HOROSPHERES IN DEGENERATE 3-MANIFOLDS

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ABSTRACT. We study horospheres in hyperbolic 3-manifolds M all whose ends are degenerate. Deciding which horospheres in M are properly embedded and which are dense reduces to
a) studying the horospherical limit set;
b) deciding which almost minimizing geodesics in M go through arbitrarily thin parts.
As an answer to (a), we show that the horospherical limit set consists precisely of the injective points of the Cannon-Thurston map. Addressing (b), we provide characterizations, sufficient conditions as well as a number of examples and counterexamples.

CONTENTS

1. Introduction

The topological study of unipotent orbits in locally symmetric spaces of finite volume has a rich history with some of the highlights being work of Hedlund [Hed36], Margulis [Mar89] and Ratner [Rat91]. However, the study in quotients of symmetric spaces by discrete subgroups of infinite covolume (c.f. [Sar14]) is in its infancy except in the special case of rank one symmetric spaces, or more generally infinite volume manifolds of pinched negative curvature, where the initial work was done by Eberlein [Ebe72, Ebe73]. Following up from work of Eberlein, a lot of work was done in dimension two, where one is interested in particular in the behavior of the horocycle flow on negatively curved surfaces of infinite genus (c.f. [DS00, Haa96, Sar10]). In this paper we initiate the detailed study of unipotent orbits (or horospheres) in hyperbolic 3-manifolds of infinite volume. We are particularly interested in degenerate hyperbolic 3-manifolds, i.e. hyperbolic 3-manifolds all whose ends are degenerate. Equivalently a degenerate hyperbolic 3-manifold is a quotient $M = \mathbb{H}^3/G$ where G is a finitely generated discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ which is not a lattice and the limit set Λ of G is the whole sphere $\mathbb{S}^2$. We will assume that our manifolds have no parabolics. We are following here the terminology introduced by Thurston in [Thu80] (as opposed to groups occurring in the Maskit slice, which have been called by similar names in the literature).

The following characterization is due to Eberlein [Ebe73], Ledrappier [Led97] and Coudène-Maucourant [CM10], (we will explain more precisely who proved what in sections 2.1 and 2.2)

Theorem 1.1. *(see Theorem 2.15) Let M be a degenerate hyperbolic 3-manifold and let $\gamma(:= \gamma(t))$ be a geodesic ray in M parametrized by arc length (and hence oriented). Let $W^{ss}(\gamma(t))$ be the strong stable manifold (i.e. stable horosphere) through $\dot{\gamma}(t)$. Then*

   (1) *$W^{ss}(\gamma(t))$ is dense in M if and only if γ is **not** almost minimizing.*
   (2) *$W^{ss}(\gamma(t))$ is recurrent but not dense in M if and only if γ is thin and almost minimizing.*
   (3) *$W^{ss}(\gamma(t))$ is properly embedded in M if and only if γ is thick and almost minimizing.*

A word about the terminology. We shall call a geodesic ray **exiting** if it is properly embedded, i.e. it is not contained in a compact set. A geodesic in M is **almost minimizing** if $|d_M(\gamma(0), \gamma(t)) - t|$ is uniformly bounded. Clearly, almost minimizing geodesics are exiting. Almost minimizing geodesics were studied by Haas [Haa96] in the context of 'flute surfaces', certain planar hyperbolic surfaces with infinitely many cusps. A version of Theorem 1.1 was deduced by Dal'bo and Starkov [DS00] in the context of infinitely geneated Schottky groups. Here we shall study almost minimizing geodesics in 3-manifolds. We will define recurrent horospheres in §2.3, for now, let us just say that they are not properly embedded. A geodesic ray is **thin** if it goes through points in M of arbitrarily small injectivity

radius, and is **thick** otherwise. Given the conclusions of Theorem 1.1 above, studying horospheres in degenerate hyperbolic 3-manifolds boils down to studying the following question:

Problem 1.2. *Describe all almost minimizing geodesics.*

Problem 1.2 has quite a satisfactory solution in terms of a necessary and sufficient condition. The ideal point of a lift of an almost minimizing geodesic ray lies in the complement $\Lambda_H^c (:= \Lambda \setminus \Lambda_H)$ of the horospherical limit set Λ_H. Another characterization can be made in term of the number of preimages under a Cannon-Thurston map. Given $M = \mathbb{H}/G$, let $i : \Gamma_G \to \mathbb{H}^3$ be the map that naturally comes from identifying the vertices of a Cayley graph of G with the orbit of a point in $\mathbb{H}^3$. Since we have assumed that G has no parabolics, Γ_G is hyperbolic. A *Cannon-Thurston map* is the restriction to the ideal boundary $\partial \hat{i} : \partial \hat{\Gamma}_G \to \mathbb{S}^2$ of a continuous extension $\hat{i} : \hat{\Gamma}_G \to \mathbb{D}^3$ of i. The existence and structure of such a map has been studied in [Mj14a, Mj10b]. Let Λ_m denote the multiple limit set, i.e. the collection of points in the limit set Λ that have more than one pre-image under the Cannon-Thurston map. The conclusion of Section 3 can be summarized as follows:

Theorem 1.3. $\Lambda_m = \Lambda_H^c$ *is the set of ideal points of lifts of almost minimizing geodesic rays.*

Theorem 1.3 answers an issue that has come up in works of several authors [Kap95, Ger12, JKLO16] who tried to relate the injective points of the Cannon-Thurston map to the *conical* limit set. They concluded that the conical limit set is strictly contained in the set of injective points of the Cannon-Thurston map. Theorem 1.3 thus shows that, in characterizing the injective points of the Cannon-Thurston map, the right limit set to be looking at is the *horospherical* rather than the conical limit set.

Then we are led to the following:

Problem 1.4. *Give conditions to determine which almost minimizing geodesics in a degenerate end of a hyperbolic 3-manifold are thick and which are thin.*

At this juncture, a kind of Murphy's Law breaks loose:

Anything that can go wrong does go wrong.

It seems difficult to solve Problem 1.4 comprehensively and we find a number of examples and counterexamples. At the end of Section 2 we give two examples: one in which all almost minimizing geodesics are thick and one in which all almost minimizing geodesics are thin. What these examples bring out is the importance of 'building blocks' in trying to solve Problem 1.4. Thus from Section 4 onwards, we attempt to address Problem 1.4 in terms of the model geometry of ends [Min10, BCM12, Mj11, Mj14a].

A number of model geometries for degenerate ends of hyperbolic 3-manifolds have come up, based primarily on Minsky's monumental work [Min94, Min99, Min10] culminating in the resolution of the Ending Lamination Conjecture by Brock-Canary-Minsky in [BCM12], and in the second author's proof of the existence of Cannon-Thurston maps [Mj14a, Mj10b]. In increasing order of complexity, these are:

(1) Bounded geometry [Min94, Mit98b, Mj10a]
(2) $i-$bounded geometry [Min99, McM01, Mj11]
(3) Amalgamation geometry [Mj16]

(4) Split geometry [Mj14a]

Of these, the first three are special and are the subject of study in Section 4, while every degenerate end E does admit a model of split geometry. The classification of these geometries depends on geometries of 'building blocks', i.e. geometries of copies of (topological) product regions $S \times I$ that are glued end-to-end to build up E. Our explorations lead to the following conclusions in the special cases of Section 4. As the reader will note, the conclusions become weaker and weaker as complexity increases.

Theorem 1.5. *(See Lemma 4.3, Proposition 4.7 and Lemma 4.11.)*

(1) *Let E be of bounded geometry. Then every exiting geodesic is thick. In particular every almost minimizing geodesic is thick.*
(2) *Let E be of i-bounded geometry. Then there exist thin exiting geodesics. However, every almost minimizing geodesic is thick.*
(3) *Any almost minimizing geodesic in an amalgamated geometry end is thick if all amalgamated blocks have bounded thickness.*

Here 'thickness' (roughly) refers to the shortest distance between the bottom and top surfaces (i.e. $S \times \{0\}$ and $S \times \{1\}$). We note here that the sufficient condition of 'bounded thickness' in Item (3) of Theorem 1.5 is quite strong.

To proceed further (and deal with the general case of split geometry) a fair bit of technical material from [MM00, Min10, Mj14a] is necessary. So as not to interrupt the flow of the paper, we proceed assuming this and relegate a summary of the relevant material to an Appendix, Section 6.

Section 5 dwells on counterexamples. Using the technology of Section 6, we find that the condition of 'bounded thickness' in Item (3) of Theorem 1.5 is not a necessary condition even in the special case of amalgamated geometry ends. The example of Section 5.1 shows that it is possible to have thick almost minimizing geodesics in manifolds of amalgamation geometry even in the presence of arbitrarily thick amalgamation blocks. Further, in the general case of split geometry, the sufficient condition of 'bounded thickness' is neither necessary nor sufficient. In Section 5.2 we provide a counterexample to show that there does exist an end of split geometry, where all the building blocks (or 'split blocks') have bounded thickness but almost minimizing geodesics are thin.

The examples of Section 5 to justify the 'Murphy's Law' that we mentioned above: as we progress to greater degrees of complexity of the geometry of ends, we tend to lose any hope of systematically characterizing which almost minimizing geodesics are thick and which are thin, i.e. we are unable to provide a satisfactory answer to Question 1.4 in the most general case (of split geometry). More precisely, the counterexamples in Section 5 show that the natural property of **thickness** of blocks fails to detect thickness or thinness of almost minimizing geodesics.

In hindsight, the difficulty in answering Question 1.4 manifests in the difference in the approaches of McMullen [McM01] and the second author [Mj14a] in proving the existence of Cannon-Thurston maps, i.e. $\pi_1(S)-$equivariant continuous maps from the (hyperbolic or relatively hyperbolic) boundary of $\pi_1(S)$ onto the limit set. McMullen finds, in the special case of a punctured torus Kleinian group, precise locations (in $\widetilde{E}$, the universal cover of an end) of geodesics parametrized by the boundary at infinity S^1_∞ (of $\pi_1(S)$). On the other hand, the approach to the general case of split geometry in [Mj14a] necessarily forgets much of the

fine structure contained within the building (split) blocks by 'electrocuting' their connected components. Punctured torus groups provide an example of $i-$bounded geometry [Mj11], which is quite special. As such an analog of McMullen's approach in the general case of split geometry is missing. The counterexamples in Section 5 indicate that even answering Question 1.4, which is a small component of the more general problem of finding precise locations of geodesics in $\widetilde{E}$, is tricky in general.

2. Dense and non-dense horospheres

We start with a few definitions. Let M be a complete hyperbolic 3-manifold, $\widetilde{M} = \mathbb{H}^3$ its universal cover and $\partial \widetilde{M}$ the boundary at infinity of $\widetilde{M}$. We will use the unit ball model and then $\partial \mathbb{H}^n$ is the unit sphere $\mathbb{S}^{n-1}$. We identify $\pi_1(M)$ with the group of deck-transformations on $\widetilde{M}$.

Let SM be the unit tangent bundle of M and $S\widetilde{M}$ the unit tangent bundle of $\tilde{M}$. Denote by $g : \mathbb{R} \times S\tilde{M} \to S\tilde{M}$ the geodesic flow. The **strong stable manifold** or **stable horosphere** through $\tilde{v} \in S\tilde{M}$ is the set $W^{ss}(\tilde{v}) = \{\tilde{w} \in S\tilde{M} : d(g(t,\tilde{v}), g(t,\tilde{w})) \longrightarrow 0 \text{ when } t \longrightarrow 0\}$. The stable horosphere $W^{ss}(v) \subset SM$ through a vector $v \in SM$ is the projection of the stable horosphere through a lift of v to $\tilde{v}$.

In the hyperbolic space $\mathbb{H}^n$ of dimension n, a horosphere $\mathcal{H}$ is the intersection of $\mathbb{H}^n$ with a round sphere tangent to the boundary at infinity (in the unit ball model of $\mathbb{H}^n$), i.e. the round sphere minus the point of tangency. A stable horosphere is the set of unit normal vectors to a horosphere $\mathcal{H}$ pointing into the round ball bounded by $\mathcal{H}$.

As stated in the Introduction, we call a hyperbolic 3-manifolds M degenerate if all its ends are degenerate. We will define the ends of 3-manifold in section 3.1, for now, let us give an alternate definition. For our purposes, a hyperbolic 3-manifold is a quotient $M = \mathbb{H}^3/G$ where G is a finitely generated discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. The manifold M is **degenerate** if G is not a lattice and its limit set Λ is the whole sphere $\mathbb{S}^2$.

Convention: Unless otherwise mentioned, all degenerate manifolds in this paper will be without parabolics. M will denote a degenerate hyperbolic 3-manifold and G the corresponding Kleinian group.

2.1. Almost minimizing geodesics and dense horospheres. In this section we will expose results of Eberlein relating dense horospheres with almost minimizing geodesic rays. We start with some definitions and properties of almost minimizing geodesic rays.

Definition 2.1. *Given $C \geq 0$, a geodesic ray $\gamma = \gamma(t) : t \in \mathbb{R}^+$ in M is called C-almost minimizing if it is has unit speed and $d_M(\gamma(0), \gamma(t)) \geq t - C$ for any $t \in \mathbb{R}^+$. A geodesic ray is almost minimizing if it is C-almost minimizing for some $C \geq 0$.*

A geodesic ray $\gamma = \gamma(t) : t \in \mathbb{R}^+$, is called asymptotically almost minimizing if it has unit speed and if for any $\delta > 0$ there exists T such that for any $s, t \in \mathbb{R}^+$ with $s, t \geq T$, $d_M(\gamma(s), \gamma(t)) \geq |s - t| - \delta$.

A point $\xi \in \partial \widetilde{M}$ is a horospherical limit point of $\pi_1(M)$, if for any base-point $o \in \widetilde{M}$ and any horoball B_ξ based at ξ, there exist infinitely many translates $g.o \in B_\xi$,

where $g \in \pi_1(M)$. *The collection of horospherical limit points of $\pi_1(M)$ is called the horospherical limit set Λ_H of $\pi_1(M)$.*

These definitions are related by the following combination of results of Eberlein and Ledrappier, who consider the much more general context of complete manifolds of pinched negative curvature.

Proposition 2.2. [Led97, Proposition 4],[EO73, Proposition 7.4] *Let γ be a geodesic ray in a negatively curved manifold M. The following are equivalent:*

(1) *γ is almost minimizing*
(2) *γ is asymptotically almost minimizing*
(3) *$\tilde{\gamma}(\infty) \in (\partial \widetilde{M} \setminus \Lambda_H)$*

In [Ebe72, Theorem 5.2], Eberlein shows the existence of dense horospheres for negatively curved manifolds satisfying Axiom 1 (any 2 points in the boundary at infinity of the universal cover are joined by at least one geodesic) for which the nonwandering set Ω is the whole unit tangent bundle SM. Manifolds satisfying Axiom 1 are called **visibility manifolds**. Complete negatively curved manifolds of pinched negative curvature are examples of visibility manifolds. Hyperbolic manifolds are, therefore, visibility manifolds. Furthermore, classical results results imply that when $M = \mathbb{H}^n/G$ then $\Omega = SM$ if and only if $\Lambda_G = \partial_\infty \mathbb{H}^n$. Thus if M is a hyperbolic 3-manifold with finitely generated fundamental group, then $\Omega = SM$ if and only if M is either degenerate or has finite volume. We state the next two Theorems, due to Eberlein, in their full generality but for our purpose in this paper, the reader can replace "Let M be a negatively curved visibility manifold such that $\Omega = SM$" with "Let M be a degenerate hyperbolic 3-manifold".

Theorem 2.3. *Let M be a negatively curved visibility manifold such that $\Omega = SM$. Then there exists a vector $v \in SM$, such that the strong stable manifold $(W^{ss}(v))$ is dense in SM.*

Going further [Ebe72, Theorem 5.5] Eberlein relates the density of horospheres to almost minimizing geodesic rays:

Theorem 2.4. *Let M be a negatively curved visibility manifold such that $\Omega = SM$. Then $(W^{ss}(v))$ is dense in SM if and only if v is **not** almost minimizing.*

Any degenerate hyperbolic 3-manifold M with infinite diameter has minimizing geodesic rays: choose a sequence p_n exiting any compact in M, and, up to extracting a subsequence, take a limit of minimizing geodesic segments $[o, p_n]$. Consequently, we have:

Remark 2.5. *Let M be a degenerate hyperbolic 3-manifold. Then there exist minimizing geodesic rays $\gamma : \mathbb{R}^+ \to M$ and hence the horospheres $(W^{ss}(\dot{\gamma}(t)))$ are **not** dense in M for $t > 0$.*

2.2. **Thick and thin geodesics.** Next we want to discuss non-dense horospheres and hence almost minimizing geodesic rays. Works of Ledrappier ([Led97]) and Coudène-Maucourant ([CM10]) relate proper and recurrent horospheres with thick and thin geodesic rays. Let us first introduce thick and thick geodesic rays.

The injectivity radius of M at a point $x \subset M$ is the maximal radius of an embedded ball centered at x. Let $p : \tilde{M} \to M$ be the covering projection, let

$\tilde{x} \subset \tilde{M}$ be a lift of x and let $B(\tilde{x}, r)$ be the ball with diameter r centered at $\tilde{x}$. Then the injectivity radius at x is:

$$\text{Inj}(x) = \sup\{r | p_{|B(\tilde{x}, r)} \text{ is an isometry}\}.$$

Definition 2.6. *A geodesic $\gamma : \mathbb{R}^+ \to M$ is said to be* **thick** *if*

$$liminf_t Inj(\gamma(t)) > 0.$$

Otherwise it is called **thin**.

It is easy to see that for a geodesic ray, the property of being almost minimizing, thin or thick (and exiting which will be defined later on) depends only on its ideal endpoint. In other word given two geodesic rays $\gamma_1, \gamma_2 : [0, \infty) \to M$ which have asymptotic lifts to $\tilde{M}$, then γ_1 is thin, thick, exiting or almost minimizing if and only if γ_2 has the same property. Thus when dealing with these properties, it seems appropriate to parametrize geodesic rays by their endpoints or the endpoints of their lifts. For future reference, let us show this fact for almost minimizing ray.

Lemma 2.7. *A geodesic ray which has a lift to $\tilde{M}$ that is asymptotic to a lift of an almost minimizing geodesic is almost minimizing.*

Proof. Assume that a lift $\tilde{\gamma}$ of a geodesic ray $\gamma \subset M$ is asymptotic to a lift $\tilde{\gamma}'$ of a C-minimizing geodesic γ'. Then by convexity of the distance between 2 geodesics, we have $d(\tilde{\gamma}(t), \tilde{\gamma}'(t)) \leq d(\tilde{\gamma}(0), \tilde{\gamma}'(0))$ for any $t > 0$. Projecting to M, we get $d_M(\gamma(t), \gamma'(t)) \leq d_M(\tilde{\gamma}(0), \tilde{\gamma}'(0))$ for any $t > 0$. The triangle inequality gives us $|d_M(\gamma(t), \gamma(0)) - d_M(\gamma'(t), \gamma'(0))| \leq 2d_M(\gamma(0), \gamma'(0))$. Thus we get $d_M(\gamma(t), \gamma(0)) \geq t - C - 2d_M(\gamma(0), \gamma'(0))$. $\square$

Remark 2.8. *On the other hand, it is easy to construct two geodesic rays with asymptotic lifts such that one is minimizing and the other one is not (for example by adding a geodesic loop at the initial point of the minimizing ray and then straightening). Thus almost minimizing geodesics depend only on the end-point on the sphere at infinity, while minimizing geodesics depend on the initial point also. This is the reason why we deal with almost minimizing rather than minimizing geodesics in most of this paper.*

When there is a positive lower bound on the injectivity radius at any point of M, then M is said to have **bounded geometry** and obviously, every geodesic ray is thick. Otherwise M is said to have **unbounded geometry** and the situation is almost opposite, i.e. almost every geodesic ray is thin:

Lemma 2.9. *Let M be a degenerate hyperbolic 3-manifold with no positive lower bound on injectivity radius. Then the preimage in the universal cover of the set of thin geodesic rays (parametrized by S_∞^2) is of full measure (for the Lebesgue measure).*

Proof. For every $e > 0$, let $M_e \subset M$ be the set of points having injectivity radius at most e and let $\widetilde{M_e}$ be the full pre-image of M_e in $\tilde{M}$ under the (universal) covering map. Fix a base-point $o \in (\tilde{M} \setminus \widetilde{M_e})$. Recall [Sul79, Sul81] that the **shadow** of a subset $K \subset \tilde{M}$ is the subset of $S_\infty^2 (= \partial \mathbf{H}^3)$ given by

$$Sh(K) = \{x \in S_\infty^2 : [o, x) \cap K \neq \emptyset\}.$$

Since $\widetilde{M_e}$ is invariant under $\pi_1(M)$, its shadow $Sh(\widetilde{M_e})$ is also invariant under the action of $\pi_1(M)$ on S^2_∞. Next, the action of $\pi_1(M)$ on S^2_∞ is ergodic [BCM12]. Also, since $Sh(\widetilde{M_e})$ contains the shadow of a lift of a single Margulis tube, it follows that $Sh(\widetilde{M_e})$ is of positive measure. By ergodicity, $Sh(\widetilde{M_e})$ is of full measure. We take intersections over $e_n = 1/n$ to conclude that thin geodesics are of full measure.

The last statement follows. $\square$

Notice that a closed geodesic is always thick and hence M always contains thick geodesic rays. As we will see later, degenerate hyperbolic 3-manifold with unbounded geometry always contain thick exiting geodesic rays (though the collection of such rays has measure zero by Lemma 2.9). On the other hand, there are hyperbolic surfaces without cusps for which all exiting geodesic rays are thin. Let us construct an example:

Lemma 2.10. *There exists a complete hyperbolic surface Σ of infinite genus such that the following dichotomy holds for geodesic rays γ on Σ:*

  (1) *Either γ lies inside a compact set*
  (2) *or γ is thin.*

Proof. Let T be a torus with two holes. Let T_n $(n > 0)$ be a hyperbolic structure on T such that the boundary components are totally geodesic and have length $\frac{1}{n}, \frac{1}{n+1}$. Also let T_0 be a hyperbolic torus with one boundary component of length 1. Attach T_i to T_{i+1} by an isometry between the boundary components of length $\frac{1}{i+1}$. We call the resulting geodesic the $(i+1)-$th neck. Let $\Sigma = \bigcup_0^\infty T_i$ be the union modulo this identification, such that the neck between the $n-$th and $(n+1)-$th torus summands has length $\frac{1}{n+1}$. Since any $i-$th neck disconnects Σ into a compact piece and a noncompact piece, it follows that any geodesic ray in Σ is either bounded or cuts every neck and is therefore thin. $\square$

2.3. **Non-dense Horospheres.** We are now ready to describe the behavior of horospheres corresponding to almost minimizing rays. Let $\Pi : \widetilde{M} \to M$ be the covering projection.

Definition 2.11. *Let $\gamma := \gamma(t)$ be a geodesic in M. Then the stable horosphere $W^{ss}(\dot\gamma(t))$ is said to be* **recurrent** *if the following holds:*
Let $\dot\gamma_1(t)$ be a lift of $\dot\gamma(t)$ to the unit tangent bundle $S\widetilde{M}$. Then for every compact $K \subset W^{ss}(\dot\gamma_1(t))(\subset S\widetilde{M})$, there exists a vector $w \in (W^{ss}(\dot\gamma_1(t)) \setminus K)$ such that its projection (under Π) is arbitrarily close to $\dot\gamma_1(t)$.

Notice that this seems to be a property of $\dot\gamma(t)$. But using the transitive action of the horocyclic flow, it is not hard to see that when the property above holds for $\dot\gamma(t)$ it holds for any vector $v \in W^{ss}(\dot\gamma(t))$. Thus it is a property of the stable horosphere.

A relation between thick geodesic rays and embedded horospheres was first established by Ledrappier in [Led97]:

Theorem 2.12. [Led97, Proposition 3] *Let M be a negatively curved manifold with bounded geometry, $\gamma = \gamma(t) : t \in \mathbb{R}^+$ an asymptotically almost minimizing geodesic. Then the strong stable leaf $(W^{ss}(\dot\gamma(t)))$ is properly embedded in M for $t > 0$.*

Ledrappier's proof (c.f. [Led97, Lemma 3]) only uses the fact that the injectivity radius $Inj(\Pi(\delta(t))$ is bounded away from zero **along** $\delta(t)$ (for $t > 0$) for any geodesic

ray δ such that $\delta(t) \subset W^{ss}(\dot{\gamma}(t))$ (for some and hence all t). As was noticed earlier this is equivalent to having the injectivity radius $Inj(\Pi(\gamma(t))$ bounded away from zero **along** $\gamma(t)$ (for $t > 0$).

Thus in the context of degenerate 3-manifolds, we can restate Ledrappier's result [Led97, Proposition 3]:

Theorem 2.13. *Let M be a degenerate hyperbolic 3-manifold and let $\gamma = \gamma(t) :$ $t \in \mathbb{R}^+$ be a thick almost minimizing geodesic ray. Then the strong stable leaf $(W^{ss}(\dot{\gamma}(t)))$ is properly embedded in M for $t > 0$.*

A converse to Ledrappier's Theorem 2.13 is furnished by Coudène and Maucourant [CM10, Section 3], this time for complete manifolds with pinched negative curvature):

Theorem 2.14. [CM10, Section 3] *Let M be a complete manifolds with pinched negative curvature and let $\gamma = \gamma(t) : t \in \mathbb{R}^+$ be a thin geodesic ray. Then the strong stable leaf $(W^{ss}(\dot{\gamma}(t)))$ for $t > 0$ is* **recurrent**.

2.4. **Summary of section 2.** Combining Theorem 2.4, Remark 2.5 and Theorems 2.13, 2.14, we get:

Theorem 2.15. *Let M be a degenerate hyperbolic 3-manifold and let γ be a geodesic ray in M. Then*

    (1) $W^{ss}(\gamma(t))$ *is dense in M if and only if γ is* **not** *almost minimizing.*
    (2) $W^{ss}(\gamma(t))$ *is recurrent but not dense in M if and only if γ is thin almost minimizing.*
    (3) $W^{ss}(\gamma(t))$ *is properly embedded in M if and only if γ is thick almost minimizing.*

This leads us to the study of Questions 1.2 and 1.4 mentioned in the Introduction. In the next two subsections, we use this result to furnish two sets of examples. In the first, all almost minimizing geodesics are thick and in the second, all almost minimizing geodesics are thin. In section 3 we will address Question 1.2. In the rest of the paper we will come back to Question 1.4 and address it in greater generality, relating it to different model geometries.

2.5. **Hyperbolic Dehn filling.** In the next two subsections, we will construct examples of thin manifolds by using the following version of Thurston's Hyperbolic Dehn Filling Theorem:

Theorem 2.16. *Let M be a geometrically finite hyperbolic 3-manifold whose convex core has totally geodesic boundary. Such a manifold M is homeomorphic to the interior of a compact manifold $\hat{M}$. The rank 1 cusps of M correspond to a pants decomposition R of the union $\partial_{\chi<0}$ of the components of $\partial\bar{M}$ with negative Euler characteristic. Let $T_0, ..., T_q$ be torus components of $\partial\bar{M}$ and let $\bar{M}(p_0, ..., p_q)$ be the manifold obtained by performing $(1, p_i)$ Dehn filling on T_i, $i = 0, \cdots, q$. Then for $p_0, ..., p_q$ large enough the interior of $\bar{M}(p_0, ..., p_q)$ admits a unique geometrically finite hyperbolic metric with totally geodesic boundary $M(p_0, ..., p_q)$ such that the rank 1 cusps correspond to R. Furthermore $M(p_0, ..., p_q)$ converges geometrically to M when $(p_0, ..., p_q) \longrightarrow (\infty, ..., \infty)$.*

Proof. Let $D\bar{M}$ be the compact 3-manifold obtained by gluing 2 copies of $\bar{M}$ along $\partial_{\chi<0}\bar{M}$ and removing a regular neighborhood of $R \subset \partial_{\chi<0}\bar{M}$. The interior of M

admits a complete hyperbolic metric with finite volume obtained by gluing 2 copies of the convex core of M along their boundaries. Let $D\bar{M}(p_0, ..., p_q)$ be the compact 3-manifold obtained by gluing 2 copies of $\bar{M}(p_0, ..., p_q)$ along $\partial_{\chi<0}\bar{M}(p_0, ..., p_q)$ and removing a regular neighborhood of $R \subset \partial_{\chi<0}\bar{M}(p_0, ..., p_q)$. It follows from Thurston's Dehn Filling Theorem [Thu86, Theorem 5.8.2] that for $p_0, ..., p_q$ large enough the interior of $D\bar{M}(p_0, ..., p_q)$ admits a hyperbolic metric with finite volume, which is unique up to isometries by Mostow-Prasad's Rigidity Theorem, let us denote by $DM(p_0, ..., p_q)$ the resulting hyperbolic manifold. Again by Mostow-Prasad's Rigidity Theorem the natural involution $\tau : D\bar{M}(p_0, ..., p_q) \to D\bar{M}(p_0, ..., p_q)$ which exchange the 2 copies of $\bar{M}(p_0, ..., p_q)$ is isotopic to an isometry. The quotient of $DM(p_0, ..., p_q)$ by this isometry is the convex core of the desired hyperbolic manifold $M(p_0, ..., p_q)$.

Still by Thurston's Dehn Filling Theorem ([Thu86, Chapter 5], see also [PP00]), $DM(p_0, ..., p_q)$ converges geometrically to DM when $(p_0, ..., p_q) \longrightarrow (\infty, ..., \infty)$ and $M(p_0, ..., p_q)$ converges geometrically to M.   □

Although we did not a find a statement containing the Theorem above in the literature, this could certainly be deduced from previous work such as [BO88] or [Bro04].

2.6. Example: A thin manifold all of whose almost minimizing geodesics are thick.

The first example is due to Thurston [Thu86] and Bonahon-Otal [BO88]. We will detail a construction explained in [Thu86] and show that the result has the expected property. Later, when we describe i-bounded geometry as a model geometry of ends, it will become clear that these examples form a special case. However, since the examples in this section can be described in a reasonably self-contained manner, we explicitly describe these below.

Let S be a closed surface and $P, Q \subset S$ two pants decompositions that fill S, i.e. the connected component of $S(P \cup Q)$ are discs.. Consider a faithful and discrete representation $\pi_1(S) \to PSL(2, \mathbb{C})$ whose convex core $C(P, Q)$ has totally geodesic boundary and cusps corresponding to P on the bottom side and to Q on the top side.

Let $\hat{M}_0$ be the manifold obtained by gluing $C(P, Q)$ on top of $C(Q, P)$ and let $\hat{M}_n$ be the manifold obtained by gluing $2n+1$ copies of $\hat{M}_0$ on top of each other. Pick a base point x_n in the middle piece of $\hat{M}_n$. By construction, $\hat{M}_i$ isometrically embeds in $\hat{M}_j$ for any $j > i$. It follows that the sequence $(\hat{M}_n, x_n)$ converges geometrically to a hyperbolic 3-manifold $\hat{M}_\infty$.

Next we fill the holes of $\hat{M}_\infty$ recursively. Let $M_0(p_0)$ be the manifold obtained by performing $(1, p_0)$ Dehn fillings on the torus cusps of M_0 (c.f. Theorem 2.16). Given $M_n(p_0, ..., p_n)$, glue a copy of $\hat{M}_0$ at the top and one at the bottom to obtain a new convex hyperbolic 3-manifold $\hat{M}_{n+1}(p_0, ..., p_n)$. Denote then by $M_{n+1}(p_0, ..., p_{n+1})$ the manifold obtained by performing $(1, p_{n+1})$ Dehn fillings on the torus cusps of $\hat{M}_{n+1}(p_0, ..., p_i)$ (c.f. Theorem 2.16). Also denote by $\hat{M}_\infty(p_0, ..., p_n)$ the manifold obtained by gluing $\hat{M}_\infty - \hat{M}_n$ along the boundary of $M_n(p_0, ..., p_n)$ (or equivalently perform $(1, p_i)$ Dehn fillings on the appropriate cusps of $\hat{M}_\infty$).

If we fix n and $p_0, ..., p_{n-1}$ and let p_n go to ∞, by Theorem 2.16, $M_n(p_0, ..., p_n)$ converges geometrically to $\hat{M}_n(p_0, ..., p_{n-1})$. It follows that for p_n large enough (depending on ϵ) there is a homeomorphism f_n between the ϵ-thick parts of $\hat{M}_\infty(p_0, ..., p_n)$

and $\hat{M}_\infty(p_0, ..., p_{n-1})$ whose restriction to $\hat{M}_\infty - \hat{M}_n$ is an isometry and restriction to $M_n(p_0, ..., p_n)$ is $K_n(p_n)$-bilipschitz, with $K_n(p_n)$ close to 1 when p_n is large. This also implies that the sum $l_n(p_n)$ of the lengths of the added geodesics is short when p_n is large.

Pick a small ϵ and choose the sequence $\{p_n\}$ so that $\Pi_n K_n(p_n)$ converges and that $l_n(p_n) \longrightarrow 0$. The map $g_n = f_1 \circ ... \circ f_n : \hat{M}_\infty(p_0, ..., p_n) \to \hat{M}_\infty$ is bilipschitz on the ϵ-thick part. It follows that $(M_n(p_0, ..., p_n), x_n)$ and $(\hat{M}_\infty(p_0, ..., p_n), x_n)$ converge to a hyperbolic manifold M_∞ homeomorphic to $S \times \mathbb{R}$ and that g_n converge to a bilipschitz map g_∞ from the ϵ-thin part of M_∞ to the ϵ-thin part of $\hat{M}_\infty$. Also since $p_n \to \infty$, the injectivity radius of M_∞ has no positive lower bound.

By construction, each cusp of $\hat{M}_\infty$ is isometric to a cusp of $\hat{M}_1$. In particular the components of the boundary of the ϵ-thick part of $\hat{M}_\infty$ have uniformly bounded diameter. Then the map g_∞ provides us with an upper bound D on the diameters of the components of the boundary of the ϵ-thick part of M_∞. Let κ be an arc in M_∞ with endpoints in the ϵ-thick part. If κ goes through the ϵ_0 thin part, it has a subsegment of length $l(\epsilon, \epsilon_0)$ in the ϵ-thin part with $l(\epsilon, \epsilon_0) \longrightarrow \infty$ when ϵ_0 tends to 0. Hence κ is not $(l(\epsilon, \epsilon_0) - D - 1)$-minimizing.

We conclude that a geodesic that goes arbitrarily deep in the thin part of M_∞ is not almost minimizing. We shall generalize this example considerably in Section 4.2.

2.7. **Example: A thin manifold all of whose almost minimizing geodesics are thin.** For the second example, we will follow the same procedure but the pieces we will glue will be different. Let $c \subset S$ be a non separating curve, $\phi : S - c \to S - c$ a pseudo-Anosov diffeomorphism and let P be a pants decomposition that crosses c. We will use $C(P, \phi^j(P))$ with larger and larger j instead of $C(P, Q)$. These pieces have the following property:

Lemma 2.17. *Given ϵ, C, there is $J = J(\epsilon, C, \phi, P)$ such that for any $j \geq J$ any C-almost minimizing segment joining the top boundary of $C(P, \phi^j(P))$ to its bottom boundary goes through the ϵ-thin part.*

Proof. Let M_ϕ be the hyperbolic manifold homeomorphic to $(S - c) \times [0, 1]/(x, 0) \sim (\phi(x), 1)$ and let $\bar{M}_\phi$ be its cyclic cover (homeomorphic to $(S - c) \times (0, 1)))$. Basic hyperbolic geometry tells us that the distance between 2 points on a horoball grows logarithmically with their distance on the horosphere. Applied to $\bar{M}_\phi$ this produces the following Claim:

Claim 2.18. *Pick a fundamental domain D for the action of $\mathbb{Z}$ on $\bar{M}_\phi$. Given ϵ, we denote by D_ϵ^k the union of k adjacent copies of the ϵ-thick part of D in $\bar{M}_\phi$. Let $x \subset \bar{M}_\phi$ be a point at the top of D_ϵ^k and $y \subset \hat{M}_\phi$ be a point at the bottom of D_ϵ^k, then for k large enough $d_{\bar{M}_\phi}(x, y) \leq 2 \log k$.* □

We will transport this property into $C(P, \phi^j(P))$ by showing that for j large enough, a large part of $C(P, \phi^j(P))$ looks like D_ϵ^k.

Claim 2.19. *Given ϵ, k, there is $I = I(\epsilon, k, \phi, P)$ such that for $j \geq I$, there is a $1 + \epsilon$-bilipschitz embedding of D_ϵ^k in $C(P, \phi^j(P))$.*

Proof. Set $j = 2i$ if j is even and $j = 2i + 1$ otherwise. Let $\rho_i : \pi_1(S) \to PSL(2, \mathbb{C})$ be a discrete and faithful representation with cusps corresponding to $\phi^{-i}(P)$ at the bottom and cusps corresponding to $\phi^i(P)$ at the top if j is even and to $\phi^{i+1}(P)$ if

j is odd. Notice that the convex core of $\mathbb{H}^3/\rho_i(\pi_1(S))$ is isometric to $C(P, \phi^j(P))$. Consider the restriction $\rho_{c,i} : \pi_1(S-c) \to PSL(2, \mathbb{C})$ of ρ_i to $\pi_1(S-c)$. By [Min00], the length of the geodesic corresponding to c in $\mathbb{H}^3/\rho_i(\pi_1(S))$ tends to 0 when i tends to ∞. It follows from a generalization of the Double Limit Theorem ([Thu86], see also [Can93b] that a subsequence of $\rho_{c,j}$ converges to a representation ρ_∞. Since its length goes to 0, c is a parabolic for ρ_∞. By [Bro00], the stable and unstable laminations of ϕ are not realized in $\mathbb{H}^3/\rho_\infty$. It follows from the Ending Lamination Theorem ([BCM12]) that $\mathbb{H}^3/\rho_\infty(\pi_1(S-c))$ is isometric to $\bar{M}_\phi$. Up to extracting a further subsequence, $\mathbb{H}^3/\rho_n(\pi_1(S-c))$ converges geometrically as well. By the Covering Theorem [Can96], $\bar{M}_\phi$ is also the geometric limit. The conclusion follows. $\qquad\square$

Combining Claims 2.18 and 2.19, we see that distances grow linearly with k in the thick part while they grow logarithmically in the thin part. Now we just need to adjust the constants to prove Lemma 2.17.

Fix ϵ, C, j and k large enough so that Claim 2.18 holds for $\frac{\epsilon}{2}$. Set ϵ_k such that a geodesic segment in $\bar{M}_\phi$ joining 2 points in $D^k_{\frac{\epsilon}{2}}$ does not enter the ϵ_k thin part. Let d be the distance between the 2 boundary components of D in the $\frac{\epsilon}{2}$-thick part. Let κ be a segment in the ϵ-thick part joining the top boundary of $C(P, \phi^j(P))$ to its bottom boundary.

By Claim 2.19, if $j \geq J(k, \epsilon_k)$, there is a 2-bilipschitz embedding $f : D^k_{\epsilon_k} \hookrightarrow C(P, \phi^j(P))$. The preimage η of $\kappa \cap f(D^k_{\epsilon_k})$ joins the top of D^k to its bottom and lies in the $\frac{\epsilon}{2}$-thick part. It follows that η has length at least kd and that $\kappa \cap f(D^k_{\epsilon_k})$ has length at least $k\frac{d}{2}$. By Claim 2.18, the endpoints of η are joined by an arc $\eta' \subset \hat{M}_\phi$ with length at most $2\log k$. Furthermore, by the choice of ϵ_k, $\eta' \subset D^k_{\epsilon_k}$. Thus $f(\eta')$ is an arc with length at most $4\log k$ joining the endpoint of $\kappa \cap f(D^k_{\epsilon_k})$. Now we can conclude that if $k\frac{d}{2} > 4\log k + C$ and $j \geq J(k, \epsilon_k)$, κ is not C-almost minimizing. $\qquad\square$

The second example is now constructed with the same steps as the first one. Let $M(j)$ be the manifold obtained by gluing $C(\phi^j(P), P)$ on top of $C(P, \phi^j(P))$. Define $M_1 = M(1)$ and define M_{n+1} recursively by gluing a copy of $M(n+1)$ at the top of M_n and one at the bottom. Pick a basepoint x_n in the middle piece of M_n. It is easy to see that (M_n, x_n) converges geometrically to a hyperbolic 3-manifold $\hat{M}_\infty$ with infinitely many rank 2 cusps. It is easy to deduce from Lemma 2.17 that any almost minimizing geodesic in $\hat{M}_\infty$ goes arbitrarily far into the thin part.

Now the manifold M_∞ is obtained as in the first example by recursively performing $(1, p_i)$-Dehn filling on the cusps of $\hat{M}_i$, with an appropriate choice of p_i so that everything converges and that the geometry is close to the geometry of $\hat{M}_\infty$. Then by Lemma 2.17, any almost minimizing geodesic in M_∞ goes arbitrarily far into the thin part.

3. The horospherical limit set

In this section, we study the interrelationships between three subsets of the limit set:

- The conical limit set Λ_c.
- The multiple limit set $\Lambda_m = \{x \in \Lambda : \#|(\partial\hat{i})^{-1}(x)| > 1\}$, where $\partial\hat{i}$ denotes the Cannon-Thurston map (defined below).

- The horospherical limit set Λ_H.

Before recalling the definitions of these sets, let us consider the topology and geometry of hyperbolic 3-manifolds.

3.1. **Ends of hyperbolic 3-manifold.** We have mentioned earlier that, for our purposes, a hyperbolic 3-manifold is a quotient $M = \mathbb{H}^3/G$ where G is a finitely generated Kleinian group. It follows from the Tameness Theorem ([Ago04], [CG06]) that M is **tame**, i.e. homeomorphic to the interior of a compact 3-manifold $\bar{M}$.

A **compact core** $C \subset M$ for M is a compact submanifold such that the inclusion $C \hookrightarrow M$ induces an isomorphism on fundamental groups. The existence of compact cores for 3-manifolds with finitely generated fundamental groups is a central result in the study of 3-manifolds due to Scott ([Sco73]). Since M is tame, there is a compact core for M homeomorphic to $\bar{M}$ so that each component of $M - C$ is homeomorphic to $S \times \mathbb{R}$ where S is a component of $\partial\bar{M} = \partial C$. We call such a component (or its closure) an **end** of M. Although this definition depends on the choice of C, it is easy to see that given a compact set $K \subset M$, we can choose C so that $K \subset C$. Thus the asymptotic behavior of the ends of M does not depend on the choice of C.

Let G be a Kleinian group. Its limit set Λ_G is the closure in the boundary at infinity $\partial\mathbb{H}^3$ of the orbit of a base point. More precisely, fix a base point $O \subset \mathbb{H}^3$ and set $GO = \{gO, g \in G\}$, then $\Lambda_G = \overline{GO} \cap \partial\mathbb{H}^3$. The convex core of $M = \mathbb{H}^3/G$ is the quotient $\text{Hull}(\Lambda_G)/G$ of the convex hull in $\mathbb{H}^3$ of the limit set. Equivalently it is the smallest convex subset of M whose inclusion induces a homotopy equivalence with M.

Let us now assume that G is finitely generated, has no parabolic elements and is not a lattice (i.e. $M = \mathbb{H}^3/G$ has infinite volume). An end of M is **degenerate** if it lies in its convex core. The manifold M is degenerate if all its end are degenerate, equivalently its convex core is the whole manifold, equivalently $\Lambda_G = \partial\mathbb{H}^3$.

Work of Thurston, Bonahon and Canary ([Thu80], [Bon86] and [Can93a]) along with tameness [Ago04, CG06] shows that degenerate ends are geometrically tame, i.e. there is a sequence of hyperbolic surfaces leaving every compact set:

Theorem 3.1. *Let $E \approx S \times \mathbb{R}$ be a degenerate end of a tame hyperbolic 3-manifold, then there is a sequence of maps $f_n : S \to E$ such that*

(1) *f_n is homotopic to the map induced by the inclusion $S \times \{1\} \subset S \times \mathbb{R}$,*
(2) *the metric induced on S by f_n is hyperbolic*
(3) *$f_n(S) \subset S \times [n, \infty)$ for any $n \in \mathbb{N}$.*

This lead to the the definition of an ending lamination associated to a degenerate end. Consider a sequence of simple closed curves $c_n \subset S$ such that $\ell_{f_n}(c_n)$ is a bounded sequence, where ℓ_{f_n} is the length associated to the metric induced on S by f_n. Extract a subsequence such that c_n converges to a projective measured geodesic lamination λ on S (see [Thu80, Chap. 8] and [CB88] for definitions and properties of measured geodesic laminations). It follows from [Bon86] that the geodesic lamination ν supporting λ does not depend on the choices of $\{f_n\}$ or $\{c_n\}$, ν is the **ending lamination of** E.

Notice that the induced metric on ∂E is not hyperbolic. We will need to define geodesic laminations on ∂E. For this purpose, we fix a reference hyperbolic metric on S. Then we have a bilipschitz homeomorphism between S and ∂E endowed with

the induced metric and a geodesic lamination on ∂E is simply defined as the image of a geodesic lamination on S.

3.2. Cannon-Thurston maps, Exiting geodesics and limit set.

Definition 3.2. *A geodesic ray in $\widetilde{M}$ is **exiting** if it is a lift of an exiting geodesic ray in M, i.e. a lift of a geodesic ray that is properly embedded in an end of M. Let E be an end of M with $S = \partial E$. A geodesic ray γ in M is **exiting in** E if γ is properly embedded and $\exists T$ such that $\gamma([T, \infty)) \subset E$. Notice that exiting geodesic must be exiting in some end. A **minimizing geodesic segment** γ through $p \in E$ is a geodesic segment between some $o \in S$ and p with length equal to $d_M(S, p)$. A lift of a minimizing geodesic segment to $\widetilde{M}$ is called a minimizing geodesic segment in $\widetilde{M}$.*

Definition 3.3. *Let X and Y be hyperbolic metric spaces and $i : Y \to X$ be an embedding. Suppose that a continuous extension $\hat{i} : \widehat{Y} \to \widehat{X}$ of i exists between their (Gromov) compactifications. Then the boundary value of $\hat{i}$, namely $\partial\hat{i} : \partial Y \to \partial X$ is called the **Cannon-Thurston map**.*

Sometimes, in the literature [Mit98b, Mit98a], $\hat{i}$ is itself called a Cannon-Thurston map. For us, Y will be a Cayley graph of Γ. We will be particularly interested in the case that Γ is a surface Kleinian group isomorphic to $\pi_1(S)$ (for S a closed surface of genus $g > 1$). Also X will be $\mathbf{H}^3$, where we identify (the vertex set of) Y with an orbit of $\pi_1(S)$ in $\mathbf{H}^3$. Equivalently (as is often done in geometric group theory) we can identify Y with $\mathbf{H}^2 = \widetilde{S}$, X with $\mathbf{H}^3$, and i with the lift to universal covers of the inclusion of S into an end E of M. Then the main Theorems of [Mj14a, DM16, Mj14b, Mj10b] gives us:

Theorem 3.4. *Let S be the boundary of a degenerate incompressible end E. A Cannon-Thurston map ∂i exists for $i : \widetilde{S} \to \widetilde{E}$. Let $\mathcal{L}_E$ denote the ending lamination corresponding to E. Then ∂i identifies $a, b \in \partial\widetilde{S}$ iff a, b are end-points of a leaf of $\mathcal{L}_E$ or boundary points of an ideal polygon whose sides are leaves of $\mathcal{L}_E$.*

More generally for a degenerate M without parabolics, let K denote a compact core. Identify the Gromov boundary $\partial\widetilde{K}$ with $\partial\Gamma$. Then a Cannon-Thurston map ∂i exists for $i : \widetilde{K} \to \widetilde{M}$. Also, ∂i identifies $a, b \in \partial\widetilde{K}$ iff a, b are end-points of a leaf of $\mathcal{L}_E$ or boundary points of an ideal polygon whose sides are leaves of $\mathcal{L}_E$ for some (lift of) an ending lamination $\mathcal{L}_E$ corresponding to an end E.

Given Theorem 3.4 we define

Definition 3.5. *The **multiple limit set** $\Lambda_m = \{x \in \Lambda : \#|(\partial i)^{-1}(x)| > 1\}$. Equivalently*

$$\Lambda_m = \{\partial i(y) : y \text{ is an end-point of a leaf of } \mathcal{L}_E \text{ for some ending lamination } \mathcal{L}_E\}$$

An infinite geodesic ray $[o, x) \subset \widetilde{M}$ (where $x \in \partial\widetilde{M}$) is said to **land** at x.

A consequence of the construction in Section 4.2 of [Mj14b] is:

Proposition 3.6. *If $x \in \Lambda_m$, then $[o, x)$ is exiting.*

In Proposition 3.10, we will prove that $[o, x)$ is almost minimizing which is a stronger conclusion. Thus Proposition 3.6 will also follow from Proposition 3.10; and we omit the proof for now.

3.3. **Relationships between limit sets.** We start with the observation that the conical limit set is contained in the complement of the multiple limit set.

Lemma 3.7. $\Lambda_c \subset \Lambda_m^c$.

In a general form, this has been proven by Jeon, Kapovich, Leininger and Ohshika [JKLO16]. We shall give a different proof specialized to our context.

Proof. Recall that a point $z \in \Lambda$ is conical if, given a base point $o \in \mathbf{H}^3$ there exists $R > 0$ such that there exist infinitely many $g \in \Gamma$ satisfying $g.o \in N_R([o, z))$. Equivalently, z is conical if and only if $[o, z) \cap N_R(\widetilde{K})$ is unbounded, i.e. $[o, z)$ visits $N_R(\widetilde{K})$ infinitely often.

Hence z is non-conical if and only if for all $R > 0$, $[o, z) \cap N_R(\widetilde{K})$ is bounded, i.e. $[o, z)$ is exiting. In particular, since $z \in \Lambda_m$ implies that $[o, z)$ is exiting (by Proposition 3.6), it follows that z is non-conical. The Lemma follows. $\square$

Proposition 3.8. Λ_c *is a proper subset of* Λ_H.

Proof. $[o, z)$ lands in the complement of Λ_c if and only if it is exiting. On the other hand, $[o, z)$ lands in the complement of Λ_H if and only if it is almost minimizing by Proposition 2.2.

It therefore suffices to find exiting rays that are not almost minimizing. Let $[o, z)$ be an almost minimizing ray in E. To construct $[o, z)$ just take a sequence z_n exiting E, join them to $S(= \partial E)$ by minimizing geodesic segments, and take a limit.

Next, choose $w_i \in [o, z)$ such that $d(w_i, w_{i+1}) > D_0$, for a large D_0 (to be fixed later). Let σ_i be closed geodesics of length at most 1 such that

  (1) If the length of σ_i is larger than the Margulis constant, then $d(w_i, \sigma_i) \leq 1$.
  (2) Else, if T_i is the Margulis tube containing σ_i, then $d(w_i, T_i) \leq 1$.

We let α_i be a loop based at w_i winding n_i times around σ_i obtained by adjoining initial and final segments of length at most one joining w_i to σ_i or T_i, and winding n_i times around σ_i in between.

Then the concatenation $\bigcup_i([w_{i-1}, w_i] \cup \alpha_i)$ is exiting in E and lifts to a quasi-geodesic η with bounded constants (provided D_0 and n_i's are sufficiently large) in $\widetilde{E}$. The geodesic tracking η is then an example of an exiting geodesic that is not almost minimizing. $\square$

Our last goal in this section is to relate the horospherical and multiple limit sets. We shall need the following consequence of Thurston's [Thu80, Chapter 9] result that the ending lamination corresponding to a degenerate end is well-defined.

Lemma 3.9. *Let* $S = \partial E$ *be the boundary of a degenerate end* E *with ending lamination* $\mathcal{L}_E$. *Assume that* S *is equipped with a hyperbolic structure. Let* α_n *be a sequence of closed geodesics in* S *whose geodesic realizations* σ_n *exit* E. *Denote by* $\alpha_n^{\pm\infty}$ *the attracting and repelling fixed points of* α *on* $\partial \widetilde{S} = S^1$. *If* p *is a limit (in* S^1) *of a (subsequence of)* α_n^{∞}, *then* p *is the end-point of a leaf of* $\mathcal{L}_E$.

Conversely, any end-point of a leaf of $\mathcal{L}_E$ *is a limit of a (subsequence of)* α_n^{∞}'s.

Proof. Let $[\alpha_n](\subset \widetilde{S})$ denote the bi-infinite geodesic with end-points $\alpha_n^{\pm\infty}$. It follows from [Thu80, Chapter 9] that any subsequential limit (in the Gromov-Hausdorff topology) of $[\alpha_n]$'s is either a leaf of $\mathcal{L}_E$ or contained (as a diagonal) in an ideal polygon in the complement of $\mathcal{L}_E$. One direction of the Lemma follows.

Further, since the Hausdorff limit of α_n's on S contains $\mathcal{L}_E$, the converse follows.
□

We shall now show:

Proposition 3.10. $z \in \Lambda_m$ *if and only if* $[o, z)$ *is almost minimizing.*

Proof of Proposition 3.10. We continue with the notation of Lemma 3.9. We first show that if $\overline{r}$ is the lift of an almost minimizing geodesic to $\widetilde{E}$, then its end-point (in $\partial \mathbf{H}^3$) belongs to Λ_m.

Let r be an almost minimizing geodesic ray in E. Then there exists $C > 0$ such that for all $t \in [0, \infty)$, there is a closed unknotted essential loop σ_t (σ_t is homotopic to a simple closed curve on S) such that $l(\sigma_t) \le C$ and $d(\sigma_t, r(t)) \le C$ (here we cannot assume that σ_t is a geodesic, the geodesic realization may be very short, enclosed in a deep Margulis tube, in which case the distance to the geodesic realization is quite large).

Let α_t be a simple closed curve on $S = \partial E$ freely homotopic to σ_t. Then there exists a geodesic segment r_t of length in the interval $[t - 2C, t + 2C]$ joining α_t, σ_t such that the Hausdorff limit (as $t \to \infty$) of r_t is asymptotic to r. Let $\overline{\alpha_t}, \overline{\sigma_t}$ be geodesic segments in $\widetilde{E}$ that are lifts of α_t, σ_t, such that their initial points and end-points are connected by lifts r_{t1}, r_{t2} of r_t. (See diagram below) Assume further that the initial point of $\overline{\alpha_t}$ (and r_{t1}) is a fixed point $o \in \widetilde{E}$ (independent of t), and let the end-point of $\overline{\alpha_t}$ be denoted by w_t.
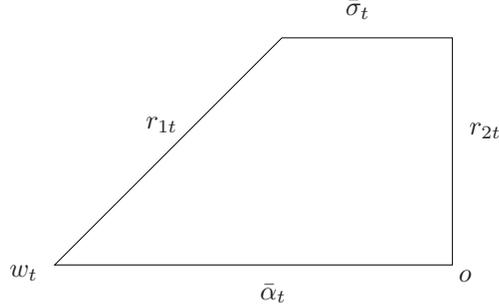
By Lemma 3.9, w_t converges (up to subsequence) to a point z which is the end-point of a leaf of $\mathcal{L}_E$. Hence by Theorem 3.4, $\hat{i}(z) \in \Lambda_m$ (recall that $\hat{i}$ denotes the Cannon-Thurston map).

The concatenation γ_t of r_{t1}, $\overline{\sigma_t}$ and r_{t2} (with orientation reversed) is a uniform quasigeodesic. The proof of this statement is a replica of the argument occurring in Lemma 3.5 of [Mit97], Proposition 3.1 of [Mj14b] or Proposition 5.2 of [Mj10b]. In the last reference a detailed proof is given and we omit the proof here.

Let $r(\infty)$ denote the terminal point of r in $S^2 = \partial \mathbf{H}^3$. Then, since γ_t is a uniform quasigeodesic containing r_{t1} as an initial segment, it follows that the end-points of r_{t1} and γ_t converge to the same point on S^2, or equivalently, $r(\infty) = \hat{i}(z)$. Hence $r(\infty) \in \Lambda_m$. Since r was arbitrary, we have shown that $\Lambda_H^c \subset \Lambda_m$.

To prove the reverse inclusion, given $z \in \Lambda_m$ we will construct a geodesic ray $\tilde{\gamma} \subset \widetilde{M}$ with endpoint z (but whose initial point may not be o) whose projection to M is minimizing. It will then follow from Lemma 2.7 that $[o, z)$ is almost minimizing.

Let (a, b) be a bi-infinite leaf of $\mathcal{L}_E$ such that $\hat{i}(a) = \hat{i}(b) = z$ (notice that there is always such a leaf according to [Mj14b] and [Mj10b]). Let α_n be a sequence of closed geodesics on $S = \partial E$ and $\tilde{\alpha}_n \subset \tilde{S}$ a leaf in the preimage of α_n such that

(1) $\tilde{\alpha}_n^{\pm\infty}$ converges to $\{a, b\}$.
(2) The geodesic realizations σ_n in M, of α_n in E, exit E.

Let $\tilde{\sigma}_n$ be the leaf of the preimage of σ_n with the same endpoints as $\tilde{\alpha}_n$. Let $\tilde{x}_n$ be a point of $\tilde{\sigma}_n$ and $\tilde{z}_n \subset \partial\tilde{E}$ be a point realizing the distance between $\tilde{x}_n$ and $\partial\tilde{E}$. The existence of $\tilde{z}_n$ comes from the properness of the embedding $\partial\tilde{E} \subset \tilde{E}$. If $\tilde{z}_n$ stays in a compact set, then, up to extracting a subsequence, the geodesic segments joining $\tilde{z}_n$ to $\tilde{x}_n$ converge to a geodesic ray $\tilde{\gamma}$ joining the limit $\tilde{z}_\infty$ of $\tilde{z}_n$ to z. Since $\tilde{z}_n$ realizes the distance between $\tilde{x}_n$ and $\partial\tilde{E}$, the projection γ of $\tilde{\gamma}$ is a minimizing geodesic ray (since, as usual, a limit of minimizing geodesic segments is a minimizing geodesic ray) and we are done.

Otherwise we pick a sequence $g_n \in \pi_1(S)$ such that $g_n \tilde{z}_n$ stays in a compact set. We are going to show that $g_n \tilde{x}_n$ also tends to z. Then up to extracting a subsequence, the geodesic segments joining $g_n \tilde{z}_n$ to $g_n \tilde{x}_n$ converges to a geodesic ray $\tilde{\gamma}$ joining the limit $\tilde{z}_\infty$ of $g_n \tilde{z}_n$ to z. Again, the projection γ of $\tilde{\gamma}$ is a minimizing geodesic ray (since a limit of minimizing geodesic segments is a minimizing geodesic ray) and we are done.

We first show that $\tilde{z}_n$ can only exit toward z. Using the continuity of Cannon-Thurston map, we will show that this imposes some restrictions on the behavior of g_n that will lead us to the expected conclusion ($g_n \tilde{x}_n$ tends to z).

Claim 3.11. *Up to extracting a subsequence, either $\tilde{z}_n$ stays in a compact set or it tends to z.*

Proof. Assume that $\tilde{z}_n$ does not stay in a compact set. Then, up to extracting a subsequence, it converges to a point $\chi \in \Lambda_G$. Seeking a contradiction, assume that $\chi \neq z$. Then the geodesic segments $[\tilde{x}_n, \tilde{z}_n]$ converge to the geodesic l with endpoints (χ, z). Pick 2 points $\tilde{y} \subset l$ and $y' \subset \partial\tilde{E}$. For n large enough we have $d(\tilde{y}, \tilde{z}_n) \geq d(\tilde{y}, \tilde{y}') + 2$. Since $[\tilde{x}_n, \tilde{z}_n]$ converge to l, for n large enough it passes nearby $\tilde{y}$ so that $d(\tilde{x}_n, \tilde{z}_n) \geq d(\tilde{x}_n, \tilde{y}) + d(\tilde{y}, \tilde{z}_n) - 1 \geq d(\tilde{x}_n, \tilde{y}) + d(\tilde{y}, \tilde{y}') + 1$. This would contradict the assumption that $\tilde{z}_n$ realizes the distance between $\tilde{x}_n$ and $\partial\tilde{E}$. $\square$

So let us assume that $\tilde{z}_n$ tends to z, in $\partial\tilde{E}$ it tends to a point c in the ideal boundary such that $i(c) = z$.

Pick a sequence $g_n \in \pi_1(S)$ such that $g_n \tilde{z}_n$ stays in a compact set and denote by (d_n, c_n) the attracting and repulsing points of g_n. Since $\tilde{z}_n$ tends to c, c_n converge to c. Extract a subsequence such that d_n converges to d. If $d = c$, then $g_n(a)$ and $g_n(b)$ converge to c. It follows that $g_n \tilde{\sigma}_n$ tends to $i(c) = z$, in particular $g_n \tilde{x}_n$ tends to z. If $d \neq b$, the axis of g_n tends to the geodesic with endpoints (c, d). In particular the distance from $g_n \tilde{z}_n$ and hence from $\tilde{z}_n$ to the axis of g_n is bounded. It follows that the distance from $\tilde{z}_n$ to $\tilde{\alpha}_n$ is bounded. Then since $g_n \tilde{z}_n$ stays in a compact set, up to extracting a subsequence, $g_n \tilde{\alpha}_n$ converges to a leaf of the preimage of $\mathcal{L}_E$. This is only possible if (a_n, b_n) and $(g_n a_n, g_n b_n)$ converge to (c, d) or (d, c). It follows that $g_n \tilde{x}_n$ tends to z.

$\square$

We should remark here that the construction of an almost minimizing geodesic in the above proof can very well furnish minimizing ones. We refer the reader to Remark 2.8 for a clarification on why we have decided to deal with almost minimizing rather than minimizing geodesics. Combining Propositions 2.2 and 3.10, we get:

Corollary 3.12. $\Lambda_H^c = \Lambda_m$.

Corollary 3.13. *For each degenerate end E with incompressible (in M) boundary S, there is an $\mathbb{R}-$tree $\mathcal{T}_E(\subset \Lambda)$ dual to $\mathcal{L}_E$ parametrizing the lifts of almost minimizing geodesics exiting E. Hence Λ_H^c is a disjoint union of $\mathcal{T}_E$'s – one for each lift of E as E ranges over the degenerate ends of M.*

Proof. By Proposition 3.12, it suffices to obtain a description of Λ_m. Also, by Proposition 3.12, and Theorem 3.4, Λ_m is the set of equivalence classes in $\mathcal{L}_E$, where $a, b \in S^1$ are equivalent if they are end-points of a leaf of $\mathcal{L}_E$ or ideal end-points of a complementary ideal polygon. By joining all such a, b by bi-infinite geodesics, and collapsing leaves and ideal complementary polygons down to points, it follows that Λ_m is the dual $\mathbb{R}-$tree $\mathcal{T}_E$ to $\mathcal{L}_E$. The last statement is an immediate consequence. □

4. Building Blocks and Model Geometries

Having discussed which geodesics are almost-minimizing as opposed to merely exiting in Section 3, it remains to discuss conditions guaranteeing thickness or thinness of almost minimizing geodesics in E. We have already seen examples in Subsections 2.6 and 2.7 where all almost minimizing geodesics are thin and an example where all almost minimizing geodesics are thick. The purpose of the rest of this paper is to furnish conditions in special cases and explore the limitations of these conditions. As the examples of subsection 2.6 and 2.7 indicate, the geometry of building blocks plays a crucial role. To proceed further, we pick up model geometries of ends of hyperbolic 3-manifolds following [Min01, Min10, BCM12, Mj10a, Mj11, Mj16, Mj14a] one by one and discuss the behavior of almost minimizing geodesics for each.

In what follows in this section we shall describe different kinds of models for building blocks of E: thick, thin, amalgamated. Each building block is homeomorphic to $S \times [0, 1]$, where S is a closed surface of genus greater than one. What is common to all these three model building blocks is that the top and bottom boundary components are uniformly bi-Lipschitz to a fixed hyperbolic S. In the next section, a more general model geometry will be described and almost minimizing geodesics in it will be analyzed.

Definition 4.1. *A model E_m is said to be built up of blocks of some prescribed geometries* **glued end to end***, if*

(1) *E_m is homeomorphic to $S \times [0, \infty)$*
(2) *There exists $L \geq 1$ such that $S \times [i, i+1]$ is $L-$bilipschitz to a block of one of the prescribed geometries*

$S \times [i, i+1]$ will be called the $(i+1)$th **block** *of the model E_m.*

The **thickness** *of the $(i+1)$th* **block is the length of the shortest path between $S \times \{i\}$ and $S \times \{i+1\}$ in $S \times [i, i+1](\subset E_m)$.**

4.1. Bounded Geometry.

Definition 4.2. *An end E of a hyperbolic M has* **bounded geometry** [Min01, Min94] *if there is a (uniform) lower bound for lengths of closed geodesics in E.*

Since E itself has bounded geometry, it follows that any exiting geodesic is thick. We note this as follows for future use:

Lemma 4.3. *Let E be of bounded geometry. Then every exiting geodesic is thick. In particular every almost minimizing geodesic is thick.*

A bi-Lipschitz model E_m for E may be described by gluing a sequence of **thick blocks** end-to-end. We describe below the construction of a thick block, as this will be used in all the model geometries that follow.

Let S be a closed hyperbolic surface with a fixed but arbitrary hyperbolic structure.

Thick Block

Fix a constant L and a hyperbolic surface S. Let $B_0 = S \times [0,1]$ be given the product metric. If B is $L-$bilipschitz homeomorphic to B_0, it is called an $L-$**thick block**.

The following statement is a consequence of [Min94] (see also [Mit98b, Mj10a]).

Remark 4.4. *For any bounded geometry end, there exists L such that E is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing $L-$thick blocks end-to-end.*

Later on in this section, we shall omit stating the constant L explicitly, but assume that, given an end E, this constant is uniform for thick blocks in E.

4.2. i-bounded Geometry.

Definition 4.5. *An end E of a hyperbolic M has* **i-bounded geometry** [Mj11] *if the boundary torus of every Margulis tube in E has bounded diameter.*

An alternate description of i-bounded geometry can be given as follows. We start with a closed hyperbolic surface S. Fix a finite collection $\mathcal{C}$ of disjoint simple closed geodesics on S and let $N_\epsilon(\sigma_i)$ denote an ϵ neighborhood of σ_i, $(\sigma_i \in \mathcal{C})$. Here ϵ is chosen small enough so that no two lifts of $N_\epsilon(\sigma_i)$ to the universal cover $\widetilde{S}$ intersect.

Thin Block

Let $I = [0,3]$. Equip $S \times I$ with the product metric. Let $B^c = (S \times I - \cup_i N_\epsilon(\sigma_i) \times [1,2]$. Equip B^c with the induced path-metric.

For each resultant torus component of the boundary of B^c, perform Dehn filling on some $(1, n_i)$ curve (the n_i's may vary from block to block but we do not add on the suffix for B to avoid cluttering notation), which goes n_i times around the meridian and once round the longitude. n_i will be called the **twist coefficient**. Foliate the torus boundary of B^c by translates of $(1, n_i)$ curves and arrange so that the solid torus Θ_i thus glued in is hyperbolic and foliated by totally geodesic disks bounding the $(1, n_i)$ curves. Θ_i equipped with this metric will be called a **Margulis tube**.

The resulting copy of $S \times I$ obtained, equipped with the metric just described, is called a **thin block**. The following statement is a consequence of [Mj11].

Proposition 4.6. *An end E of a hyperbolic 3-manifold M has **i-bounded ge-ometry** if and only if it is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing thick and thin blocks end-to-end.*

The figure below illustrates a model E_m, where the black squares denote Margulis tubes and the (long) rectangles without black squares represent thick blocks.
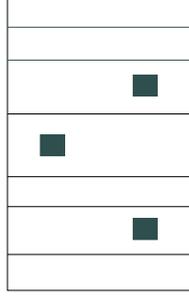
Model of i-bounded geometry (schematic)

Proposition 4.7. *Let E be of i-bounded geometry. Then there exist thin exiting geodesics. However, every almost minimizing geodesic is thick.*

Proof. **Existence of thin exiting geodesics:** The proof of this is similar to Proposition 3.8. Let B_{n_i} be the thin blocks with $n_i < n_{i+1}$. Let T_{n_i} and $T_{n_{i+1}}$ be Margulis tubes in these blocks. We choose thick minimizing geodesics λ_i between T_{n_i} and $T_{n_{i+1}}$, so that the length of λ_i is given by $l(\lambda_i) = d(T_{n_i}, T_{n_{i+1}})$.

Now consider long geodesic paths $\mu_i \subset T_{n_i}$ in Margulis tubes winding m_i times around the core of T_{n_i}, where $l(\mu_i) \to \infty$ as $i \to \infty$. The rest of the proof is as in Proposition 3.8. The concatenation $\bigcup_i (\lambda_i \cup \mu_i)$ is exiting and lifts to a (uniform) exiting quasigeodesic which is thin. Hence the exiting geodesic that tracks it is thin.

Thickness of almost minimizing geodesics: Let λ be an almost minimizing geodesic and $\lambda_i' = \lambda \cap B_i$ be the piece(s) of it within the block B_i. Let λ_i be the geodesic subsegment of λ between the first intersection point of λ with $S \times \{i\}$ and the last intersection point of λ with $S \times \{i\}$. Since there exists $L \geq 1$ such that each $S \times \{i\}$ is $L-$bilipschitz homeomorphic to S, it follows that there exists L_0 such that for any $p_i \in S \times \{i\}$ and $p_{i+1} \in S \times \{i\}$, there exists a path of length at most L_0 joining p_i to p_{i+1}. To see this, choose $x \in S_i \setminus (\bigcup_j N_\epsilon(\sigma_{i_j}))$, where the σ_{i_j}'s correspond to the Margulis tubes in B_{i+1}. Then $x \times I$ is a thick path of length at most $3L$, where $x \times \{0\}$ corresponds to $x \in S_i$ and $x \times \{3\}$ lies on S_{i+1}. Since S_i's are of bounded geometry, i.e. there exists $D > 0$ such that the diameter of S_i is bounded by D for all i, it follows that there is a path between p_i and p_{i+1} of length at most $(2D + 3L)$. We choose $L_0 = (2D + 3L)$.

Since λ is almost minimizing, there exists $C \geq 0$ such that $l(\lambda_i) \leq (L_0 + C)$ for all i. In particular, λ_i cannot go arbitrarily deep into the Margulis tube T_i in case B_i is thin. It follows that λ is thick. □

4.3. **Amalgamation Geometry.** As before, we start with a closed hyperbolic surface S. An amalgamated geometry block is similar to a thin block, except that we impose no control on the geometry of $S \times [1,2] \setminus (\bigcup_j N_\epsilon(\sigma_{i_j}) \times [1,2])$.

Definition 4.8. Amalgamated Block As before $I = [0, 3]$. We will describe a geometry on $S \times I$. There exist ϵ, L (these constants will be uniform over blocks of the model E_m) such that

(1) $B = S \times I$. Let $K = S \times [1, 2]$ under the identification of B with $S \times I$.
(2) We call K the geometric core. In its intrinsic path metric, it is $L-$bilipschitz to a convex hyperbolic manifold with boundary consisting of surfaces L-bilipschitz to a fixed hyperbolic surface. It follows that there exists $D > 0$ such that the diameter of $S \times \{i\}$ is bounded above by D (for $i = 1, 2$).
(3) There exists a simple closed multicurve on S each component of which has a geodesic realization on $S \times \{i\}$ for $i = \{1, 2\}$ with (total) length at most ϵ_0. Let γ denote its geodesic realization in K.
(4) There exists a regular neighborhood $N_k(\gamma) \subset K$ of γ which is homeomorphic to a union of disjoint solid tori, such that $N_k(\gamma) \cap S \times \{i\}$ is homeomorphic to a union of disjoint open annuli for $i = 1, 2$. Denote $N_k(\gamma)$ by T_γ and call it the Margulis tube(s) corresponding to γ.
(5) $S \times [0, 1]$ and $S \times [2, 3]$ are given the product structures corresponding to the bounded geometry structures on $S \times \{i\}$, for $i = 1, 2$ respectively.

In [Mj16] we had imposed further restrictions on the geometry of the geometric core K. But for the purposes of this paper, the above is enough.

Definition 4.9. *An end E of a hyperbolic 3-manifold M has* **amalgamated geometry** *if it is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing thick and amalgamated blocks end-to-end.*

Remark 4.10. *Note that with the above Definition, i-bounded geometry becomes a special case of amalgamated geometry. The difference is that amalgamated geometry imposes no conditions on the geometry of the complement $K - T_\gamma$. The component of $K - T_\gamma$ shall be called* **amalgamation components** *of K.*

The figure below illustrates schematically what the model looks like. Filled squares correspond to solid tori along which amalgamation occurs. The adjoining piece(s) denote amalgamation blocks of K. The blocks which have no filled squares are the *thick blocks* and those with filled squares are the *amalgamated blocks*
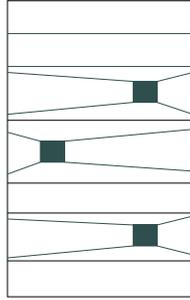
Model of amalgamated geometry (schematic)

4.3.1. *Almost minimizing geodesics.* Recall that the thickness of the $(i+1)$th block is the length of the shortest path between $S \times \{i\}$ and $S \times \{i + 1\}$ in $S \times [i, i + 1]$.

Lemma 4.11. *Any almost minimizing geodesic in an amalgamated geometry end is thick if all amalgamated blocks have bounded thickness.*

Proof. The proof is similar to the second part of Proposition 4.6.

Let λ be an almost minimizing geodesic and λ_i be the geodesic subsegment of λ between the first intersection point of λ with $S \times \{i\}$ and the last intersection point of λ with $S \times \{i+1\}$. If each B_i has thickness bounded by D_0 then as in the proof of Proposition 4.6, there exists L_0 such that for any $p_i \in S \times \{i\}$ and $p_{i+1} \in S \times \{i+1\}$ there is a path of length at most L_0 joining p_i to p_{i+1}. To see this, note first that there is a path of length at most D_0 from $S \times \{i\}$ to $S \times \{i+1\}$. Further, since each $S \times \{i\}$ has diameter bounded by some L, $L_0 = D_0 + 2L$ will suffice.

Since λ is almost minimizing, there exists $C \geq 0$ such that $l(\lambda_i) \leq (L_0 + C)$ for all i. In particular, λ_i cannot go arbitrarily deep into the Margulis tube T_i in case B_i is an amalgamated block. It follows that λ is thick. $\square$

## 5. Counterexamples: Bounded thickness neither necessary nor sufficient

The converse to Lemma 4.11 is not true. In particular, as we shall see in Example 5.1 below, it is possible to have all almost minimizing geodesics thick in manifolds of amalgamation geometry even in the presence of arbitrarily thick amalgamation blocks. Further, for more general geometries of ends ((than amalgamated geometry), bounded thickness of blocks is not sufficient to guarantee that almost minimizing geodesics are thick. In Section 5.2 we shall provide a counterexample. Thus, for general geometries, bounded thickness of blocks is neither necessary nor sufficient to ensure thickness of almost minimizing geodesics. For both these counterexamples, we shall need the more general technology of split geometry. This is summarized in the Appendix to the paper (Section 6). For convenience of the reader we shall refer to specific sections of the Appendix that are used.

5.1. **Example: Converse to Lemma 4.11 is false.** We furnish here a counterexample to the converse direction to Lemma 4.11 as follows. We shall proceed in two steps:

(1) Construct an amalgamation geometry block.
(2) Describe how to glue a sequence of such blocks together.

The gluing method will construct for us a hierarchy, which suffices ([BCM12], see Theorem 6.8 in the Appendix) to furnish the example we seek.

5.1.1. *The Amalgamation Geometry Block.* Instead of constructing a complete amalgamation geometry block, we shall describe the construction only in an amalgamation component. See figure below.
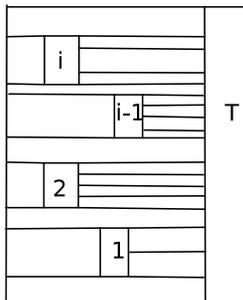
Let K be an amalgamation component (homeomorphic to $S_K \times I$) bounding a thin Margulis tube T which begins and ends at bounded geometry surfaces. The left vertical boundary of T has length $\sum_{j=1}^{i}(m_j + n_j)$ corresponding to Minsky blocks (Section 6.7) abutting on it. We shall say below what the m_j, n_j are.

Let v be the curve corresponding to T on the surface S. There are $i(= i(K))$ hierarchy curves $v_1, v_2, \cdots, v_{i-1}, v_i$ corresponding to Margulis tubes $T_1, T_2, \cdots, T_{i-1}, T_i$ (labeled $1, 2, \cdots, i-1, i$ in the figure above) inside the amalgamation component.

$S_K \setminus \{v_j\}$ has two components W_j and V_j which are component domains for hierarchy geodesics (see Section 6.1). The component domain W_j with v, v_j as boundary components supports a thick hierarchy geodesic of length n_j. The other component domain V_j (with only v_j as its boundary, lying to the left of v_j in the picture) has a length one (or uniformly bounded length in general) hierarchy geodesic segment supported on it.

Further, between the last split surface containing v, v_j and the first split surface containing v, v_{j+1} all split surfaces are thick and the component domain $S_K \setminus \{v\}$ supports a thick hierarchy geodesic of length m_j.

This forces the left vertical boundary of T to have length $\sum_{j=1}^{i}(m_j + n_j)$. On the other hand the length of a path η (say) in K in the left part of the picture is of the order of $\sum_j (m_j + 1)$.

The shortest path through the Margulis tube T has length of the order of $log(\sum_{j=1}^{i}(m_j + n_j))$ and by choosing n_i large one can make the length of η (i.e. $\sum_j(m_j + 1)$) the thickness of K.

A similar construction is done to the right of T with similar estimates, so as to get finally an amalgamated block B which has thickness of the order of $\sum_j(m_j + 1)$.

5.1.2. *Gluing Blocks Together and producing a hierarchy.* We finally indicate how to glue several such blocks together to give a model for a hyperbolic manifold. The hierarchy machinery comes quite handy here. Instead of blocks, we describe a hierarchy path, the correspondence between these two descriptions being given by [Min10, BCM12] as summarized in Theorem 6.8 and Definition 6.14 in the Appendix.

Thus, we choose amalgamation geometry blocks $B_1, \cdots, B_l, \cdots$ and between the top boundary of B_l and the bottom boundary of B_{l+1} we glue a sequence of thick blocks. In the hierarchy, this corresponds to a thick Teichmuller geodesic μ_l of length d_l, which remains thick in the curve complex $\mathcal{C}(S)$. Thus, successive μ_l, μ_{l+1} are attached at a vertex v_l (corresponding to the Margulis tube T in the above construction). However (a la [DGO11]), the incoming μ_l and the outgoing μ_{l+1} make an 'angle' at v_l approximately equal in size to the length of the hierarchy path corresponding to B_l in the link of v_l. Since this 'angle' is large, the concatenation of the μ_l's along with the hierarchy paths corresponding to B_l is a quasigeodesic r in $\mathcal{C}(S)$ ensuring that the gluing of the blocks in order actually approximates the split geometry model of the hyperbolic 3-manifold whose ending lamination is given by $r(\infty) \in \partial CC(S) = \mathcal{EL}(S)$ (c.f. [Kla99]).

A word about the hierarchy within the amalgamation geometry block is necessary here. The gluing pattern constructed in Section 5.1.1 also constructs a quasigeodesic in the relevant component domain. Hence the *qualitative* features of the constants used survive when we pass to the hierarchy determined by the ending lamination.

We note, in particular, that even after passing to the actual hierarchy the *difference* between the lengths of the left vertical boundary and the right vertical boundary does not go to infinity. This is because there is a bilipschitz map between the model built from the quasigeodesic approximation of the hierarchy (given by the gluing pattern) and that built from the actual hierarchy. Further there is a uniform constant depending only on that of the quasigeodesics. Hence, a posteriori (using Theorem 6.8), the additive errors in passing from the quasigeodesic approximation of the hierarchy to the actual hierarchy are all uniformly bounded.

Summary:
We summarize the features of the above example:

(1) The boundaries of the amalgamation blocks have (uniformly) bounded geometry as required in a model of amalgamated geometry.
(2) the almost minimizing paths are thick
(3) the amalgamation geometry blocks have unbounded thickness.

Hence thick almost minimizing paths may exist even when amalgamation geometry blocks have unbounded thickness. This example shows that the converse to Lemma 4.11 is false.

5.2. **Bounded thickness does not imply thick almost minimizing geodesics.**
In this subsection we give an example to show that even when all split blocks have bounded thickness, it is not necessary that almost minimizing geodesics be thick. Thus, the analog of Lemma 4.11 is false in the general case of split geometry and hence by Theorem 6.15 for degenerate hyperbolic 3-manifolds in general. As in Section 5.1 we shall build an approximation to a hierarchy whose qualitative features pass to the genuine hierarchy determined by the ending lamination corresponding to the base quasigeodesic of the approximate hierarchy.

Here is the idea of the construction using the model of split geometry from Theorem 6.8 and Definition 6.14 in the Appendix. We work with a sphere with n holes for convenience. It is straightforward to generalize the construction below to closed surfaces. The aim is to first construct a split geometry model such that

(1) All the blocks are of split geometry
(2) Each split geometry block B_i has a Margulis tube T_i corresponding to curve v_i splitting it into an $S_{(0,3)i}$ and an $S_{(0,n-1)i}$. We call them A_i, C_i for convenience. Thus the vertical boundary of T_i has two sides. The *short* vertical boundary abutting A_i has length one and the *long* vertical boundary abutting C_i has length m_i.
(3) There is a hanging tube $T_{i,i+1}$ (see Section 6.3.1) denoted H_i corresponding to curve $v_{i,i+1}$ intersecting only B_i and B_{i+1} separating the successive A_i and A_{i+1} so that the shortest path between A_i and A_{i+1} necessarily passes through H_i.
(4) Arrange so that for all i, any geodesic in the hierarchy supported in a proper subsurface of $S \setminus \{v_i, v_{i+1}, v_{i,i+1}\}$ is of bounded length; equivalently all curves other than $\cup_i \{v_i, v_{i+1}, v_{i,i+1}\}$ have geodesic realization in M of length at least $\epsilon_0 > 0$.

Once this is done, we see that

(1) Each split block is of bounded thickness (given by the part of S corresponding to A_i).

(2) Any almost minimizing geodesic necessarily passes deep into H_i and is therefore thin.

Towards this, it suffices to construct a hierarchy path (see Section 6.1 in the Appendix) such that conditions 3 and 4 above are satisfied. We translate this into the language of resolutions of a hierarchy to obtain the qualitative properties of the resolution we want. We require that

(1) The geodesic in the hierarchy on the subsurface $\Sigma_{i,i+1}$ bounded by $\{v_i, v_{i,i+1}\}$ is long and thick, for instance such that $\Sigma_{i,i+1}$ is $S_{0,4}$ and the corresponding blocks are thick.
(2) The same for the geodesic in the hierarchy on the subsurface $\Sigma_{i+1,i}$ bounded by $\{v_{i+1}, v_{i,i+1}\}$

Towards this we outline the construction in blocks B_i and the split surfaces in them. We will have numbers m_i, l_i, n_i corresponding to block B_i and then determine quite flexible conditions on them to satisfy the above requirements.

(1) The length of the long side of T_i is n_i (equivalently, the long side of T_i has n_i Minsky blocks abutting it).
(2) There is a constant L such that the number of $L-$bilipschitz split surfaces having v_i and $v_{i-1,i}$ as boundary curves is l_i and successive such split surfaces (corresponding to the resolution) bound between them a region that is $L-$bilipschitz to the product $\Sigma_{i,i-1} \times I$.
(3) Similarly, the number of $L-$bilipschitz split surfaces having v_i and $v_{i,i+1}$ as boundary curves is m_i and successive such split surfaces (corresponding to the resolution) bound between them a region that is $L-$bilipschitz to the product $\Sigma_{i+1,i} \times I$.
(4) $n_i \geq (l_i + m_i)$. This ensures that there is an $L-$bilipschitz split surface with only v_i as its boundary component, situated between the top of the lower hanging tube and the bottom of the higher hanging tube.
(5) There exist $L-$bilipschitz split surfaces having only $v_{i-1,i}$ as boundary curves (these correspond to the split surfaces abutting on the vertical boundary of $T_{i-1,i}$ opposite to the ones considered in (2) above). These are also l_i in number and successive ones cobound a region $L-$bilipschitz to a product.
(6) There exist $L-$bilipschitz split surfaces having only $v_{i,i+1}$ as boundary curves (these correspond to the split surfaces abutting on the vertical boundary of $T_{i,i+1}$ opposite to the ones considered in (3) above). These are also m_i in number and successive ones cobound a region $L-$bilipschitz to a product.
(7) $l_i, m_i \to \infty$ as $i \to \infty$.

See figure below.
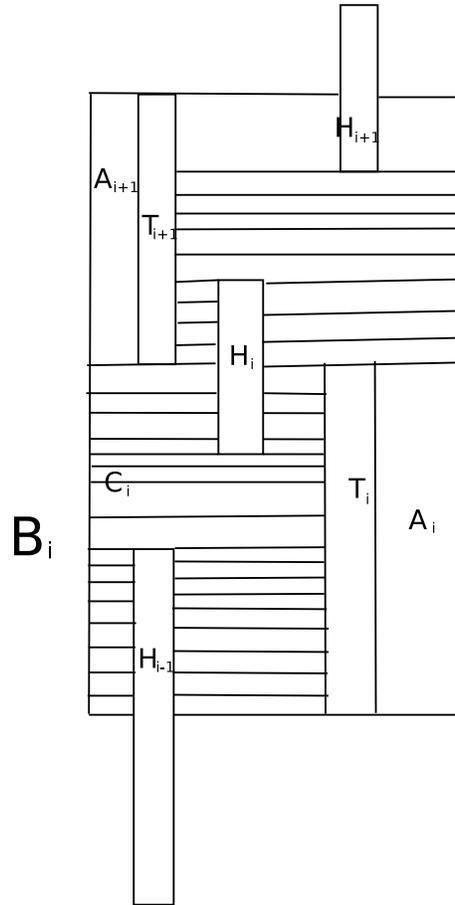
The Estimates:
With the conditions above satisfied, the promised counterexample is a consequence of the following argument. The distance from the bottom of B_i to the top of B_{i+1} is approximately $1 + log(2l_i)$ if one cuts across the hanging tube. Else, any thick path has length at least m_{i+1}. We can make m_{i+1} and l_i comparable, forcing the thin path to be shorter (as it is logarithmic in the length of the shortest thick path).

Constructing slices and the hierarchy
It therefore suffices to find a geodesic in the curve complex such that its associated hierarchy path satisfies the above conditions. Let us start at the middle of block B_i to see how to build up the hierarchy (the hierarchy described below is a somewhat more sophisticated version of the well-known 'chariot-wheel' example [MM00, Min10]):

(1) The sequence of split surfaces (or equivalently, slices of the hierarchy) give a thick geodesic supported in C_i

(2) The geodesic stops at a slice containing $v_i, v_{i,i+1}$. This corresponds to the slice through the lower boundary of the hanging tube H_i in the picture.

(3) Next we have two thick geodesics in a pair of component domains properly contained in C_i. These component domains correspond to the two components of $C_i \setminus v_{i,i+1}$.

(4) In the resolution, the hierarchy geodesics of the previous item end at the last slice containing v_i and corresponds to the top boundary of B_i.

(5) At this stage v_i is replaced by v_{i+1} and we have two thick geodesics in a pair of component domains properly contained in C_{i+1}. These component domains correspond to the two components of $C_{i+1} \setminus v_{i,i+1}$.

(6) In the resolution, the hierarchy geodesics of the previous item ends with the last slice containing both $v_{i+1}, v_{i,i+1}$ and corresponds to the slice through the upper boundary of H_i in the picture.

(7) The next set of slices of the hierarchy give a thick geodesic supported in C_{i+1}

(8) We can now go back to item (1), replacing i by $i+1$ and proceeding as above.

It is now possible to construct a geodesic path exhibiting the above behavior by choosing the geodesics in the links of v_i and $\{v_i, v_{i,i+1}\}$ according to the requirements given by Items (1) and (3) above. This completes the construction of our counterexample.

Acknowledgments

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6. Appendix: Hierarchies and Split Geometry

We recapitulate the essential aspects of hierarchies and split geometry from [MM00, Min10, Mj14a]. The definitions here follow [Mj14a].

6.1. Hierarchies. We fix some notation first:

- $\xi(S_{g,b}) = 3g + b$ is the *complexity* of a compact surface $S = S_{g,b}$ of genus g and b boundary components
- For an essential subsurface Y of S, $\mathcal{C}(Y)$ will be its curve complex and $\mathcal{P}(Y)$ its pants complex.
- γ_α will be a collection of disjoint simple closed curves on S corresponding to a simplex $\alpha \in \mathcal{C}(Y)$
- α, β in $\mathcal{C}(Y)$ fill an essential subsurface Y of S if all non-trivial non-peripheral curves in Y have essential intersection with at least one of γ_α or γ_β, where γ_α and γ_β are chosen to intersect minimally.
- Given α, β in $\mathcal{C}(S)$, form a regular neighborhood of $\gamma_\alpha \cup \gamma_\beta$, and fill in all disks and one-holed disks to obtain Y which is said to be *filled* by α, β.
- For an essential subsurface $X \subset Z$ let $\partial_Z(X)$ denote the *relative boundary* of X in Z, i.e. those boundary components of X that are non-peripheral in Z.
- A **pants decomposition** of a compact surface S, possibly with boundary, is a disjoint collection of 3-holed spheres $P_1, \cdots, P_n$ embedded in S such that $S \setminus \bigcup_i P_i$ is a disjoint collection of non-peripheral annuli in S, no two of which are homotopic.
- A **tube** in an end $E \subset N$ is a regular $R-$neighborhood $N(\gamma, R)$ of an unknotted geodesic γ in E.

Definition 6.1. Tight Geodesics and Component Domains
Let Y be an essential subsurface in S. If $\xi(Y) > 4$, a **tight** sequence of simplices $\{v_i\}_{i \in \mathcal{I}} \subset \mathcal{C}(Y)$ (where $\mathcal{I}$ is a finite or infinite interval in $\mathbb{Z}$) satisfies the following:
1) For any vertices w_i of v_i and w_j of v_j where $i \neq j$, $d_{\mathcal{C}(Y)}(w_i, w_j) = |i - j|$,
2) For $\{i-1, i, i+1\} \subset \mathcal{I}$, v_i equals $\partial_Y F(v_{i-1}, v_{i+1})$.
If $\xi(Y) = 4$ then a tight sequence is the vertex sequence of a geodesic in $\mathcal{C}(Y)$.
A *tight geodesic* g in $\mathcal{C}(Y)$ consists of a tight sequence $v_0, \cdots, v_n$, and two simplices in $\mathcal{C}(Y)$, $\mathbf{I} = \mathbf{I}(g)$ and $\mathbf{T} = \mathbf{T}(g)$, called its initial and terminal markings such that v_0 (resp. v_n) is a sub-simplex of $\mathbf{I}$ (resp. $\mathbf{T}$). The length of g is n. v_i is called a simplex of g. Y is called the *domain or support* of g and is denoted as $Y = D(g)$. g is said to be supported in $D(g)$.

For a surface W with $\xi(W) \geq 4$ and v a simplex of $\mathcal{C}(W)$ we say that Y is a *component domain of* (W, v) if Y is a component of $W \setminus \mathbf{collar}(v)$, where $\mathbf{collar}(v)$ is a small tubular neighborhood of the simple closed curves.

If g is a tight geodesic with domain $D(g)$, we call $Y \subset S$ a *component domain of* g if for some simplex v_j of g, Y is a component domain of $(D(g), v_j)$.

Definition 6.2. Hierarchies
A **hierarchy path** in $\mathcal{P}(S)$ joining pants decompositions P_1 and P_2 is a path $\rho : [0, n] \to P(S)$ joining $\rho(0) = P_1$ to $\rho(n) = P_2$ such that
1) There is a collection $\{Y\}$ of essential, non-annular subsurfaces of S, called component domains for ρ, such that for each component domain Y there is a connected

interval $J_Y \subset [0, n]$ with $\partial Y \subset \rho(j)$ for each $j \in J_Y$.

2) For a component domain Y, there exists a tight geodesic g_Y supported in Y such that for each $j \in J_Y$, there is an $\alpha \in g_Y$ with $\alpha \in \rho(j)$.

A **hierarchy path** in $\mathcal{P}(S)$ is a sequence $\{P_n\}_n$ of pants decompositions of S such that for any $P_i, P_j \in \{P_n\}_n$, $i \leq j$, the finite sequence $P_i, P_{i+1}, \cdots, P_{j-1}, P_j$ is a hierarchy path joining pants decompositions P_i and P_j.

The collection H of tight geodesics g_Y supported in component domains Y of ρ will be called the **hierarchy** of tight geodesics associated to ρ.

Definition 6.3. A **slice** of a hierarchy H associated to a hierarchy path ρ is a set τ of pairs (h, v), where $h \in H$ and v is a simplex of h, satisfying the following properties:

(1) A geodesic h appears in at most one pair in τ.
(2) There is a distinguished pair (h_τ, v_τ) in τ, called the bottom pair of τ. We call h_τ the bottom geodesic.
(3) For every $(k, w) \in \tau$ other than the bottom pair, $D(k)$ is a component domain of $(D(h), v)$ for some $(h, v) \in \tau$.

A **resolution** of a hierarchy H associated to a hierarchy path $\rho : I \to \mathcal{P}(S)$ is a sequence of slices $\tau_i = \{(h_{i1}, v_{i1}), (h_{i2}, v_{i2}), \cdots, (h_{in_i}, v_{in_i})\}$ (for $i \in I$, the same indexing set) such that the set of vertices of the simplices $\{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ is the same as the set of the non-peripheral boundary curves of the pairs of pants in $\rho(i) \in \mathcal{P}(S)$.

6.2. **Split level Surfaces.** Let E be a degenerate end of a hyperbolic 3-manifold N. Let $\mathcal{T}$ denote a collection of disjoint, uniformly separated tubes in ends of N such that

(1) All Margulis tubes in E belong to $\mathcal{T}$.
(2) there exists $\epsilon_0 > 0$ such that the injectivity radius $injrad_x(E) > \epsilon_0$ for all $x \in E \setminus \bigcup_{T \in \mathcal{T}} Int(T)$.

In [Min10], Minsky constructs a model manifold M bilipschitz homeomorphic to N and equipped with a piecewise Riemannian structure.

Let $(Q, \partial Q)$ be the unique hyperbolic pair of pants such that each component of ∂Q has length one. Q will be called the *standard* pair of pants. An isometrically embedded copy of $(Q, \partial Q)$ in $(M(0), \partial M(0))$ will be said to be *flat*.

Definition 6.4. A **split level surface** associated to a pants decomposition $\{Q_1, \cdots, Q_n\}$ of a compact surface S (possibly with boundary) in $M(0) \subset M$ is an embedding $f : \cup_i (Q_i, \partial Q_i) \to (M(0), \partial M(0))$ such that

1) Each $f(Q_i, \partial Q_i)$ is flat

2) f extends to an embedding (also denoted f) of S into M such that the interior of each annulus component of $f(S \setminus \bigcup_i Q_i)$ lies entirely in $F(\bigcup_{T \in \mathcal{T}} Int(T))$.

Let S_i^s denote the union of the collection of flat pairs of pants in the image of the embedding S_i. Note that $S_i \setminus S_i^s$ consists of annuli properly embedded in Margulis tubes.

The class of *all* topological embeddings from S to M that agree with a split level surface f associated to a pants decomposition $\{Q_1, \cdots, Q_n\}$ on $Q_1 \cup \cdots \cup Q_n$ will be denoted by $[f]$.

We define a partial order $\leq_E$ on the collection of split level surfaces in an end E of M as follows:

$f_1 \leq_E f_2$ if there exist $g_i \in [f_i]$, $i = 1, 2$, such that $g_2(S)$ lies in the unbounded component of $E \setminus g_1(S)$.

A sequence S_i of split level surfaces is said to exit an end E if $i < j$ implies $S_i \leq_E S_j$ and further for all compact subsets $B \subset E$, there exists $L > 0$ such that $S_i \cap B = \emptyset$ for all $i \geq L$.

Definition 6.5. *A curve v in $S \subset E$ is l-**thin** if the core curve of the Margulis tube $T_v(\subset E \subset N)$ has length less than or equal to l. A tube $T \in \mathcal{T}$ is l-thin if its core curve is l-thin. A tube $T \in \mathcal{T}$ is l-thick if it is not l-thin.*
*A curve v is said to split a pair of split level surfaces S_i and S_j $(i < j)$ if v occurs as a boundary curve of both S_i and S_j. A pair of split level surfaces S_i and S_j $(i < j)$ is said to be an l-**thin pair** if there exists an l-thin curve v splitting both S_i and S_j.*

The collection of all l-thin tubes is denoted as $\mathcal{T}_l$. The union of all l-thick tubes along with $M(0)$ is denoted as $M(l)$.

Definition 6.6. *A pair of split level surfaces S_i and S_j $(i < j)$ is said to be k-**separated** if*
a) for all $x \in S_i^s$, $d_M(x, S_j^s) \geq k$
b) Similarly, for all $x \in S_j^s$, $d_M(x, S_i^s) \geq k$.

Definition 6.7. Minsky Blocks *(Section 8.1 of* [Min10]*)*
A tight geodesic in H supported in a component domain of complexity 4 is called a 4-geodesic and an edge of a 4-geodesic in H is called a 4-edge.

Given a 4-edge e in H, let g be the 4-geodesic containing it, and let $D(e)$ be the domain $D(g)$. Let e^- and e^+ denote the initial and terminal vertices of e. Also **collar** v *denotes a small tubular neighborhood of v in $D(e)$.*

To each e a Minsky block $B(e)$ is assigned as as follows:

$$B(e) = (D(e) \times [-1, 1]) \setminus (\mathbf{collar}(e^-) \times [-1, -1/2] \cup \mathbf{collar}(e^+) \times (1/2, 1]).$$

The horizontal boundary *of $B(e)$ is*

$$\partial_\pm B(e) \equiv (D(e) \setminus \mathbf{collar}(e^\pm)) \times \{\pm 1\}.$$

The horizontal boundary is a union of three-holed spheres. The rest of the boundary is a union of annuli and is called the vertical boundary. The top (resp. bottom) horizontal boundaries of $B(e)$ are $(D(e) \setminus \mathbf{collar}(e^+)) \times \{1\}$ (resp. $(D(e) \setminus \mathbf{collar}(e^-)) \times \{-1\}$.

6.2.1. *The Model and the bi-Lipschitz Model Theorem.*

Theorem 6.8. [Min10] [BCM12] *Let N be the convex core of a simply or doubly degenerate hyperbolic 3-manifold minus an open neighborhood of the cusp(s). Let S be a compact surface, possibly with boundary, such that N is homeomorphic to $S \times [0, \infty)$ or $S \times \mathbb{R}$ according as N is simply or doubly degenerate. There exist $L \geq 1$, $\theta, \omega, \epsilon, \epsilon_1 > 0$, a collection $\mathcal{T}$ of (θ, ω)-thin tubes containing all Margulis tubes in N, a 3-manifold M, and an L-bilipschitz homeomorphism $F : N \to M$ such that the following holds.*
Let $M(0) = F(N \setminus \bigcup_{T \in \mathcal{T}} Int(T))$ and let $F(\mathcal{T})$ denote the image of the collection $\mathcal{T}$ under F. Let $\leq_E$ denote the partial order on the collection of split level surfaces

in an end E of M. Then there exists a sequence S_i of split level surfaces associated to pants decompositions P_i exiting E such that

(1) *$S_i \leq_E S_j$ if $i \leq j$.*
(2) *The sequence $\{P_i\}$ is a hierarchy path in $\mathcal{P}(S)$.*
(3) *If $P_i \cap P_j = \{Q_1, \cdots Q_l\}$ then $f_i(Q_k) = f_j(Q_k)$ for $k = 1 \cdots l$, where f_i, f_j are the embeddings defining the split level surfaces S_i, S_j respectively.*
(4) *For all i, $P_i \cap P_{i+1} = \{Q_{i,1}, \cdots Q_{i,l}\}$ consists of a collection of l pairs of pants, such that $S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l})$ has a single non-annular component of complexity 4. Further, there exists a Minsky block W_i and an isometric map G_i of W_i into $M(0)$ such that $f_i(S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l}))$ (resp. $f_{i+1}(S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l}))$) is contained in the bottom (resp. top) gluing boundary of W_i.*
(5) *For each flat pair of pants Q in a split level surface S_i there exists an isometric embedding of $Q \times [-\epsilon, \epsilon]$ into $M(0)$ such that the embedding restricted to $Q \times \{0\}$ agrees with f_i restricted to Q.*
(6) *For each $T \in \mathcal{T}$, there exists a split level surface S_i associated to pants decompositions P_i such that the core curve of T is isotopic to a non-peripheral boundary curve of P_i. The boundary $F(\partial T)$ of $F(T)$ with the induced metric d_T from $M(0)$ is a Euclidean torus equipped with a product structure $S^1 \times S_v^1$, where any circle of the form $S^1 \times \{t\} \subset S^1 \times S_v^1$ is a round circle of unit length and is called a horizontal circle; and any circle of the form $\{t\} \times S_v^1$ is a round circle of length l_v and is called a vertical circle.*
(7) *Let g be a tight geodesic other than the bottom geodesic in the hierarchy H associated to the hierarchy path $\{P_i\}$, let $D(g)$ be the support of g and let v be a boundary curve of $D(g)$. Let T_v be the tube in $\mathcal{T}$ such that the core curve of T_v is isotopic to v. If a vertical circle of $(F(\partial T_v), d_{T_v})$ has length l_v less than $n\epsilon_1$, then the length of g is less than n.*

6.2.2. *Tori and Meridinal Coefficients.* Let T be the boundary of a Margulis tube in M. The boundary of a Margulis tube has the structure of a Euclidean torus and gives a unique point ω_T in the upper half plane, the Teichmuller space of the torus. The real and imaginary components of ω_T have a geometric interpretation.

Suppose that the Margulis tube T corresponds to a vertex $v \in \mathcal{C}(S)$. Let tw_T be the signed length of the annulus geodesic corresponding to v, i.e. it counts with sign the number of Dehn twists about the curve represented by v. Next, note that by the construction of the Minsky model, the **vertical** boundary of T consists of two sides - the **left vertical boundary** and **right vertical boundary**. Each is attached to vertical boundaries of Minsky blocks. Let the total number of blocks whose vertical boundaries, are glued to the vertical boundary of T be n_T. Similarly, let the total number of blocks whose vertical boundaries, are glued to the left (resp. right) vertical boundary of T be n_{Tl} (resp. n_{Tr}) so that $n_T = n_{Tl} + n_{Tr}$.

In Section 9 of [Min10], Minsky shows:

Theorem 6.9. [Min10] *There exists $C_0 \geq 0$, such that the following holds.*
$\omega - (tw_T + in_T) \leq C_0$

6.2.3. *Consequences.* [Mj14a] Two consequences of Theorem 6.8 that we shall need are given below.

Lemma 6.10. [Mj14a, Lemma 3.6] *Given $l > 0$ there exists $n \in \mathbb{N}$ such that the following holds.*

Let v be a vertex in the hierarchy H such that the length of the core curve of the Margulis tube T_v corresponding to v is greater than l. Next suppose $(h, v) \in \tau_i$ for some slice τ_i of the hierarchy H such that h is supported on Y, and D is a component of $Y \setminus \mathbf{collar}\, v$. Also suppose that $h_1 \in H$ such that D is the support of h_1. Then the length of h_1 is at most n.

Lemma 6.11. [Mj14a, Lemma 3.7] *Given $l > 0$ and $n \in \mathbb{N}$, there exists $L_2 \geq 1$ such that the following holds:*
Let S_i, S_j $(i < j)$ be split level surfaces associated to pants decompositions P_i, P_j such that
a) $(j - i) \leq n$
b) $P_i \cap P_j$ is a (possibly empty) pants decomposition of $S \setminus W$, where W is an essential (possibly disconnected) subsurface of S such that each component W_k of W has complexity $\xi(W_k) \geq 4$.
c)For any k with $i < k < j$, and $(g_D, v) \in \tau_k$ for $D \subset W_i$ for some i, no curve in v has a geodesic realization in N of length less than l.
Then there exists an L_2-bilipschitz embedding $G : W \times [-1, 1] \to M$, such that
1) W admits a hyperbolic metric given by $W = Q_1 \cup \cdots \cup Q_m$ where each Q_i is a flat pair of pants.
2) $W \times [-1, 1]$ is given the product metric.
3) $f_i(P_i \setminus P_i \cap P_j) \subset W \times \{-1\}$ and $f_j(P_j \setminus P_i \cap P_i) \subset W \times \{1\}$.

6.3. **Split surfaces and weak split geometry.**

Definition 6.12. An L-bi-Lipschitz **split surface** in $M(l)$ associated to a pants decomposition $\{Q_1, \cdots, Q_n\}$ of S and a collection $\{A_1, \cdots, A_m\}$ of complementary annuli (not necessarily all of them) in S is an embedding $f : \cup_i Q_i \bigcup \cup_i A_i \to M(l)$ such that
1) the restriction $f : \cup_i(Q_i, \partial Q_i) \to (M(0), \partial M(0))$ is a split level surface
2) the restriction $f : A_i \to M(l)$ is an L-bi-Lipschitz embedding.
3) f extends to an embedding (also denoted f) of S into M such that the interior of each annulus component of $f(S \setminus (\cup_i Q_i \bigcup \cup_i A_i))$ lies entirely in $F(\bigcup_{T \in \mathcal{T}_l} Int(T))$.

A split level surface differs from a split surface in that the latter may contain bi-Lipschitz annuli in addition to flat pairs of pants. We denote split surfaces by Σ_i. Let Σ_i^s denote the union of the collection of flat pairs of pants and bi-Lipschitz annuli in the image of the split surface (embedding) Σ_i.

The next Theorem is one of the technical tools from [Mj14a].

Theorem 6.13. [Mj14a, Theorem 4.8] *Let $N, M, M(0), S, F$ be as in Theorem 6.8 and E an end of M. For any l less than the Margulis constant, let $M(l) = \{F(x) : \operatorname{injrad}_x(N) \geq l\}$. Fix a hyperbolic metric on S such that each component of ∂S is totally geodesic of length one (this is a normalization condition). There exist $L_1 \geq 1$, $\epsilon_1 > 0$, $n \in \mathbb{N}$, and a sequence Σ_i of L_1-bilipschitz, ϵ_1-separated split surfaces exiting the end E of M such that for all i, one of the following occurs:*

  (1) *An l-thin curve splits the pair (Σ_i, Σ_{i+1}), i.e. the associated split level surfaces form an l-thin pair.*
  (2) *there exists an L_1-bilipschitz embedding*

$$G_i : (S \times [0, 1], (\partial S) \times [0, 1]) \to (M, \partial M)$$

  such that $\Sigma_i^s = G_i(S \times \{0\})$ and $\Sigma_{i+1}^s = G_i(S \times \{1\})$

Finally, each l-thin curve in S splits at most n split level surfaces in the sequence $\{\Sigma_i\}$.

Pairs of split surfaces satisfying Alternative (1) of Theorem 6.13 will be called an *l*-**thin pair** of split surfaces (or simply a thin pair if *l* is understood). Similarly, pairs of split surfaces satisfying Alternative (2) of Theorem 6.13 will be called an *l*-**thick pair** (or simply a thick pair) of split surfaces.

Definition 6.14. *A model manifold satisfying the following conditions is said to have* **weak split geometry***:*

(1) *A sequence of split surfaces S_i^s exiting the end(s) of M, where M is marked with a homeomorphism to $S \times J$ (J is $\mathbb{R}$ or $[0, \infty)$ according as M is totally or simply degenerate). $S_i^s \subset S \times \{i\}$.*

(2) *A collection of Margulis tubes $\mathcal{T}$ in N with image $F(\mathcal{T})$ in M (under the bilipschitz homeomorphism between N and M). We refer to the elements of $F(\mathcal{T})$ also as Margulis tubes.*

(3) *For each complementary annulus of S_i^s with core σ, there is a Margulis tube $T \in \mathcal{T}$ whose core is freely homotopic to σ such that $F(T)$ intersects S_i^s at the boundary. (What this roughly means is that there is an $F(T)$ that contains the complementary annulus.) We say that $F(T)$ splits S_i^s.*

(4) *There exist constants $\epsilon_0 > 0, K_0 > 1$ such that for all i, either there exists a Margulis tube splitting both S_i^s and S_{i+1}^s, or else $S_i(= S_i^s)$ and $S_{i+1}(= S_{i+1}^s)$ have injectivity radius bounded below by ϵ_0 and bound a* **thick block** *B_i, where a thick block is defined to be a K_0-bilipschitz homeomorphic image of $S \times I$.*

(5) *$F(T) \cap S_i^s$ is either empty or consists of a pair of boundary components of S_i^s that are parallel in S_i.*

(6) *There is a uniform upper bound $n = n(M)$ on the number of surfaces that $F(T)$ splits.*

Theorem 6.15. [Mj14a] *Any degenerate end of a hyperbolic 3-manifold is bi-Lipschitz homeomorphic to a Minsky model and hence to a model of weak split geometry.*

6.3.1. *Split Blocks and Hanging Tubes.*

Definition 6.16. *Let $(\Sigma_i^s, \Sigma_{i+1}^s)$ be a thick pair of split surfaces in M. The closure of the bounded component of $M \setminus (\Sigma_i^s \cup \Sigma_{i+1}^s)$ between $\Sigma_i^s, \Sigma_{i+1}^s$ will be called a thick block.*

Note that a thick block is uniformly bi-Lipschitz to the product $S \times [0, 1]$ and that its boundary components are $\Sigma_i^s, \Sigma_{i+1}^s$.

Definition 6.17. *Let $(\Sigma_i^s, \Sigma_{i+1}^s)$ be an l-thin pair of split surfaces in M and $F(\mathcal{T}_i)$ be the collection of l-thin Margulis tubes that split both $\Sigma_i^s, \Sigma_{i+1}^s$. The closure of the union of the bounded components of $M \setminus ((\Sigma_i^s \cup \Sigma_{i+1}^s) \bigcup_{F(T) \in F(\mathcal{T}_i)} F(T))$ between $\Sigma_i^s, \Sigma_{i+1}^s$ will be called a* **split block***.*

The closure of any bounded component is called a **split component***.*

Each split component may contain Margulis tubes, which we shall call **hanging tubes** (see below) that *do not* split both $\Sigma_i^s, \Sigma_{i+1}^s$.

Topologically, a **split block** $B^s \subset B = S \times I$ is a topological product $S^s \times I$ for some *connected* S^s. However, the upper and lower boundaries of B^s need only be be split subsurfaces of S^s. This is to allow for Margulis tubes starting (or ending) within the split block. Such tubes would split one of the horizontal boundaries but not both. We shall call such tubes **hanging tubes**. See figure below:
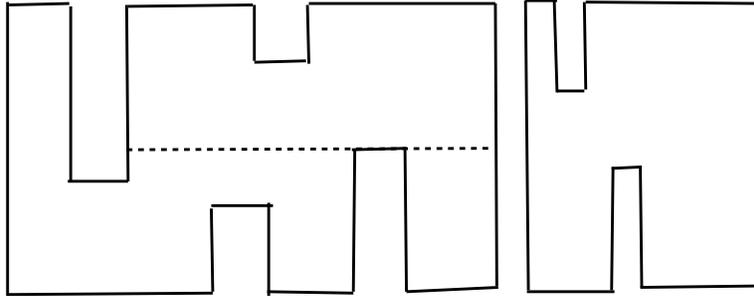
Split Block with hanging tubes

The vertical lengths of *hanging tubes* are further required to be *uniformly bounded below by some $\eta_0 > 0$*. Further, each such annulus has cross section a round circle of length ϵ_0.

Definition 6.18. *Hanging tubes intersecting the upper (resp. lower) boundaries of a split block are called upper (resp. lower) hanging tubes.*

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