Sums of reciprocals of fractional parts and applications to Diophantine approximation: Part 1

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Joint work with Victor Beresnevich and Alan Haynes (York)
The sums of interest

We investigate the sums

\[ S_N(\alpha, \gamma) := \sum_{n=1}^{N} \frac{1}{n \| n\alpha - \gamma \|} \quad \text{and} \quad R_N(\alpha, \gamma) := \sum_{n=1}^{N} \frac{1}{\| n\alpha - \gamma \|} , \]

where \( \alpha \) and \( \gamma \) are real parameters and \( \| \cdot \| \) is the distance to the nearest integer. The sums are related (via partial summation):

\[ S_N(\alpha, \gamma) = \sum_{n=1}^{N} \frac{R_n(\alpha, \gamma)}{n(n+1)} + \frac{R_N(\alpha, \gamma)}{N+1} . \]
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Schmidt (1964): for any \( \gamma \in \mathbb{R} \) and for any \( \varepsilon > 0 \)

\[ (\log N)^2 \ll S_N(\alpha, \gamma) \ll (\log N)^{2+\varepsilon}, \]

for almost all \( \alpha \in \mathbb{R} \).
Schmidt (1964): for any $\gamma \in \mathbb{R}$ and for any $\varepsilon > 0$

$$(\log N)^2 \ll S_N(\alpha, \gamma) \ll (\log N)^{2+\varepsilon}, \quad (1)$$

for almost all $\alpha \in \mathbb{R}$. In the homogeneous case ($\gamma = 0$), easy to see that the $\varepsilon$ term in (1) can be removed if $\alpha$ is badly approximable.
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for almost all $\alpha \in \mathbb{R}$. In the homogeneous case ($\gamma = 0$), easy to see that the $\varepsilon$ term in (1) can be removed if $\alpha$ is badly approximable.

We show that when $\gamma = 0$, the l.h.s. (1) is true for all irrationals while the r.h.s. (1) is true with $\varepsilon = 0$ for a.a. irrationals.

More precisely:
Theorem. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then for $N \geq N_0$

$$\frac{1}{2} (\log N)^2 \leq \forall S_N(\alpha, 0) := \sum_{n=1}^{N} \frac{1}{n \|n\alpha\|} \leq 34 (\log N)^2.$$ 

In fact, the upper bound is valid for any $\alpha := [a_1, a_2, \ldots]$ such that

$$A_k(\alpha) := \sum_{i=1}^{k} a_i = o(k^2).$$

(Diamond + Vaaler: For a.a. $\alpha$, for $k$ sufficiently large $A_k \leq k^{1+\varepsilon}$.)
Homogeneous results: $R_N(\alpha, 0)$

**Theorem.** Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then for $N \geq N_0$

$$N \log N \ll R_N(\alpha, 0) := \sum_{n=1}^{N} \frac{1}{\|n\alpha\|}.$$ 

The fact that above is valid for any irrational $\alpha$ is crucial for the applications in mind. (Independently: Lê + Vaaler)
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Hardy + Wright: $R_N(\alpha, 0) \ll N \log N$ for badly approximable $\alpha$. In general, not even true a.a. Indeed:

$$ N \log N \log \log N \quad a.a. \quad R_N(\alpha, 0) \quad a.a. \quad N \log N \left(\log \log N\right)^{1+\epsilon}. $$

Now to some inhomogeneous statements.
Theorem. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then for $N \geq N_0$

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Now to some inhomogeneous statements.
Theorem. For each $\gamma \in \mathbb{R}$ there exists a set $A_\gamma \subset \mathbb{R}$ of full measure such that for all $\alpha \in A_\gamma$ and all sufficiently large $N$

$$S_N(\alpha, \gamma) := \sum_{1 \leq n \leq N} \frac{1}{n\|n\alpha - \gamma\|} \ll (\log N)^2.$$ 

The result removes the ‘epsilon’ term in Schmidt’s upper bound.
Inhomogeneous results: a taster

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The result removes the ‘epsilon’ term in Schmidt’s upper bound.

**Theorem.** Let $\alpha \in \mathbb{R} \setminus (\mathcal{L} \cup \mathbb{Q})$. Then, for all sufficiently large $N$

and any $\gamma \in \mathbb{R}$

\[
S_N(\alpha, \gamma) \gg (\log N)^2.
\]

Schmidt’s lower bound is a.a. and depends on $\gamma$. The above is for all irrationals except possibly for Liouville numbers $\mathcal{L}$.
In the lower bound result for $S_N(\alpha, \gamma)$ we are not sure if we need to exclude Liouville numbers. However, it is necessary when dealing with $R_N(\alpha, \gamma)$.

**Theorem.** Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then, $\alpha \notin \mathcal{L}$ if and only if for any $\gamma \in \mathbb{R},$

$$R_N(\alpha, \gamma) := \sum_{1 \leq n \leq N} \frac{1}{\|n\alpha - \gamma\|} \gg N \log N \quad \text{for } N \geq 2.$$
The counting function:  \( \#\{1 \leq n \leq N : \|n\alpha - \gamma\| < \varepsilon \} \)

Related to the sums, given \( \alpha, \gamma \in \mathbb{R} \), \( N \in \mathbb{N} \) and \( \varepsilon > 0 \), we consider the cardinality of

\[
N_\gamma(\alpha, \varepsilon) := \{n \in \mathbb{N} : \|n\alpha - \gamma\| < \varepsilon, \ n \leq N\}.
\]

Observing that in the homogeneous case, when \( \varepsilon N \geq 1 \), Minkowski’s Theorem for convex bodies, implies that

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\#N(\alpha, \varepsilon) := \#N_0(\alpha, \varepsilon) \geq \lfloor \varepsilon N \rfloor.
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Under which conditions can this bound can be reversed?
The homogeneous counting results

**Theorem.** Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let $(q_k)_{k \geq 0}$ be the sequence of denominators of the convergents of $\alpha$. Let $N \in \mathbb{N}$ and $\varepsilon > 0$ such that $0 < 2\varepsilon < \|q_2\alpha\|$. Suppose that

$$\frac{1}{2\varepsilon} \leq q_k \leq N$$

for some integer $k$. Then

$$\lfloor \varepsilon N \rfloor \leq \#N(\alpha, \varepsilon) \leq 32 \varepsilon N.$$  \hspace{1cm} (2)
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**Theorem.** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) and let \((q_k)_{k \geq 0}\) be the sequence of denominators of the convergents of \( \alpha \). Let \( N \in \mathbb{N} \) and \( \varepsilon > 0 \) such that \( 0 < 2\varepsilon < \|q_2\alpha\| \). Suppose that \( \frac{1}{2\varepsilon} \leq q_k \leq N \) for some integer \( k \). Then

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\]

In terms of the Diophantine exponent \( \tau(\alpha) \): Let \( \alpha \not\in \mathcal{L} \cup \mathbb{Q} \) and let \( \nu \in \mathbb{R} \) satisfy

\[
0 < \nu < \frac{1}{\tau(\alpha)}.
\]

Then, \( \exists \varepsilon_0 = \varepsilon_0(\alpha) > 0 \) such that for any sufficiently large \( N \) and any \( \varepsilon \) with \( N^{-\nu} < \varepsilon < \varepsilon_0 \), estimate (2) is satisfied.
The inhomogeneous counting results

Estimates for $\#N_\gamma(\alpha, \varepsilon)$ are obtained from the homogenous results via the following.

**Theorem.** For any $\varepsilon > 0$ and $N \in \mathbb{N}$, we have that

$$\#N_\gamma(\alpha, \varepsilon) \leq \#N(\alpha, 2\varepsilon) + 1.$$ 

If $N'_\gamma(\alpha, \varepsilon') \neq \emptyset$, where $N' := \frac{1}{2}N$ and $\varepsilon' := \frac{1}{2}\varepsilon$, then

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Estimates for \( \#N_\gamma(\alpha, \varepsilon) \) are obtained from the homogenous results via the following.

**Theorem.** For any \( \varepsilon > 0 \) and \( N \in \mathbb{N} \), we have that

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If \( N_\gamma'(\alpha, \varepsilon') \neq \emptyset \), where \( N' := \frac{1}{2}N \) and \( \varepsilon' := \frac{1}{2}\varepsilon \), then

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\#N_\gamma(\alpha, \varepsilon) \geq \#N'(\alpha, \varepsilon') + 1.
\]

**UPSHOT:** If \( \#\{1 \leq n \leq N/2 : \|n\alpha - \gamma\| < \varepsilon/2\} > 0 \), then under the conditions of the homogeneous results

\[
\left\lfloor \frac{1}{4}\varepsilon N \right\rfloor \leq \#N_\gamma(\alpha, \varepsilon) \leq 64\varepsilon N + 1.
\]
Ostrowski expansion of real numbers: Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then, for every $n \in \mathbb{N}$ there is a unique integer $K \geq 0$ such that

$$q_K \leq n < q_{K+1},$$

and a unique sequence $\{c_{k+1}\}_{k=0}^{\infty}$ of integers such that

$$n = \sum_{k=0}^{\infty} c_{k+1} q_k, \quad (3)$$

$$0 \leq c_1 < a_1 \quad \text{and} \quad 0 \leq c_{k+1} \leq a_{k+1} \quad \forall \ k \geq 1,$$

$$c_k = 0 \quad \text{whenever} \quad c_{k+1} = a_{k+1} \quad \forall \ k \geq 1,$$

$$c_{k+1} = 0 \quad \forall \ k > K.$$
Let $\alpha \in [0, 1) \setminus \mathbb{Q}$, $n \in \mathbb{N}$ and, let $m$ be the smallest integer such that $c_{m+1} \neq 0$ in the Ostrowski expansion of $n$. If $m \geq 2$, then

$$\| n\alpha \| = \left| \sum_{k=m}^{\infty} c_{k+1} D_k \right| \quad \left( D_k := q_k \alpha - p_k \ (k \geq 0) \right)$$

In particular

$$(c_{m+1} - 1)|D_m| \leq \| n\alpha \| \leq (c_{m+1} + 1)|D_m|.$$  (4)
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In particular

$$(c_{m+1} - 1)|D_m| \leq \|n\alpha\| \leq (c_{m+1} + 1)|D_m|.$$ \hspace{1cm} (4)

Since $\frac{1}{2} \leq q_{k+1}|D_k| \leq 1$, it follows that

$$\frac{1}{2}(c_{m+1} - 1) \leq q_{k+1}\|n\alpha\| \leq (c_{m+1} + 1).$$
Why should we care?

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- They have elegant applications to metrical Diophantine approximation; in particular the multiplicative theory.
Khintchine’s Theorem: 1-dimensional

Let $I := [0, 1]$, $\psi : \mathbb{N} \to \mathbb{R}^+$ be a real positive function and

$$W(\psi) := \{x \in I : \|qx\| \leq \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}$$

– the set of $\psi$-well approximable numbers.

**Khintchine’s Theorem (1924)** If $\psi$ is monotonic, then

$$m(W(\psi)) = \begin{cases} 
0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty, \\
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- Divergence part requires monotonicity.

- Put $\psi(q) = \frac{1}{q \log q}$. Divergent part implies for almost all $x \exists$ infinitely many $q > 0$ such that $q \|qx\| \leq 1/\log q$. 
The Duffin-Schaeffer Conjecture

Can we remove the monotonicity assumption in Khintchine?

Duffin & Schaeffer constructed a non-monotonic $\psi$ such that
$$\sum_{q=1}^{\infty} \psi(q) = \infty$$
but
$$m(W(\psi)) = 0.$$

Idea is to keep using the same rational; i.e. $p/q, 2p/2q, \ldots$

Overcome this by insisting that $p, q$ are co-prime: let $W'(\psi)$ be
the set of $x \in I$ such that $|qx - p| \leq \psi(q)$ for infinitely many $p/q$ with $(p, q) = 1$. 

The Duffin-Schaeffer Conjecture (1941)

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a real positive function. Then
$$m(W'(\psi)) = 1$$
if
$$\sum_{q=1}^{\infty} \frac{\phi(q)}{q} \psi(q) = \infty.$$

Gallagher (1965): $m(W'(\psi)) = 0$ or $1$

Various partial results are known:
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The Duffin-Schaeffer Conjecture (1941) Let \( \psi : \mathbb{N} \to \mathbb{R}^+ \) be a
real positive function. Then
\[ m(W'(\psi)) = 1 \text{ if } \sum_{q=1}^{\infty} \frac{\varphi(q)\psi(q)}{q} = \infty . \]

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The Duffin-Schaeffer Theorem (1941) Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a real positive function. Then $m(W'(\psi)) = 1$ if

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and

$$\limsup_{N\to\infty} \left( \sum_{q=1}^{N} \frac{\varphi(q)}{q} \psi(q) \right) \left( \sum_{q=1}^{N} \psi(q) \right)^{-1} > 0. \quad (5)$$

Note that (5) implies that the convergence/divergence behavior of $\sum_{q=1}^{\infty} (\varphi(q)\psi(q))/q$ and $\sum_{q=1}^{\infty} \psi(q)$ are equivalent.
Let $\gamma \in \mathbb{R}$ and let $W'(\psi, \gamma)$ be the set of $x \in I$ such that $|qx - p - \gamma| \leq \psi(q)$ for infinitely many $p/q$ with $(p, q) = 1$.

**Inhomogeneous D-S Conjecture.** Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a real positive function. Then

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- No inhomogeneous analogue of Gallagher’s $0 - 1$ law. Ramírez (2015): $\exists$ integer $t \geq 1$ so that $m(W'(\psi, t\gamma)) = 0$ or $1$.
- No inhomogeneous analogue of the Duffin-Schaeffer Theorem.
The Inhomogeneous Duffin-Schaeffer Conjecture

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- No inhomogeneous analogue of the Duffin-Schaeffer Theorem. (We show that such a theorem would imply inhomogeneous Gallagher for multiplicative approximation)
Khintchine in $\mathbb{R}^2$: the statement

Let $I^2 = [0,1)^2$ and given $\psi : \mathbb{N} \to \mathbb{R}^+$ let

$$W(2, \psi) := \{(\alpha, \beta) \in I^2 : \max\{\|qa\|, \|q\beta\|\} < \psi(q) \text{ for i. m. } q \in \mathbb{N}\}.$$ 

Throughout, $m_2$ will denote 2-dimensional Lebesgue measure.

**Khintchine in $\mathbb{R}^2$. If $\psi$ is monotonic, then**

$$m_2(W(2, \psi)) = \begin{cases} 0 & \text{ if } \sum_{q=1}^{\infty} \psi^2(q) < \infty, \\ 1 & \text{ if } \sum_{q=1}^{\infty} \psi^2(q) = \infty. \end{cases}$$
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- Gallagher: monotonicity not required.
- Convergence not true in general for fixed $\alpha$. 
Convergence not true for rational $\alpha$

Let $L_\alpha$ be the line parallel to the $y$-axis passing through the point $(\alpha,0)$. Suppose $\alpha = \frac{a}{b}$.
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Let $L_\alpha$ be the line parallel to the $y$-axis passing through the point $(\alpha, 0)$. Suppose $\alpha = \frac{a}{b}$. Then, by Dirichlet’s theorem, for any $\beta$ there exists infinitely many $q \in \mathbb{N}$ such that $\|q\beta\| < q^{-1}$ and so it follows that

$$
\|bq\beta\| < \frac{b}{q} = \frac{b^2}{bq} \quad \text{and} \quad \|bq\alpha\| = \|aq\| = 0 < \frac{b^2}{bq}.
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Convergence not true for rational \( \alpha \)

Let \( L_\alpha \) be the line parallel to the \( y \)-axis passing through the point \((\alpha, 0)\). Suppose \( \alpha = \frac{a}{b} \). Then, by Dirichlet’s theorem, for any \( \beta \) there exists infinitely many \( q \in \mathbb{N} \) such that \( \|q\beta\| < q^{-1} \) and so it follows that

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\]

The upshot of this is that every point on the rational vertical line \( L_\alpha \) is \( \psi(q) = b^2 \ q^{-1} \) - approximable and so

\[
m_1(W(2, \psi) \cap L_\alpha) = 1 \quad \text{but} \quad \sum_{q=1}^{\infty} \psi(q)^2 = b^4 \sum_{q=1}^{\infty} q^{-2} < \infty.
\]
Khintchine in $\mathbb{R}^2$: the statement

Let $I^2 = [0, 1)^2$ and given $\psi : \mathbb{N} \to \mathbb{R}^+$ let

$$W(2, \psi) := \{(\alpha, \beta) \in I^2: \max\{\|q\alpha\|, \|q\beta\|\} < \psi(q) \text{ for i. m. } q \in \mathbb{N}\}.$$  

**Khintchine in $\mathbb{R}^2$. If $\psi$ is monotonic, then**

$$m_2(W(2, \psi)) = \begin{cases} 
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\end{cases}$$

- Convergence not true in general for fixed $\alpha$.
- Is divergence true for fixed $\alpha$?
Fix $\alpha \in I$ and let $L_{\alpha}$ be the line parallel to the $y$-axis passing through the point $(\alpha, 0)$ and let $\psi$ is monotonic. The claim is that

$$m_1(W(2, \psi) \cap L_{\alpha}) = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} \psi^2(q) = \infty. \quad (6)$$

**Theorem (Ramírez, Simmons, Süess)**

A. If $\tau(\alpha) < 2$, then (6) is true.

B. If $\tau(\alpha) > 2$ and for $\epsilon > 0$, $\psi(q) > q^{-\frac{1}{2} - \epsilon}$ for $q$ large, then $W(2, \psi) \cap L_{\alpha} = I^2 \cap L_{\alpha}$. In particular, $m_1(W(2, \psi) \cap L_{\alpha}) = 1$.

(A) requires estimates for $\#\{q \leq N : \|q\alpha\| \leq \psi(q)\}$
Littlewood Conjecture (c. 1930): For every \((\alpha, \beta) \in \mathbb{I}^2\)

\[
\liminf_{q \to \infty} q \| q\alpha \| \| q\beta \| = 0.
\]

Khintchine’s theorem implies that

\[
\liminf_{q \to \infty} q \log q \| q\alpha \| \| q\beta \| = 0 \quad \forall \alpha \in \mathbb{R} \text{ and for almost all } \beta \in \mathbb{R}.
\]
Given $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$ let

$$W^\times (\psi) := \{ (\alpha, \beta) \in I^2 : \| q\alpha \| \| q\beta \| < \psi(q) \text{ for i. m. } q \in \mathbb{N} \} .$$
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The following result is the analogue of Khintchine’s simultaneous approximation theorem within the multiplicative setup.

**Theorem (Gallagher, 1962)**

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonic function. Then

$$m_2(W^\times(\psi)) = \begin{cases} 
0 & \text{if } \sum_{q=1}^{\infty} \psi(q) \log q < \infty, \\
1 & \text{if } \sum_{q=1}^{\infty} \psi(q) \log q = \infty.
\end{cases}$$
Multiplicative approximation: a moments reflection

Gallagher’s theorem implies that

\[ \lim \inf_{q \to \infty} q \log^2 q \|q\alpha\| \|q\beta\| = 0 \quad \text{for a.a.} \quad \alpha \in \mathbb{R} \quad \text{and for a.a.} \quad \beta \in \mathbb{R}. \]

(7)
Gallagher’s theorem implies that

\[ \liminf_{q \to \infty} q \log^2 q \| q\alpha \| \| q\beta \| = 0 \quad \text{for a.a. } \alpha \in \mathbb{R} \text{ and for a.a. } \beta \in \mathbb{R}. \] (7)

Khintchine’s theorem implies that

\[ \liminf_{q \to \infty} q \log q \| q\alpha \| \| q\beta \| = 0 \quad \forall \alpha \in \mathbb{R} \text{ and for a.a. } \beta \in \mathbb{R}. \] (8)

The extra log factor from Gallagher comes at a cost of having to sacrifice a set of measure zero on the \( \alpha \) side. Thus, unlike with (8) which is valid for any \( \alpha \), we are unable to claim that the stronger ‘log squared’ statement (7) is true for say when \( \alpha = \sqrt{2} \).

This raises the natural question of whether (7) holds for every \( \alpha \).
Multiplicative approximation: a moments reflection

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The extra log factor from Gallagher comes at a cost of having to sacrifice a set of measure zero on the \(\alpha\) side. Thus, unlike with (8) which is valid for any \(\alpha\), we are unable to claim that the stronger ‘log squared’ statement (7) is true for say when \(\alpha = \sqrt{2}\). This raises the natural question of whether (7) holds for every \(\alpha\).
Theorem (Beresnevich-Haynes-V, 2015)

Let $\alpha \in I$ and $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonic function such that
\[
\sum_{q=1}^{\infty} \psi(q) \log q = \infty \quad (9)
\]
and
\[
\exists \delta > 0 \liminf_{n \to \infty} q_n^{3-\delta} \psi(q_n) \geq 1, \quad (10)
\]
where $q_n$ denotes the denominators of the convergents of $\alpha$. Then for almost every $\beta \in I$, there exists infinitely many $q \in \mathbb{N}$ such that
\[
\|q\alpha\| \|q\beta\| < \psi(q). \quad (11)
\]

Condition (10) holds for all $\alpha$ with Diophantine exponent $\tau(\alpha) < 3$. Note that $\dim \{ \alpha \in \mathbb{R} : \tau(\alpha) \geq 3 \} = 1/2$. 
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It follows that for every $\alpha \in \mathbb{R}$
\[
\liminf_{q \to \infty} q \log^2 q \|q\alpha\| \|q\beta\| = 0 \quad \text{for almost all } \beta \in \mathbb{R}.
\]
Pseudo sketch proof of divergent Gallagher on fibres

Given $\alpha$ and monotonic $\psi$, consider

$$\|q\beta\| < \Psi_\alpha(q) \quad \text{where} \quad \Psi_\alpha(q) := \frac{\psi(q)}{\|q\alpha\|}.$$
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$$\|q\beta\| < \Psi_{\alpha}(q) \quad \text{where} \quad \Psi_{\alpha}(q) := \frac{\psi(q)}{\|q\alpha\|}.$$ 

Suppose Khintchine’s Theorem is true for functions $\Psi_{\alpha}$, then:

$$m_1(W(\Psi_{\alpha})) = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} \Psi_{\alpha}(q) = \infty.$$
Pseudo sketch proof of divergent Gallagher on fibres

Given $\alpha$ and monotonic $\psi$, consider

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Suppose Khintchine’s Theorem is true for functions $\Psi_\alpha$, then:

$$m_1(\mathcal{W}(\Psi_\alpha)) = 1 \quad \text{if} \quad \sum_{q=1}^\infty \Psi_\alpha(q) = \infty.$$  

We need to show that

$$\sum_{q=1}^\infty \psi(q) \log q = \infty \quad \implies \quad \sum_{q=1}^\infty \Psi_\alpha(q) = \infty.$$  

This follows by partial summation and the fact that for any irrational $\alpha$ and $Q \geq 2$

$$R_Q(\alpha; 0) := \sum_{q=1}^Q \frac{1}{\|q\alpha\|} \gg Q \log Q.$$
Theorem (Beresnevich-Haynes-V, 2015)

Let $\gamma, \delta \in \mathbb{R}$ and $\alpha \in \mathbb{I}$ be irrational and let $\psi : \mathbb{N} \to \mathbb{R}^+$ be such that $\sum \psi(q) \log q$ converges. Furthermore, assume either:

(i) $n \mapsto n\psi(n)$ is decreasing and

\[ S_N(\alpha; \gamma) \ll (\log N)^2 \quad \text{for all } N \geq 2; \]

(ii) $n \mapsto \psi(n)$ is decreasing and

\[ R_N(\alpha; \gamma) \ll N \log N \quad \text{for all } N \geq 2. \]

Then for almost all $\beta \in \mathbb{I}$, there exist only finitely many $q \in \mathbb{N}$ such that

\[ \| q\alpha - \gamma \| \| q\beta - \delta \| < \psi(q) \]

(i.e. $m_1(W^\times(\psi, \gamma, \delta) \cap L_{\alpha}) = 0$).

Taking $\alpha \in \text{Bad}$ and $\gamma = 0$ works.
**Conjecture.** Let $\gamma, \delta \in \mathbb{R}$ and let $\psi : \mathbb{N} \to \mathbb{R}^+$ be monotonic. Then

$$m_2(W^x(\psi, \gamma, \delta)) = 1 \text{ if } \sum_{q=1}^{\infty} \psi(q) \log q = \infty$$

i.e. for almost all $(\alpha, \beta) \in \mathbb{I}^2$, there exist infinitely many $q \in \mathbb{N}$ such that

$$\| q\alpha - \gamma \| \| q\beta - \delta \| < \psi(q).$$
Inhomogeneous Divergent Gallagher

**Conjecture.** Let \( \gamma, \delta \in \mathbb{R} \) and let \( \psi : \mathbb{N} \to \mathbb{R}^+ \) be monotonic. Then

\[
m_2(\mathcal{W}^\times(\psi, \gamma, \delta)) = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} \psi(q) \log q = \infty
\]

i.e. for almost all \((\alpha, \beta) \in I^2\), there exist infinitely many \(q \in \mathbb{N}\) such that

\[
\|q\alpha - \gamma\| \|q\beta - \delta\| < \psi(q).
\]

- Duffin-Schaeffer Theorem implies Conjecture true with \(\delta = 0\).
- Inhomogeneous Duffin-Schaeffer Theorem (20??) implies Conjecture true in general.