Chai and Why?

THE TAU OF RAMANUJAN

August 1, 2010
Prithvi Theater

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School of Maths
TIFR, Mumbai
What is Mathematics?

Mathematics is an expression of the human mind that reflects the active will, the contemplative reason, and the desire for aesthetic perfection.

- Courant and Robbins, 1941
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But, more simply....
Math is fun: Like Sudoku

There are 6,670,903,752,021,072,936,960 Sudoku puzzles.

Is there a puzzle with 16 starting entries?
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BSD Conjecture:

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\frac{L^{(r)}(E, 1)}{r!} = \frac{\#\Sha(E) \Omega_E R_E \prod_{p | N} c_p}{(\#E_{\text{Tor}})^2}
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Everest Base Camp

And not without pitfalls
And math is beautiful: Like Chopin’s music
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Quadratic reciprocity:

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\left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{\frac{p-1}{2} \frac{q-1}{2}}
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1810 - 1849 Fantasie Impromptu
Today our story starts with the mathematician Euler.
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1707 - 1783
Bridges of Königsberg

Citizens: Is it possible to walk over all the bridges, crossing each bridge only once?

At each vertex, # In = # Out, for there to be a path. So the number of edges meeting at every vertex must be EVEN.

Euler: There is NO such walk!

Topology is born.
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Infinite Sums

Euler was also interested in infinite sums.
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\[\sum_{n=1}^{\infty} \frac{1}{n} = \infty\]

Euler did all sums with even powers. Odd powers still open!
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\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots =
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Euler’s phi-function

Euler was also interested in infinite products.
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\[ \phi(n) = \prod_{\text{prime } p} \left(1 - \frac{1}{p}\right) \]

Here \( x \) is a formal variable.

To understand this consider the finite products:

\[ (1 - x)(1 - x^2)= 1 - x - x^2 + x^3 \]
\[ (1 - x)(1 - x^2)(1 - x^3) = 1 - x - 2x^2 + 2x^4 \]
Euler’s phi-function

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\[ (1 - x)(1 - x^2)(1 - x^3)(1 - x^4) \cdots \]

It expands to:

\[ 1 - x - x^2 + x^3 - x^4 + x^5 - x^6 + \cdots \]
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Computing some more we get:

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The terms stabilize, so the infinite product makes sense. There are many cancellations, so that the coefficients are always +1, 0, or \(-1\).
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- The terms stabilize, so the infinite product makes sense.
- There are many cancellations, so that the coefficients are always +1, 0, or -1.
Infinite product

We get:

\[ \phi(x) = 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - x^{70} - x^{77} \ldots \]
Infinite product

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- There are lots of zeros (and a few \(-\) and \(\+)
  signs)
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- There are lots of zeros (and a few \(-\) and \(+\) signs)
- Always get two \(-\) signs followed by two \(+\) signs
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- There are lots of zeros (and a few $-$ and $+$ signs)
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- The difference between the numbers in each pair is 1, 2, 3, 4, \ldots
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- There are lots of zeros (and a few $-$ and $+$ signs)
- Always get two $-$ signs followed by two $+$ signs
- The difference between the numbers in each pair is $1, 2, 3, 4, \ldots$
- The first number in each pair are the numbers $1, 5, 12, 22, 35, \ldots$
Infinite product

We get:

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\phi(x) = 1 - x^1 - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + x^{51} + x^{57} - x^{70} - x^{77} \ldots
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- There are lots of zeros (and a few – and + signs)
- Always get two – signs followed by two + signs
- The difference between the numbers in each pair is 1, 2, 3, 4, \ldots
- The first number in each pair are the numbers 1, 5, 12, 22, 35, \ldots

But what are these last numbers? Are they interesting?
Pentagonal numbers

The numbers 1, 5, 12, 22, 35, ... are exactly the pentagonal numbers $3n^2 - n$ for $n = 1, 2, 3, ...$
Pentagonal numbers

The numbers 1, 5, 12, 22, 35, ... are exactly the pentagonal numbers

\[
\frac{3n^2 - n}{2}
\]

for \( n = 1, 2, 3, \ldots \)
Pentagonal number identity

Theorem (Euler)

\[ \prod_{n=1}^{\infty} (1 - x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{3n^2-n}{2}}. \]
Pentagonal number identity

**Theorem (Euler)**

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\prod_{n=1}^{\infty} (1 - x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{3n^2-n}{2}}.
\]

Infinite product gives pentagonal numbers with $+$ and $-$ signs!
Application to Partitions

A partition of a number is the obvious thing. For example:

\[ 5 = 1 + 1 + 1 + 1 + 1 \]
\[ = 1 + 1 + 1 + 2 \]
\[ = 1 + 2 + 2 \]
\[ = 1 + 1 + 3 \]
\[ = 2 + 3 \]
\[ = 1 + 4 \]
\[ = 5 \]

Let \( p(n) \) be the number of partitions of \( n \).

\( p(5) = 7. \)

So not a good idea to count partitions by brute force!
Application to Partitions

A partition of a number is the obvious thing. For example:

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Application to Partitions

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Let \( p(n) \) be the number of partitions of \( n \). So \( p(5) = 7 \).

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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>…</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(n)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>7</td>
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Use the Euler $\phi$-function instead

Let $p(x) = 1 + \sum_{n=1}^{\infty} p(n) x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \cdots$

Then it can be easily shown that $\phi(p(x)) = 1$.

So can use pentagonal numbers to compute partitions:

$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) + \cdots$

For example if $n = 5$:

$p(5) = p(4) + p(3) - p(0) = 5 + 3 - 1 = 7$. 
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The next character in our story is Gauss.

1777-1855
A child prodigy

Teacher: Add 1 + 2 + \cdots + 100, and tell me the answer.

Gauss: (almost instantly) 5,050.

Idea:
Write the second 50 numbers backwards, under the first 50:

\[1 + 2 + 3 + \cdots + 49 + 50\]
\[100 + 99 + 98 \cdots + 52 + 51\]

Each column adds up to 101, and there are 50 columns, so the answer is \(50 \times 101 = 5,050!\)
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\[50 \times 101 = 5,050!\]
Gauss studied the cube of the Euler $\phi$-function:

$$\phi(x)^3 = (1 - x)^3(1 - x^2)^3(1 - x^3)^3(1 - x^4)^3 \ldots$$
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Again, what are these last numbers?
Triangular numbers

1, 3, 6, 10, ...

are exactly the triangular numbers $n^2 + \frac{n(n-1)}{2}$ for $n = 1, 2, 3, ...$. 
Triangular numbers

1, 3, 6, 10, ... are exactly the triangular numbers

\[ \frac{n^2 + n}{2} \]

for \( n = 1, 2, 3, \ldots \).
Triangular number identity

Theorem (Gauss)

\[ \prod_{n=1}^{\infty} (1 - x^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) x^{\frac{n^2+n}{2}}. \]
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The infinite product cubed gives the triangular numbers!
Other powers (and some general culture)

Are there formulas for other powers $\phi(x)^d = \cdots$ of the $\phi$-function?

If there are lots of zeros, then there probably is a formula!

Let's count the number of zeros in the first 500 powers of $x$:

<table>
<thead>
<tr>
<th>$d$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$8$</th>
<th>$9$</th>
<th>$10$</th>
<th>$11$</th>
<th>$12$</th>
<th>$13$</th>
<th>$14$</th>
<th>$15$</th>
<th>$16$-$25$</th>
<th>$26$</th>
<th>$27$-$35$</th>
<th>$36$</th>
</tr>
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<tbody>
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<tr>
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<td></td>
<td></td>
<td></td>
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<td>0</td>
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Dyson and MacDonald proved formulas for the powers $d = 3, 8, 10, 14, 15, 21, 24, 26, 28, 35, 36, \ldots$

These come from the theory of Lie algebras, giving a connection to a new field of mathematics and to physics.
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<tr>
<th>$d$</th>
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<th>11,12,13</th>
<th>14</th>
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We get the tau-function!
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\[ \phi(x) = 1 - 24x + 252x^2 - 1472x^3 + 4830x^4 - 6048x^5 + \cdots \]

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<th>5</th>
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<th>7</th>
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<tbody>
<tr>
<td>( \tau(n) )</td>
<td>1</td>
<td>-24</td>
<td>252</td>
<td>-1472</td>
<td>4830</td>
<td>-6048</td>
<td>-16744</td>
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Tau never vanishes

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**Conjecture (Lehmer)**

The numbers $\tau(n)$ for $n = 1, 2, 3, \ldots$ are never equal to 0.
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But these computations do not constitute a proof, and the problem is still open!
Ramanujan

Ramanujan extensively studied the $\tau$-function.
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1887-1920
The Legend of Ramanujan

Hardy (visiting Ramanujan in hospital):
1729, what a boring taxi number...

Ramanujan: Not at all!
It is the smallest number to be written as the sum of two cubes in two different ways.

$1^3 + 12^3 = 9^3 + 10^3$. 
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Tau is multiplicative

Let’s now focus on some patterns that Ramanujan found in studying the $\tau$-function.

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2. Also, for powers of a prime $p$:

$$\tau(p^n) = \tau(p)\tau(p^{n-1}) - p^{11}\tau(p^{n-2}).$$
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E.g.: $\tau(2^2) = -1472 = \tau(2)\tau(2) - 2^{11} = (-24)^2 - 2^{11}$. 
These two properties of $\tau(n)$ were proved by Mordell in 1917. Later Hecke showed that these properties follow from a much more general theory of MODULAR FORMS.

In any case: Since every number is a product of primes $12 = 4 \times 3 = 2 \times 3$, knowing $\tau(p)$ for all primes $p$, is the same as knowing $\tau(n)$ for all $n$, e.g.,

$$\tau(12) = \tau(4) \times \tau(3) = (\tau(2)^2 - 2^{11}) \times \tau(3) = -1472 \times 252 = -370,944.$$
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The Magic of 23

It turns out that for half the primes $p$, $\tau(p)$ is divisible by 23.

Examples:

$\tau(5) = 4$, $5 = 23 \times 210$

$\tau(7) = -16$, $744 = 23 \times -728$

$\tau(11) = 534$, $612 = 23 \times 23 \times 244$

This is now well understood (using Galois representations).
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The divisor function

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The first few values are:

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\begin{align*}
\sigma_{11}(1) &= 1^{11} = 1 \\
\sigma_{11}(2) &= 1^{11} + 2^{11} = 2,049 \\
\sigma_{11}(3) &= 1^{11} + 3^{11} = 177,148 \\
\sigma_{11}(4) &= 1^{11} + 2^{11} + 4^{11} = 4,196,353 \quad \text{etc.}
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Compared to $\tau(n)$, the function $\sigma_{11}(n)$ is quite well understood.
Ramanujan discovered an amazing connection between $\tau(n)$, $\sigma_{11}(n)$ and the prime **691**.
Tau and 691

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**Theorem (Ramanujan)**

For all numbers $n = 1, 2, 3, \ldots$, the difference between

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Can \( p \) divide \( \tau(p) \)?
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Similarly:

3 divides $\tau(3) = 252$
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7 divides $\tau(7) = -16,744$, but then, for no prime $p$ up to a million (1,000,000), does $p$ divide $\tau(p)$, except:

2411 divides $\tau(2411)$!

The prime 2411 was found in 1972 by Newman.

Are there any other such 'non-ordinary' primes?
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There is another!

Amazingly this was found just this year (2010!)

Theorem (Lygeros, Rozier)

The prime $7$, $758$, $337$, $633$ divides $\tau(7, 758, 337, 633)$

and this is now the largest known prime with this property.

Are there any more such primes?

There should be infinitely many, but they appear very slowly (log log philosophy!)
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For all primes $p$,

$$-2 \sqrt{p^{11}} \leq \tau(p) \leq 2 \sqrt{p^{11}}.$$
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This was proved by Deligne, as a consequence of a much more general result, for which he got the Fields Medal in 1978.
Tau as an Error

Every integer is a sum of 4 squares!

Let $r_{24}(n)$ be the number of ways to write $n$ as a sum of 24 squares. One knows that, for odd primes $p$:

$$r_{24}(p) = 16 \cdot 691 \cdot \sigma_{11}(p) + 33 \cdot 152 \cdot 691 \cdot \tau(p).$$

Since $\sigma_{11}(p)$ is close to $p^{11}$ and $\tau(p)$ is bounded by (twice) the square-root of $p^{11}$, we see that $\tau(p)$ is much smaller than $\sigma_{11}(p)$, and so may be thought of as an ERROR term when computing $r_{24}(p)$. 
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Distribution of error terms

Ramanujan and Deligne say $-1 \leq \tau(p) \sqrt{p} \leq 1$ for all primes $p$.

How are these (scaled) error terms distributed?

Numerical data: Probability distribution of (scaled) error terms
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![Probability distribution of (scaled) error terms](image)
Sato-Tate distribution

The following theorem, which was conjectured by Sato and Tate, was proved just this year (2010!).
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**Theorem (Barnet-Lamb, Geraghty, Harris, Taylor)**

The numbers

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are equidistributed with respect to the measure

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So the tau-function continues to tantalize us to this day!
Thank you

See you at the next ICM in Hyderabad!
Acknowledgements

1. A. Bhattacharya, S. Kulkarni, R. Rao, *Chai and Why?* team
2. L. Vepstas, cover image of the values of the Delta function, http://linas.org
5. R. Patnaik, Photo of Everest Base Camp, Kala Pattar (1991)
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