

# A SPLITTING THEOREM FOR GOOD COMPLEXIFICATIONS

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ABSTRACT. The purpose of this paper is to produce restrictions on fundamental groups of manifolds admitting good complexifications by proving the following Cheeger-Gromoll type splitting theorem: Any closed manifold  $M$  admitting a good complexification has a finite-sheeted regular covering  $M_1$  such that  $M_1$  admits a fiber bundle structure with base  $(S^1)^k$  and fiber  $N$  that admits a good complexification and also has zero virtual first Betti number. We give several applications to manifolds of dimension at most 5.

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## 1. INTRODUCTION

A good complexification of a closed smooth manifold  $M$  is defined to be a smooth affine algebraic variety  $U$  over the real numbers such that  $M$  is diffeomorphic to  $U(\mathbb{R})$  and the inclusion  $U(\mathbb{R}) \rightarrow U(\mathbb{C})$  is a homotopy equivalence [To], [Ku]. A good complexification comes naturally equipped with a natural antiholomorphic involution  $A$  on  $U(\mathbb{C})$  whose fixed point set is precisely the set of real points  $U(\mathbb{R})$ . Kulkarni [Ku] and Totaro [To] investigate the topology of good complexifications using characteristic classes and Euler characteristic. In this paper we prove a Cheeger-Gromoll type splitting theorem and initiate a systematic study of fundamental groups of good complexifications.

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In [To, p. 69, 2nd para], Totaro asks the following question:

**Question 1.1.** If a closed smooth manifold  $M$  admits a good complexification, does  $M$  also admit a metric of non-negative curvature?

The following Cheeger–Gromoll type splitting theorem is the main result proved here.

**Theorem 1.2** (See Theorem 2.8). *Let  $M$  be a closed manifold admitting a good complexification. Then  $M$  has a finite-sheeted regular covering  $M_1$  satisfying the following:*

- (1)  $M_1$  admits a fiber bundle structure with fiber  $N$  and base  $(S^1)^d$ . Here  $d$  denotes the (real) Albanese dimension of  $M_1$ .
- (2) The first virtual Betti number  $vb_1(N) = 0$ ,
- (3)  $N$  admits a good complexification.

Let us spell out the analogy with the Cheeger–Gromoll splitting theorem [Pe, Theorem 69, page 288]: A closed manifold of non-negative curvature has a finite sheeted cover of the form  $N \times (S^1)^d$ , where  $N$  is simply connected. Theorem 1.2 similarly furnishes a fibering over a torus  $(S^1)^d$  with fiber  $N$  having  $vb_1(N) = 0$ .

Gromov [Gr81] proves that if a closed smooth manifold  $M$  of dimension  $n$  admits a metric of non-negative curvature, then there is an upper bound, that depends only on  $n$ , on the sum of the Betti numbers of  $M$ . He further conjectures, [Gr14, Section 7], that  $b_i(M) \leq b_i((S^1)^n)$ . Theorem 1.3 below furnishes positive evidence towards a combination of Question 1.1 with this conjecture of Gromov by giving an affirmative answer for the first Betti number of manifolds admitting good complexifications.

We shall say that a finitely presented group  $G$  is a **good complexification group** if  $G$  can be realized as the fundamental group of a closed smooth manifold admitting a good complexification (see also [ABCKT]). We deduce from Theorem 1.2 the following critical restriction on good complexification groups (See Theorem 3.1):

**Theorem 1.3.** *Let  $G$  be a good complexification group. Then there exists a finite index subgroup  $G_1$  of  $G$  such that two following statements hold:*

- (1) *There is an exact sequence:*

$$1 \longrightarrow H \longrightarrow G_1 \longrightarrow \mathbb{Z}^k \longrightarrow 1,$$

*where  $k$  can be zero.*

- (2) *The above  $H$  is a finitely presented good complexification group with  $vb_1(H) = 0$ , where  $vb_1(H)$  denotes the virtual first Betti number of  $H$ .*

Recall that for a group  $H$ , the **virtual first Betti number**  $vb_1(H)$  is the supremum of first Betti numbers  $b_1(H_1)$  as  $H_1$  runs over finite index subgroups of  $H$ .

The following classes of groups are then ruled out as good complexification groups (See Section 3, especially Corollary 3.2):

- (1) Groups with infinite  $vb_1$ , in particular large groups.
- (2) Hyperbolic CAT(0) cubulated groups.
- (3) Solvable groups that are not virtually abelian.
- (4) 2- and 3-manifold groups that are not virtually abelian.
- (5) any group admitting a surjection onto any of the above.

Question 1.1 has an affirmative answer for 2-manifolds; this is probably classical but follows also from [Ku, To]. An affirmative answer to Question 1.1 for 3-manifolds was given in [BiMj]. As a consequence of the above restrictions, Question 1.1 has an affirmative answer for 2 and 3-manifold (see Sections 4.1 and 4.2). Thus a new self-contained proof of the main Theorem of [BiMj] on good complexifications is obtained. In Section 4 we give a number of applications to low-dimensional manifolds.

**Theorem 1.4.**

- (1) *Question 1.1 has an affirmative answer for 2-manifolds.*

- (2) *Question 1.1 has an affirmative answer for 3-manifolds [BiMj].*
- (3) *Let  $M$  be a closed simply connected 4-manifold admitting a symplectic good complexification. Then  $M$  admits a metric of non-negative curvature.*
- (4) *Let  $M$  be a closed 4-manifold admitting a good complexification. Further suppose that  $\pi_1(M)$  is infinite, torsion-free abelian. Then  $\pi_1(M)$  is isomorphic to  $\mathbb{Z}^d$ , where  $d = 1, 2$  or  $4$ . Moreover, the manifold  $M$  admits a finite-sheeted cover with a metric of non-negative curvature (i.e., Question 1.1 has an affirmative answer up to finite-sheeted covering).*
- (5) *Let  $M$  be a closed 5-manifold admitting a good complexification. Further suppose that  $\pi_1(M)$  is infinite, torsion-free abelian. Then  $\pi_1(M)$  is isomorphic to  $\mathbb{Z}^d$ , where  $d = 1, 2, 3$  or  $5$ . Further, if  $d = 2, 3$  or  $5$ , then  $M$  admits a finite-sheeted cover  $M_1$  homeomorphic to  $S^2 \times T^3$  or  $S^3 \times T^2$  or  $T^5$ . In particular,  $M_1$  admits a metric of non-negative curvature.*

For an algebraic variety  $X$ , we shall denote the real (respectively, complex) points by  $X_{\mathbb{R}}$  (respectively,  $X_{\mathbb{C}}$ ).

**Definition 1.5.** We say that an algebraic map  $f : X \rightarrow Y$  between good complexifications is a **good complexification map** if there is a commutative diagram

$$(1.1) \quad \begin{array}{ccc} X_{\mathbb{R}} & \subset & X_{\mathbb{C}} \\ \downarrow f_{\mathbb{R}} & & \downarrow f_{\mathbb{C}} \\ Y_{\mathbb{R}} & \subset & Y_{\mathbb{C}} \end{array}$$

such that  $f_{\mathbb{C}}$  is equivariant with respect to the antiholomorphic involutions  $A_X, A_Y$ , meaning  $f_{\mathbb{C}} \circ A_X = A_Y \circ f_{\mathbb{C}}$ .

## 2. FIBRATIONS AND THE ALBANESE

We need to first introduce some terminology following [Ti]. Let  $f : X \rightarrow \mathbb{C}^*$  be an algebraic map of quasiprojective varieties. This  $f$  is said to be topologically trivial at  $z \in \mathbb{C}^*$  if there is a neighborhood  $D$  of  $z$  such that the restriction of  $f$  from  $f^{-1}(D)$  to  $D$  is a topologically trivial fibration. If  $z$  does not satisfy this property, then we say that  $z$  is an **atypical value** and that  $f^{-1}(z)$  is an **atypical fiber**. Clearly, critical values (respectively, critical fibers) are atypical values (respectively, atypical fibers). The book [Ti] gives examples of atypical values that are not critical.

**2.1. Fibration over  $\mathbb{C}^*$ .** The punctured plane  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  is the standard good complexification of the unit circle  $S^1$  with the antiholomorphic involution given by  $z \rightarrow \frac{1}{\bar{z}}$ . In this section we study good complexification maps  $f_{\mathbb{C}} : X \rightarrow \mathbb{C}^*$  restricting to  $f_{\mathbb{R}} := f_{\mathbb{C}}|_{X_{\mathbb{R}}} : X_{\mathbb{R}} \rightarrow S^1$  and prove that such maps have to be topologically trivial fibrations.

**Theorem 2.1.** *Let  $X$  be a good complexification of  $M$ , and let  $\mathbb{C}^*$  be the standard good complexification of  $S^1$ . Let  $f : X \rightarrow \mathbb{C}^*$  be a good complexification map between the good complexifications of  $M$  and  $S^1$ . The corresponding map between real loci*

$$f_{\mathbb{R}} := f|_M : M \rightarrow S^1$$

*is a smooth fiber bundle. If  $N$  denotes the fiber of  $f_{\mathbb{R}}$ , then  $N$  admits a good complexification.*

*Proof.* The proof will be divided into a number of steps. We fix some notation first. The map  $f$  will also be denoted by  $f_{\mathbb{C}}$ . The restriction of  $f$  to  $f^{-1}(S^1) \subset X_{\mathbb{C}}$  will be denoted as  $\hat{f}$ .

In what follows a *small perturbation* will refer to a small perturbation in the real algebraic category.

**Step 1:  $f_{\mathbb{R}}$  has no critical points after small perturbation:**

First, note that since  $b_1(M) = b_1(X) \neq 0$ , it follows from [Ku, To] that the Euler characteristic  $\chi(M) = \chi(X) = 0$ . Next, assume for the time being, that  $f$  can be perturbed to a function  $g$  with isolated quadratic (i.e., Morse-type) singularities in a small tubular neighborhood  $X_M$  of  $M = X(\mathbb{R})$  such that the image of  $g$  is an annular neighborhood  $A_{\mathbb{C}}$  of  $S^1$ . Let  $F$  denote the general fiber of  $g$ .

Then the Euler characteristics of  $F$ ,  $X_M$  and  $A_{\mathbb{C}}$  are related by the following formula due to Suzuki [Su] (Suzuki emphasizes dimension 2, though the formula works in the general context of Stein manifolds and Stein morphisms):

$$\chi(X_M) = \chi(F) \cdot \chi(A_{\mathbb{C}}) + \sum_{x \in A_{\mathbb{C}}} (\chi(F_x) - \chi(F)),$$

where  $F_x$  denotes the fiber of  $g$  over  $x$ . Let  $c_x$  be the defect in the Euler characteristic given by  $(\chi(F_x) - \chi(F))$ . Since  $g$  is defined in the neighborhood of  $M$ , the only exceptional fibers of  $g$  are critical (i.e. atypical non-critical fibers do not exist). Let  $c_i$  denote the finitely many non-zero defects. Each isolated quadratic critical point contributes a cell of a fixed real dimension  $m$  (one more than the complex dimension of the fiber). Hence all the  $c_i$ 's are either  $+1$  or  $-1$  depending on the parity of  $m$ . Since  $\chi(X) = 0 = \chi(\mathbb{C}^*)$ , it follows from the Euler characteristic formula above that  $g$  has no critical points.

It remains to show that  $f$  can indeed be perturbed to a function  $g$  with isolated quadratic (i.e., Morse-type) singularities locally. Since  $f$  is algebraic onto  $\mathbb{C}^*$ , it has finitely many critical and atypical values. Restricted to a small tubular neighborhood of  $M$ ,  $f$  has only critical points. Let  $A_{\mathbb{C}}$  denote a small annular neighborhood of  $S^1$  containing the corresponding critical values. Any small perturbation of the annulus  $A_{\mathbb{C}}$  is again an annulus (unlike  $\mathbb{C}^*$ , which can be perturbed slightly to  $\mathbb{C}$ ). We note that  $f$  is an affine and hence a Stein morphism, and  $A_{\mathbb{C}}$  is Stein.  $X_M$  is also Stein. Since  $f$  is a Stein morphism between Stein manifolds with only critical points, it can be perturbed slightly to a Stein morphism with isolated quadratic singularities, with all singular values in  $A_{\mathbb{C}}$ .

**Step 2: Existence of critical points of  $f_{\mathbb{C}}$  is stable under perturbation; hence  $f_{\mathbb{R}}$  has no critical points:**

Further, if  $f_{\mathbb{C}}$  has critical points inside  $X_M$ , then any small perturbation of  $f_{\mathbb{C}}$  must also have critical points. This is clear for isolated critical points. In general, the set of critical points  $Z(f_{\mathbb{C}})$  is a variety and taking a local (algebraic) section  $\sigma$  of  $f_{\mathbb{C}}$  with the image of  $\sigma$  lying in a neighborhood of  $Z(f_{\mathbb{C}})$ , we can ensure that the intersection  $Z(f_{\mathbb{C}}) \cap \sigma$  is an isolated point. Restricting  $f_{\mathbb{C}}$  to  $Z(f_{\mathbb{C}}) \cap \sigma$  we observe that a small perturbation of  $f_{\mathbb{C}}$  must have critical points in  $Z(f_{\mathbb{C}}) \cap \sigma$ .

Since a small perturbation  $g$  of  $f$  has no critical points inside  $X_M$ , it follows that  $f = f_{\mathbb{C}}$  has no critical points inside  $X_M$ . If  $f_{\mathbb{R}}$  has critical points, so does  $f_{\mathbb{C}}$  inside  $X_M$ . It follows that  $f_{\mathbb{R}}$  has no critical points.

**Step 3:  $f_{\mathbb{R}}$  is a smooth fiber bundle map:**

We get as a direct consequence of Step 2 and Ehresmann's fibration theorem that  $f_{\mathbb{R}}$  is a smooth fiber bundle map.

**Step 4:  $f_{\mathbb{C}}$  is a smooth homotopy fibration:**

In view of Steps 1, 2 and 3, we may, and we will, assume that  $f_{\mathbb{R}}$  has no critical points. It remains to handle the atypical (including critical) values. We assume that the atypical values  $z_1, \dots, z_k$  of  $f_{\mathbb{C}}$  are isolated points in  $\mathbb{C}^*$ . Let  $D_1, \dots, D_k$  be small (disjoint) analytic neighborhoods of  $z_1, \dots, z_k$  respectively. We join  $D_i$  by non-intersecting simple arcs  $\alpha_i$  to  $S^1$  and define

$$K := S^1 \bigcup_{i=1}^k \alpha_i \bigcup_{i=1}^k D_i$$

and  $W := f^{-1}(K)$ . If some  $z_i \in S^1$ , we assume that  $\alpha_i$  is the constant arc at  $z_i$ . Then  $X_{\mathbb{C}}$  deformation retracts onto  $W$ , and hence  $W$  has the same homotopy type as  $X_{\mathbb{R}}$  (since  $X$  is a good complexification).

The relative homotopy type of  $(W, f^{-1}(S^1))$  is then given by the local topology changes at the critical fibers  $f^{-1}(z_i)$  (cf. [Lam]). We shall now need to combine the above observation with the conclusion of Step 2.

Let  $F_{\mathbb{R}}$  denote the fiber of  $f_{\mathbb{R}} : X_{\mathbb{R}} \rightarrow S^1$ , and let  $F_{\mathbb{C}}$  denote the general fiber of  $f_{\mathbb{C}}$ . Then  $F_{\mathbb{C}}$  is also the general fiber of  $\hat{f}$ . Pass to the cover  $\tilde{X}_{\mathbb{C}}$  of  $X_{\mathbb{C}}$  corresponding to  $\pi_1(F_{\mathbb{R}})$  (so  $\tilde{X}_{\mathbb{C}}$  is the fiber product of  $X_{\mathbb{C}}$  with the universal cover of  $\mathbb{C}^*$ ), and let

$$P : \tilde{X}_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$$

denote the covering. Then  $P^{-1}(X_{\mathbb{R}})$  is homeomorphic to  $F_{\mathbb{R}} \times \mathbb{R}$ . We note that  $f_{\mathbb{C}}$  lifts to a map

$$\widetilde{f}_{\mathbb{C}} : \widetilde{X}_{\mathbb{C}} \longrightarrow \mathbb{C},$$

where  $\mathbb{C}$  is identified with the universal cover of  $\mathbb{C}^*$ , and  $\widetilde{f}_{\mathbb{C}}$  restricts to a map  $\widetilde{f}_{\mathbb{R}} : \widetilde{X}_{\mathbb{R}} \longrightarrow \mathbb{R}$ , where  $\widetilde{X}_{\mathbb{R}} = P^{-1}(X_{\mathbb{R}})$ , and  $\mathbb{R}$  is identified with the universal cover of  $S^1$ .

Fix fibers  $F_{\mathbb{R}} \subset F_{\mathbb{C}} \subset \widetilde{X}_{\mathbb{C}}$  as above. Then the quotient space  $\widetilde{X}_{\mathbb{C}}/F_{\mathbb{R}}$  is contractible by the definition of a good complexification. Further,  $F_{\mathbb{C}}$  has the homotopy type of a finite  $CW$  complex  $H$  because it is a quasiprojective variety. Without loss of generality, we may assume that  $H$  is a compact subset of  $F_{\mathbb{C}}$ . Let  $F$  denote the (compact finite) quotient complex  $H/F_{\mathbb{R}} \subset \widetilde{X}_{\mathbb{C}}/F_{\mathbb{R}}$ . Each  $z_i \in \mathbb{C}^*$  lifts to a countably infinite collection of atypical points in  $\mathbb{C}$ . Hence (in its cell complex structure)  $\widetilde{X}_{\mathbb{C}}/F_{\mathbb{R}}$  has the same homotopy type as  $H/F_{\mathbb{R}}$  with infinitely many cells attached, one for every lift of  $z_i$ ,  $i = 1, \dots, k$ . Since  $\widetilde{X}_{\mathbb{C}}/F_{\mathbb{R}}$  is contractible, it follows that

- (1)  $k = 0$ , and
- (2)  $H/F_{\mathbb{R}}$  is contractible, in particular,  $F_{\mathbb{C}}$  has the same homotopy type as  $F_{\mathbb{R}}$ .

We summarize the conclusion of Step 4 as follows:

The map  $f_{\mathbb{C}}$  is a smooth homotopy fibration over  $\mathbb{C}^*$ , where all fibers are smooth manifolds of the same homotopy type.

**Step 5: The fiber of  $f_{\mathbb{R}}$  admits a good complexification:**

Let  $N$  denote a general fiber of  $f_{\mathbb{R}}$ . The antiholomorphic involution  $A$  preserves the fibers  $f_{\mathbb{C}}^{-1}(p)$  for  $p \in S^1$ . Let  $X_p = f_{\mathbb{C}}^{-1}(p)$  and  $N_p = f_{\mathbb{R}}^{-1}(p)$ . Then  $N_p$  is diffeomorphic to  $N$  for  $p \in S^1$ . It suffices to show that  $N_p$  is homotopy equivalent to  $X_p$ .

Now, by Step 4, the map  $f_{\mathbb{C}}$  is a smooth homotopy fibration over  $\mathbb{C}^*$ , where all fibers are smooth manifolds of the same homotopy type as  $X_p$ . Hence we get the following commutative diagram of homotopy exact sequences; the vertical arrows in the diagram are maps induced by inclusion:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & \pi_i(M) & \longrightarrow & \pi_i(S^1) & \longrightarrow & \pi_{i-1}(N) & \longrightarrow & \pi_{i-1}(M) & \longrightarrow & \pi_{i-1}(S^1) & \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \pi_i(X) & \longrightarrow & \pi_i(\mathbb{C}^*) & \longrightarrow & \pi_{i-1}(X_p) & \longrightarrow & \pi_{i-1}(X) & \longrightarrow & \pi_{i-1}(\mathbb{C}^*) & \longrightarrow \cdots \end{array}$$

Since the homomorphism  $\pi_i(S^1) \longrightarrow \pi_i(\mathbb{C}^*)$  is an isomorphism for all  $i$  (in fact they vanish for  $i > 1$ ), and since the homomorphism  $\pi_i(M) \longrightarrow \pi_i(X)$  is an isomorphism by the definition of a good complexification, it follows that the inclusion of  $N$  into  $X_p$  induces isomorphisms  $\pi_i(N) \longrightarrow \pi_i(X_p)$  for all  $i > 1$  (by the 5-Lemma, using the fact that higher homotopy groups are abelian). Note further that the homomorphism  $\pi_1(N) \longrightarrow \pi_1(X_p)$  coincides with the identifications of the kernels of  $\pi_1(M) \longrightarrow \pi_1(S^1)$  and  $\pi_1(X) \longrightarrow \pi_1(\mathbb{C}^*)$ . Also,  $\pi_0(N) \longrightarrow \pi_0(X_p)$  is an isomorphism. Hence by the Whitehead Theorem, the inclusion of  $N$  into  $X_p$  is a homotopy equivalence. This concludes Step 6 and proves the Theorem.  $\square$

We point out a couple of consequences of the above proof of Theorem 2.1.

By Step 3, the map  $f_{\mathbb{R}}$  has no critical points, and by Step 6,  $f$  has no critical points in  $f^{-1}(S^1)$ . Further, by Step 6, the fiber  $F_{\mathbb{C}}$  has the same homotopy type as  $F_{\mathbb{R}}$ . Thus we have the following:

**Corollary 2.2.** *Let  $f : X \longrightarrow \mathbb{C}^*$  be a good complexification map between good complexifications  $X, \mathbb{C}^*$ , and denote the antiholomorphic involution of  $X$  by  $A_X$ . Then*

- $A_X(f^{-1}(p)) = f^{-1}(p)$  for all  $p \in S^1$ , and
- $A_X(f^{-1}(p))$  is a good complexification of  $f^{-1}(p) \cap X_{\mathbb{R}}$ .

*Proof.* The only point to note is that  $A_X$  preserves the fibers  $f^{-1}(p)$ . This in turn follows from the condition  $f \circ A_X = A_X \circ f$  in the definition of a good complexification map.  $\square$

Combining Theorem 2.1 and Corollary 2.2, we obtain the following:

**Corollary 2.3.** *Let  $M$  be a closed smooth manifold admitting a good complexification. Then there exists a good complexification map  $f : X \rightarrow \mathbb{C}^*$  between good complexifications  $X$  (of  $M$ ) and  $\mathbb{C}^*$  (of  $S^1$ ) such that  $f$  is a smooth homotopy fibration. Further,*

- (1)  $M$  admits a smooth fibration over  $S^1$  with fiber  $N$  such that  $f_{\mathbb{R}}$  is a projection onto the base  $S^1$ , and
- (2)  $X$  admits a homotopy fibration over  $\mathbb{C}^*$  with generic fiber  $Y$  such that  $f$  is a projection onto  $\mathbb{C}^*$  and  $Y$  is a good complexification of  $N$ .

**2.2. The Albanese.** We refer the reader to [Se], [It], [SpSz] for quasi-Albanese maps (see also [Fu]), and to [NWy] for details on the semiabelian varieties (also called quasi-abelian varieties). For a smooth quasiprojective variety  $Z$ , the quasi-Albanese map

$$(2.1) \quad a_Z : Z \rightarrow \text{Alb}(Z)$$

is constructed by choosing a point  $z_0 \in Z$  (the map on the degree zero part of  $\text{CH}_0(Z)$  does not depend on the choice of the point  $z_0$ ). We will replace the quasi-Albanese  $\text{Alb}(Z)$  by a torsor for it so that the construction of the map from  $Z$  does not require choosing a base point. Introduce the following equivalence relation on  $Z \times \text{Alb}(Z)$ :

$$(z_1, \alpha_1) \equiv (z_2, \alpha_2) \quad \text{if} \quad a_Z(z_1 - z_2) = \alpha_2 - \alpha_1;$$

as mentioned above, the map  $a_X$  on the degree zero part of  $\text{CH}_0(Z)$  does not depend on the choice of base point, so  $a_Z(z_1 - z_2)$  does not depend on  $z_0$ . The corresponding quotient  $\text{Alb}(Z)^1$  of  $Z \times \text{Alb}(Z)$  is a torsor for  $\text{Alb}(Z)$ . Now consider the map

$$(2.2) \quad Z \rightarrow \text{Alb}(Z)^1, \quad x \mapsto (z_0, a_Z(x)),$$

where  $a_Z$  is the map in (2.1) constructed using  $z_0$ . It is straight-forward check that this map does not depend on the choice of  $z_0$ . Henceforth, by the quasi-Albanese of  $Z$  we will mean  $\text{Alb}(Z)^1$  constructed above, and by quasi-Albanese map for  $Z$  we will mean the map in (2.2).

If  $Z$  is defined over  $\mathbb{R}$ , then both  $\text{Alb}(Z)^1$  and  $a_Z$  in (2.2) are also defined over  $\mathbb{R}$ .

**Lemma 2.4.** *Let  $X_{\mathbb{C}}$  be a good complexification of  $X_{\mathbb{R}}$ . Then the quasi-Albanese of  $X_{\mathbb{C}}$  is  $(\mathbb{C}^*)^n$ , where  $n = b_1(X_{\mathbb{C}})$ .*

*Proof.* Let  $W_{\mathbb{C}}$  denote the quasi-Albanese of  $X_{\mathbb{C}}$ , and let

$$\alpha : X_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$$

be the Albanese map. The variety  $W_{\mathbb{C}}$  being defined over  $\mathbb{R}$ , is equipped with an antiholomorphic involution  $A_a$ , and  $\alpha$  intertwines the antiholomorphic involutions of  $X_{\mathbb{C}}$  and  $W_{\mathbb{C}}$ , or in other words, we have

$$(2.3) \quad A_a \circ \alpha = \alpha \circ A_X.$$

Let  $W_{\mathbb{R}} \subset W_{\mathbb{C}}$  denote the fixed-point locus of the antiholomorphic involution. We have  $\alpha(W_{\mathbb{C}}) \subset W_{\mathbb{R}}$ .

Since the Albanese  $W_{\mathbb{C}}$  is a semi-abelian variety, there is an exact sequence

$$1 \rightarrow (\mathbb{C}^*)^m \rightarrow W_{\mathbb{C}} \rightarrow Q \rightarrow 1,$$

where  $Q$  is an abelian variety.

The antiholomorphic complex conjugation  $A_a$  on  $W_{\mathbb{C}}$  descends to an antiholomorphic complex conjugation  $A_q$  on  $Q$  whose fixed point set has dimension  $\dim_{\mathbb{C}}(Q)$  (note that the fixed point set is nonempty because the identity element is fixed). Hence  $\alpha(X_{\mathbb{R}})$  has dimension at most  $m + \dim_{\mathbb{C}}(Q)$  which is less than  $\dim_{\mathbb{C}}(W_{\mathbb{C}})$  unless  $\dim_{\mathbb{C}}(Q) = 0$ . Since  $\dim_{\mathbb{C}}(W_{\mathbb{C}}) = \dim_{\mathbb{R}}(W_{\mathbb{R}})$ , it follows that  $\dim_{\mathbb{C}}(Q) = 0$ . Hence the quasi-Albanese of  $X_{\mathbb{C}}$  is  $(\mathbb{C}^*)^n$ , where  $n = b_1(X_{\mathbb{C}})$ .  $\square$

**Corollary 2.5.** *Let  $X_{\mathbb{C}}$  be a good complexification of  $X_{\mathbb{R}}$ . Let  $\alpha$  denote the Albanese map. Let*

$$\Pi : (\mathbb{C}^*)^n \rightarrow \mathbb{C}^*$$

*be a projection of the Albanese  $W_{\mathbb{C}} = (\mathbb{C}^*)^n$  onto one of the factors. Then  $\Pi \circ \alpha$  is a good complexification map.*

*Proof.* The quasi-Albanese of  $X_{\mathbb{C}}$  is  $(\mathbb{C}^*)^n$  by Lemma 2.4, and  $\Pi$  is clearly a good complexification map. Therefore, from (2.3) It follows that  $\Pi \circ \alpha$  intertwines the antiholomorphic involution  $A_X$  on  $X_{\mathbb{C}}$  and the standard antiholomorphic involution  $A_{\mathbb{C}^*}$  on  $\mathbb{C}^*$ . Consequently,  $\Pi \circ \alpha$  is a good complexification map.  $\square$

**2.3. Albanese Fibrations.** The next lemma is a generalization of Case 3 B of Proposition 4.3 of [BiMj].

**Lemma 2.6.** *Let  $f : X \rightarrow \mathbb{C}^*$  be a fibration that is a good complexification map between good complexifications  $X, \mathbb{C}^*$  such that the general fiber is  $(\mathbb{C}^*)^m$ . Then  $f$  is a trivial fibration.*

*Proof.* First note that  $X$  may be regarded as a principal  $(\mathbb{C}^*)^m$ -bundle over  $\mathbb{C}^*$ . Let  $V$  denote the vector bundle on  $\mathbb{C}^*$  of rank  $m$  associated to the principal  $(\mathbb{C}^*)^m$ -bundle  $X$  for the natural action of  $(\mathbb{C}^*)^m$  on  $\mathbb{C}^{\oplus m}$ . So  $V$  splits as a direct sum of  $m$  line bundles  $L_i$ , i.e.,  $V = \bigoplus_{i=1}^m L_i$ . It suffices to show that  $L_i$  is a trivial line bundle or equivalently that the first Chern class  $c_1(L_i) = 0$  for all  $i$ .

Take any algebraic line bundle  $L$  over  $\mathbb{C}^*$ . The line bundle  $L$  extends to an algebraic line bundle over the  $\mathbb{P}^1$ . To see this, take the image in  $\mathbb{P}^1$  of any divisor in  $\mathbb{C}^*$  representing  $L$  by the homomorphism defined by the inclusion map  $\iota : \mathbb{C}^* \hookrightarrow \mathbb{P}^1$ . Let  $L' \rightarrow \mathbb{P}^1$  be an extension of  $L$ . Therefore,  $c_1(L) = \iota^* c_1(L')$ . But

$$\iota^*(H^2(\mathbb{P}^1, \mathbb{Z})) = 0.$$

Therefore, we conclude that  $c_1(L) = 0$ .  $\square$

Essentially the same proof works when the base  $\mathbb{C}^*$  is replaced by  $(\mathbb{C}^*)^n$ . The only change is that we choose  $(\mathbb{P}^1)^n$  to be the compactification of  $(\mathbb{C}^*)^n$ . This gives us the following:

**Corollary 2.7.** *Let  $f : X \rightarrow (\mathbb{C}^*)^n$  be a fibration that is a good complexification map between good complexifications  $X, (\mathbb{C}^*)^n$  such that the general fiber is  $(\mathbb{C}^*)^m$ . Then  $f$  is a trivial fibration.*

We shall need the notion of a relative Albanese in the following. Let  $f : X \rightarrow Y$  be a fibration. Then the relative Albanese is a fibration  $f_{\alpha} : X_{\alpha} \rightarrow Y$ , where the fiber  $f_{\alpha}^{-1}(p)$  is the Albanese of  $f^{-1}(p)$  for  $p \in Y$ . Note that the relative Albanese is a bundle of torsors, with canonically defined fibers, and hence in our case, a principal  $(\mathbb{C}^*)^k$ -bundle for some integer  $k \geq 0$  (multiplication is well-defined in local trivializations and the fact that the fibers are canonically defined shows that multiplication agrees on overlaps). The following is the main technical tool of this paper.

**Theorem 2.8.** *Let  $M$  be a closed manifold admitting a good complexification. Then there exists a good complexification  $X_{\mathbb{C}}$  of  $M = X_{\mathbb{R}}$ , such that the following holds:*

*Let  $W_{\mathbb{C}}$  be the quasi-Albanese of  $X_{\mathbb{C}}$ , and let  $\alpha : X_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$  be the quasi-Albanese map. Then  $\alpha$  is a homotopy smooth fibration with no critical points. Further, if  $F$  denote the general fiber, then*

- $b_1(F) = 0$ , and
- $F$  is a good complexification of the intersection  $F \cap X_{\mathbb{R}}$ .

*Proof.* If  $W_{\mathbb{C}}$  is zero dimensional, then there is nothing to prove, so assume that  $\dim W_{\mathbb{C}} \geq 1$ .

We now proceed by induction on  $b_1(X_{\mathbb{C}})$ . By Lemma 2.4, we have  $W_{\mathbb{C}} = (\mathbb{C}^*)^n$ . Let  $\Pi : (\mathbb{C}^*)^n \rightarrow \mathbb{C}^*$  be a projection of the Albanese  $W_{\mathbb{C}} = (\mathbb{C}^*)^n$  onto one of the factors. Then  $\Pi \circ \alpha$  is a good complexification map by Corollary 2.5.

Hence, by Theorem 2.1, the composition is a homotopy smooth fibration with no critical points. Further, by Corollary 2.3, the fiber of  $\Pi \circ \alpha$  over any point of  $S^1$  is a good complexifications of the intersection of the fiber with  $X_{\mathbb{R}}$ .

Then  $\Pi \circ \alpha : X_{\mathbb{C}} \rightarrow \mathbb{C}^*$  induces a good complexification map with fiber  $F_z$  (say) over  $z$ . Let  $V_z$  denote the Albanese of the fiber  $F_z = (\Pi \circ \alpha)^{-1}(z)$ , and let

$$\widehat{\Pi \circ \alpha} : V \rightarrow \mathbb{C}^*$$

be the bundle  $\{V_z \mid z \in \mathbb{C}^*\}$ . For any such fibration, we have  $b_1(V_z) \geq (b_1(X_{\mathbb{C}}) - 1)$ .

By Theorem 2.1, the above map  $\widehat{\Pi \circ \alpha}$  defines a homotopy smooth fibration and the fibers  $V_z$  are of the form  $(\mathbb{C}^*)^k$  by Lemma 2.4, where  $k = b_1(V_z)$ . Since  $V$  is a trivial bundle by Lemma 2.6, we have  $(k+1) \leq b_1(X_{\mathbb{C}})$ . It follows that  $k = (b_1(X_{\mathbb{C}}) - 1)$ .

By induction on the number of factors of  $\mathbb{C}^*$  it follows that  $F_z \rightarrow V_z$  is a homotopy smooth fibration with no critical points. Hence  $\alpha$  is a homotopy smooth fibration with no critical points.

Let  $F$  denote the general fiber of  $\alpha$ . It remains to show that  $b_1(F) = 0$ . As above, pass to the relative Albanese bundle  $W_F$  of Albanese tori. Then  $W_F$  is a trivial smooth fibration over  $W_{\mathbb{C}} = (\mathbb{C}^*)^n$  by Lemma 2.7 and hence has  $b_1(W_F) > n$  unless  $b_1(F) = 0$ . Further, since  $W_F$  is a semi-abelian variety, we have  $b_1(W_F) \leq n$ . Therefore, it now follows that  $b_1(F) = 0$ .

That  $F$  is itself a good complexification now follows from Corollary 2.3 by induction the number of factors of  $\mathbb{C}^*$ .  $\square$

The virtual first Betti number  $vb_1(M)$  is defined to be the supremum of the first Betti numbers of finite sheeted covers of  $M$ . It is known, [BiMj, Lemma 4.1], that if  $M$  admits a good complexification and  $M_1$  is a finite-sheeted cover of  $M$ , then  $M_1$  admits a good complexification. We have the following immediate Corollary of Theorem 2.8:

**Corollary 2.9.** *Suppose  $M$  is a closed  $n$ -dimensional manifold admitting a good complexification. Then  $vb_1(M) \leq n$ .*

*Proof.* Let  $M_1$  be a finite-sheeted cover of  $M$ . Then  $M_1$  admits a good complexification [BiMj, Lemma 4.1]. Let  $X$  be a good complexification of  $M_1$ . By Theorem 2.8 this  $X$  admits a smooth homotopy fibration with no critical points over the Albanese  $W$  of  $X$ . In particular, we have

$$n = \dim_{\mathbb{C}}(X) \geq b_1(X) = b_1(M_1).$$

Since  $M_1$  is arbitrary, the corollary follows.  $\square$

### 3. GOOD COMPLEXIFICATION GROUPS

**Theorem 3.1.** *Let  $G$  be a good complexification group. Then there exists a finite index subgroup  $G_1$  of  $G$  such that*

- (1) *There is an exact sequence:*

$$1 \longrightarrow H \longrightarrow G_1 \longrightarrow \mathbb{Z}^k \longrightarrow 1,$$

*where  $k$  can be zero.*

- (2)  *$H$  is a finitely presented good complexification group with  $vb_1(H) = 0$ , where  $vb_1(H)$  denotes the virtual first Betti number of  $H$ .*

*Proof.* Let  $M$  be a closed smooth  $n$ -manifold admitting a good complexification with  $\pi_1(M) = G$ . By Corollary 2.9, there is a finite sheeted cover  $M_1$  of  $M$  such that

$$vb_1(M) = b_1(M_1) = m \leq n.$$

Also by Theorem 2.8,  $M_1$  fibers over  $(S^1)^m$  with fiber a closed  $(n-m)$ -manifold  $N$ . Further  $N$  admits a good complexification by Theorem 2.8. Hence  $vb_1(N) = 0$ ; indeed, if  $vb_1(N) \neq 0$ , by applying Theorem 2.8 again,  $vb_1(M) = vb_1(M_1) > m$ . Defining  $H = \pi_1(N)$  and  $G_1 = \pi_1(M_1)$ , we obtain the following homotopy long exact sequence:

$$1 \longrightarrow HG_1 \longrightarrow \mathbb{Z}^{k'}.$$

Hence the theorem follows.  $\square$

**Corollary 3.2.** *The following are not good complexification groups:*

- (1) *Groups with infinite  $vb_1$ , in particular, large groups.*
- (2) *Hyperbolic CAT(0) cubulated groups; in particular, infinite hyperbolic Coxeter groups.*
- (3) *Solvable groups that are not virtually abelian.*
- (4) *2-manifold groups that are not virtually abelian.*
- (5) *3-manifold groups that are not virtually abelian.*



*Proof.* (1) Groups with infinite  $vb_1$  cannot be good complexification groups by Theorem 3.1. Recall that  $G$  is large if  $G$  has a finite index subgroup that surjects onto the free group  $F_2$  of two generators [BiMj]. Clearly, large groups have infinite  $vb_1$ .

(2) We refer the reader to [Gr85, Fa, Gr93] for generalities on hyperbolic and relatively hyperbolic groups, to [Sa, Wi] for generalities on CAT(0) cube-complexes and their properties and to [HW] for the theory of virtually special complexes.

A hyperbolic CAT(0) cubulated group  $G$  is virtually special and hence has infinite  $vb_1(G)$  [Ag, Wi]. We are thus reduced to Case (1) above.

The second statement follows from [HW] where the authors show that infinite Coxeter groups are CAT(0) cubulated virtually special.

Non-elementary hyperbolic, or relatively hyperbolic, groups have asymptotic cones with cut points. On the other hand, a good complexification group with positive  $vb_1$  must be virtually of the form  $G_1 = \mathbb{Z}^k \times H$  by Theorem 3.1. Such a group has as asymptotic cone

$$AC(G) = AC(G_1) = \mathbb{R}^k \times AC(H),$$

where  $AC(H)$  is an asymptotic cone of  $H$ , and  $k \geq 1$ . Hence  $AC(G)$  can have a cut-point if and only if  $H$  is finite and  $k = 1$ , which means that  $G$  is virtually cyclic.

(3) A solvable group  $G$  that is not virtually abelian admits, up to finite index, an exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow \mathbb{Z}^k \longrightarrow 1,$$

where  $k$  is maximal, and  $b_1(N) > 0$ . This contradicts Theorem 3.1. Note that we avoided using [ArNo] in this context.

(4) Since 2-manifold groups  $G$  that are not virtually abelian have  $vb_1(G) = \infty$ , this follows from Case 1 above.

(5) By work of a large number of people culminating in the work of Agol and Wise [Ag, Wi] (see [AFW], especially Diagram 1 in page 36 for details), a 3-manifold group  $G = \pi_1(M^3)$  is large unless  $G$  is virtually solvable, or equivalently,  $M^3$  has geometry modelled on one of  $S^3$ ,  $E^3$ ,  $S^2 \times \mathbb{R}$ ,  $Nil$  and  $Sol$ . Now, by Case 2 above, the group  $G$  must be virtually abelian. Hence  $M$  must have geometry modelled on one of  $S^3$ ,  $E^3$  and  $S^2 \times \mathbb{R}$ .  $\square$

**Remark 3.3.**

- (1) Case 3 of Corollary 3.2 implies that any unipotent representation of a good complexification group is virtually abelian.
- (2) It also follows from Theorem 3.1 that the commutator subgroup of a finite index subgroup of a good complexification group is finitely presented.
- (3) Several cases of Corollary 3.2 above may be strengthened by saying that the groups listed there cannot even appear as quotients of good complexification groups. The proof involves applying Theorem 3.1 as done in Corollary 3.2.

4. EXAMPLES AND APPLICATIONS

In this Section, we apply Theorem 2.8 to furnish conclusions for low-dimensional examples.

4.1. 2 Manifolds.

**Lemma 4.1.** *Question 1.1 has an affirmative answer for 2-manifolds.*

*Proof.* By Case 4 of Corollary 3.2, the only 2-manifolds that can possibly admit good complexifications are those finite covered by the sphere  $S^2$  or the torus  $S^1 \times S^1$ . That all such manifolds do admit metrics of non-negative curvature is classical. Hence Question 1.1 has an affirmative answer for 2-manifolds.  $\square$

#### 4.2. 3 Manifolds.

**Theorem 4.2** ([BiMj]). *Question 1.1 has an affirmative answer for 3-manifolds.*

*Proof.* By Case 5 of Corollary 3.2, the only 3-manifolds that can possibly admit good complexifications are those finite covered by  $S^3$ ,  $S^2 \times S^1$  or  $S^1 \times S^1 \times S^1$ . That all such manifolds do admit metrics of non-negative curvature is a consequence of the Geometrization Theorem of Perelman.  $\square$

**4.3. Simply Connected 4-manifolds.** A stronger notion of good complexification appears implicitly in the work of McLean [Mc]. There, the author observes first that any smooth complex affine variety  $X$  has a natural structure of a symplectic manifold (up to symplectomorphisms) when regarded as a real smooth manifold. The cotangent bundle  $T^*M$  of a closed manifold  $M$  carries a natural symplectic structure (it is called the Liouville symplectic form). Corollary 1.3 of [Mc] implies that if  $X$  is *symplectomorphic* to  $T^*M$  for some closed manifold  $M$  (in particular,  $M$  admits a good complexification in the sense of [To] (recalled in Section 1)), then  $M$  is rationally elliptic. We say that  $M$  admits a **symplectic good complexification** if  $T^*M$  with its natural symplectic structure is symplectomorphic to a smooth complex affine variety  $X$  with its natural underlying symplectic structure.

**Proposition 4.3.** *Let  $M$  be a closed simply connected 4-manifold admitting a symplectic good complexification. Then  $M$  admits a metric of non-negative curvature (i.e., Question 1.1 has an affirmative answer).*

*Proof.* From [Mc, Corollary 1.3], the manifold  $M$  is rationally elliptic. Hence  $b_2(M) \leq 2$ . By the classification of simply connected 4-manifolds,  $M$  is homeomorphic to one of the following:

$$S^4, \mathbb{C}P^2, \mathbb{C}P^2 \# \mathbb{C}P^2, \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, S^2 \times S^2.$$

That each of these admits a metric of non-negative curvature follows from the examples constructed by Totaro in [To].  $\square$

#### 4.4. 4 manifolds with fundamental group $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ .

We shall need the following theorem.

**Theorem 4.4** (Smale, Hatcher).

- (1)  $\text{Diff}(S^2)$  is homotopy equivalent to  $\text{SO}(3)$  [Sm].
- (2)  $\text{Diff}(S^3)$  is homotopy equivalent to  $\text{SO}(4)$  [Ha].

**Corollary 4.5.**

- (a) An  $S^2$  bundle over  $T^2$  is trivial after (at most) passing to a double cover of  $T^2$ .
- (b) An  $S^2$  bundle over  $T^3$  is trivial after (at most) passing to a finite-sheeted cover of  $T^3$ .
- (c) An  $S^3$  bundle over  $T^2$  is trivial after (at most) passing to a double cover of  $T^2$ .
- (d) An  $S^3$  bundle over  $T^3$  is trivial after (at most) passing to a finite-sheeted cover of  $T^3$ .

*Proof.*

(a) The collection of distinct  $S^2$ -bundles over  $T^2$  is in bijective correspondence with homotopy classes of maps  $[T^2, \text{BDiff}(S^2)]$ . By the first statement of Theorem 4.4 this is equivalent to  $[T^2, \text{BSO}(3)]$ . By the homotopy long exact sequence,  $\pi_n(\text{BSO}(3)) = \pi_{n-1}(\text{SO}(3))$ . Since  $\text{BSO}(3)$  is simply connected, it follows that (after equipping  $T^2$  with the standard CW complex structure consisting of one 0-cell, two 1-cells and one 2-cell) any map from  $T^2$  to  $\text{BSO}(3)$  induces a map from  $S^2$  to  $\text{BSO}(3)$ , where  $S^2$  is the quotient of  $T^2$  obtained by collapsing its one skeleton to a point. Hence  $[T^2, \text{BDiff}(S^2)] = \pi_2(\text{BSO}(3)) = \pi_1(\text{SO}(3)) = \mathbb{Z}/2\mathbb{Z}$ . It follows that after passing to a double cover of  $T^2$  if necessary, the pullback  $S^2$  bundle becomes trivial.

(b) Repeating the argument for (a) above, we find that any  $S^2$  bundle over  $T^3$  can be made trivial over the 2-skeleton after passing to a finite-sheeted cover. Hence any map from  $T^3$  to  $\text{BSO}(3)$  induces a map from  $S^3$  to  $\text{BSO}(3)$ , where  $S^3$  is the quotient of  $T^3$  obtained by collapsing its two skeleton to a point. Hence  $[T^3, \text{BDiff}(S^2)] = \pi_3(\text{BSO}(3)) = \pi_2(\text{SO}(3)) = 0$ . It follows that after passing to a finite-sheeted cover of  $T^3$  if necessary, the pullback  $S^2$  bundle becomes trivial.

(c) We repeat the argument in (a), replacing  $\text{BSO}(3)$  with  $\text{BSO}(4)$  (and using the second statement of Theorem 4.4) to obtain  $[T^2, \text{BDiff}(S^3)] = \pi_2(\text{BSO}(4)) = \pi_1(\text{SO}(4)) = \mathbb{Z}/2\mathbb{Z}$ . Here the last statement follows from the fact that  $\text{SO}(4)$  is diffeomorphic to  $\text{SO}(3) \times S^3$ . The rest of the argument is identical to that in (a).

(d) We repeat the argument in (b) to obtain  $[T^3, \text{BDiff}(S^3)] = \pi_3(\text{BSO}(4)) = \pi_2(\text{SO}(4)) = 0$ . The Corollary follows.  $\square$

**Proposition 4.6.** *Let  $M$  be a closed 4-manifold admitting a good complexification. Further suppose that  $\pi_1(M)$  is infinite, torsion-free abelian. Then  $\pi_1(M)$  is isomorphic to  $\mathbb{Z}^d$ , where  $d = 1, 2$  or  $4$ . Further, the manifold  $M$  admits a finite-sheeted cover with a metric of non-negative curvature (i.e., Question 1.1 has an affirmative answer up to finite-sheeted covering).*

*Proof.* By Theorem 2.8, the 4-manifold  $M$  admits a fiber bundle structure over  $(S^1)^d$ , where  $1 \leq d \leq 4$  (by Corollary 2.9). Let  $N$  denote the fiber. Then  $b_1(N) = 0$  (by Theorem 2.8 again). Hence  $N$  cannot be one-dimensional, and hence  $d = 1, 2$  or  $4$ .

If  $d = 4$ , then  $M$  is diffeomorphic to  $(S^1)^4$ .

If  $d = 2$ , then  $M$  fibers over  $(S^1)^2$  with fiber  $N$  a 2-manifold admitting a good complexification and having  $b_1(N) = 0$ . Hence  $M$  is an  $S^2$ -bundle over  $T^2$  and is finitely covered by  $S^2 \times T^2$ . Clearly,  $S^2 \times T^2$  admits a metric of non-negative curvature.

If  $d = 1$ , then  $M$  fibers over  $S^1$  with fiber  $N$  a 3-manifold admitting a good complexification and having  $b_1(N) = 0$ . By Theorem 4.2,  $N$  must be finitely covered by  $(S^1)^3$  or  $S^1 \times S^2$  or  $S^3$ . Hence after passing to a finite-sheeted cover if necessary,  $M$  is a fiber bundle over  $B$  with fiber  $N_1$ , where  $B, N_1$  satisfy one of the following (Corollary 4.5):

- (1)  $B = (S^1)^4$ , and  $N_1$  is a point. In this case,  $M$  admits a finite-sheeted cover diffeomorphic to  $(S^1)^4$ .
- (2)  $B = (S^1)^2$ , and  $N_1 = S^2$ . In this case,  $M$  admits a finite-sheeted cover diffeomorphic to  $(S^1)^2 \times S^2$  as for the case  $d = 2$  above.
- (3)  $B = S^1$ , and  $N_1 = S^3$ . In this case,  $M$  admits a finite-sheeted cover diffeomorphic to  $S^1 \times S^3$ .

The Theorem follows.  $\square$

**4.5. 5 Manifolds.** The argument for Proposition 4.6 coupled with Corollary 4.5 furnishes the following fact for 5-manifolds as well.

**Proposition 4.7.** *Let  $M$  be a closed 5-manifold admitting a good complexification. Further suppose that  $\pi_1(M)$  is infinite, torsion-free abelian. Then  $\pi_1(M)$  is isomorphic to  $\mathbb{Z}^d$ , where  $d = 1, 2, 3$  or  $5$ . Further, if  $d = 2, 3$  or  $5$ , then  $M$  admits a finite-sheeted cover  $M_1$  homeomorphic to  $S^2 \times T^3$  or  $S^3 \times T^2$  or  $T^5$ . In particular,  $M_1$  admits a metric of non-negative curvature.*

We do not repeat the proof of Proposition 4.6 here but observe that (after passing to a finite-sheeted cover),  $d = 2$  corresponds to an  $S^3$ -bundle over  $T^2$ ,  $d = 3$  corresponds to an  $S^2$ -bundle over  $T^3$ ,  $d = 5$  corresponds to  $T^5$ .

**4.6. Branched Covers.**

**Proposition 4.8.** *Let  $f : X \rightarrow Y$  be a good complexification map between good complexifications  $X, Y$  such that  $f$  is a branched cover. Then  $f$  is actually étale.*

*Proof.* The branch locus of  $f_{\mathbb{C}}$  has to be of complex codimension one in  $Y_{\mathbb{C}}$ . Hence the branch locus  $B_{\mathbb{R}} \subset Y_{\mathbb{R}}$  of  $f_{\mathbb{R}}$  is of real codimension one. If  $B_{\mathbb{R}} \neq \emptyset$  (or equivalently, if  $f_{\mathbb{R}}$  is not étale) then  $X_{\mathbb{R}}$  cannot be a manifold because the local model of  $X_{\mathbb{R}}$  around a preimage of a point in  $B_{\mathbb{R}}$  will consist of more than two half-spaces glued along a codimension one hyperplane. It follows that  $B_{\mathbb{R}} = \emptyset$  and hence  $f_{\mathbb{R}}$  is étale.

It remains to show that  $f_{\mathbb{C}}$  is étale. First, since  $f_{\mathbb{R}}$  is étale, and  $Y_{\mathbb{R}} \subset Y_{\mathbb{C}}$  is a homotopy equivalence, we can construct a new good complexification  $X_{\mathbb{C}}^1$  of  $X_{\mathbb{R}}$  by simply taking  $X_{\mathbb{C}}^1$  to be the cover of  $Y_{\mathbb{C}}$  corresponding to  $f_{\mathbb{R},*}(\pi_1(X_{\mathbb{R}}))$ . Hence by lifting the map  $f_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$  to  $X_{\mathbb{C}}^1$ , we have an algebraic homotopy equivalence  $h$  between  $X_{\mathbb{C}}$  and  $X_{\mathbb{C}}^1$  which is the identity on  $X_{\mathbb{R}}$ . Since  $X_{\mathbb{R}}$  is Zariski-dense in  $X_{\mathbb{C}}$ ,  $h$  must be the identity map. i.e.,  $X_{\mathbb{C}} = X_{\mathbb{C}}^1$ .  $\square$

**4.7. Questions.** In order to answer Question 1.1 affirmatively, it suffices by Theorems 2.8, 3.1, to answer the following questions affirmatively:

**Question 4.9.**

- (a) Suppose that the virtual first Betti number of a closed manifold  $M$  is zero:  $vb_1(M) = 0$ . If  $M$  admits a good complexification, then does  $M$  admit a metric of non-negative curvature?
- (b) If  $M$  is a fiber bundle over  $(S^1)^d$  admitting a good complexification, is the bundle trivial (at least up to finite sheeted covers)?

We divide Question 4.9 (a) into two further questions:

**Question 4.10.**

- (a) Suppose that  $G$  is a good complexification group with  $vb_1(G) = 0$ , is  $G$  finite?
- (b) If a closed manifold  $M$  with finite fundamental group admits a good complexification, then does  $M$  admit a metric, at least virtually, of non-negative curvature?

**Remark 4.11.** An affirmative answer to both parts of Questions 4.9 and 4.10 exist if and only if Question 1.1 has an affirmative answer.

To see this, note that if  $M$  admits a metric of non-negative curvature, then the universal cover splits (by the Cheeger-Gromoll splitting Theorem [Pe, Theorem 69, page 288]) as a metric product  $N \times \mathbb{R}^k$  and hence  $G = \pi_1(M)$  is virtually  $\mathbb{Z}^k$ .

Conversely, let  $G = \pi_1(M)$  such that  $M$  admits a good complexification. By Theorem 3.1,  $G$  is virtually of the form  $H \rtimes \mathbb{Z}^k$ , where  $H = \pi_1(N)$  where  $N$  admits a good complexification and  $vb_1(N) = 0$ . By a positive answer to Question 4.10(a),  $H$  must be finite. Hence  $N$  admits a finite simply connected cover  $N_1$ , which in turn admits a good complexification [BiMj, Lemma 4.1]. Hence  $N_1$  admits a metric of non-negative curvature by a positive answer to Question 4.10(b). Hence  $N_1 \times (S^1)^k$  admits a metric of non-negative curvature. Consequently, by a positive answer to Question 4.9(b),  $M$  admits a finite sheeted cover with a metric of non-negative curvature.

Since every finite group is a subgroup of some  $SU(n)$ , every finite group is a good complexification group [To], and also the fundamental group of a closed manifold of non-negative curvature.

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