

# SEMICONJUGACIES BETWEEN RELATIVELY HYPERBOLIC BOUNDARIES

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ABSTRACT. We prove the existence of Cannon-Thurston maps for Kleinian groups corresponding to pared manifolds whose boundary is incompressible away from cusps. We also describe the structure of these maps in terms of ending laminations.

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## 1. INTRODUCTION

The aim of this paper is threefold:

- 1) To extend the main Theorems of [Mj06], [Mj07] [DM10] (which prove the existence and structure of Cannon-Thurston maps for surface groups without accidental parabolics) to Kleinian groups corresponding to *pared manifolds whose boundary is incompressible away from cusps*.<sup>1</sup> This is the content of Theorem 3.11.
- 2) To give a considerably shorter and more streamlined proof of the main step of [Mj09]. This is the content of Theorem 3.5.
- 3) To generalize a reduction Theorem of Klarreich [Kla99] to the context of relative hyperbolicity. This is the content of Theorem 3.1.

The main tool, Theorem 3.1, is a ‘reduction Theorem’ ((3) above) which allows us to deduce the existence and structure of Cannon-Thurston maps for the inclusion of one relatively hyperbolic metric space into another, once we know the existence

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<sup>1</sup>A considerably more elaborate and somewhat clumsier proof had been sketched in an earlier version of [Mj06]. This proof has been excised from the present version of [Mj06].

and structure of Cannon-Thurston maps for inclusions of certain relatively quasiconvex subspaces into *ends*. The exact statement of Theorem 3.1 is somewhat technical. Suffice to say, this is the appropriate relative hyperbolic generalization of inclusions of geometrically finite hyperbolic 3-manifolds  $M_{gf}$  into degenerate hyperbolic 3-manifolds  $N^h$  such that

- a) the inclusion of a boundary component  $S_{gf}$  of  $M_{gf}$  into the end  $E^h$  of  $N^h$  it bounds is a homotopy equivalence.
- b) Each  $S_{gf}$  is incompressible in  $M_{gf}$ .

We give the main application below.

**Theorem 3.11:** Suppose that  $N^h \in H(M, P)$  is a hyperbolic structure on a pared manifold  $(M, P)$  with incompressible boundary  $\partial_0 M$ . Let  $M_{gf}$  denotes a geometrically finite hyperbolic structure adapted to  $(M, P)$ . Then the map  $i : \widehat{M}_{gf} \rightarrow \widehat{N}^h$  extends continuously to the boundary  $\hat{i} : \widehat{M}_{gf} \rightarrow \widehat{N}^h$ . Further, the Cannon-Thurston map  $\partial i$  identifies precisely the end-points of leaves of the ending lamination.

The last step of the programme of proving the existence of Cannon-Thurston maps for arbitrary finitely generated Kleinian groups and describing their structure is dealt with in [Mj10].

## 2. BACKGROUND

**2.1. Relative Hyperbolicity and Quasiconvexity.** Let  $(X, d)$  be a path metric space. A collection of closed subsets  $\mathcal{H} = \{H_\alpha\}$  of  $X$  will be said to be **uniformly separated** if there exists  $\epsilon > 0$  such that  $d(H_1, H_2) \geq \epsilon$  for all distinct  $H_1, H_2 \in \mathcal{H}$ .

**Definition 2.1.** (Farb [Far98]) *The electric space (or coned-off space)  $\mathcal{E}(X, \mathcal{H})$  corresponding to the pair  $(X, \mathcal{H})$  is a metric space which consists of  $X$  and a collection of vertices  $v_\alpha$  (one for each  $H_\alpha \in \mathcal{H}$ ) such that each point of  $H_\alpha$  is joined to (coned off at)  $v_\alpha$  by an edge of length  $\frac{1}{2}$ . The sets  $H_\alpha$  shall be referred to as horosphere-like sets and the vertices  $v_\alpha$  as cone-points.*

$X$  is said to be **weakly hyperbolic** relative to the collection  $\mathcal{H}$  if  $\mathcal{E}(X, \mathcal{H})$  is a hyperbolic metric space.

**Definition 2.2.** *A path  $\gamma$  in  $\mathcal{E}(X, \mathcal{H})$  is said to be an electric geodesic (resp. electric  $K$ -quasigeodesic) if it is a geodesic (resp.  $K$ -quasigeodesic) in  $\mathcal{E}(X, \mathcal{H})$ .*

$\gamma$  is said to be an electric  $K$ -quasigeodesic in (the electric space)  $\mathcal{E}(X, \mathcal{H})$  without backtracking if  $\gamma$  is an electric  $K$ -quasigeodesic in  $\mathcal{E}(X, \mathcal{H})$  and  $\gamma$  does not return to any horosphere-like set  $H_\alpha$  after leaving it.

Let  $i : X \rightarrow \mathcal{E}(X, \mathcal{H})$  denotes the natural inclusion of spaces. Then for a path  $\gamma \subset X$ , the path  $i(\gamma)$  lies in  $\mathcal{E}(X, \mathcal{H})$ . Replacing maximal subsegments  $[a, b]$  of  $i(\gamma)$  lying in a particular  $H_\alpha$  by a path that goes from  $a$  to  $v_\alpha$  and then from  $v_\alpha$  to  $b$ , and repeating this for every  $H_\alpha$  that  $i(\gamma)$  meets we obtain a new path  $\hat{\gamma}$ . If  $\hat{\gamma}$  is an electric geodesic (resp.  $P$ -quasigeodesic),  $\gamma$  is called a *relative geodesic* (resp. *relative  $P$ -quasigeodesic*). We shall usually be concerned with the case that  $\gamma$  is an ambient geodesic/quasigeodesic without backtracking.

**Definition 2.3.** *Relative  $P$ -quasigeodesics in  $(X, \mathcal{H})$  are said to satisfy **bounded region penetration** if for any two relative  $P$ -quasigeodesics without backtracking  $\beta, \gamma$ , joining  $x, y \in X$  there exists  $B = B(P)$  such that*

**Similar Intersection Patterns 1:** if precisely one of  $\{\beta, \gamma\}$  meets a horosphere-like set  $H_\alpha$ , then the length (measured in the intrinsic path-metric on  $H_\alpha$ ) from the first (entry) point to the last (exit) point (of the relevant path) is at most  $B$ .

**Similar Intersection Patterns 2:** if both  $\{\beta, \gamma\}$  meet some  $H_\alpha$  then the length (measured in the intrinsic path-metric on  $H_\alpha$ ) from the entry point of  $\beta$  to that of  $\gamma$  is at most  $B$ ; similarly for exit points.

Replacing ‘ $P$ -quasigeodesic’ by ‘geodesic’ in the above definition, we obtain the notion of relative geodesics in  $(X, \mathcal{H})$  satisfying bounded region penetration.

Families of paths which enjoy the above properties shall be said to have similar intersection patterns with horospheres.

**Definition 2.4.** (Farb [Far98])  $X$  is said to be hyperbolic relative to the uniformly separated collection  $\mathcal{H}$  if

- 1)  $X$  is weakly hyperbolic relative to  $\mathcal{H}$
  - 2) For all  $P \geq 1$ , relative  $P$ -quasigeodesics without backtracking satisfy the bounded penetration property
- Elements of  $\mathcal{H}$  will be referred to as **horosphere-like sets**.

**Gromov’s definition of relative hyperbolicity** [Gro85] :

**Definition 2.5. (Gromov)** For any geodesic metric space  $(H, d)$ , the hyperbolic cone (analog of a horoball)  $H^h$  is the metric space  $H \times [0, \infty) = H^h$  equipped with the path metric  $d_h$  obtained from two pieces of data

- 1)  $d_{h,t}((x, t), (y, t)) = 2^{-t}d_H(x, y)$ , where  $d_{h,t}$  is the induced path metric on  $H \times \{t\}$ . Paths joining  $(x, t), (y, t)$  and lying on  $H \times \{t\}$  are called horizontal paths.
- 2)  $d_h((x, t), (x, s)) = |t - s|$  for all  $x \in H$  and for all  $t, s \in [0, \infty)$ , and the corresponding paths are called vertical paths.
- 3) for all  $x, y \in H^h$ ,  $d_h(x, y)$  is the path metric induced by the collection of horizontal and vertical paths.

**Definition 2.6.** Let  $X$  be a geodesic metric space and  $\mathcal{H}$  be a collection of mutually disjoint uniformly separated subsets of  $X$ .  $X$  is said to be hyperbolic relative to  $\mathcal{H}$  in the sense of Gromov, if the quotient space  $\mathcal{G}(X, \mathcal{H})$ , obtained by attaching the hyperbolic cones  $H^h$  to  $H \in \mathcal{H}$  by identifying  $(z, 0)$  with  $z$  for all  $H \in \mathcal{H}$  and  $z \in H$ , is a complete hyperbolic metric space. The collection  $\{H^h : H \in \mathcal{H}\}$  is denoted as  $\mathcal{H}^h$ . The induced path metric is denoted as  $d_h$ .

**Theorem 2.7.** (Bowditch [Bow97]) The following are equivalent:

- 1)  $X$  is hyperbolic relative to the collection  $\mathcal{H}$  of uniformly separated subsets of  $X$
- 2)  $X$  is hyperbolic relative to the collection  $\mathcal{H}$  of uniformly separated subsets of  $X$  in the sense of Gromov
- 3)  $\mathcal{G}(X, \mathcal{H})$  is hyperbolic relative to the collection  $\mathcal{H}^h$

**Definition 2.8.** Let  $X$  be hyperbolic relative to the collection  $\mathcal{H}$ . We call a set  $W \subset X$  relatively  $K$ -quasiconvex if

- 1)  $W$  is hyperbolic relative to the collection  $\mathcal{W} = \{W \cap H : H \in \mathcal{H}\}$
- 2)  $\mathcal{E}(W, \mathcal{W})$  is  $K$ -quasiconvex in  $\mathcal{E}(X, \mathcal{H})$ .

$W \subset X$  is relatively quasiconvex if it is relatively  $K$ -quasiconvex for some  $K$ .

**Ends:**

Let  $X$  be hyperbolic rel.  $\mathcal{H}$ . Now let  $\mathcal{B} = \{B_\alpha\}, \alpha \in \Lambda$ , for some indexing set  $\Lambda$ , be a collection of uniformly relatively quasiconvex sets inside  $X$ . Here each  $B_\alpha$  is relatively quasiconvex with respect to the collection  $\{B_{\alpha\beta}\}, B_{\alpha\beta} = B_\alpha \cap H_\beta$ . The sets  $H_\beta$  are also uniformly  $D$ -separated.

**Definition 2.9.** *Let  $Y$  be hyperbolic relative to the collection  $\mathcal{H}$  and  $X$  be strongly hyperbolic with respect to a collection  $\mathcal{J}$ . A map  $i : Y \rightarrow X$  is said to be strictly type-preserving if:*

- 1) For all  $H \in \mathcal{H}$ ,  $i(H) \subset J_H$  for some  $J_H \in \mathcal{J}$  and if
- 2) For all  $J \in \mathcal{J}$ ,  $i^{-1}(J) = \emptyset$  OR  $H_J$  for some  $H_J \in \mathcal{H}$

**Definition 2.10.** *A strictly type-preserving inclusion  $i : Y \subset X$  of relatively hyperbolic metric spaces is said to be an **ends-inclusion** if*

- 1) There exist collections  $\mathcal{J}, \mathcal{H}$  such that  $X$  is hyperbolic rel.  $\mathcal{J}$  and  $Y$  is hyperbolic rel.  $\mathcal{H}$ .
- 2)  $\mathcal{B}$  is a collection of relatively quasiconvex subsets of  $Y$ .
- 3) There exists a collection of relatively hyperbolic sets  $\mathcal{F} = \{F_\alpha \subset X\}, \alpha \in \Lambda$ , (thought of as **ends** of  $X$ ) such that  $B_\alpha = F_\alpha \cap Y, \forall \alpha$  and  $X = Y \cup \{\bigcup_\alpha F_\alpha\}$ .
- 4) Each  $F_\alpha$  is strongly hyperbolic relative to a collection  $\{F_{\alpha\beta}\}$ .
- 5) If  $\mathcal{H}_0$  be the subcollection of elements  $H_\gamma \in \mathcal{H}$  such that  $H_\gamma \cap F_\alpha = \emptyset$  for all  $F_\alpha$ , then  $\mathcal{J} = \mathcal{H}_0 \bigcup_{\alpha, \beta} \{F_{\alpha\beta}\}$ .

We let  $\mathcal{H}_1 = \mathcal{H} \setminus \mathcal{H}_0$ .

**Lemma 2.11.** [Bow97] *Let  $X$  be a hyperbolic metric space and let  $\mathcal{B}$  be a collection of uniformly separated quasiconvex sets. Then  $X$  is weakly hyperbolic relative to the collection  $\mathcal{B}$ .*

Let  $X$  be a  $\delta$ -hyperbolic metric space, and  $\mathcal{B}$  a family of  $C$ -quasiconvex,  $D$ -separated, collection of subsets. Then by Lemma 2.11 (see also [Far98]),  $X_{el} = \mathcal{E}(X, \mathcal{B})$  obtained by electrocuting the subsets in  $\mathcal{B}$  is a  $\Delta = \Delta(\delta, C, D)$ -hyperbolic metric space. Now, let  $\alpha = [a, b]$  be a hyperbolic geodesic in  $X$  and  $\beta$  be an electric  $P$ -quasigeodesic without backtracking joining  $a, b$ . Replace each maximal subsegment, (with end-points  $p, q$ , say) starting from the left of  $\beta$  lying within some  $H \in \mathcal{H}$  by a hyperbolic geodesic  $[p, q]$ . The resulting **connected** path  $\beta_{ea}$  is called an *electro-ambient path representative* in  $X$ .

Note that  $\beta_{ea}$  need not be a hyperbolic quasigeodesic. However, the proof of Proposition 4.3 of Klarreich [Kla99] gives the following:

**Lemma 2.12.** (See Proposition 4.3 of [Kla99]) *Given  $\delta, C, D, P$  there exists  $C_3$  such that the following holds:*

*Let  $(X, d)$  be a  $\delta$ -hyperbolic metric space and  $\mathcal{H}$  a family of  $C$ -quasiconvex,  $D$ -separated collection of quasiconvex subsets. Let  $(X, d_e)$  denote the electric space obtained by electrocuting elements of  $\mathcal{H}$ . Then, if  $\alpha, \beta_{ea}$  denote respectively a hyperbolic geodesic and an electro-ambient  $P$ -quasigeodesic with the same end-points, then  $\alpha$  lies in a (hyperbolic)  $C_3$  neighborhood of  $\beta_{ea}$ .*

We shall describe this cryptically as:

*Hyperbolic geodesics lies hyperbolically close to Electro-ambient representatives of electric geodesics joining their end-points.*

**2.2. Cannon-Thurston Maps.** For a hyperbolic metric space  $X$ , the Gromov bordification will be denoted by  $\overline{X}$ .

**Definition 2.13.** *Let  $X$  and  $Y$  be hyperbolic metric spaces and  $i : Y \rightarrow X$  be an embedding. A **Cannon-Thurston map**  $\bar{i}$  from  $\overline{Y}$  to  $\overline{X}$  is a continuous extension of  $i$  to the Gromov bordifications  $\overline{X}$  and  $\overline{Y}$ .*

The following lemma from [Mit98] gives a necessary and sufficient condition for the existence of Cannon-Thurston maps.

**Lemma 2.14.** [Mit98] *A Cannon-Thurston map  $\bar{i}$  from  $\overline{Y}$  to  $\overline{X}$  exists for the proper embedding  $i : Y \rightarrow X$  if and only if there exists a non-negative function  $M(N)$  with  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$  such that the following holds:*

*Given  $y_0 \in Y$ , for all geodesic segments  $\lambda$  in  $Y$  lying outside an  $N$ -ball around  $y_0 \in Y$  any geodesic segment in  $X$  joining the end points of  $i(\lambda)$  lies outside the  $M(N)$ -ball around  $i(y_0) \in X$ .*

Note that due to stability of quasigeodesics, the above statement is also true if geodesics are replaced by uniform quasigeodesics.

Let  $X$  and  $Y$  be hyperbolic relative to the collections  $\mathcal{H}_X$  and  $\mathcal{H}_Y$  respectively. Let  $i : Y \rightarrow X$  be a strictly type-preserving proper embedding. Then the proper embedding  $i : Y \rightarrow X$  induces a proper embedding  $i_h : \mathcal{G}(Y, \mathcal{H}_Y) \rightarrow \mathcal{G}(X, \mathcal{H}_X)$  and a map  $\hat{i} : \hat{X} \rightarrow \hat{Y}$ .

**Definition 2.15.** *A Cannon-Thurston map is said to exist for the pair  $X, Y$  of relatively hyperbolic metric spaces and a strictly type-preserving inclusion  $i : Y \rightarrow X$  if a Cannon-Thurston map exists for the induced map  $i_h : \mathcal{G}(Y, \mathcal{H}_Y) \rightarrow \mathcal{G}(X, \mathcal{H}_X)$  between the respective hyperbolic cones.*

In [MP10] Lemma 2.14 was generalized to relatively hyperbolic metric spaces as follows.

**Lemma 2.16.** ([MP10] Lemma 1.28) *Let  $Y, X$  be hyperbolic rel.  $\mathcal{Y}, \mathcal{X}$  respectively. Let  $Y^h = \mathcal{G}(Y, \mathcal{Y}), \hat{Y} = \mathcal{E}(Y, \mathcal{Y})$  and  $X^h = \mathcal{G}(X, \mathcal{X}), \hat{X} = \mathcal{E}(X, \mathcal{X})$ . A Cannon-Thurston map for  $i : Y \rightarrow X$  exists if and only if there exists a non-negative function  $M(N)$  with  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$  such that the following holds:*

*Suppose  $y_0 \in Y$ , and  $\hat{\lambda}$  in  $\hat{Y}$  is an electric geodesic segment starting and ending outside horospheres. If  $\lambda^b = \hat{\lambda} \setminus \bigcup_{K \in \mathcal{Y}} K$  lies outside  $B_N(y_0) \subset Y$ , then for any electric quasigeodesic  $\hat{\beta}$  joining the end points of  $\hat{i}(\hat{\lambda})$  in  $\hat{X}$ ,  $\beta^b = \hat{\beta} \setminus \bigcup_{H \in \mathcal{X}} H$  lies outside  $B_{M(N)}(i(y_0)) \subset X$ .*

The above necessary and sufficient condition for existence of Cannon-Thurston map for relatively hyperbolic spaces can also be used as a definition of Cannon-Thurston map for relatively hyperbolic spaces. Hence the following definition makes sense.

**Definition 2.17.** *A collection of proper, strictly type preserving embedding  $i_\alpha : Y_\alpha \rightarrow X_\alpha$  of relatively hyperbolic spaces is said to extend to a collection of **uniform***

**Cannon-Thurston maps** if there exists  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$  such that the functions  $M_\alpha(N)$  (obtained in Lemma 2.16 above) satisfy  $M_\alpha(N) \geq M(N)$  for all  $\alpha$ .

Lemma 2.16 says that it is enough to consider only the ‘bounded’-part of the electric quasigeodesic in a relatively hyperbolic space  $X$  in order to prove existence of **CT** map, For ease of reference below, we make a definition.

**Definition 2.18.** Let  $X$  be hyperbolic rel.  $\mathcal{X}$ . If  $\sigma$  is a path in  $X$ , the **bounded part**  $\sigma^b = \sigma \setminus \bigcup_{H \in \mathcal{X}} H$ .

We shall use the notion of electro-ambient path representatives to obtain an alternate criterion for the existence of Cannon-Thurston maps in the case of an **ends-inclusion**. Combining Lemma 2.16 with Lemma 2.12 we have the following.

**Lemma 2.19.** Let  $X, Y$  be hyperbolic rel.  $\mathcal{J}, \mathcal{H}$  respectively and  $i : Y \rightarrow X$  be an ends-inclusion of relatively hyperbolic spaces. A Cannon-Thurston map for  $i : Y \hookrightarrow X$  exists if and only there exists a non-negative function  $M(N)$  with  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$  such that the following holds.

Suppose  $y \in Y$ , and  $\hat{\lambda}$  in  $\hat{Y}$  is an electric geodesic segment starting and ending outside horospheres, such that  $\lambda^b = \hat{\lambda} \setminus \bigcup_{K \in \mathcal{H}} K$ , the bounded part of  $\hat{\lambda}$  lies outside  $B_N(y) \subset Y$ .

Then for some electric quasigeodesic  $\hat{\rho}$  joining the end points of  $\hat{i}(\hat{\lambda})$  in  $\hat{X}$ , the bounded part  $\rho_{ea}^b = \hat{\rho}_{ea} \setminus \bigcup_{H \in \mathcal{J}} H$  of the electro-ambient representative  $\rho_{ea}$  (of  $\hat{\rho}$ ) lies outside  $B_{M(N)}(i(y)) \subset X$ .

### 3. REDUCTION THEOREM

#### 3.1. The Main Theorem.

**Theorem 3.1.** Let  $Y, X$  be hyperbolic relative to  $\mathcal{H}$  and  $\mathcal{J}$  respectively and  $i : Y \rightarrow X$  be an **ends-inclusion** of spaces. Let  $\mathcal{B} = \{B_\alpha\}$ ,  $B_\alpha \subset Y$  and  $\mathcal{F} = \{F_\alpha\}$ ,  $F_\alpha \subset X$ , be collections such that:

- 1)  $B_\alpha \in \mathcal{B}$  are relatively quasiconvex sets in  $Y$ , and  $B_\alpha = F_\alpha \cap Y$ ,  $\forall \alpha$ , where  $F_\alpha \in \mathcal{F}$ .  $X = Y \cup_{\alpha \in \Lambda} F_\alpha$ .
- 2) Each  $F_\alpha$  is hyperbolic relative to a collection  $\mathcal{F}_\alpha = \{F_{\alpha\beta}\}$ .
- 3)  $\mathcal{J} = \mathcal{H}_0 \cup_{\alpha \in \Lambda} \mathcal{F}_\alpha$ , where  $\mathcal{H}_0 \subset \mathcal{H}$  is the collection of  $H_\gamma$  such that  $H_\gamma \cap F_\alpha = \emptyset$ ,  $\forall \alpha$ .

Then the ends inclusion  $i : Y \rightarrow X$  extends to a Cannon-Thurston Map if the inclusions  $i_\alpha : B_\alpha \rightarrow F_\alpha$  extend **uniformly** to Cannon-Thurston maps for all  $\alpha$ .

**Proof:** Fix a base point  $y \in Y$  and consider a large enough ball  $U_N(y) \subset Y$ , Let  $\hat{\eta} \subset \hat{Y} = \mathcal{E}(Y, \mathcal{H})$  be an electric geodesic segment, starting and ending outside elements of  $\mathcal{H}$ . Let  $\eta^b$  denote the bounded part of  $\hat{\eta}$ , and assume that it lies outside  $U_N(y) \subset Y$ , i.e  $\eta^b \cap U_N(y) = \emptyset$ .

Let  $\mathcal{B}_0 = \{B_\nu \in \mathcal{B} : \eta^b \cap B_\nu \neq \emptyset \text{ and } U_N(y) \cap B_\nu \neq \emptyset\}$ . For each  $B_\nu \in \mathcal{B}_0$  let  $\eta^b(\nu) = \eta^b \cap B_\nu$ . Then  $\eta^b(\nu)$  lies outside  $U_N(y) \cap B_\nu$ . Let  $y_\nu$  be the nearest point projection of  $y$  on  $B_\nu$  in the metric  $d_Y$  of  $Y$ . Since  $F_\nu \cap Y = B_\nu$  it follows that  $y_\nu$  is also the nearest point projection of  $y$  on  $F_\nu$  in the metric  $d_X$  on  $X$ . Then  $y_\nu \in U_N(y) \cap B_\nu$ . Let  $d_Y(y, y_\nu) = R_\nu$ . Consider the ball  $U_{(N-R_\nu)}(y_\nu)$ , of radius  $N - R_\nu$  about  $y_\nu$ .  $U_{(N-R_\nu)}(y_\nu) \cap B_\nu$  is a ball in  $B_\nu$  of radius  $N - R_\nu$  based at  $y_\nu$ . We denote this ball as  $U(\nu)$ . Then  $\eta^b(\nu) \subset B_\nu \setminus U(\nu)$ .

Let  $\hat{\rho}$  be the electric geodesic in  $\hat{X} = \mathcal{E}(X, \mathcal{J})$  joining the end points of  $\hat{i}(\hat{\eta})$ . Since  $\hat{Y}$  is weakly hyperbolic rel.  $\mathcal{B}$ , it follows that  $\hat{X}$  is weakly hyperbolic rel.  $\mathcal{F}$ . Let the electroambient path representative of  $\hat{\rho}$  with respect to  $\mathcal{F}$  be  $\hat{\rho}_{ea}$ . Let  $\rho_{ea}^b = \hat{\rho}_{ea} \setminus \cup_{H \in \mathcal{J}} H$  be the bounded part of  $\hat{\rho}_{ea}$ . By Condition (1) in the hypothesis, we may assume that  $\rho_{ea}^b \setminus \cup_{F_\alpha \in \mathcal{F}} F_\alpha = \eta^b \setminus \cup_{B_\alpha \in \mathcal{B}} B_\alpha$ , i.e.  $\rho_{ea}^b$  and  $\eta^b$  coincide outside  $\mathcal{F}$ .

As per hypothesis, **CT** maps exist *uniformly* for each  $B_\nu \hookrightarrow F_\nu$ . By Lemma 2.19, there exists a function  $M(N) \rightarrow \infty$  as  $N \rightarrow \infty$  such that  $\rho_{ea}^b(\nu)$  lies outside  $U_{M(N-R_\nu)}(y_\nu)$ ,  $\forall \nu$ . It is worth noting that the function  $M(N)$  is independent of  $\nu$  by definition of uniformity.

Since  $Y$  is properly embedded in  $X$ , it follows that there exists a function  $M_1(N) \rightarrow \infty$  as  $N \rightarrow \infty$  such that if  $x, y \in Y$  and  $d_Y(x, y) \geq N$  then  $d_X(i(x), i(y)) \geq M_1(N)$ . It follows immediately that  $\rho_{ea}^b \setminus \cup_{F_\alpha \in \mathcal{F}} F_\alpha$  lies outside  $U_{M_1(N)}^X(i(y))$ .

Hence  $\rho_{ea}^b(\nu)$  lies outside  $U_{M_1(R_\nu)+M(N-R_\nu)}^X(i(y))$  – a ball of radius  $M_1(R_\nu) + M(N - R_\nu)$  in  $X$ , i.e.

$$d_X((\rho_{ea}^b(\nu), i(y))) \geq M_1(R_\nu) + M(N - R_\nu)$$

for all  $\nu$ .

Let  $M_2(N) = \inf_\nu (M_1(R_\nu) + M(N - R_\nu))$ , and  $M_3(N) = \min(M_1(N), M_2(N))$ , which is again a proper function of  $N$ , i.e.  $M_3(N) \rightarrow \infty$  as  $N \rightarrow \infty$ . This proves that  $\eta^b$  and  $\rho_{ea}^b$  satisfy the criteria of Lemma 2.19.

Hence the theorem.  $\square$

An important fact we used in the above proof is that  $\rho_{ea}^b \setminus \cup_{F_\alpha \in \mathcal{F}} F_\alpha = \eta^b \setminus \cup_{B_\alpha \in \mathcal{B}} B_\alpha$ , i.e.  $\rho_{ea}^b$  and  $\eta^b$  coincide outside  $\mathcal{F}$ . This followed from Condition 1 of Theorem 3.1.

Now let  $\partial i$  denote the Cannon-Thurston map on the boundary  $\partial \mathcal{G}(Y, \mathcal{H})$  obtained in Theorem 3.1 and suppose  $\partial i(a) = \partial i(b)$  for  $a \neq b \in \partial \mathcal{G}(Y, \mathcal{H})$  then the geodesic  $\eta = (a, b) \subset \partial \mathcal{G}(Y, \mathcal{H})$  satisfies the following:

- If  $a_n, b_n \in (a, b) \subset \mathcal{G}(Y, \mathcal{H})$  such that  $a_n \rightarrow a, b_n \rightarrow b$ ,  $\eta_n$  is the subsegment of  $\eta$  joining  $a_n, b_n$ , and  $\rho_n$  is the geodesic in  $\mathcal{G}(X, \mathcal{J})$  joining  $a_n, b_n$ , then  $d_X(i(y), \rho_n^b) \rightarrow \infty$  as  $n \rightarrow \infty$ . Also since  $\rho_n^b$  and  $\eta_n^b$  coincide outside  $\mathcal{F}$ , it follows that  $\eta$  lies wholly inside some particular  $B_\alpha \in \mathcal{B}$ .

We have thus shown:

**Corollary 3.2.** *Let  $Y, X$  be hyperbolic relative to  $\mathcal{H}$  and  $\mathcal{J}$  respectively and  $i : Y \rightarrow X$  be an **ends-inclusion** of spaces. Let  $\mathcal{B} = \{B_\alpha\}$ ,  $B_\alpha \subset Y$  and  $\mathcal{F} = \{F_\alpha\}$ ,  $F_\alpha \subset X$ , be collections such that:*

- 1)  $B_\alpha \in \mathcal{B}$  are relatively quasiconvex sets in  $Y$ , and  $B_\alpha = F_\alpha \cap Y$ ,  $\forall \alpha$ , where  $F_\alpha \in \mathcal{F}$ .  $X = Y \cup_{\alpha \in \Lambda} F_\alpha$ .
- 2) Each  $F_\alpha$  is hyperbolic relative to a collection  $\mathcal{F}_\alpha = \{F_{\alpha\beta}\}$ .
- 3)  $\mathcal{J} = \mathcal{H}_0 \cup_{\alpha \in \Lambda} \mathcal{F}_\alpha$ , where  $\mathcal{H}_0 \subset \mathcal{H}$  is the collection of  $H_\gamma$  such that  $H_\gamma \cap F_\alpha = \emptyset$ ,  $\forall \alpha$ .

*Also, let  $\partial i : \partial \mathcal{G}(Y, \mathcal{H}) \rightarrow \partial \mathcal{G}(X, \mathcal{J})$  be the induced Cannon-Thurston map on relative hyperbolic boundaries. Then  $\partial i(a) = \partial i(b)$  for  $a \neq b \in \partial \mathcal{G}(Y, \mathcal{H})$  implies that the geodesic  $(a, b) \subset \mathcal{G}(Y, \mathcal{H})$  lies in a bounded neighborhood of  $\mathcal{G}(B_\alpha, \{B_\alpha \cap H : H \in \mathcal{H}\})$  for some fixed  $B_\alpha \in \mathcal{B}$ .*

**3.2. Kleinian Groups with no Accidental Parabolics.** A first application of Theorem 3.1 is to prove the existence of Cannon-Thurston maps for pared manifolds with incompressible boundary and no accidental parabolics.

**Definition 3.3.** A **pared manifold** is a pair  $(M, P)$ , where  $P \subset \delta M$  is a (possibly empty) 2-dimensional submanifold with boundary such that

- (1) the fundamental group of each component of  $P$  injects into the fundamental group of  $M$
- (2) the fundamental group of each component of  $P$  contains an abelian subgroup with finite index.
- (3) any cylinder  $C : (S^1 \times I, \delta S^1 \times I) \rightarrow (M, P)$  such that  $\pi_1(C)$  is injective is homotopic rel boundary to  $P$ .
- (4)  $P$  contains every component of  $\delta M$  which has an abelian subgroup of finite index.

A pared manifold  $(M, P)$  is said to have **incompressible boundary** if each component of  $\partial_0 M = \partial M \setminus P$  is incompressible in  $M$ .

Further,  $(M, P)$  is said to have **no accidental parabolics** if

- (1) it has incompressible boundary
- (2) if some curve  $\sigma$  on a component of  $\partial_0 M$  is freely homotopic in  $M$  to a curve  $\alpha$  on a component of  $P$ , then  $\sigma$  is homotopic to  $\alpha$  in  $\partial M$ .

The following is the main Theorem of [Mj06].

**Theorem 3.4.** [Mj06] *Let  $\rho$  be a representation of a surface group  $H$  (corresponding to the surface  $S$ ) into  $PSL_2(C)$  without accidental parabolics. Let  $M$  denote the (convex core of)  $\mathbb{H}^3/\rho(H)$ . Further suppose that  $i : S \rightarrow M$ , taking parabolic to parabolics, induces a homotopy equivalence. Then the inclusion  $\tilde{i} : \tilde{S} \rightarrow \tilde{M}$  extends continuously to a map of the compactifications  $\bar{i} : \bar{S} \rightarrow \bar{M}$ .*

As an immediate consequence of Theorems 3.1 and 3.4 we have the following.

**Theorem 3.5.** *Suppose that  $N^h \in H(M, P)$  is a hyperbolic structure on a pared manifold  $(M, P)$  with no accidental parabolics. Let  $M_{gf}$  denotes a geometrically finite hyperbolic structure adapted to  $(M, P)$ , then the map  $i : \widetilde{M}_{gf} \rightarrow \widetilde{N}^h$  extends continuously to the boundary  $\bar{i} : \overline{M}_{gf} \rightarrow \overline{N}^h$ .*

In [Mj07] and [DM10] we also identify the point pre-images of the Cannon-Thurston map.

**Theorem 3.6.** [Mj07] [DM10] *Let  $G$  be a surface Kleinian group without accidental parabolics. Then the Cannon-Thurston map from the (relative) hyperbolic boundary of  $G$  to its limit set identifies precisely the end-points of leaves of the ending laminations.*

Hence combining Theorem 3.5, Corollary 3.2 and Theorem 3.6 we get:

**Theorem 3.7.** *Suppose that  $N^h \in H(M, P)$  is a hyperbolic structure on a pared manifold  $(M, P)$  with no accidental parabolics. Let  $M_{gf}$  denotes a geometrically finite hyperbolic structure adapted to  $(M, P)$ . Let  $\bar{i} : \overline{M}_{gf} \rightarrow \overline{N}^h$  be the Cannon-Thurston map extending  $i : \widetilde{M}_{gf} \rightarrow \widetilde{N}^h$ . Then  $\bar{i}$  identifies precisely the end-points of leaves of the ending laminations.*

**3.3. Accidental Parabolics.** The last step in proving the main Theorem 3.11 of this paper is to remove the restriction on accidental parabolics from Theorem 3.5. This step is dealt with in Section 5.4 of [Mj09]. The argument in Section 5.4 of [Mj09] is independent of the hypothesis of ‘bounded geometry of ends’ used in that paper. We sketch it for completeness.

**Definition 3.8.** A cusp of  $M_{gf}$  is said to be **exceptional** if there exist closed curves carried by the cusp (i.e. lying on its boundary horocycle or horosphere) which are homotopic to non-peripheral curves on some other boundary component of  $M_{gf}$ .

**Exceptional horoballs** are lifts of exceptional cusps.

A geodesic is said to **penetrate a horoball  $H$  by at most  $D$**  if any subsegment of it lying inside  $H$  has length less than or equal to  $D$ .

We will use the following notation in this section to make it consistent with [Mj09]:

- $\lambda^h$  = Hyperbolic geodesic in  $\widetilde{M}_{gf}$  lying outside  $U_N(y) \subset \widetilde{M}_{gf}$ .
- $\beta^h$  = Hyperbolic geodesic in  $\widetilde{N}^h$  joining the endpoints of  $\lambda^h$ .

With this notation we have the following.

**Proposition 3.9.** (Corollary 5.10 of [Mj09]) Suppose  $(M, P)$  has incompressible boundary  $\partial_0 P$ . Given  $D, n$  there exist  $m(n, D)$  such that the following holds:

Let  $\lambda^h, \beta^h$  be as above. If  $\lambda^h$  penetrates exceptional horoballs by at most  $D$  and if  $\lambda^h$  lies outside  $B_n(p)$  in  $\widetilde{M}_{gf}$  then  $\beta^h$  lies outside a ball of radius  $m(n, D)$ , for some function  $m(n, D)$  where  $m(n, D) \rightarrow \infty$  as  $n \rightarrow \infty$  for each fixed  $D$ .

We also recall the following Lemma that follows essentially from [Gro85] p.187, Prop. 7.2C or [GdlH90].

**Lemma 3.10.** (Proposition 5.11 of [Mj09]) There exist  $D, K, \epsilon$  such that the following holds:

Suppose  $\beta^x$  is a path in  $\widetilde{N}^h$  such that  $\beta^x$  can be decomposed into finitely many geodesic segments  $\beta_1, \dots, \beta_k$ . Further suppose that the starting and ending point of each  $\beta_i$  lie on exceptional horospheres (except possibly the starting point of  $\beta_1$  and the ending-point of  $\beta_k$ ). Also suppose that the ‘even segments’  $\beta_{2i}$  lie entirely within exceptional horoballs and have length greater than  $D$ . Then  $\beta^x$  is a  $(K, \epsilon)$  quasigeodesic.

As in [Mj09] (Section 5.4 p. 240), break  $\lambda^h$  into pieces  $\lambda_1^h, \lambda_2^h, \dots$  such that  $\lambda_{2i}^h$  are the maximal subsegments of  $\lambda^h$  lying inside exceptional horoballs and having length greater than some large enough  $D_0$ . Let  $\lambda_{2i-1}^h$  denote the complementary segments. Now, let  $\overline{\beta_{2i-1}^h}$  be the hyperbolic geodesic in  $\widetilde{N}^h$  joining the endpoints of  $\lambda_{2i-1}^h$ . Then the entry point of  $\overline{\beta_{2i-1}^h}$  into the *exceptional horoball  $\mathbf{H}$*  it terminates on, lies at a distance greater than  $D_1 = D_1(D_0)$  from the exit point of  $\overline{\beta_{2i+1}^h}$  from the same horoball  $\mathbf{H}$ . Shorten the hyperbolic geodesic  $\overline{\beta_{2i-1}^h}$  if necessary by cutting it off at the entry point into  $\mathbf{H}$ . Let  $\beta_{2i-1}^h$  denote the resultant geodesic. By Corollary 3.9  $\beta_{2i-1}^h$  being a subsegment of  $\overline{\beta_{2i-1}^h}$  lies at a distance of at least  $M_1(N)$  from the reference point  $i(y)$  for some proper function  $M_1(N)$  of  $N$ .

Next denote by  $\beta_{2i}^h$  the hyperbolic geodesic lying entirely within  $\mathbf{H}$  joining the entry point of  $\beta_{2i-1}^h$  into  $\mathbf{H}$  to the exit point of  $\beta_{2i+1}^h$  from  $\mathbf{H}$ . The initial and terminal points of  $\beta_{2i}^h$  lying on  $\mathbf{H}$  are at a distance of at least  $M_1(N)$  from  $i(y)$ . Therefore each  $\beta_{2i}^h$  and hence the union of all the  $\beta_i^h$  lie outside an  $M_2(N)$  ball about  $p$  where  $M_2(N) \rightarrow \infty$  as  $N \rightarrow \infty$ . Further, the union of the segments  $\beta_i^h$  is a hyperbolic quasigeodesic by Lemma 3.10 and hence lies at a bounded distance  $D'$  from the hyperbolic geodesic  $\beta^h$  joining the endpoints of  $\lambda$  (i.e. the end-points of  $\lambda^h$ ). Let  $M_3(N) = M_2(N) - D'$ . We have thus shown:

- If  $\lambda^h$  lies outside  $U_N(y) \subset \widetilde{M}_{gf}$ , then  $\beta^h$  lies outside a ball of radius  $M_3(N)$  about  $i(y) \in \widetilde{N}^h$ .
- $M_3(N) \rightarrow \infty$  as  $N \rightarrow \infty$

Coupled with Lemma 2.16 this proves the main theorem of this paper given below:

**Theorem 3.11.** *Suppose that  $N^h \in H(M, P)$  is a hyperbolic structure on a pared manifold  $(M, P)$  with incompressible boundary  $\partial_0 M$ . Let  $M_{gf}$  denotes a geometrically finite hyperbolic structure adapted to  $(M, P)$ . Then the map  $i : \widetilde{M}_{gf} \rightarrow \widetilde{N}^h$  extends continuously to the boundary  $\hat{i} : \widehat{M}_{gf} \rightarrow \widehat{N}^h$ . Further, the Cannon-Thurston map  $\partial i$  identifies precisely the end-points of leaves of the ending lamination.*

The last statement follows from the structure of the Cannon-Thurston map given by Theorem 3.7.

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