N \gg 0$, t^N has large enough translation length and hence has a conjugate contained in a closed surface subgroups $G < \pi_1(M)$ by the previous paragraph. □

By setting t to be a sufficiently high power of a correctly chosen lift of the stable letter in $\pi_1(M)$, we obtain the following:

Corollary 6.10. *There exists closed surface subgroup $G < \pi_1(M)$ and a nontrivial element $t \in G$ such that t has at least four fixed points on S^1.*

We remark that geometrically, the group G in Corollary 6.10 is forced to be quasi–Fuchsian, as opposed to a virtual fiber, by [25].

For $H \leqslant \mathrm{Homeo}(S^1)$, the minimal H–invariant nonempty closed set is called as the *limit set* of H, and denoted as $\Lambda(H)$.

Theorem 6.11. *Let $\phi\colon \pi_1(M) \to \mathrm{Homeo}^+(S^1)$ be the natural action induced by the fibration*

$$S \to M \to S^1,$$

with $t \in \pi_1(M)$ identified with a suitable power of the stable letter so that t has at least four fixed points on S^1. Then there exist a closed surface subgroup $G \cong \pi_1(S_g)$ of $\pi_1(M)$ containing a nonzero power of t such that the following hold:

(i) The action of G on S^1 via ϕ has marked rotation spectrum identically zero;
(ii) The action of G on S^1 via ϕ is not semi-conjugate to a projective action of $\pi_1(S_g)$.

Proof of Theorem 6.11. Part (i) is the content of Lemma 6.8, and is true for all subgroups of $\pi_1(M)$, after passing to a finite index subgroup if necessary.

For part (ii), we may first note from Lemma 6.1 that G is not semi-conjugate to an indiscrete projective action. Now there are two separate cases to consider. In the first, we have that the action of G on S^1 is minimal. In this case, Lemma 2.10

implies that any semi-conjugacy between the action of G and an projective action of $\pi_1(S_g)$ would have to be a conjugacy. Since $t^N \in G$ acts with at least four fixed points on S^1, it follows that the action of t^N is not conjugate to the action of any element of $\mathrm{PSL}_2(\mathbb{R})$ on S^1. In particular, the G–action on S^1 is not semi–conjugate to a projective action.

So, we may assume that G does not act minimally on S^1. Let $\{z_1, \ldots, z_k\}$ be the fixed points of the generator t on S^1 and set $H = G \cap \pi_1(S)$, the kernel of the map $\pi_1(M) \to \mathbb{Z}$ induced by the fibration $M \to S^1$ when restricted to G. Set $\Lambda = \Lambda(H)$ to be the limit set of H, which we may assume is properly contained in S^1. Since for each i the point z_i is attracting for either t or t^{-1} and because H is normal in G, we have that

$$\{z_1, \ldots, z_k\} \subset \Lambda.$$

By collapsing wandering domains (using minimalization as in Section 2), we may apply a semi-conjugacy ψ to the action of G to replace it by a minimal action. If no two distinct points in $\{z_1, \ldots, z_k\}$ are identified after ψ then it follows as in the case of a minimal G action that the action is not semi-conjugate to an projective action of $\pi_1(S_g)$.

It therefore suffices to show that no two such attracting fixed points of t, say z_1 and z_2, are identified ψ. Observe that there is a repelling fixed point z_3 lying between z_1 and z_2, which also lies in Λ. So, if z_1 and z_2 are identified via ψ then z_3 is also identified with them. This cannot happen under the minimalization, since the minimalization is either two–to–one or one–to–one at every point in the limit set. Precisely, the minimalization is given by the "devil's staircase map" or "Nielsen's Cannon–Thurston map" (see [32]).

Alternatively, we can use the fact that the action of H on its limit set is minimal, and that the semi-conjugacy ψ between the G–action and a minimal one is G–equivariant. It follows that an open subset of Λ is collapsed to a point under ψ which in turn implies that ψ is a semi-conjugacy to a constant map. In particular, the G action is not semi-conjugate to a projective action of $\pi_1(S_g)$. □

6.4. **Free groups.** For free groups, we perform same analysis as for surface groups in the previous subsection.

6.4.1. *Projective actions.*

Theorem 6.12. *There are only finitely many semi-conjugacy classes of faithful projective actions ϕ of F_n such that $\Sigma(\phi) = \{0\}$. Such a representation ϕ is necessarily discrete.*

Proof. The case $n = 1$ is trivial by Lemma 2.16. So we suppose $n \geqslant 2$. We let

$$\Phi = \{\phi \in \mathrm{Hom}(F_n, \mathrm{PSL}_2(\mathbb{R})) \mid \Sigma(\phi) = \{0\} \text{ and } \phi \text{ is faithful}\}.$$

By Lemma 6.1, each $\phi \in \Phi$ is discrete. In particular, $\mathbb{H}^2/\phi(F_n)$ is a non–compact hyperbolic surface.

Claim. *If $\phi \in \Phi$, then $\phi \sim_s \psi$ for some $\psi \in \Phi$ such that $\Lambda(\psi) = S^1$.*

Suppose $\Lambda(\phi) \neq S^1$. In this case, $\mathbb{H}^2/\phi(F_n)$ is a non–compact quotient surface with geodesic cuffs and flaring ends. Degenerating these cuff lengths to zero results in a free group action on S^1 which is semi-conjugate to the original one, but whose limit set is all of S^1, since the quotient has finite volume. The details of this construction are worked out by Floyd in [32].

Now let $\phi \in \Phi$ satisfy $\Lambda(\phi) = S^1$. Then $S = \mathbb{H}^2/\phi(F_n)$ is a finite-volume hyperbolic surface with cusps. Since we have

$$\chi(S) = 1 - n,$$

there are finitely many homeomorphism types of such a surface S. Each homeomorphism type corresponds to two conjugacy classes (depending on the choice of the orientation) of actions by Lemma 6.3. In the case when the limit set is S^1, we know the notion of semi-conjugacy coincides with that of conjugacy by Lemma 2.10. It follows that $\Phi/\sim_s$ is finite. □

Question 6.13. *How many semi-conjugacy classes of projective free group actions are there on the circle with a given nonzero marked rotation spectrum?*

6.4.2. *Nonlinear actions.* As described in the introduction, analytic actions of free groups which are not semi-conjugate to projective actions are produced in [1]. Here, we follow the discussion above and prove the corresponding analogue of Theorem 6.11:

Theorem 6.14. *There exist a faithful representation of a finitely generated free group into the group* $\mathrm{Homeo}^+(S^1)$ *such that the following hold.*

(i) The actions have marked rotation spectrum identically zero;
(ii) The actions are not semi-conjugate to a projective action of a free group.

Proof. We write

$$\phi\colon G \to \mathrm{Homeo}^+(S^1)$$

for a representation as furnished by Theorem 6.11, and we let $t \in G$ be a power of the stable letter. Let $K < G$ be a finitely generated free subgroup of G which contains a nonzero power of t such that $K \cap \pi_1(S)$ is nonabelian. Then the same argument as in Theorem 6.11 shows that K is not semi-conjugate to a projective action (The necessary facts about Cannon-Thurston maps in the punctured surface case may be found in [9, 59, 60]). □

6.5. **Free products of finite cyclic groups.**

Theorem 6.15. *For all pairs of distinct integers $n, m > 1$, there exists a faithful C^∞ action*

$$\phi\colon \mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z} \to \mathrm{Diff}^\infty(S^1)$$

such that $\mathrm{Fix}\,\phi(g) = \varnothing$ *for some infinite order*

$$g \in \mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}.$$

In particular, the image of ϕ is not semi-conjugate to a discrete projective action of $\mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}$, and the image of ϕ is not conjugate into a subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

Proof. We let D and T be rotations of the circle of order n and m respectively. We choose an element $\psi \in \mathrm{Diff}^\infty_+(S^1)$ which is very close to the identity and such that

$$\langle D, T^\psi \rangle \cong \mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z}.$$

Such an element exists by Theorem 3.30.

Writing d and t for order n and order m generators of

$$\mathbb{Z}/n\mathbb{Z} * \mathbb{Z}/m\mathbb{Z},$$

we note that $d \cdot t$ has infinite order. We claim that if ψ is sufficiently close to the identity then $D \cdot T^\psi$ has no fixed points in S^1 and is therefore an infinite order element of $\mathrm{Diff}^\infty_+(S^1)$ with nonzero rotation number.

View S^1 as the unit complex numbers. Suppose

$$D \cdot T^\psi(x) = x$$

for some $x \in S^1$. Then

$$T^\psi(x) = \zeta \cdot x$$

for some primitive n^{th} root of unity ζ. The equation

$$T(x) = \zeta \cdot x$$

has no solutions, since T acts by multiplication by a primitive m^{th} root of unity, where $n \neq m$. We now have that the function $S^1 \to \mathbb{C}$ given by

$$x \mapsto T(x) - \zeta \cdot x$$

does not attain the value 0. Since S^1 is compact, it follows that if ψ is close enough to the identity then the function

$$T^\psi(x) - \zeta \cdot x$$

also does not attain the value 0. Therefore, $D \cdot T^\psi$ has no fixed points in S^1.

For the second part of the conclusion, we choose ψ so that T^ψ preserves the set of m–th roots of unity but such that T^ψ is not a rotation about the center. □

We remark that an easy application of the Kurosh Subgroup Theorem shows that given a finite free product G of cyclic groups (finite or infinite), there exists a free product of two cyclic groups which contains a copy of G.

Question 6.16. *Are the actions in Theorem 6.15 semi-conjugate to (possibly indiscrete) projective actions?*

We suspect not.

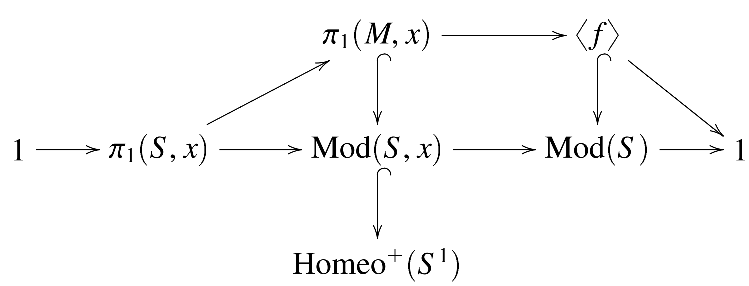
7. Bounded mapping class group actions on the circle

In this section, we shift the focus to exotic actions of mapping class groups on the circle, where here "exotic" means "not conjugate to Nielsen's standard action" (see [23] for a detailed discussion of Nielsen's action).

7.1. **The universal circle and Nielsen's action.** The actions of hyperbolic 3–manifold groups which we discussed above, particularly in Theorem 6.11, generalize naturally to mapping class group actions on the circle. Let S be a closed hyperbolic surface, let $f \in \mathrm{Mod}(S)$ be a pseudo-Anosov mapping class of S, and let $x \in S$ be a distinguished marked point. We lift f to the mapping class group $\mathrm{Mod}(S, x)$ fixing x and we abuse notation by calling this lift f as well. Let M denotes the mapping torus of S with the gluing map f. By choosing the preferred lift of f to $\mathbb{H}^2$ which fixes a chosen preimage of the basepoint $\widetilde{x} \in \mathbb{H}^2$, we obtain an action of

$$\pi_1(M, x) \cong \pi_1(S, x) \rtimes_f \mathbb{Z}$$

on $\mathbb{H}^2$ and consequently on S^1, as discussed in Section 6. For compactness of notation, we call this action the *universal circle action* of $\pi_1(M)$. We refer to the action of $\mathrm{Mod}(S)$ on S^1 given by choosing the preferred lift of a given mapping class which fixed $\widetilde{x}$ by *Nielsen's action*. We have the following diagram:



The commutativity of the diagram above is probably well–known (cf. [28], for instance, as well as [20, 18]), but we include a proof for the convenience of the reader.

Proposition 7.1. *The universal circle action*

$$\pi_1(M, x) \to \mathrm{Homeo}^+(S^1)$$

extends to Nielsen's action

$$\mathrm{Mod}(S, x) \to \mathrm{Homeo}^+(S^1).$$

Proof. This follows more or less from the definition of the universal circle action and Nielsen's action. The group $\pi_1(S, x)$ acts as usual by deck transformations on $\mathbb{H}^2$, permuting the lifts of x. To build the mapping torus M and to compute its fundamental group, one first lifts the mapping class f to a homeomorphism F of S fixing x, so that arbitrary lifts of F preserve the total preimage of x. Different lifts of F are obtained by precomposing or by composing with a deck transformation, so there always exists a preferred lift of F once a point $\widetilde{x}$ in the preimage of x is fixed. The action of $\pi_1(M)$ on the circle one obtains this way is independent of the choice of lift.

Now consider Nielsen's action of $\mathrm{Mod}(S, x)$ on S^1. This action is defined by taking preferred lifts of mapping classes to $\mathbb{H}^2$ which fix $\widetilde{x}$, and extending their actions to S^1. The natural copy

$$\pi_1(S, x) < \mathrm{Mod}(S, x)$$

which is the kernel of the forgetful map

$$\mathrm{Mod}(S, x) \to \mathrm{Mod}(S)$$

is the point pushing subgroup. It is well–known that the action of the point pushing subgroup of $\mathrm{Mod}(S, x)$ acts on S^1 in a way which is identical to the deck group action of $\pi_1(S)$ on S^1 (see [31] for instance). It follows that if

$$f \in \mathrm{Mod}(S, x) \backslash \pi_1(S, x)$$

then the action of $\langle \pi_1(S, x), f \rangle$ coincides with the universal circle action of

$$\pi_1(S) \rtimes_f \mathbb{Z}.$$

Since $f \in \mathrm{Mod}(S)$ was assumed to be nontrivial, we have that any lift of f to $\mathrm{Mod}(S, x)$ avoids $\pi_1(S, x)$. This completes the proof of the result. □

We remark that from the discussion above, it is easy to see that Nielsen's action on S^1 is minimal.

7.2. **Exotic mapping class group actions.** Let S be a compact hyperbolic surface with nonempty boundary and with one marked point x in the interior of S. In this section, we show there are a pair of representations

$$\mathrm{Mod}(S, x) \to \mathrm{Homeo}^+(S^1)$$

which are inequivalent, as well as consider certain representations

$$\mathrm{Mod}(S) \to \mathrm{Homeo}^+(\mathbb{R}).$$

We note that the mapping class group with at least one marked point admits many other interesting actions on the circle, which we will not consider here. See [10], for instance.

We can define *weak semi-conjugacy* for two representations

$$G \to \mathrm{Homeo}^+(\mathbb{R}),$$

where we only require $h\colon \mathbb{R} \to \mathbb{R}$ in the diagram in Section 2 to be non-constant and monotone increasing. For a representation ρ, we let $\mathrm{Fix}\,\rho$ mean the fixed point set of ρ. The following complements Theorem 2.13.

Lemma 7.2. *Let*

$$\rho_1, \rho_2\colon G \to \mathrm{Homeo}^+(\mathbb{R})$$

be representations of a group G such that $\rho_1 \geqslant_h \rho_2$ for some non-constant monotone increasing map h. If $\mathrm{Fix}\,\rho_1 = \varnothing$, *then* $\mathrm{Fix}\,\rho_2 = \varnothing$.

Proof of Lemma 7.2. Let us assume the contrary and pick $y \in \mathrm{Fix}\,\rho_2$. Then $J = h^{-1}(y)$ is a $\rho_1(G)$–invariant proper connected subset of M. Since ∂J consists of global fixed points of ρ_1, it follows that J is empty. □

Let $S = S_{g,0,b}$ where $g \geqslant 1$ and $b \geqslant 1$. Endow S with a compact hyperbolic metric so that ∂S is geodesic. We fix an interior point $x \in S$. We denote by $p\colon \tilde{S} \to S$ the universal cover of S. We can identify $\tilde{S}$ with a convex subset of the hyperbolic plane.

Nielsen's action gives us a representation

$$\rho_1\colon\ \mathrm{Mod}(S, x) \to \mathrm{Homeo}^+(S^1).$$

Thurston constructed a faithful action

$$\mathrm{Mod}(S) \to \mathrm{Homeo}^+(\mathbb{R})$$

by considering lifts of self-homeomorphisms of S to the universal cover $\tilde{S}$. Such lifts are chosen to fix one distinguished lift $\tilde{\beta}$ of β, so that $\mathrm{Mod}(S)$ acts on

$$(\partial\tilde{S} \setminus \tilde{\beta}) \cup \partial\pi_1(S, x) \approx \mathbb{R}.$$

See [65] and [38] for more detail. Thurston's action similarly yields

$$\mathrm{Mod}(S \setminus x) \to \mathrm{Homeo}^+(\mathbb{R})$$

using the universal cover of $S \setminus x$. So we have a faithful representation

$$\rho_2\colon\ \mathrm{Mod}(S, x) \to \mathrm{Homeo}^+(S^1).$$

defined by the composition

$$\mathrm{Mod}(S, x) \xrightarrow{\cong} \mathrm{Mod}(S \setminus x) \to \mathrm{Homeo}^+(\mathbb{R}) \hookrightarrow \mathrm{Homeo}^+(S^1).$$

We denote Thurston's action of $\mathrm{Mod}(S)$ by

$$\tau_1\colon\ \mathrm{Mod}(S) \to \mathrm{Homeo}^+(\mathbb{R}).$$

Let us denote by A a once-punctured annulus. If A is glued to S along a boundary component, the resulting surface is homeomorphic to $S\backslash x$. Hence we have the following *inclusion homomorphism*, as in [31]:

$$i\colon\ \mathrm{Mod}(S) \to \mathrm{Mod}(S\backslash x).$$

Note that the inclusion homomorphism splits the Birman exact sequence:

$$1 \to \pi_1(S,q) \to \mathrm{Mod}(S\backslash x) \to \mathrm{Mod}(S) \to 1,$$

where

$$\pi_1(S,x) \to \mathrm{Mod}(S\backslash x) \cong \mathrm{Mod}(S,x)$$

is the point–pushing map. By composing the maps

$$\mathrm{Mod}(S) \to \mathrm{Mod}(S\backslash x) \to \mathrm{Homeo}^+(\mathbb{R}),$$

we have a faithful action

$$\tau_2\colon\ \mathrm{Mod}(S) \to \mathrm{Homeo}^+(\mathbb{R}).$$

We can summarize the aforementioned maps in the following diagram:

$$\begin{array}{ccccc}
\mathrm{Homeo}^+(S^1) & \longleftarrow & & & \mathrm{Homeo}^+(\mathbb{R}) \\
\rho_1 \uparrow\uparrow \rho_2 & & & \nearrow & \tau_2 \uparrow\uparrow \tau_1 \\
\mathrm{Mod}(S,x) & \xrightarrow{\cong} & \mathrm{Mod}(S\backslash x) & \xleftarrow{\ i\ } & \mathrm{Mod}(S)
\end{array}$$

Note that

$$\mathrm{Fix}\,\rho_1 = \varnothing \neq \mathrm{Fix}\,\rho_2$$

and that

$$\mathrm{Fix}\,\tau_1 = \varnothing \neq \mathrm{Fix}\,\tau_2.$$

It is immediate from Theorem 2.13 that

$$\rho_1^*\mathrm{eu}_b \neq 0 = \rho_2^*\mathrm{eu}_b.$$

In fact, we have that ρ_1 and ρ_2 (also, τ_1 and τ_2) do not merely fail to be semi-conjugate, but actually inequivalent.

We immediately obtain the following proposition:

Proposition 7.3. *The following statements hold:*

(1) The actions ρ_1 and ρ_2 are inequivalent;
(2) The actions τ_1 and $\tau_2\circ\alpha$ are not weakly semi-conjugate for all $\alpha \in \mathrm{Aut}(\mathrm{Mod}(S))$.

In the spirit of Section 4, we ask:

Question 7.4. *Do mapping class groups admit uncountably many inequivalent actions on the circle?*

Appendix A. Equivalent of notions of semi-conjugacy

The aim of this appendix is to illustrate the equivalence of the different notions of semi-conjugacy in Theorem 2.17. We will mainly give an account of the fact that monotone equivalence and minimalization equivalence both coincide with the other notions of semi-conjugacy. We will also guide the reader through the literature for a proof that the rest of the notions of semi-conjugacy are equivalent to each other.

The facts in this section are mostly based on Ghys' original ideas in [36, 38]. Readers are also referred to very recent surveys by Mann [53] and by Bucher–Frigerio–Hartnick [14].

Monotone invariance of the Euler class

Throughout this appendix, we let G be a countable group.

Let $h\colon S^1 \to S^1$ be a monotone degree one map. Following the notation from [19], we let $\mathrm{Gap}(h)$ denote the set of locally constant points of h, and

$$\mathrm{Core}(h) = S^1 \backslash \mathrm{Gap}(h).$$

The set $\mathrm{Gap}(h)$ is a countable disjoint union of open intervals such that no two intervals have a common endpoint. Let

$$\mathrm{Gap}_\partial(h) = \{x \in S^1 \mid x \in \partial J \text{ for some connected component } J \text{ of } \mathrm{Gap}(h)\}.$$

Lemma A.1. *For a surjective monotone degree one map h on S^1, the set*

$$\mathrm{Core}(h)\backslash \mathrm{Gap}_\partial(h)$$

is uncountable.

Proof. It is proved in [19, Lemma 2.14] that $\mathrm{Core}(h)$ is perfect. This also follows from the fact that if A is a countable disjoint union of open intervals in S^1 such that no two intervals have a common endpoint, then $S^1\backslash A$ is perfect. So in particular, $\mathrm{Core}(h)$ is uncountable. Note $\mathrm{Gap}_\partial(h)$ is countable. ◻

Let us now explain the invariance of the bounded Euler class, closely following [19]. Recall our notation $\mathscr{H} = \mathrm{Homeo}^+(S^1)$. A *circle action* of G means a group homomorphism $G \to \mathrm{Hom}(G, \mathscr{H})$.

Lemma A.2 ([19])**.** *Let ρ_1 and ρ_2 be circle actions of G. If $\rho_1 \succcurlyeq_h \rho_2$ for some surjective monotone degree one map h, then $\rho_1^* \mathrm{eu}_b = \rho_2^* \mathrm{eu}_b$.*

Proof. Using Lemma A.1, we may assume $0 \in \mathrm{Core}(h)\backslash \mathrm{Gap}_\partial(h)$.

Choose an arbitrary continuous monotone degree one map $\tilde{h} \in \tilde{\mathscr{H}}$ lifting h and let $p = \tilde{h}(0)$. Recall from Section 2 that we have sections

$$s, s^p \colon \mathscr{H} \to \tilde{\mathscr{H}},$$

denoted by $sf = \tilde{f}$ and $s^p f = \hat{f}$. We have that

$$\tilde{f}(0) \in [0,1)$$

and that

$$\hat{f}(p) \in [p, p+1)$$

for each $f \in \mathscr{H}$. Then the following diagram commutes modulo $\mathbb{Z}$:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widetilde{\rho_1(a)}} & \mathbb{R} \\ \downarrow{\tilde{h}} & & \downarrow{\tilde{h}} \\ \mathbb{R} & \xrightarrow{\widehat{\rho_2(a)}} & \mathbb{R}. \end{array}$$

We claim that the above diagram commutes on the nose. Evaluating at 0, we see that

$$\widehat{\rho_2(a)}\tilde{h}(0) = \widehat{\rho_2(a)}p \in [p, p+1), \quad \text{and that} \quad p = \tilde{h}(0) \leqslant \tilde{h}\widetilde{\rho_1(a)}(0) \leqslant \tilde{h}(1) = p+1.$$

Moreover, we have

$$\tilde{h}\widetilde{\rho_1(a)}(0) \neq \tilde{h}(1)$$

since

$$0 \in \operatorname{Core}(h) \backslash \operatorname{Gap}_\partial(h),$$

which establishes the commutativity of the diagram.

For $a, b \in G$, we have that

$$\begin{aligned} \rho_1^* \mathrm{eu}(a,b) = 1 &\Leftrightarrow \widetilde{\rho_1(a)}\widetilde{\rho_1(b)}0 \geqslant 1 \\ &\Leftrightarrow \tilde{h}\widetilde{\rho_1(a)}\widetilde{\rho_1(b)}0 \geqslant \tilde{h}1 = p+1 \\ &\Leftrightarrow \widehat{\rho_2(a)}\widehat{\rho_2(b)}p \geqslant p+1 \\ &\Leftrightarrow \rho_2^* \mathrm{eu}^p(a,b) = 1. \end{aligned}$$

Note we used again the condition that 0 is not a locally constant point of h. It follows that

$$\rho_1^* \mathrm{eu} = \rho_2^* \mathrm{eu}^p$$

as cocycles. By the basepoint independence of the bounded Euler class, we have

$$\rho_1^* \mathrm{eu}_b = \rho_2^* \mathrm{eu}_b$$

in $H_b^2(G; \mathbb{Z})$. □

Lemma A.3. *Let ρ_1 and ρ_2 be circle actions of G. Suppose that $h\colon \mathbb{R} \to \mathbb{R}$ is a (possibly discontinuous) monotone degree one map such that $\rho_1 \succcurlyeq_h \rho_2$. Then there exist a representation*

$$\rho \in \mathrm{Hom}(G, \tilde{\mathscr{H}})$$

and continuous monotone degree one maps

$$h_1, h_2\colon \mathbb{R} \to \mathbb{R}$$

such that $\rho \succcurlyeq_{h_i} \rho_i$ for $i =, 2$.

Proof. For each $x \in \mathbb{R}$, we define

$$h(x-) = \lim_{t \to x-0} h(t) \quad \text{and} \quad h(x+) = \lim_{t \to x+0} h(t).$$

Let us consider the strip

$$X = \mathbb{R} \times [1, 2] \subseteq \mathbb{R}^2.$$

For two points $p, q \in X$, we denote by $[p, q]$ the segment joining p and q. Define a singular foliation of X:

$$\mathscr{F} = \{[(x, 1), (y, 2)] \colon x \in \mathbb{R} \text{ and } y \in [h(x-), h(x+)]\}.$$

Each point $(x, 3/2) \in X$ belongs to a unique leaf, say $L(x) \in \mathscr{F}$. So there exists maps $h_i\colon \mathbb{R} \to \mathbb{R}$ uniquely determined by the condition:

$$L(x) = [(h_1(x), 1), (h_2(x), 2)].$$

The map h_i is monotone increasing, since leaves to not intersect in the interior of X. Also

$$h \circ T = T \circ h$$

implies that

$$h_i \circ T = T \circ h_i$$

for each $i = 1, 2$. Whenever $\{x_n\}$ converges to x, the sequence of leaves $\{L(x_n)\}$ converges to $L(x)$ in the Hausdorff distance. So h_i is continuous.

For each $g \in G$ and $x \in \mathbb{R}$, we define $\rho(g)(x) \in \mathbb{R}$ by the formula:

$$L(\rho(g)(x)) = [(\rho_1(g) \circ h_1(x), 1), (\rho_2(g) \circ h_2(x), 2)].$$

Then it is routine to check that $\rho(g) \in \tilde{\mathscr{H}}$ for each $g \in G$ and $\rho\colon G \to \tilde{\mathscr{H}}$ is a group homomorphism. Since

$$[(\rho_1(g) \circ h_1(x), 1), (\rho_2(g) \circ h_2(x), 2)]$$

is the unique leaf containing $(\rho(g)(x), 3/2)$, the dominance $\rho \succcurlyeq_{h_i} \rho_i$ follows. □

Alternative proof of Lemma A.3. One can give an elementary algebraic proof based on the above idea. Define two strictly increasing maps

$$f^-(y) = \frac{y + h(y-)}{2}$$

and

$$f^+(y) = \frac{y + h(y+)}{2}.$$

Then f^- and f^+ are left- and right-continuous, respectively. Moreover, $f^- = f^+$ except for the (countably many) jump discontinuities of h. So there uniquely exists a continuous monotone map h_1 which is the "inverse" of $f^\pm$ in the following sense:

$$f^- \circ h_1(x) \leqslant x \leqslant f^+ \circ h_1(x), \text{ for all } x.$$

Define

$$h_2(x) = 2x - h_1(x).$$

It is immediate that h_1 and h_2 are continuous and

$$h_i \circ T = T \circ h_i.$$

If $x < y$, then

$$h_2(x) = 2x - h_1(x) \leqslant h(h_1(x)+) \leqslant h(h_1(y)-) \leqslant 2y - h_1(y) = h_2(y).$$

So h_2 is also monotone.

For each $a \in G$, define

$$\rho(a)(x) = \frac{1}{2}\left(\rho_1(a)h_1(x) + \rho_2(a)h_2(x)\right).$$

We have seen that

$$h_2(x) \in [h(h_1(x)-), h(h_1(x)+)].$$

Therefore we can write

$$\rho(a)(x) \in [p^-, p^+],$$

where

$$\begin{aligned} p^\pm &= \frac{1}{2}\left(\rho_1(a)h_1(x) + \rho_2(a) \circ h(h_1(x)\pm)\right) \\ &= \frac{1}{2}\left(\rho_1(a)h_1(x) + h(\rho_1(a) \circ h_1(x)\pm)\right) = f^\pm(\rho_1(a)h_1(x)). \end{aligned}$$

By the definition of h_1, we have

$$h_1\rho(a)(x) = \rho_1(a)h_1(x).$$

Moreover,

$$h_2\rho(a)(x) = 2\rho(a)(x) - h_1\rho(a)(x) = 2\rho(a)(x) - \rho_1 h_1(x) = \rho_2(a)h_2(x).$$

We see

$$\rho(a)(\rho(b)x) = \frac{1}{2}\left(\rho_1(a)h_1\rho(b)(x) + \rho_2(a)h_2\rho(b)(x)\right)$$
$$= \frac{1}{2}\left(\rho_1(a)\rho_1(b)h_1(x) + \rho_2(a)\rho_2(b)h_2(x)\right) = \rho(ab)(x).$$

Hence ρ is a desired group action. □

Lemma A.4. *Let ρ_1 and ρ_2 be minimal circle actions of G. If $\rho_1 \succcurlyeq \rho_2$, then ρ_1 and ρ_2 are conjugate.*

Proof. Let $\rho_1 \succcurlyeq_h \rho_2$ for some monotone degree one map h on S^1. Note that

$$\rho_2(G)\left(h(S^1)\right) = h\left(\rho_1(G)(S^1)\right) = h(S^1).$$

So $h(S^1)$ is $\rho_2(G)$–invariant. By minimality, we see $h(S^1)$ is dense in S^1. It follows that h does not have jump discontinuities, and so surjective and continuous. For each open nonempty interval J in $\mathrm{Gap}(h)$, we see

$$h \circ \rho_1(g) \restriction_J = \rho_2(g) \circ h \restriction_J$$

is constant. So $\mathrm{Gap}(h)$ is $\rho_1(G)$–invariant. Since

$$\mathrm{Core}(h) = S^1 \backslash \mathrm{Gap}(h)$$

is a closed, uncountable (Lemma A.1) and $\rho_1(G)$–invariant, we see that $\mathrm{Core}(h) = S^1$. So h is injective. This shows that h is a homeomorphism. □

Finite orbits

Note that a circle action ρ of G has a finite orbit if and only if a finite-index subgroup of G has a global fixed point. For each

$$\theta \in S^1 = \mathbb{R}/\mathbb{Z},$$

let us also use $T(\theta)$ to denote the projection (to S^1) of the map $x \mapsto x + \theta$ on $\mathbb{R}$.

Denote by $\mathrm{eu}_b^{\mathbb{R}} \in H_b^2(G;\mathbb{R})$ the image of $\mathrm{eu}_b \in H_b^2(G;\mathbb{Z})$ under the map induced by $\mathbb{Z} \hookrightarrow \mathbb{R}$.

Lemma A.5. *Let ρ be a circle action of G.*

(1) Then ρ has a finite orbit of cardinality N if and only if the exponent of $\rho^*\mathrm{eu}_b$ is N in $H_b^2(G;\mathbb{Z})$.
(2) If ρ has a finite orbit, then the map $\bar\rho = T \circ \mathrm{rot} \circ \rho$ is a group action of G and satisfies $\bar\rho^*\mathrm{eu}_b = \rho^*\mathrm{eu}_b$.

Proof. (1) ($\Rightarrow$) Suppose

$$\rho(G)p = \{p_0 = p, p_1, \ldots, p_{N-1}\}$$

in the cyclic order of S^1. Then we have a surjective group homomorphism $\beta\colon G \to \mathbb{Z}/N\mathbb{Z}$ such that $\rho(g)p_i = p_{i+\beta(g)}$. We may assume $p = 0$. Fix a lift

$$\tilde{\beta}\colon G \to \{0, 1, \ldots, N-1\}.$$

Then

$$\rho^*\mathrm{eu}(a,b) = \frac{1}{N}\left(\tilde{\beta}(a) + \tilde{\beta}(b) - \tilde{\beta}(ab)\right) = \frac{1}{N}\partial\tilde{\beta}(a,b).$$

So we have $N\rho^*\mathrm{eu}_b = 0$.

Let M be the exponent of $\rho^*\mathrm{eu}_b = 0$. We have $M\rho^*\mathrm{eu} = \partial\tilde{\gamma}$ for some bounded map $\tilde{\gamma}\colon G \to \mathbb{Z}$. We can write $N = Mk$ for some $k > 0$ by the minimality of M. We see

$$\partial\left(\tilde{\beta} - k\tilde{\gamma}\right) = N\left(\frac{1}{N}\partial\tilde{\beta} - \frac{1}{M}\partial\tilde{\gamma}\right) = 0.$$

So $\tilde{\beta} - k\tilde{\gamma}\colon G \to \mathbb{Z}$ is a homomorphism, which is bounded. It follows that $\tilde{\beta} = k\tilde{\gamma}$. Since $\tilde{\beta}$ is surjective, we have $k = 1$ and $M = N$.

(1) ($\Leftarrow$) Suppose N is the exponent of $\rho^*\mathrm{eu}$. Note that we allow $N = 1$. Then

$$N\rho^*\mathrm{eu} = \partial\tilde{\beta}$$

for some bounded map

$$\tilde{\beta}\colon G \to \mathbb{Z}.$$

We have a group homomorphism

$$\beta\colon G \to \mathbb{Z}/N\mathbb{Z}$$

from the post-composition $\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$. The minimality of N implies that β is surjective.

Set $H = \ker\beta$ and $\sigma = \rho\restriction_H$. Note that

$$\tilde{\beta}(a)/N \in \mathbb{Z}$$

for each $a \in H$. By writing

$$\sigma^*\mathrm{eu}(a,b) = \tilde{\beta}(a)/N + \tilde{\beta}(b)/N - \tilde{\beta}(ab)/N$$

we see that

$$\sigma^*\mathrm{eu}_b = 0 \in H^2_b(H;\mathbb{Z}).$$

Write the cocycle $\sigma^*\mathrm{eu} = \partial\gamma$ for some bounded map γ on H. We can lift σ to $\tilde{\sigma}\colon H \to \mathscr{H}$ by the formula

$$\tilde{\sigma}(g) = T(-\gamma(g))s \circ \sigma(g);$$

see Section 2. Then

$$\tilde{\sigma}(g)(0) = s \circ \sigma(g)(0) - \gamma(0) \in [-\|\gamma\|_\infty, \|\gamma\|_\infty + 1).$$

So we can find

$$p = \sup\{\tilde{\sigma}(g)(0) \mid g \in G\} \in \operatorname{Fix}\tilde{\sigma}.$$

The projection of p is a global fixed point of σ. Moreover,

$$|\rho(G)p| = |G/H| = N.$$

(2) Let $\rho(G)p$ be a finite orbit. Using the notations from the proof of (1), we observe

$$\operatorname{rot}\circ\rho(g) = \beta(g)/N.$$

So $\bar{\rho}$ is a group homomorphism. While proving the direction ($\Rightarrow$) in (1), we have already seen that

$$\rho^*\mathrm{eu}^p(a,b) = \frac{1}{N}\left(\tilde{\beta}(a) + \tilde{\beta}(b) - \tilde{\beta}(ab)\right) = \bar{\rho}^*\mathrm{eu}^p(a,b)$$

as cocycles. □

Lemma A.6. *Let ρ_1, ρ_2 be circle actions of G. Suppose either*

$$\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b$$

or

$$\rho_1 \sim_{\min} \rho_2.$$

If ρ_1 has a finite orbit, then so does ρ_2.

Proof. If

$$\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b,$$

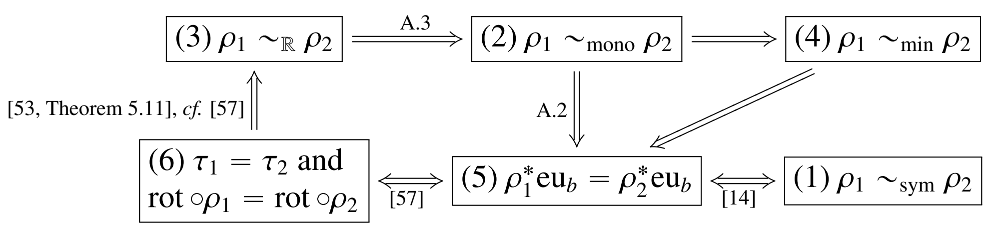
then Lemma A.5 (1) implies the conclusion. In the case where

$$\rho_1 \sim_{\min} \rho_2,$$

recall that a minimalization of ρ_i is minimal if and only if ρ does not have a finite orbit. □

Proof of Theorem 2.17

Proof. We will sketch a proof of Theorem 2.17 by giving the arguments for the implications in the following diagram.



Conditions (1) and (5) are equivalent by [14, Theorem 1.4]. Condition (6) implies (3), as elementarily (without using cohomology) proven in [53, Theorem 5.11]. Matsumoto proved that Condition (6) is equivalent to (5) [57]. Condition (2) implies (5) by Lemma A.2. Also (3) implies (2) by Lemma A.3.

Let ρ_1 and ρ_2 be minimalization equivalent, and let $\bar{\rho}$ be a common minimalization. By Lemma A.6, we have two cases. The first case is when they have finite orbits. Then Lemma A.5 (2) implies that

$$\rho_1^* \mathrm{eu}_b = \bar{\rho}^* \mathrm{eu}_b = \rho_2^* \mathrm{eu}_b.$$

The second case is when both actions have no finite orbits. There exist a surjective monotone degree one map h_i such that $\rho_i \succcurlyeq_{h_i} \bar{\rho}$ for $i = 1, 2$. Lemma A.2 implies that $\rho_1^* \mathrm{eu}_b = \rho_2^* \mathrm{eu}_b$.

Let ρ_1 and ρ_2 be monotone equivalent, and $\bar{\rho}_i$ is a minimalization of ρ_i. We already saw that $\rho_1^* \mathrm{eu}_b = \rho_2^* \mathrm{eu}_b$ from Lemma A.2. Again by Lemma A.5, we have two cases. If both have finite orbits, then Lemma 2.12 implies that the minimalizations $\bar{\rho}_i = T \circ \mathrm{rot} \circ \rho_i$ for $i = 1, 2$ satisfy $\bar{\rho}_1 = \bar{\rho}_2$. So we may assume neither have finite orbits. We have an action ρ such that

$$\rho \succcurlyeq_{h_i} \rho_i \succcurlyeq_{g_i} \bar{\rho}_i$$

for some surjective monotone degree one maps g_i, h_i. Moreover, $\bar{\rho}_i$ is a minimal action dominated by ρ for $i = 1, 2$. By Lemma A.4, we see that $\bar{\rho}_1$ and $\bar{\rho}_2$ are conjugate. □

Remark A.7. Let us define a *weak monotone equivalence* as the equivalence relation *generated* by a dominance relation using a continuous surjective monotone degree one map. This equivalence was originally called "monotone equivalence" in [19]. The agreement of the two notions of monotone equivalence is noted in [19, Lemma 2.22]. This coincidence also follows from Theorem 2.17, since weak monotone equivalence implies $\rho_1^* \mathrm{eu}_b = \rho_2^* \mathrm{eu}_b$ by Lemma A.2. The agreement between weak monotone equivalence and symmetric semi-conjugacy appears in [14].

Acknowledgments

The authors thank M. Bestvina, B. Bowditch, D. Calegari, M. Casals-Ruiz, K. Fujiwara, J. Kahn, I. Kim, Z. Sela, M. Triestino, T. N. Venkataramana and H. Wilton for helpful discussions. The authors thank M. Triestino for sharing the draft of [1] and for pointing out the reference [3]. The authors thank M. Wolff for sharing [72] and for several helpful comments. The authors are particularly grateful to K. Mann for a number of insightful comments and for pointing out several references, particularly [27]. The second and third authors thank the Tata Institute for Fundamental Research in Mumbai, and the first and second authors thank the Institute for Mathematical Sciences in Singapore, where parts of this research were conducted. The authors were supported by the National Science Foundation under Grant No. DMS-1440140 at the Mathematical Sciences Research Institute in Berkeley during Fall 2016, where this research was completed. The first author is supported by Samsung Science and Technology Foundation (SSTF-BA1301-06). The second author is partially supported by Simons Foundation Collaboration Grant number 429836. The third author is supported in part by a Department of Science and Technology J.C. Bose Fellowship grant.

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