SUPERCUSPIDAL RAMIFICATION OF MODULAR ENDOMORPHISM ALGEBRAS

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Abstract. The endomorphism algebra X_f attached to a non-CM primitive cusp form f of weight at least two is a 2-torsion element in the Brauer group of a number field F. We give formulas for the ramification of X_f locally at primes lying above the odd supercuspidal primes of f. We show that the local Brauer class is determined by the underlying local Galois representation together with an auxiliary Fourier coefficient.

1. Introduction

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(N, \epsilon)$ be a primitive non-CM cusp form of weight $k \geq 2$ and level $N \geq 1$. Let M_f denote the abelian variety ($k = 2$) or the Grothendieck motive ($k > 2$) attached to f. The $\mathbb{Q}$-algebra of endomorphisms of M_f is denoted by

$$X = X_f := \mathrm{End}_{\bar{\mathbb{Q}}}(M_f) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The Hecke field $E = \mathbb{Q}(a_n)$ is either a totally real or CM number field. Let F denote the subfield of E generated by all the elements of the form $a_n^2 \epsilon^{-1}(n)$ with $(n, N) = 1$. Then F is a totally real number field, and may be thought of as the Hecke field of the adjoint lift of f. The algebra X has the structure of a crossed product algebra over F defined in terms of the inner twists of f, as proved in [Ri80] and [Ri81] for $k = 2$, and generalized to higher weight forms in [BG03] and [GGQ05]. The class of X is a 2-torsion element in the Brauer group $\mathrm{Br}(F)$ of F. Ribet has asked whether one can determine the class $[X] \in \mathrm{Br}(F)$ explicitly.

The Brauer class of X can be studied locally under the map $\mathrm{Br}(F) \hookrightarrow \oplus_v \mathrm{Br}(F_v)$, where v varies over all the primes in F. The algebra $X_v := X \otimes_F F_v$ is central simple over F_v and its class $[X_v] \in {}_2\mathrm{Br}(F_v) \cong \mathbb{Z}/2$. When this class is trivial, X_v is a matrix algebra over F_v and X_v is said to be unramified, and when it is non-trivial, X_v is Brauer-equivalent to a quaternion division algebra over F_v and X_v is said to be ramified. The Brauer class of X is determined by the Brauer classes of all the X_v, only finitely many of which can be non-trivial.

For v lying above a prime p of good reduction, a Steinberg prime p, or a ramified principal series prime p, with $a_p \neq 0$, the class $[X_v]$ is essentially determined by the parity of the slope at p of the twisted adjoint lift of f. For instance, if $p \nmid N$, then $[X_v]$ is essentially determined by the v-adic valuation of $a_p^2 \epsilon^{-1}(p) \in F^*$. See [BG13]. In particular, if ρ_f is the ℓ-adic Galois representation attached to f, then the $\bar{\mathbb{Q}}_\ell$-isomorphism class of the local representation $\rho_f|_{G_p}$ at p essentially determines the Brauer class $[X_v]$, for $\ell \neq p$. Thus, if the Fourier coefficient $a_p \neq 0$, then it essentially determines $[X_v]$.

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Assume, therefore, that $a_p = 0$. For v lying above a prime p of good reduction with $a_p = 0$, the slope of the twisted adjoint lift of f at p is not finite. Moreover, the local ℓ-adic Galois representation $\rho_f|_{G_p}$ does not determine the class $[X_v]$, even if $\ell = p$. However, in [BG13, Thm. 11] it is shown that $[X_v]$ is essentially determined by a Fourier coefficient $a_{p^\dagger} \neq 0$ at an auxiliary prime $p^\dagger \nmid N$. More precisely, $[X_v]$ is essentially determined by the v-adic valuation of $a^2_{p^\dagger}\epsilon^{-1}(p^\dagger)$, the Fourier coefficient of the twisted adjoint lift of f at $p^\dagger$.

Assume now that v lies over a prime p of bad reduction with $a_p = 0$. Since the Brauer class of $[X_f]$ is invariant under twisting f by Dirichlet characters, we may as well assume that f is p-minimal, in which case p is a supercuspidal prime. Almost nothing is known about the ramification of X_v in this case. Again the $\bar{\mathbb{Q}}_\ell$-isomorphism class of $\rho_f|_{G_p}$ does not determine $[X_v]$; see Example 1 in Section 8. In this paper, we obtain formulas for $[X_v]$ in terms of a Fourier coefficient at an auxiliary prime related to p, as in the good reduction case. It turns out that the $\bar{\mathbb{Q}}_\ell$-isomorphism class of $\rho_f|_{G_p}$, for $\ell \neq p$, together with this coefficient, determines $[X_v]$ completely. This solves the problem of determining the Brauer classes at the odd supercuspidal primes (under an extra hypothesis in the level 0 case).

2. Statement of results

Throughout this article, p will denote an odd prime. Let $f \in S_k(N, \epsilon)$ be a non-CM primitive cusp form of weight $k \geq 2$, level N and nebentypus ϵ. We write $N = p^{N_p}N'$, such that $p \nmid N'$, and $\epsilon = \epsilon_p \cdot \epsilon'$, where ϵ_p is a Dirichlet character of conductor p^{C_p} for some $C_p \leq N_p$, and the conductor of ϵ' divides N'. If f is p-minimal and if $C_p < N_p \geq 2$, then p is a supercuspidal prime for f and $a_p = 0$. Let (K, θ) be an admissible pair attached to f at p, where K is a quadratic extension of $\mathbb{Q}_p$ and θ is a continuous character of K^*. The prime p is called a ramified (or unramified) supercuspidal prime for f, if $K|\mathbb{Q}_p$ is a ramified (or unramified) extension. When $C_p < N_p = 2$, then p is an unramified supercuspidal prime and θ is a tamely ramified character of K^*. The corresponding local automorphic representation has level 0, and we say p is supercuspidal of level 0. When $K|\mathbb{Q}_p$ is unramified, we let s denote a fixed primitive $(p^2 - 1)$-th root of unity in K^*. For any quadratic field extension $K_1|K_2$, and for an arbitrary $x \in K_2^*$, let

$$(x, K_1|K_2) := \begin{cases} -1, & \text{if } x \notin N_{K_1|K_2}(K_1)^*, \\ 1, & \text{otherwise.} \end{cases}$$

For a prime v in F lying above p, let $f_v := f(F_v|\mathbb{Q}_p)$ be the residue degree and let $e_v := e(F_v|\mathbb{Q}_p)$ be the ramification index. Let $\upsilon : F_v^* \to \mathbb{Z}$ be the standard surjective valuation.

Since f is non-CM, there exist infinitely many primes with non-zero Fourier coefficient in any congruence class modulo N. Let us choose two primes p' and p'', both coprime to N, satisfying:

($*$) $p' \equiv 1 \mod p^{N_p}$, $p' \equiv p \mod N'$ and $a_{p'} \neq 0$.

($**$) $p'' \equiv 1 \mod N'$, p'' has order $(p-1)$ in $\left(\mathbb{Z}/p^{N_p}\mathbb{Z}\right)^*$ and $a_{p''} \neq 0$.

The Fourier coefficients of the twisted adjoint lift of f at p' and p'', namely $a^2_{p'}\epsilon^{-1}(p')$ and $a^2_{p''}\epsilon^{-1}(p'')$, give elements of $F^*/(F^*)^2$ which do not depend on the choice of p' and p''. Let us write

$$[X_v] \sim \begin{cases} -1, & \text{if } X_v \text{ is ramified}, \\ 1, & \text{if } X_v \text{ is unramified.} \end{cases}$$

With notation as above, our main result is as follows.

Theorem 2.1. *Let v be a prime of F lying over a prime $p \neq 2$.*

(a) *If p is an unramified supercuspidal prime, and $\theta(s) + \theta(s)^p \neq 0$ if the p-minimal twist of f is of level* 0*, then*

$$[X_v] \sim (-1)^{f_v \cdot \upsilon(a_{p'}^2 \epsilon^{-1}(p'))}.$$

(b) *If $p \equiv 1 \mod 4$ is a ramified supercuspidal prime, then*

$$[X_v] \sim (-1)^{f_v \cdot \upsilon(a_{p'}^2 \epsilon^{-1}(p'))}.$$

(c) *If $p \equiv 3 \mod 4$ is a ramified supercuspidal prime, then*

$$[X_v] \sim \left((-1)^k \cdot a_{p''}^2 \epsilon^{-1}(p''), KF_v|F_v\right).$$

Remark One reason behind the similarity and simplicity of the formulas in (a) and (b) above is that in both cases the extension $KF_v|F_v$ turns out to be unramified, and hence the symbol $\left(a_{p'}^2 \epsilon^{-1}(p'), KF_v|F_v\right) = (-1)^{\upsilon(a_{p'}^2 \epsilon^{-1}(p'))}$. On the other hand, we will see that in case (c), the extension $KF_v|F_v$ can be ramified.

We list some interesting consequences of the theorem.

Corollary 2.2. *Suppose we are in case* (a) *or* (b) *of the theorem. If either*

(1) *$N' = 1$ (i.e., N is a prime power) or if p has odd order in $\left(\frac{\mathbb{Z}}{N'\mathbb{Z}}\right)^*$, or;*

(2) *$v \nmid \mathrm{disc}(E|F)$,*

then X_v is unramified.

Corollary 2.3. *Suppose we are in case* (a) *or* (c) *of the theorem. If $K \subseteq F_v$, then X_v is unramified.*

We will show that $(p-1)$ divides $2e_v$, if p is a ramified supercuspidal prime.

Corollary 2.4. *Suppose we are in case* (b) *of the theorem. If $K = \mathbb{Q}_p(\sqrt{p})$, then*

$$[X_v] \sim \epsilon_p(-1)^{2[F_v:\mathbb{Q}_p]/(p-1)}.$$

3. Endomorphism algebra and its cocycle class

Let f and E be as above. Then $\mathrm{Aut}(E)$ contains an abelian subgroup defined by $\Gamma := \{\gamma \in \mathrm{Aut}(E) \mid \exists \text{ a Dirichlet character } \chi_\gamma \text{ such that } a_p^\gamma = a_p \cdot \chi_\gamma(p), \forall p \nmid N\}$. The subfield of E fixed by Γ equals the field F mentioned in Section 1. As f is non-CM, for each $\gamma \in \Gamma$, the character χ_γ is unique, and is called an 'inner twist' of f. For a fixed $g \in G_{\mathbb{Q}}$, the map $\gamma \mapsto \chi_\gamma(g)$ is a 1-cocycle. By Hilbert's theorem 90, $\exists\, \alpha(g) \in E^*$ such that for all $\gamma \in \Gamma$,

$$\chi_\gamma(g) = \alpha(g)^{\gamma - 1}. \tag{1}$$

Thus we get a well-defined continuous character $\tilde{\alpha} : G_{\mathbb{Q}} \to E^*/F^*$, sending each g to $\alpha(g) \mod F^*$. Let ρ_f denote the λ-adic representation attached to f by Deligne, for some prime $\lambda \mid \ell$ of E. The following result of Ribet [Ri04, Thm. 5.5] for $k = 2$, holds for all weights $k \geq 2$.

Proposition 3.1. *Let α be any lift of the character $\tilde{\alpha}$ to E^*. Then*

(1) *For all $g \in G_{\mathbb{Q}}$, $\alpha^2(g) \equiv \epsilon(g) \mod F^*$.*

(2) *For all $g \in G_{\mathbb{Q}}$, $\alpha(g) \equiv \mathrm{Trace}(\rho_f(g)) \mod F^*$, provided that the trace is non-zero.*

For any continuous lift α of $\tilde{\alpha}$ to E^*, the 2-cocycle c_α defined by $c_\alpha(g,h) = \alpha(g)\alpha(h)\alpha^{-1}(gh)$ gives a 2-torsion class $[c_\alpha]$ in $\mathrm{H}^2(G_F, \bar{F}^*) \cong \mathrm{Br}(F)$. The class $[c_\alpha] \in \mathrm{H}^2(G_F, \bar{F}^*)$ corresponds to the global Brauer class $[X_f] \in \mathrm{Br}(F)$. Similarly the local Brauer class $[X_v]$ is determined by the restriction $[c_\alpha|_{G_{F_v}}] \in \mathrm{H}^2(G_{F_v}, \bar{F}_v^*)$, for any prime v of F.

Let G_p be the decomposition group at p. If p is a supercuspidal prime, then the local Galois representation $\rho_f|_{G_p}$ is induced from a character of G_K for some quadratic extension K of $\mathbb{Q}_p$. This implies that for any lift $g \in G_p$ of the generator of $\mathrm{Gal}(K|\mathbb{Q}_p)$, the trace in part (2) always vanishes, so one cannot use Proposition 3.1 to compute $\alpha(g)$. In fact, the auxiliary primes p' and p'' introduced in the previous section are chosen in such a way that there is a lift g for which $\alpha(g) \equiv a_{p'}$ or $a_{p''}$ mod F^*.

4. Galois representations and Langlands correspondence

The pair (K, θ) is called an 'admissible pair', if K is a quadratic extension of $\mathbb{Q}_p$ and $\theta : K^* \to \bar{\mathbb{Q}}^* \subseteq \bar{\mathbb{Q}}_\ell^*$ is a (continuous) character of K^* satisfying

(1) θ does not factor through the norm map $N_{K|\mathbb{Q}_p} : K^* \to \mathbb{Q}_p^*$,
(2) If $K|\mathbb{Q}_p$ is ramified, then even $\theta|_{U_K^{(1)}}$ does not factor through $N_{K|\mathbb{Q}_p}$.

Here $U_K^{(i)}$ denotes the i-th step in the standard filtration of the units of K. Two pairs (K_1, θ_1) and (K_2, θ_2) are equivalent if there is a $\mathbb{Q}_p$-isomorphism $\iota : K_1 \to K_2$ with $\theta_2 \circ \iota = \theta_1$. The equivalence classes of admissible pairs are in bijection with $\bar{\mathbb{Q}}_\ell$-isomorphism classes of irreducible 2-dimensional representations of G_p as well as with $\bar{\mathbb{Q}}_\ell$-isomorphism classes of supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ (for $p \neq 2$).

If p is a supercuspidal prime for f, then $\rho_f|_{G_p}$ is absolutely irreducible. We call (K, θ) to be an admissible pair attached to f at p, if

$$\rho_f|_{G_p} \sim \mathrm{Ind}_{G_K}^{G_p} \theta.$$

Given a supercuspidal prime $p\,(\neq 2)$ for f, the quadratic extension K is unique, though it is not easy to determine the admissible pair (K, θ) attached to f at p explicitly, see [LW12].

For $L|\mathbb{Q}_p$ a finite extension and an ℓ-adic representation ρ of G_L, let $c(\rho)$ be the conductor of the corresponding representation of the Weil group of L. It equals the Artin conductor of a suitable unramified twist of ρ. For the character θ of G_K, it equals the usual $c(\theta) = \min\{i : \theta|_{U_K^{(i)}} \equiv 1\}$. By [CF10, Prop. 4(b), §4.3, Ch. 6], we get $c\left(\mathrm{Ind}_{G_K}^{G_p} \theta\right) = \upsilon_p(\delta(K|\mathbb{Q}_p)) + f(K|\mathbb{Q}_p)c(\theta)$, where υ_p is the normalized valuation on $\mathbb{Q}_p^*$, $\delta(K|\mathbb{Q}_p)$ stands for the discriminant, and $f(K|\mathbb{Q}_p)$ is the residue degree. Applying this formula in our setting, we get

$$N_p = \begin{cases} 2c(\theta), & \text{if } K = \mathbb{Q}_{p^2} \text{ is unramified,} \\ 1 + c(\theta), & \text{if } K|\mathbb{Q}_p \text{ is ramified.} \end{cases} \tag{2}$$

Thus if $K = \mathbb{Q}_{p^2}$, then N_p is even. A newform f of level $N = p^{N_p}N'$ is said to be p-minimal, if N_p is the smallest among all possible twists $f \otimes \chi$ of f by Dirichlet characters χ. If f is a p-minimal form for which p is a supercuspidal prime, then by [AL78, Thms. 4.3, 4.3′], $C_p \leq \lfloor N_p/2 \rfloor \geq 1$; moreover, if N_p is even, then $K|\mathbb{Q}_p$ is unramified.

While proving the ℓ-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, the 'supercuspidals of level zero' are always treated separately. They are exactly the supercuspidal representations attached to some admissible pairs (K, θ) with $c(\theta) = 1$, hence $N_p = 2$. By the definition of an admissible pair, $c(\theta) = 1$ forces K to be unramified. For the exact definition of the level of a supercuspidal representation we refer to [BH06, §12.6]. The next lemma will be used to investigate the case of supercuspidal primes of positive level.

Let σ be any lift of the generator of $\mathrm{Gal}(K|\mathbb{Q}_p)$ to G_p, and let θ^σ denote the conjugate character of θ by σ. Then we have

Lemma 4.1. *Let p be a supercuspidal prime for f with $N_p \geq 3$ and suppose ϵ is tamely ramified at p. Then there is an element $\tau \in I_w(K)$, the wild inertia group of K, satisfying:*

(1) *$\theta(\tau) = \zeta_p$ and $\theta^\sigma(\tau) = \zeta_p^{-1}$, where ζ_p is some primitive p-th root of unity.*
(2) *$\alpha(\tau) \equiv (\zeta_p + \zeta_p^{-1}) \equiv 1 \mod F^*$.*

Thus $\mathbb{Q}(\zeta_p + \zeta_p^{-1}) \subseteq F$. In particular, $(p-1)/2$ divides $e(F_v|\mathbb{Q}_p) = e_v$, for all $v \mid p$.

Proof. By equation (2) above, $N_p \geq 3 \Rightarrow c(\theta) > 1$, so $\theta|_{U_K^{(1)}} \neq 1$. So $\exists\, \tau \in I_w(K)$ with $\theta(\tau) = \zeta_p$, for some primitive p-th root of unity ζ_p. As ϵ_p is tame, $\epsilon(\tau) = 1$, and $\chi_\gamma^2(\tau) = \epsilon(\tau)^{\gamma-1} = 1$, for all $\gamma \in \Gamma$. But since τ is an element of a pro-p group, and p is odd, we conclude that $\chi_\gamma(\tau) = 1$, for all $\gamma \in \Gamma$. In other words, $\alpha(\tau) \equiv 1 \mod F^*$.

Both $\theta(\tau)$ and $\theta^\sigma(\tau)$ have some p-power order. The sum of two roots of unity of odd order cannot be zero. Hence by Proposition 3.1, $\alpha(\tau) \equiv (\theta + \theta^\sigma)(\tau) \equiv 1 \mod F^*$. Since F is totally real, $\theta(\tau)$ and $\theta^\sigma(\tau)$ are two roots of unity whose sum is a non-zero real number. Hence they are complex conjugates. □

Lemma 4.2. *If $p \neq 2$ is a supercuspidal prime of level > 0 for a p-minimal f, then F contains $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ and e_v is a multiple of $(p-1)/2$, for each prime v in F above p.*

Proof. Since p is odd, we can twist f by a suitable character to make the nebentypus tame at p, without changing the field F. As f was p-minimal, $N_p \geq 3$ must hold even after twisting. Now apply the previous lemma. □

We know that $\det(\rho_f) = \chi_\ell^{k-1}\epsilon$, where χ_ℓ is the ℓ-adic cyclotomic character. The nebentypus character ϵ may be adelized as follows. For $x \in \mathbb{Q}_p^*$, let $[x]$ denote the corresponding element $(1, \cdots, x, \cdots, 1) \in \mathbb{A}_{\mathbb{Q}}^*$. Then the idèlic character ϵ, restricted to $\mathbb{Q}_p^*$ is given by the following formula. For any $m \in \mathbb{Z}$ and $u \in \mathbb{Z}_p^*$,

$$\epsilon([p^m u]) = \epsilon'(p)^m \epsilon_p(u)^{-1}. \tag{3}$$

The Galois character $\epsilon|_{G_p}$ is also determined by this formula via the norm residue map of class field theory, which maps $\mathbb{Q}_p^* \subseteq \mathbb{A}_{\mathbb{Q}}^*$ onto a dense subset of the decomposition group G_p at p.

The local Langlands correspondence for GL_2 is described using the theory of admissible pairs. In [BH06], for any admissible pair (K, θ) the authors construct an automorphic representation π_θ with central character θ, as well as a 2-dimensional Galois representation, say ρ_θ, both unique up to isomorphism. Let p be a supercuspidal prime for f and (K, θ) be an admissible pair attached to f at p. There exists a character Δ_θ of K^*, such that $(K, \theta\Delta_\theta)$ is also an admissible pair, and the supercuspidal representation $\pi_{\theta\Delta_\theta}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ is in Langlands correspondence with $\rho_f|_{G_p}$. Equating the central character with the determinant on the Galois side, we get

$$(\theta\Delta_\theta)|_{\mathbb{Q}_p^*} = (\chi_\ell^{k-1}\epsilon)|_{\mathbb{Q}_p^*}.$$

We refer to [BH06, §34], for the explicit description of Δ_θ. But let us mention here that it is a quadratic character unless we are in case (c) of Theorem 2.1 (in which case Δ_θ has order 4), and that $\Delta_\theta|_{\mathbb{Q}_p^*}$ is always the unique non-trivial character factoring through $N_{K|\mathbb{Q}_p}(K^*)$. Using the equations above, we get that for any $u \in \mathbb{Z}_p^*$,

$$\theta(u) = \begin{cases} \epsilon_p(u)^{-1}, & \text{if } K = \mathbb{Q}_{p^2}, \\ \left(u, K|\mathbb{Q}_p\right) \cdot \epsilon_p(u)^{-1}, & \text{if } K|\mathbb{Q}_p \text{ is ramified.} \end{cases} \tag{4}$$

5. Local symbols

We use the same notations as in the previous section. Consider the nebentypus ϵ as a Galois character, and fix a square root $\sqrt{\epsilon(g)}$, for each $g \in G_{\mathbb{Q}}$. For all $\gamma \in G_F$, there exists a unique quadratic (of order 1 or 2) Dirichlet character ψ_γ, such that $\forall\, g \in G_{\mathbb{Q}}$,

$$\chi_\gamma(g) = \sqrt{\epsilon(g)}^{\gamma-1} \psi_\gamma(g). \tag{5}$$

Note that here χ_γ means the inner twist corresponding to the image of $\gamma \in G_F$ in its quotient $\Gamma = \mathrm{Gal}(E|F)$. For any $\gamma \in G_F$, let t_γ denote the fundamental discriminant corresponding to the character ψ_γ. The set $A = \{\psi_\gamma : \gamma \in G_F\}$ is an elementary 2-group. Let $\Gamma_0 \subseteq G_F$ be a fixed subset such that $\{\psi_\gamma : \gamma \in \Gamma_0\}$ forms a basis for the group A. For each $\gamma \in \Gamma_0$, choose a square-free positive integer n_γ prime to N, with $a_{n_\gamma} \neq 0$, and such that for all $\gamma' \in \Gamma_0$,

$$\psi_{\gamma'}(n_\gamma) = \begin{cases} -1, & \text{if } \gamma' = \gamma \\ 1, & \text{otherwise.} \end{cases} \tag{6}$$

For each n_γ, set $z_{n_\gamma} := a_{n_\gamma}^2 \epsilon^{-1}(n_\gamma) \in F^*$. Let v be a prime in F above some odd prime p. Let $[c_\epsilon]_v$ denote the class of the cocycle $c_\epsilon \in Z^2(G_{F_v}, \{\pm 1\})$ defined by $c_\epsilon(g,h) = \sqrt{\epsilon(g)}\sqrt{\epsilon(h)}\sqrt{\epsilon(gh)}^{-1}$. It follows from [Qu98] that,

$$[c_\epsilon]_v \sim \epsilon_v(-1) = \epsilon_p(-1)^{[F_v:\mathbb{Q}_p]}. \tag{7}$$

For any two elements $a, b \in F_v^*$, let us write $a = \pi_v^{v(a)} \cdot a'$ and $b = \pi_v^{v(b)} \cdot b'$, where π_v is a uniformizer in F_v. Then the local symbol $(a,b)_v$, which is independent of the choice of π_v, is given by the following equation:

$$(a,b)_v = (-1)^{v(a)v(b)(Nv-1)/2} \cdot \left(\frac{b'}{v}\right)^{v(a)} \cdot \left(\frac{a'}{v}\right)^{v(b)}, \tag{8}$$

where $\left(\frac{\cdot}{v}\right)$ is the standard quadratic residue symbol in the residue field of F_v. The next formula expressing the local Brauer class in terms of symbols follows from [GGQ05, Thm. 4.1].

Theorem 5.1. *The local Brauer class is given by*

$$[X_v] \sim [c_\epsilon]_v \otimes \bigotimes_{\gamma \in \Gamma_0} (z_{n_\gamma}, t_\gamma)_v.$$

Note that each t_γ must divide N. If p divides N, $p^* := \left(\frac{-1}{p}\right) \cdot p$ may or may not divide t_γ for a given $\gamma \in \Gamma_0$. For a fixed set Γ_0 as above, let $S := \{t_\gamma : \gamma \in \Gamma_0\}$. Let us write S as a disjoint union of three sets: $S = S_p \cup S^- \cup S^+$, where

$$S_p = \{t_\gamma \in S : p^* \mid t_\gamma\}, S^- = \{t_\gamma \in S \backslash S_p : \left(\frac{t_\gamma}{p}\right) = -1\}, S^+ = \{t_\gamma \in S \backslash S_p : \left(\frac{t_\gamma}{p}\right) = 1\}.$$

The statement of Theorem 5.1 is true for any choice of $\Gamma_0 \subseteq G_F$, as long as $\{\psi_\gamma : \gamma \in \Gamma_0\}$ forms a basis of the elementary 2-group A. If S_p consists of more than one element, we can choose some $t_{\gamma_0} \in S_p$ and multiply the (quadratic) characters corresponding to all other elements of S_p by ψ_{γ_0}, to construct a new basis of A, for which S_p is singleton. Similarly we can multiply ψ_{γ_0} by a character corresponding to any element of S^-, if necessary, and then assume that Γ_0 satisfies the following two conditions:

(a) If $S_p \neq \emptyset$, then it is singleton; denote the unique element of S_p by t_{γ_0},

(b) If $S_p = \{t_{\gamma_0}\} \neq \emptyset$ and $S^- \neq \emptyset$ too, then $\left(\frac{\tilde{t}_{\gamma_0}}{p}\right) = 1$, where $\tilde{t}_{\gamma_0} := t_{\gamma_0}/p^*$.

When $S_p \neq \emptyset$, we split the symbol $(z_{n_{\gamma_0}}, t_{\gamma_0})_v$ involved in the statement of Theorem 5.1 into a p-part and a prime-to-p part

$$(z_{n_{\gamma_0}}, t_{\gamma_0})_v = (z_{n_{\gamma_0}}, p^*)_v \cdot \left(\frac{\tilde{t}_{\gamma_0}}{v}\right)^{\upsilon(z_{n_{\gamma_0}})}.$$

Let p' be as in Section 2, satisfying $(*)$. Then the following general lemma computes the prime-to-p part of the formula for $[X_v]$ in Theorem 5.1.

Lemma 5.2. *If* Γ_0 *satisfies the conditions* (a) *and* (b) *above, then we have*

$$\left(\frac{\tilde{t}_{\gamma_0}}{v}\right)^{\upsilon(z_{n_{\gamma_0}})} \otimes \bigotimes_{t_\gamma \in S \setminus S_p} (z_{n_\gamma}, t_\gamma)_v = (-1)^{f_v \cdot \upsilon(a_{p'}^2 \epsilon^{-1}(p'))}. \tag{9}$$

(If $S_p = \emptyset$, then the first symbol on the left hand side is assumed to be 1.)

Proof. Note that if $t_\gamma \notin S_p$, then $\upsilon(t_\gamma) = 0$, hence $(z_{n_\gamma}, t_\gamma)_v = \left(\frac{t_\gamma}{v}\right)^{\upsilon(z_{n_\gamma})} = \left(\frac{t_\gamma}{p}\right)^{f_v \cdot \upsilon(z_{n_\gamma})}$, by equation (8).

First we consider the case where $S^- = \emptyset$. Hence $S \setminus S_p = S^+$, and so $(z_{n_\gamma}, t_\gamma)_v = \left(\frac{t_\gamma}{p}\right)^{f_v \cdot \upsilon(z_{n_\gamma})} = 1$, for all $t_\gamma \in S \setminus S_p$. Thus the left hand side of (9) reduces to just the first symbol, which is $\left(\frac{\tilde{t}_{\gamma_0}}{v}\right)^{\upsilon(z_{n_{\gamma_0}})} = \left(\frac{\tilde{t}_{\gamma_0}}{p}\right)^{f_v \cdot \upsilon(z_{n_{\gamma_0}})}$. If either $S_p = \emptyset$ or if $S_p = \{t_{\gamma_0}\}$ with $\left(\frac{\tilde{t}_{\gamma_0}}{p}\right) = 1$, then we use $(*)$ and the condition (a) on Γ_0 to check that $\psi_\gamma(p') = \left(\frac{t_\gamma}{p'}\right) = 1$, $\forall\, \gamma \in \Gamma_0$. Hence $\psi_\gamma(p') \overset{(5)}{=} \left(\frac{\chi_\gamma(p')}{\sqrt{\epsilon(p')}}\right)^{\gamma-1} = \left(\frac{a_{p'}}{\sqrt{\epsilon(p')}}\right)^{\gamma-1} = 1$, $\forall\, \gamma \in G_F$, as the characters ψ_γ for $\gamma \in \Gamma_0$ generates the group $\{\psi_\gamma : \gamma \in G_F\}$. Therefore $\frac{a_{p'}}{\sqrt{\epsilon(p')}} \in F^*$ and $\upsilon(a_{p'}^2 \epsilon^{-1}(p')) \equiv 0 \mod 2$. Thus, both sides of equation (9) equal 1. On the other hand, if $S_p \neq \emptyset$ and $\left(\frac{\tilde{t}_{\gamma_0}}{p}\right) = -1$, then one checks that p' is a candidate for the integer n_{γ_0} defined in (6). Hence $\left(\frac{\tilde{t}_{\gamma_0}}{p}\right)^{f_v \cdot \upsilon(z_{n_{\gamma_0}})} = (-1)^{f_v \cdot \upsilon(a_{p'}^2 \epsilon^{-1}(p'))}$, as desired.

So now assume that $S^- = \{t_{\gamma_1} = t_1, \cdots, t_{\gamma_m} = t_m\} \neq \emptyset$. By condition (b) on Γ_0, we have the first symbol $\left(\frac{\tilde{t}_{\gamma_0}}{v}\right)^{\upsilon(z_{n_{\gamma_0}})} = 1$ in this case. Choose distinct primes r_j, with $a_{r_j} \neq 0$, for $j = 0, 1, 2, \cdots, m-1$ recursively, satisfying the following properties:

(1) $r_0 = p'$,

(2) $\left(\frac{t_\gamma}{r_j}\right) = 1$, for all $t_\gamma \in S^+ \cup S_p$,

(3) $\left(\frac{t_i}{r_j}\right) = (-1)^{\delta_{ij}} \left(\frac{t_i}{r_{j-1}}\right)$, for all $i = 1, 2, \cdots, m$ (and $j \geq 1$).

Note that our choice of r_0 is consistent with property (2) above. Indeed, if $S_p \neq \emptyset$, then by condition (b), $\left(\frac{t_{\gamma_0}}{p'}\right) \overset{(*)}{=} \left(\frac{\tilde{t}_{\gamma_0}}{p}\right) = 1$, and if $t_\gamma \in S^+$, then $\left(\frac{t_\gamma}{p'}\right) \overset{(*)}{=} \left(\frac{t_\gamma}{p}\right) = 1$.

Next we define $n_i := \begin{cases} r_{i-1} r_i, & \text{if } 1 \leq i \leq m-1, \\ r_{m-1}, & \text{if } i = m. \end{cases}$

It can be checked that each n_i satisfies the criterion given in (6) for n_{γ_i}. By the telescoping argument used in the proof of [GGQ05, Thm. 4.3] or [BG03, Thm. 4.1.11], we get

$$\bigotimes_{t_\gamma \in S \setminus S_p} (z_{n_\gamma}, t_\gamma)_v = \prod_{t_\gamma \in S^- \cup S^+} \left(\frac{t_\gamma}{p}\right)^{f_v \cdot v(z_{n_\gamma})} = \prod_{i=1}^{m} (-1)^{f_v \cdot v(z_{n_i})} = (-1)^{f_v \cdot v(a_{p'}^2, \epsilon^{-1}(p'))}.$$

□

Applying equation (7) and Lemma 5.2, we get the following simplification of Theorem 5.1:

$$[X_v] \sim \epsilon_p (-1)^{[F_v:\mathbb{Q}_p]} \cdot (z_{n_{\gamma_0}}, p^*)_v \cdot (-1)^{f_v \cdot v(a_{p'}^2, \epsilon^{-1}(p'))}, \tag{10}$$

where, by equation (8), the middle symbol

$$(z_{n_{\gamma_0}}, p^*)_v = (-1)^{v(z_{n_{\gamma_0}}) e_v (p^{f_v}-1)/2} \cdot \left(\frac{(p^*)'}{v}\right)^{v(z_{n_{\gamma_0}})} \cdot \left(\frac{z'_{n_{\gamma_0}}}{v}\right)^{e_v}, \tag{11}$$

when $S_p = \{t_{\gamma_0}\}$, and is taken to be trivial when $S_p = \emptyset$. The formulas (10) and (11) will be the starting point in our computation of the local Brauer class at a supercuspidal prime. Note that these formulas depend on the choice of Γ_0 satisfying the conditions (a) and (b) stated before Lemma 5.2.

6. Unramified supercuspidal primes

In this section, we will prove Theorem 2.1 (a) in two parts, and then study some of its consequences.

Theorem 6.1. *Let p be an odd unramified supercuspidal prime with $\theta(s) + \theta(s)^p \neq 0$. Then, we have $[X_v] \sim (-1)^{f_v \cdot v(a_{p'}^2, \epsilon^{-1}(p'))}$, for $v \mid p$.*

Proof. Let the set Γ_0 satisfy the two conditions before Lemma 5.2.

Suppose $S_p = \{t_{\gamma_0}\} \neq \emptyset$, then by equation (10) it is enough to show that $\epsilon_p(-1)^{[F_v:\mathbb{Q}_p]} \cdot (z_{n_{\gamma_0}}, p^*)_v = 1$. Let $g_s \in G_{p^2}$ be an element which is mapped to $s \in \mathbb{Q}_{p^2}^*$ under the reciprocity map. By local class field theory, we have $\psi(g_s) = \psi([N_{\mathbb{Q}_{p^2}|\mathbb{Q}_p}(s)])$, for any Dirichlet (or idèlic) character ψ. Note that p divides the conductor of a quadratic character ψ if and only if ψ is non-trivial on $(\mathbb{Z}/p\mathbb{Z})^*$. The norm $N_{\mathbb{Q}_{p^2}|\mathbb{Q}_p}(s) = s^{p+1} = x$, say, is a generator of the group $(\mathbb{Z}/p\mathbb{Z})^*$ inside $\mathbb{Z}_p^*$. Hence for any $\gamma \in \Gamma_0$, $\psi_\gamma(g_s) = -1 \iff p^* | t_\gamma \iff \gamma = \gamma_0$, where the last implication follows from condition (a) on Γ_0. Looking at the definition of n_{γ_0} given in (6), we conclude that $\forall\, \gamma \in \Gamma_0$ and hence $\forall\, \gamma \in G_F$, $\psi_\gamma(g_s) = \psi_\gamma(n_{\gamma_0})$. Hence it follows from (1) and (5) that $\left(\frac{\alpha(g_s)}{\sqrt{\epsilon(g_s)}}\right)^{\gamma-1} = \left(\frac{a_{n_{\gamma_0}}}{\sqrt{\epsilon(n_{\gamma_0})}}\right)^{\gamma-1}$, for all $\gamma \in G_F$. So we have

$$z_{n_{\gamma_0}} = \frac{a_{n_{\gamma_0}}^2}{\epsilon(n_{\gamma_0})} \equiv \frac{\alpha^2(g_s)}{\epsilon(g_s)} \equiv \frac{(\theta(s) + \theta(s)^p)^2}{\epsilon_p^{-1}(x)} \mod F^{*2},$$

where the last congruence is by part (2) of Proposition 3.1, noting that the trace $\theta(s) + \theta(s)^p$ is non-zero by assumption. As $\theta(s) + \theta(s)^p = \mathrm{Tr}_{\mathbb{Q}_{p^2}|\mathbb{Q}_p}(\theta(s)) \in \mathbb{Q}_p^* \subseteq F_v^*$, we get $z_{n_{\gamma_0}} \equiv \epsilon_p(x) \mod (F_v^*)^2$, and $\upsilon(z_{n_{\gamma_0}}) \equiv 0 \mod 2$. By equation (11), we get $(z_{n_{\gamma_0}}, p^*)_v = \left(\frac{z'_{n_{\gamma_0}}}{v}\right)^{e_v} = \left(\frac{\epsilon_p(x)}{v}\right)^{e_v} = \left(\frac{\epsilon_p(x)}{p}\right)^{f_v e_v} = \left(\epsilon_p(x)^{(p-1)/2}\right)^{e_v f_v} = \epsilon_p(-1)^{e_v f_v}$, therefore $\epsilon_p(-1)^{[F_v:\mathbb{Q}_p]} \cdot (z_{n_{\gamma_0}}, p^*)_v = \left(\epsilon_p(-1)^{e_v f_v}\right)^2 = 1$.

If $S_p = \emptyset$, then $\psi_\gamma(g_s) = 1$ for all $\gamma \in G_F$, and a similar argument shows that $(\theta(s) + \theta(s)^p)^2 \equiv \epsilon_p^{-1}(x) \mod F^{*2}$. Thus we get $\epsilon_p(x) \in F_v^{*2}$, which implies that $\left(\frac{\epsilon_p(x)}{v}\right) = \left(\frac{\epsilon_p(x)}{p}\right)^{f_v} = \epsilon_p(-1)^{f_v} = 1$. Hence $\epsilon_p(-1)^{[F_v:\mathbb{Q}_p]} = 1$, and the result follows by equation (10). □

Note that if p is a supercuspidal prime of level zero, then by the regularity condition (1) on θ in the definition of an admissible pair, we have $\theta(s)^{p-1} \neq 1$. The condition on $\theta(s)$ in the hypothesis of Theorem 6.1 is equivalent to $\theta(s)^{p-1} \neq -1$. However, for an unramified supercuspidal prime of *positive* level we remove this condition now.

Theorem 6.2. *Let p be an odd unramified supercuspidal prime of level > 0 for a p-minimal non-CM newform f. Then, we have $[X_v] \sim (-1)^{f_v \cdot \upsilon(a_{p'}^2, \epsilon^{-1}(p'))}$, for $v \mid p$.*

Proof. Since p is an unramified supercuspidal prime of level > 0, $N_p \geq 4$. We twist f by a suitable character at p if necessary, and assume that ϵ_p is tame. Note that the formula to be proved is invariant under twist by a character of p-power conductor. As f was p-minimal to begin with, N_p cannot decrease by twisting. Thus the hypothesis of Lemma 4.1 is satisfied. If $\theta(s) + \theta(s)^p \neq 0$, we are done by Theorem 6.1. So assume $\theta(s) + \theta(s)^p = 0$, i.e., $\theta(s)^{p-1} = -1$.

As explained in the proof of Theorem 6.1, if $S_p \neq \emptyset$, then

$$z_{n_{\gamma_0}} = \frac{a_{n_{\gamma_0}}^2}{\epsilon(n_{\gamma_0})} \equiv \frac{\alpha^2(g_s)}{\epsilon(g_s)} \mod (F^*)^2. \tag{12}$$

Note that $\theta^\sigma(s) = \theta(s)^p = -\theta(s)$. By Lemma 4.1, $\exists\, \tau \in I_w(K)$, such that

$$\alpha(g_s) \equiv \alpha(g_s\tau) \equiv (\theta + \theta^\sigma)(g_s\tau) \equiv \theta(s)(\zeta_p - \zeta_p^{-1}) \mod F^*. \tag{13}$$

Using equations (3) and (4), we get

$$\epsilon(g_s) = \epsilon([N_{\mathbb{Q}_{p^2}|\mathbb{Q}_p}(s)]) = \epsilon_p^{-1}(s^{p+1}) = \theta(s^{p+1}). \tag{14}$$

By the equations (12), (13), and (14) and using that $\theta(s)^{p-1} = -1$, we get

$$z_{n_{\gamma_0}} \equiv -(\zeta_p - \zeta_p^{-1})^2 \equiv -p^2(\zeta_p - \zeta_p^{-1})^2 \mod (F^*)^2. \tag{15}$$

Suppose $p \equiv 3 \mod 4$, then $\sqrt{-p}(\zeta_p - \zeta_p^{-1})$ is a totally real element in $\mathbb{Q}(\zeta_p)$, hence it is contained in the field $\mathbb{Q}(\zeta_p + \zeta_p^{-1}) \subseteq F$, by Lemma 4.1. Thus we have $-p(\zeta_p - \zeta_p^{-1})^2 \in (F^*)^2$, hence

$$\left(\frac{z'_{n_{\gamma_0}}}{v}\right) = \left(\frac{(-p^2(\zeta_p - \zeta_p^{-1})^2)'}{v}\right) = \left(\frac{(p)'}{v}\right), \tag{16}$$

where we write $(p)'$ for the prime-to-π_v part of p to distinguish it from the auxiliary prime p'. From equation (15), we have the valuation

$$\upsilon(z_{n_{\gamma_0}}) \equiv \upsilon\left(-(\zeta_p - \zeta_p^{-1})^2\right) = 2e_v/(p-1) \equiv e_v \bmod 2. \tag{17}$$

As $p \equiv 3 \mod 4$, $(Nv - 1)/2 = (p^{f_v} - 1)/2 \equiv f_v \mod 2$. Hence by (11) and (16), $(z_{n_{\gamma_0}}, p^*)_v = (-1)^{e_v e_v f_v} \cdot \left(\frac{(-p)'}{v}\right)^{e_v} \cdot \left(\frac{(p)'}{v}\right)^{e_v} = (-1)^{e_v f_v} \cdot \left(\frac{-1}{v}\right)^{e_v} = (-1)^{e_v f_v} \cdot \left(\frac{-1}{p}\right)^{e_v f_v} = 1.$

By (4), $\epsilon_p(-1) = \theta(-1) = \theta(s)^{(p^2-1)/2} = (\theta(s)^{p-1})^{(p+1)/2} = (-1)^{(p+1)/2} = 1$, and therefore by (10), we get $[X_v] \sim (-1)^{f_v \cdot \upsilon(a_{p'}^2, \epsilon^{-1}(p'))}$.

Now suppose $p \equiv 1 \mod 4$. By Lemma 4.1, we get that e_v is even, and $\sqrt{p} = \sqrt{p^*} \in \mathbb{Q}_p(\zeta_p + \zeta_p^{-1})^* \subseteq F_v^*$, hence $\left(\frac{(p^*)'}{v}\right) = 1$. As all the other symbols involved in $\epsilon_p(-1)^{[F_v:\mathbb{Q}_p]} \cdot (z_{n_{\gamma_0}}, p^*)_v$ have an e_v in the exponent, we get $[X_v] \sim (-1)^{f_v \cdot \upsilon(a_{p'}^2, \epsilon^{-1}(p'))}$, by equation (10).

If $S_p = \emptyset$, then the result follows trivially from the fact that $\epsilon_p(-1)^{[F_v:\mathbb{Q}_p]} = 1$ in all cases, as shown above. □

Remark The result above is false in general for supercuspidal primes of level zero. There exist p-minimal newforms with $C_p < N_p = 2$, with $\theta(s) + \theta(s)^p = 0$, such that f_v is even, but X_v is ramified at some $v \mid p$; see Example 5 in the last section.

Theorems 6.1 and 6.2 imply Theorem 2.1 (a). Indeed, if $\theta(s) + \theta(s)^p \neq 0$, the first theorem applies. Otherwise, $\theta(s) + \theta(s)^p$ vanishes. By the hypothesis made in Theorem 2.1, we may assume that the p-minimal twist of f has positive level, and the second theorem applies. We remark here that the vanishing of $\theta(s) + \theta(s)^p$ does not depend on the twist.

Corollary 6.3. *Let p be an odd unramified supercuspidal prime for f. Also assume that $\theta(s) + \theta(s)^p \neq 0$, if the p-minimal twist of f is of level* 0*.*

(a) *If $N' = 1$, or if p has odd order in $(\mathbb{Z}/N'\mathbb{Z})^*$, then X_v is a matrix algebra over F_v.*
(b) *If $\mathbb{Q}_{p^2} \subseteq F_v$, then X_v is a matrix algebra over F_v.*
(c) *If X_v has non-trivial Brauer class in* $\mathrm{Br}(F_v)$*, then v must divide* $\mathrm{disc}(E|F)$*.*

Proof. We have:

(a) For any Dirichlet character χ, let χ' denote its prime-to-p component.
If $N' = 1$, then $\chi_\gamma(p') \stackrel{(*)}{=} \chi'_\gamma(p') = 1$, for all $\gamma \in \Gamma$. Hence $a_{p'} \in F^*$.
If $N' \neq 1$, let $n = 2m + 1$ be the order of $p \mod N'$, i.e., $p^n \equiv 1 \mod N'$. Hence we have $\chi_\gamma(p')^n = \chi'_\gamma(p^n) = 1$, for all $\gamma \in \Gamma$. We know that $\chi'_\gamma(p)^2 = \chi_\gamma(p')^2 = \epsilon(p')^{\gamma-1}$, for all $\gamma \in \Gamma$. Hence $\chi_\gamma(p') = \chi'_\gamma(p)^{n-2m} = \left(\chi'_\gamma(p)^2\right)^{-m} = (\epsilon(p')^{\gamma-1})^{-m}$, which implies that $a_{p'}^{\gamma-1} = (\epsilon(p')^{-m})^{\gamma-1}$, for all $\gamma \in \Gamma$. So $a_{p'} \equiv \epsilon(p')^{-m} \mod F^*$.
Now use Theorems 6.1 and 6.2 to get the result in both cases.
(b) $\mathbb{Q}_{p^2} \subseteq F_v$ would imply that f_v is even. Now apply Theorems 6.1 and 6.2. Note that here $K = \mathbb{Q}_{p^2}$, so we have proved a part of Corollary 2.3.
(c) By Theorems 6.1 and 6.2, $[X_v] \sim -1$ implies that $\upsilon(a_{p'}^2, \epsilon^{-1}(p'))$ is odd. If υ is extended to a valuation on E^*, then clearly $\upsilon(a_{p'})$ is not an integer. Hence v ramifies in E, or $v \mid \mathrm{disc}(E|F)$.

□

7. Ramified supercuspidal primes

In this section we will prove part (b) and (c) of Theorem 2.1, and derive some consequences.

Let p be an odd ramified supercuspidal prime for f. As $K|\mathbb{Q}_p$ is ramified, there are two possible choices for K. If $(p, K|\mathbb{Q}_p) = 1$, then $K = \mathbb{Q}_p(\sqrt{-p})$, and if $(p, K|\mathbb{Q}_p) = -1$, then $K = \mathbb{Q}_p(\sqrt{-p\zeta_{p-1}})$. Depending on K, we can always choose a uniformizer $\pi = \sqrt{-p}$ or $\sqrt{-p\zeta_{p-1}}$, and write $K = \mathbb{Q}_p(\pi)$. For $\sigma \in G_p$, any lift of the generator of $\mathrm{Gal}(K|\mathbb{Q}_p)$, we have $\pi^\sigma = -\pi$, and $N_{K|\mathbb{Q}_p}(\pi) = -\pi^2$.

Let p' and p'' be as in Section 2, satisfying the properties $(*)$ and $(**)$ respectively.

Theorem 7.1. *Let $p \equiv 1 \mod 4$ be a ramified supercuspidal prime. Then, for any prime $v \mid p$ in F, the Brauer class is given by $[X_v] \sim (-1)^{f_v \cdot v(a_{p'}^2, \epsilon^{-1}(p'))}$.*

Proof. The formula to be proved is invariant under twist by a Dirichlet character of p-power conductor. So we twist f by a suitable character, if necessary, and then assume that f is p-minimal. Since p is a ramified supercuspidal prime, even after applying the twist the level will still be positive, i.e., the hypothesis of Lemma 4.2 is satisfied.

Since $p \equiv 1 \mod 4$, e_v is even and $\sqrt{p} \in \mathbb{Q}(\zeta_p + \zeta_p^{-1})^* \subseteq F^*$, by Lemma 4.2. As $p^* = p$ is a square in F_v^*, we get $\left(\frac{(p^*)'}{v}\right) = 1$. Hence by (10) and (11), we get $[X_v] \sim (-1)^{f_v \cdot v(a_{p'}^2, \epsilon^{-1}(p'))}$. □

Remark Since the formula turns out to be the same as in the case of unramified supercuspidal primes, Corollary 6.3 also holds for ramified supercuspidal primes $p \equiv 1 \mod 4$, as stated in Corollary 2.2. But note that none of the results in Corollary 2.2 are true for ramified supercuspidal primes $p \equiv 3 \mod 4$; see the examples in the last section.

Lemma 7.2. *For any odd ramified supercuspidal prime p, the valuations of $a_{p'}$ and $a_{p''}$ are related by the following equation:*

$$(-1)^{v(a_{p'}^2, \epsilon^{-1}(p'))} = (-1)^{e_v(k-1)} \cdot \left(\left(\frac{-1}{p}\right)\epsilon_p(-1)\right)^{2e_v/(p-1)} \cdot (p, K|\mathbb{Q}_p)^{v(a_{p''}^2, \epsilon^{-1}(p''))}.$$

Proof. The equation to be proved is invariant under twist by a character of p-power conductor. So without loss of generality we assume ϵ_p to be tame, so that we can apply Lemma 4.1 . We write the ramified quadratic extension K as $K = \mathbb{Q}_p(\pi)$, where either $\pi = \sqrt{-p}$, or $\pi = \sqrt{-p\zeta_{p-1}}$.

Let $g_\pi \in G_K$ be an element whose image under the reciprocity map is $\pi \in K^*$. Taking the determinant of ρ_f (described in Section 4) at g_π, we get

$$\theta(\pi)\theta^\sigma(\pi) = \theta(\pi)\theta(-\pi) = \chi_\ell^{k-1}(g_\pi)\epsilon(g_\pi)$$

$$\Longrightarrow \theta(\pi)^2\epsilon^{-1}(g_\pi) = \theta(-1)p^{k-1}. \tag{18}$$

If $\theta(-1) = 1$, then $\alpha(g_\pi) \equiv (\theta + \theta^\sigma)(\pi) = 2\theta(\pi) \equiv \theta(\pi) \mod F^*$. So by (18), we have $\alpha^2(g_\pi)\epsilon^{-1}(g_\pi) \equiv p^{k-1} \mod (F^*)^2$. If $\theta(-1) = -1$, we use the element $\tau \in I_w(K)$ given by Lemma 4.1, to get $\alpha(g_\pi) \equiv \alpha(g_\pi\tau) \equiv (\theta + \theta^\sigma)(g_\pi\tau) \equiv \theta(\pi)(\zeta_p - \zeta_p^{-1}) \mod F^*$, and by (18), we have $\alpha^2(g_\pi)\epsilon^{-1}(g_\pi) \equiv -p^{k-1}(\zeta_p - \zeta_p^{-1})^2 \mod (F^*)^2$. Therefore using $v(p) = e_v$ and $v\left((\zeta_p - \zeta_p^{-1})^2\right) = 2e_v/(p-1)$, we get

$$v\left(\alpha^2(g_\pi)\epsilon^{-1}(g_\pi)\right) \equiv \begin{cases} e_v(k-1) \mod 2, & \text{if } \theta(-1) = 1, \\ e_v(k-1) + 2e_v/(p-1) \mod 2, & \text{if } \theta(-1) = -1. \end{cases}$$

By (4), $\theta(-1) = \left(\frac{-1}{p}\right)\epsilon_p(-1)$, so the congruence above can be written as

$$(-1)^{v(\alpha^2(g_\pi)\epsilon^{-1}(g_\pi))} = (-1)^{e_v(k-1)} \cdot \left(\left(\frac{-1}{p}\right)\epsilon_p(-1)\right)^{2e_v/(p-1)}. \tag{19}$$

By class field theory, $g_\pi \in G_p \subseteq G_\mathbb{Q}$ is mapped to $[N_{K|\mathbb{Q}_p}(\pi)] \in \mathbb{Q}_p^* \subseteq \mathbb{A}_\mathbb{Q}^*$. Each χ_γ is realized as an idèlic character as in equation (3), and hence as a Galois character.

If $(p, K|\mathbb{Q}_p) = 1$, then $N_{K|\mathbb{Q}_p}(\pi) = p$, and for all $\gamma \in \Gamma$, we have $\chi_\gamma(g_\pi) = \chi_\gamma([p]) = \chi'_\gamma(p) \overset{(*)}{=} \chi_\gamma(p')$. Hence applying (1) and (3), we get that

$$\alpha^2(g_\pi)\epsilon^{-1}(g_\pi) \equiv a_{p'}^2\epsilon^{-1}(p') \mod (F^*)^2.$$

If $(p, K|\mathbb{Q}_p) = -1$, then $N_{K|\mathbb{Q}_p}(\pi) = p\zeta_{p-1}$. So for all $\gamma \in \Gamma$, we have $\chi_\gamma(g_\pi) = \chi_\gamma([p\zeta_{p-1}]) \overset{(**)}{=} \chi'_\gamma(p)\chi_{\gamma,p}^{-1}(p'') \overset{(*)}{=} \chi_\gamma(p')\chi_\gamma^{-1}(p'')$. Applying (1) and (3), we get

$$\left(\alpha^2(g_\pi)\epsilon^{-1}(g_\pi)\right) \cdot \left(a_{p''}^2\epsilon^{-1}(p'')\right) \equiv a_{p'}^2\epsilon^{-1}(p') \mod (F^*)^2.$$

Now the result follows from equation (19) above. □

Corollary 7.3. *Let $p \equiv 1 \mod 4$ be a ramified supercuspidal prime such that $K = \mathbb{Q}_p(\sqrt{p})$. Then, for any prime $v \mid p$ in F, we have $[X_v] \sim \epsilon_p(-1)^{2[F_v:\mathbb{Q}_p]/(p-1)}$.*

Proof. As explained in the proof of Theorem 7.1, without loss of generality we may assume that f is p-minimal, where p is a supercuspidal prime of level > 0. By the hypothesis $K = \mathbb{Q}_p(\sqrt{p}) = \mathbb{Q}_p(\sqrt{-p})$, as $p \equiv 1 \mod 4$. Hence we have $\left(\frac{-1}{p}\right) = 1 = (p, K|\mathbb{Q}_p)$. By Lemma 4.2, e_v is even. Now we apply Lemma 7.2 to the statement of Theorem 7.1 to get the result. □

Theorem 7.4. *Let $p \equiv 3 \mod 4$ be a ramified supercuspidal prime. Let v be a prime in F such that e_v is odd. Then $[X_v] \sim \left((-1)^k a_{p''}^2\epsilon^{-1}(p''),\ KF_v|F_v\right)$.*

Proof. Since e_v is odd, $K \not\subseteq F_v$ and $KF_v|F_v$ is a ramified proper quadratic extension. So we can and do choose a uniformizer $\pi_v \in N_{KF_v|F_v}(KF_v)^* \subseteq F_v^*$. Note that in this case, we have $N_{KF_v|F_v}(O_{KF_v}^*) = O_{F_v}^{*2}$, where O^* denotes units. Hence for any $a = \pi_v^{v(a)} \cdot a' \in F_v^*$, we have $\left(\frac{a'}{v}\right) = (a, KF_v|F_v)$.

If $(p, K|\mathbb{Q}_p) = 1$, then $\left(\frac{(p)'}{v}\right) = (p,\ KF_v|F_v) = 1$. If $(p, K|\mathbb{Q}_p) = -1$, then $K = \mathbb{Q}_p(\sqrt{-p\zeta_{p-1}})$ and $N_{KF_v|F_v}(\sqrt{-p\zeta_{p-1}}) = p\zeta_{p-1}$, so $\left(\frac{(p\zeta_{p-1})'}{v}\right) = (p\zeta_{p-1}, KF_v|F_v) = 1$, which implies that $\left(\frac{(p)'}{v}\right) = \left(\frac{\zeta_{p-1}}{v}\right) = (-1)^{f_v}$. Combining both the cases we get

$$\left(\frac{(p^*)'}{v}\right) = \left(\frac{-1}{v}\right)\left(\frac{(p)'}{v}\right) = \left(\frac{-1}{p}\right)^{f_v}\left((p, K|\mathbb{Q}_p)\right)^{f_v} = \left(-(p, K|\mathbb{Q}_p)\right)^{f_v}. \tag{20}$$

If $\emptyset \neq S_p = \{t_{\gamma_0}\}$, then as in the proof of Theorem 6.1, we use (6) and $(**)$ to choose n_{γ_0} to be p'', hence $z_{n_{\gamma_0}} = a_{p''}^2\epsilon^{-1}(p'')$. Now using (10), (11), (20), Lemma 7.2, and the facts that e_v is odd, $(p-1)/2$ is odd and so $(p^{f_v} - 1)/2 \equiv f_v \mod 2$, we get

$$\begin{aligned}
[X_v] &\overset{(10)}{\sim} \epsilon_p(-1)^{[F_v:\mathbb{Q}_p]}\cdot(z_{n_{\gamma_0}},p^*)_v\cdot(-1)^{f_v\cdot v(a_{p'}^2\epsilon^{-1}(p'))}\\
&\overset{(11)}{=} \epsilon_p(-1)^{e_vf_v}\cdot(-1)^{v(z_{n_{\gamma_0}})\cdot e_v\cdot(p^{f_v}-1)/2}\left(\frac{(p^*)'}{v}\right)^{v(z_{n_{\gamma_0}})}\left(\frac{(z_{n_{\gamma_0}})'}{v}\right)^{e_v}\\
&\quad\cdot(-1)^{f_v\cdot v(a_{p'}^2\epsilon^{-1}(p'))}\\
&\overset{(20),\,7.2}{=} \epsilon_p(-1)^{f_v}\cdot(-1)^{f_v\cdot v(z_{n_{\gamma_0}})}\left(-(p,K|\mathbb{Q}_p)\right)^{f_v\cdot v(z_{n_{\gamma_0}})}\left(\frac{(z_{n_{\gamma_0}})'}{v}\right)\\
&\quad\cdot\left((-1)^{(k-1)}(-\epsilon_p(-1))(p,K|\mathbb{Q}_p)^{v(a_{p''}^2\epsilon^{-1}(p''))}\right)^{f_v}\\
&= (-1)^{kf_v}\cdot\left(\frac{(z_{n_{\gamma_0}})'}{v}\right)\sim\left(\frac{((-1)^kz_{n_{\gamma_0}})'}{v}\right)\\
&= \left((-1)^kz_{n_{\gamma_0}},\ KF_v|F_v\right)\\
&= \left((-1)^ka_{p''}^2\epsilon^{-1}(p''),\ KF_v|F_v\right).
\end{aligned}$$

If $S_p=\emptyset$, then $\psi_\gamma(p'')=1,\ \forall\,\gamma\in G_F$, so it follows from (5) that $a_{p''}^2\epsilon^{-1}(p'')\in F^{*2}\subseteq F_v^{*2}$. Therefore the symbol $(z_{n_{\gamma_0}},p^*)_v$ in equation (10) can be replaced by the trivial symbol $(a_{p''}^2\epsilon^{-1}(p''),p^*)_v$, and then the same proof (as above) works. □

The next lemma is a basic application of algebraic number theory and hence we state it without proof.

Lemma 7.5. *Let ϖ be any uniformizer in $\mathbb{Z}_p$. If e_v is even, then either $\sqrt{\varpi}\in F_v^*$ or $\sqrt{\varpi\zeta_{p^{f_v}-1}}\in F_v^*$. Moreover, if $K|\mathbb{Q}_p$ is a ramified quadratic extension, then $KF_v|F_v$ is an unramified extension of degree 1 or 2.*

Theorem 7.6. *Let $p\equiv 3\mod 4$ be a ramified supercuspidal prime and suppose that e_v is even. Then, the formula $[X_v]\sim\left((-1)^ka_{p''}^2\epsilon^{-1}(p''),\ KF_v|F_v\right)$ still holds.*

Proof. As p'' is a candidate for the integer n_{γ_0}, by (10), (11), Lemma 7.2 and the assumption that e_v is even, we get that if $S_p\neq\emptyset$, then

(21)
$$[X_v]\sim\left(\frac{(p^*)'}{v}\right)^{v(a_{p''}^2\epsilon^{-1}(p''))}\cdot(-1)^{f_v\cdot v(a_{p'}^2\epsilon^{-1}(p'))}\sim\left(\left(\frac{(-p)'}{v}\right)\cdot(p,K|\mathbb{Q}_p)^{f_v}\right)^{v(a_{p''}^2\epsilon^{-1}(p''))}.$$

So we compute the symbol $\left(\frac{(-p)'}{v}\right)$ case by case. We use the fact that since $p\equiv 3\mod 4$, the possibilities for K are $\mathbb{Q}_p(\sqrt{-p})$ and $\mathbb{Q}_p(\sqrt{p})=\mathbb{Q}_p(\sqrt{-p\zeta_{p-1}})$, and these occur exactly when $(p,K|\mathbb{Q}_p)=1$ or -1, respectively.

Case 1: Assume $K\subseteq F_v$. If $(p,K|\mathbb{Q}_p)=1$, then $\sqrt{-p}\in K\subseteq F_v\Rightarrow\left(\frac{(-p)'}{v}\right)=1$. If $(p,K|\mathbb{Q}_p)=-1$, then $\sqrt{p}\in K\subseteq F_v\Rightarrow\left(\frac{(p)'}{v}\right)=1\Rightarrow\left(\frac{(-p)'}{v}\right)=\left(\frac{-1}{v}\right)=(-1)^{f_v}$.

Case 2: Assume $K\nsubseteq F_v$. By Lemma 7.5, $KF_v|F_v$ is a proper quadratic unramified extension. If $(p,K|\mathbb{Q}_p)=1$, then $\sqrt{-p}\notin F_v$, so by Lemma 7.5 with $\varpi=-p$,

$$\sqrt{-p\zeta_{p^{f_v}-1}}\in F_v\Rightarrow\left(\frac{(-p\zeta_{p^{f_v}-1})'}{v}\right)=1\Rightarrow\left(\frac{(-p)'}{v}\right)=\left(\frac{\zeta_{p^{f_v}-1}}{v}\right)=-1.$$

If $(p,K|\mathbb{Q}_p)=-1$, then $\sqrt{p}\notin F_v$, so by Lemma 7.5 with $\varpi=p$,

$$\sqrt{p\zeta_{p^{f_v}-1}}\in F_v\Rightarrow\left(\frac{(p\zeta_{p^{f_v}-1})'}{v}\right)=1\Rightarrow\left(\frac{(-p)'}{v}\right)=\left(\frac{-\zeta_{p^{f_v}-1}}{v}\right)=(-1)^{f_v+1}.$$

Applying these to equation (21), we get

$$[X_v] \sim \begin{cases} 1, & \text{if } K \subseteq F_v, \\ (-1)^{v(a_{p''}^2 \epsilon^{-1}(p''))}, & \text{otherwise.} \end{cases}$$

It can be checked that this formula is valid even if $S_p = \emptyset$, as in the proof of Theorem 7.4. Equivalently, $[X_v] \sim \left((-1)^k a_{p''}^2 \epsilon^{-1}(p''),\ KF_v|F_v\right)$, as $KF_v|F_v$ is unramified. □

Remark Note that the factor $(-1)^k$ above does not have any significance, since $KF_v|F_v$ is unramified. We keep it only to get a uniform formula for all e_v, even or odd.

8. Numerical examples

Here are some numerical examples in support of our results. We used the program `Sage` to compute the admissible pair (K, θ) attached to a supercuspidal prime of a newform in some cases. One may also directly check the hypothesis on $\theta(s)$ in Theorem 6.1 in the case of level zero supercuspidals, using equation (4), together with the fact that the order of $\theta(s)$ divides $p^2 - 1$, but not $p - 1$. To determine the local Brauer class $[X_v]$, we used the program `Endohecke` for newforms with quadratic character, and the data from the tables in [GGQ05] and [Qu05] for newforms with arbitrary character.

(1) $f \in S_5(75, [1, 0])$, $E = \mathbb{Q}(\sqrt{-35})$, $F = \mathbb{Q}$; $p = 5$ is an unramified supercuspidal prime of level zero with $\theta(s) + \theta(s)^5 \neq 0$. For $p = 5$, $p' = 101$ satisfies $(*)$ and we computed $a_{101}^2 = -1800^2 \cdot 35$. So $f_v \cdot v\left(a_{p'}^2 \epsilon^{-1}(p')\right) = 1 \cdot v_5(1800^2 \cdot 35) = 5 \equiv 1 \mod 2$. By Thm. 6.1, X_5 is ramified.

There exists another newform, say $g \in S_5(75, [1, 0])$, with $F = \mathbb{Q}$ and $E = \mathbb{Q}(\sqrt{-14})$, such that $p = 5$ is an unramified supercuspidal prime of level zero with $\theta(s) + \theta(s)^5 \neq 0$. By Cor. 6.3 (c), X_5 has to be unramified. But, the admissible pairs attached to f and g at p are equivalent. This shows that the $\bar{\mathbb{Q}}_\ell$-isomorphism class of the local ℓ-adic Galois representation at a supercuspidal prime p fails to predict the Brauer class $[X_v]$ above p, even in the simplest case $F = \mathbb{Q}$.

(2) $f \in S_2(72, [1, 1, 3])$, $E = \mathbb{Q}(\sqrt{-2}, \sqrt{-3})$, $F = \mathbb{Q}$. Here $p = 3$ is an unramified supercuspidal prime of level zero with order $o(\theta(s)) = 8$, so $\theta(s) + \theta(s)^3 \neq 0$. We choose $p' = 19$ and checked that $a_{19} = -4$. Clearly $v_3\left(a_{19}^2 \epsilon^{-1}(19)\right) = 0$, so X_3 is unramified. Note that 3 divides $\mathrm{disc}(E|F)$, thus the converse of Cor. 6.3 (c) is false.

(3) $f \in S_2(405, [0, 2])$ is 3-minimal with $E = \mathbb{Q}(\sqrt{-2}, \sqrt{3})$ and $F = \mathbb{Q}$. $p = 3$ is an unramified supercuspidal prime of positive level, with $o(\theta(s)) \mid 4$, so $\theta(s) + \theta(s)^3$ may vanish. But since the level > 0, we can still apply Thm. 6.2. We choose $p' = 163$ and checked that $a_{p'}^2 \epsilon^{-1}(p') \equiv 6 \mod \mathbb{Q}^{*2}$. Hence X_3 ramifies.

(4) $f \in S_2(99, [3, 5])$, $E = \mathbb{Q}(\sqrt{-2}, \sqrt{3})$, $F = \mathbb{Q}$. $p = 3$ is an unramified supercuspidal prime of level zero with $\theta(s) + \theta(s)^3 \neq 0$. The order of $3 \in (\mathbb{Z}/11\mathbb{Z})^*$ is 5, hence by Cor. 6.3 (a), X_3 is unramified.

(5) $f \in S_2(99, [0, 2])$, $[E : \mathbb{Q}] = 8$ and $F = \mathbb{Q}(\sqrt{5})$. $p = 3$ is an unramified supercuspidal prime of level zero. The order of $\theta(s)$ is 4, so $\theta(s) + \theta(s)^3 = 0$, and we cannot apply Thms. 6.1 or 6.2. Note that $v = 3$ is the unique prime

in F lying above 3, and $F_v = \mathbb{Q}_3(\sqrt{5}) = \mathbb{Q}_{3^2}$. Thus $f_v = 2$, but still $[X_v] \sim -1$. This proves the necessity of the condition on $\theta(s)$ in Thm. 6.1.

(6) $f \in S_2(375, [1, 25])$, $[E : \mathbb{Q}] = 16$ and $F = \mathbb{Q}(\sqrt{5})$. $p = 5 \equiv 1 \mod 4$ is a ramified supercuspidal prime for f, and $K = \mathbb{Q}_5(\sqrt{5})$. There is a unique prime v in F above 5, and $[F_v : \mathbb{Q}_5] = 2$. As $\epsilon_p(-1)^{2[F_v:\mathbb{Q}_p]/(p-1)} = -1$, X_v is ramified by Cor. 7.3. In fact, X_f is ramified at the primes above 5, 89.

(7) $f \in S_5(27, [9])$ with $E = \mathbb{Q}(\sqrt{-1})$ and $F = \mathbb{Q}$. $p = 3$ is a ramified supercuspidal prime for f. For $p = 3$, $p'' = 53$ satisfies $(**)$, and we checked that $a_{53}^2 = -2537649 = -1593^2$. So $(-1)^k a_{p''}^2 \epsilon^{-1}(p'') = a_{53}^2 \equiv -1 \mod N_{K|\mathbb{Q}_p} K^*$. For any ramified quadratic extension K of $\mathbb{Q}_3$, $(-1, K|\mathbb{Q}_3) = -1$. Hence by Thm. 7.4, X_3 is ramified. Note that here $3 \nmid \mathrm{disc}(E|F)$, and $N' = 1$, but still $[X_3] \sim -1$. So the analogue of Cor. 6.3 does not hold in this case.

(8) $f \in S_5(27, [9])$ with $E = \mathbb{Q}(\sqrt{-6})$, and $F = \mathbb{Q}$. $p = 3$ is a ramified supercuspidal prime and $K = \mathbb{Q}_3(\sqrt{-3})$. We choose $p'' = 53$ as before, and compute $a_{p''}^2 = a_{53}^2 = -8468064 = -6 \cdot 1188^2$. Hence $(-1)^k a_{p''}^2 \epsilon^{-1}(p'') = a_{p''}^2 \equiv -6 \equiv 3 \mod N_{K|\mathbb{Q}_3} K^*$. But $(3, K|\mathbb{Q}_3) = 1$, so X_3 is unramified by Thm. 7.4. In fact, X_f is ramified only at 2 and ∞.

Errata to [BG13]

(1) Page 517, line 8: The condition "$N_p \geq 2 > C_p$" should be replaced by "$N_p \geq 2, N_p > C_p$".
(2) Page 522, line 4: In the definition of $m_v^\dagger$, "$[F_v : \mathbb{Q}_{p^\dagger}]$" should be replaced by "$[F_v : \mathbb{Q}_p]$".
(3) Page 523, line 1: The inequality "$\leq$" should be replaced by "$<$".
(4) Page 539, Prop. 33: One should add the following condition in the hypotheses: "If $K/\mathbb{Q}_p$ is unramified, then assume $\chi(g_s) + \chi(g_s)^p \neq 0$, where $g_s \in G_K$ corresponds to a primitive $(p^2 - 1)$-th root of unity $s \in K^*$." Without this assumption, the usual argument referred to on line 16 of page 540 does not work.

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