ON NON-ADMISSIBLE IRREDUCIBLE MODULO $p$
REPRESENTATIONS OF $GL_2(\mathbb{Q}_p^2)$

SUR LES REPRÉSENTATIONS IRRÉDUCTIBLES NON-ADMISSIBLES
MODULO $p$ DE $GL_2(\mathbb{Q}_p^2)$

EKNATH GHATE AND MIHIR SHETH

Abstract. We use a Diamond diagram attached to a 2-dimensional reducible split mod $p$ Galois representation of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to construct a non-admissible smooth irreducible mod $p$ representation of $GL_2(\mathbb{Q}_p^2)$ following the approach of Daniel Le.

Résumé. Nous utilisons un diagramme de Diamond attaché à une représentation galoisienne mod $p$ semi-simple réductible de dimension 2 de $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ pour construire une représentation mod $p$ non-admissible irréductible lisse de $GL_2(\mathbb{Q}_p^2)$ en suivant l’approche de Daniel Le.

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1. Introduction

Let $p$ be a prime number, $\mathbb{Q}_p$ be the field of $p$-adic numbers, and $\overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$ of cardinality $p$. The study of the admissibility of smooth irreducible representations of connected reductive $p$-adic groups goes back to Harish-Chandra ([6]). Building upon his work, Jacquet proved that every such representation over the field of complex numbers is admissible ([3], see also [9]). This result was extended by Vignéras to smooth irreducible representations over any algebraically closed field of characteristic not equal to $p$ ([12]). In the note [11], the authors ask whether this is true for smooth irreducible representations over algebraically closed fields of characteristic $p$. It is known that every smooth irreducible representation of $GL_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$ is admissible (see [2]). However, Daniel Le recently constructed non-admissible smooth irreducible $\overline{\mathbb{F}}_p$-linear representations

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of \( \text{GL}_2(F) \), for \( F \) a finite unramified extension of \( \mathbb{Q}_p \) of degree at least 3 and for \( p > 2 \), providing a negative answer to the question raised above (\[12\]). In this paper, we follow Le’s approach and construct non-admissible irreducible representations of \( \text{GL}_2(\mathbb{Q}_p^2) \) where \( \mathbb{Q}_p^2 \) is the unramified extension of \( \mathbb{Q}_p \) of degree 2. These results support the viewpoint of Breuil and Paškūnas that the mod \( p \) (and \( p \)-adic) representation theory of \( \text{GL}_2(F) \) becomes more complicated as soon as \( F \neq \mathbb{Q}_p ^2 \) (\[5\], see also \[11\]).

Let \( G = \text{GL}_2(\mathbb{Q}_p^2) \), \( K = \text{GL}_2(\mathbb{Z}_p^2) \), and \( \Gamma = \text{GL}_2(\mathbb{F}_p^2) \), where \( \mathbb{Z}_p^2 \) is the ring of integers of \( \mathbb{Q}_p^2 \) with residue field \( \mathbb{F}_p^2 \). Fix an embedding \( \mathbb{F}_p^2 \hookrightarrow \mathbb{F}_p \). Let \( I \) and \( I_1 \) denote the Iwahori and the pro-\( p \) Iwahori subgroups of \( K \) respectively, and \( K_1 \) denote the first principal congruence subgroup of \( K \). Write \( N \) for the normalizer of \( I \) (and of \( I_1 \)) in \( G \). As a group, \( N \) is generated by \( I \), the center \( Z \) of \( G \), and by the element \( \Pi = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \). All representations considered in this paper from now on are over \( \overline{\mathbb{F}}_p \)-vector spaces. For a character \( \chi \) of \( I \), \( \chi^s \) denotes its \( \Pi \)-conjugate sending \( g \) in \( I \) to \( \chi(\Pi g \Pi^{-1}) \).

A weight is a smooth irreducible representation of \( K \). The \( K \)-action on such a representation factors through \( \Gamma \) and thus any weight is described by a 2-tuple \((r_0, r_1) \otimes \text{det}^m := \text{Sym}^r \mathbb{F}_p^m \otimes (\text{Sym}^1 \mathbb{F}_p^m)^\text{prob} \otimes \text{det}^m \) of integers with \( 0 \leq r_0, r_1 \leq p - 1 \) together with a determinant twist for some \( 0 \leq m < p^2 - 1 \) (\[1\], Lemma 2.16 and Proposition 2.17). Given a weight \( \sigma \), its subspace \( \sigma^I \) of \( I_1 \)-invariants has dimension 1. If \( \chi_\sigma \) denotes the corresponding smooth character of \( I \) and \( \chi_\sigma \neq \chi_{\sigma^I} \), then there exists a unique weight \( \sigma^* \) such that \( \chi_{\sigma^*} = \chi_\sigma^I \) (\[10\], Theorem 3.1.1).

A basic 0-diagram is a triplet \((D_0, D_1, r)\) consisting of a smooth \( K \)-representation \( D_0 \), a smooth \( N \)-representation \( D_1 \) and an \( IZ \)-equivariant isomorphism \( r : D_1 \xrightarrow{\sim} D_0^I \otimes \text{det}^m \) with the trivial action of \( p \) on \( D_0 \) and \( D_1 \). Given such a diagram such that \( D_0^I \) has finite dimension, the smooth injective \( K \)-envelope \( \text{inj}_K D_0 \) admits a non-canonical \( N \)-action which glue together with the \( K \)-action to give a smooth \( G \)-action on \( \text{inj}_KD_0 \) (\[5\], Theorem 9.8). The \( G \)-subrepresentation of \( \text{inj}_KD_0 \) generated by \( D_0 \) is smooth admissible and its \( K \)-socle equals the \( K \)-socle \( \text{soc}_K D_0 \) of \( D_0 \).

From now on, assume that \( p \) is odd. Let \( \rho : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^2) \to \text{GL}_2(\overline{\mathbb{F}}_p) \) be a continuous generic Galois representation such that \( p \) acts trivially on its determinant and \( \mathcal{D}(\rho) \) be the set of weights, called Diamond weights, associated to \( \rho \) as described in \[3\], Section 11. Breuil and Paškūnas attach a family of basic 0-diagrams \((D_0(\rho), D_1(\rho), r)\), called Diamond diagrams, to \( \rho \) such that \( \text{soc}_K D_0(\rho) = \bigoplus_{\sigma \in \mathcal{D}(\rho)} \sigma \) (\[5\], Theorem 13.8).

For a finite unramified extension \( F \) of \( \mathbb{Q}_p \) of degree at least 3, Le uses a Diamond diagram attached to an irreducible \( \rho : \text{Gal}(\overline{\mathbb{Q}}_p/F) \to \text{GL}_2(\overline{\mathbb{F}}_p) \) to construct an infinite dimensional diagram which gives rise to a non-admissible smooth irreducible representation of \( \text{GL}_2(F) \) (\[12\]). His strategy does not work for a Diamond diagram attached to an irreducible Galois representation of \( \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^2) \) because such a diagram does not have suitable \( \Pi \)-action dynamics. However, for \( F = \mathbb{Q}_p^2 \), we observe that a Diamond diagram attached to a reducible split \( \rho \) has an indecomposable subdiagram with suitable \( \Pi \)-action dynamics so that Le’s method can be used to obtain a non-admissible irreducible representation of \( G = \text{GL}_2(\mathbb{Q}_p^2) \).
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2. Reducible Diamond diagram

Let $\omega_2$ be Serre’s fundamental character of level 2 for the fixed embedding $F_p^2 \hookrightarrow \overline{F}_p$, and let $\rho : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^2) \to \text{GL}_2(\overline{F}_p)$ be a continuous reducible split generic Galois representation. The restriction of $\rho$ to the inertia subgroup is, up to a twist by some character, isomorphic to

$$
\left( \begin{array}{cc}
\omega_2^{r_0+1+(r_1+1)p} & 0 \\
0 & 1
\end{array} \right)
$$

for some $0 \leq r_0, r_1 \leq p-3$, not both equal to 0 or equal to $p-3$ ([4], Corollary 2.9 (i) and [5], Definition 11.7 (i)). Define the weight

$$
\sigma := (r_0 + 1, p - 2 - r_1) \otimes \text{det}^{p-1+r_1 p}.
$$

Then the set of Diamond weights for $\rho$ is given by

$$
\mathcal{D}(\rho) = \{ (r_0, r_1), \sigma, \sigma^s, (p - 3 - r_0, p - 3 - r_1) \otimes \text{det}^{r_0+1+(r_1+1)p} \}$$

([5], Lemma 11.2 or Section 16, Example (ii)). Fix a Diamond diagram $(D_0(\rho), D_1(\rho), r)$ attached to $\rho$, and identify $D_1(\rho)$ with $D_0(\rho)$ as $\mathbb{I} \mathbb{Z}$-representations via $r$. There is a direct sum decomposition $D_0(\rho) = \bigoplus_{\nu \in \mathcal{D}(\rho)} D_{0,\nu}(\rho)$ of $K$-representations with $\text{soc}_K D_{0,\nu}(\rho) = \nu$ ([5], Proposition 13.4).

Now define

$$
D_0 := D_{0,\sigma}(\rho) \oplus D_{0,\sigma^s}(\rho) \quad \text{and} \quad D_1 := D_1^H.
$$

It follows from [5], Theorem 15.4 (ii) that $(D_0, D_1, r)$ is an indecomposable subdiagram of $(D_0(\rho), D_1(\rho), r)$. Set

$$
\tau := (r_0 + 2, r_1) \otimes \text{det}^{p-2+(p-1)p} \quad \text{and} \quad \tau' := (p - 1 - r_0, p - 3 - r_1) \otimes \text{det}^{r_0+(r_1+1)p}.
$$

The graded pieces of the socle filtrations of $D_{0,\sigma}(\rho)$ and $D_{0,\sigma^s}(\rho)$ are as follows ([5], Theorem 14.8 or Section 16, Example (ii)):

$$
\begin{align*}
D_{0,\sigma}(\rho) : & \quad \sigma \quad \tau \oplus \tau^s \quad (p - 4 - r_0, r_1 - 1) \otimes \text{det}^{r_0+2} \\
D_{0,\sigma^s}(\rho) : & \quad \sigma^s \quad \tau' \oplus \tau'^s \quad (r_0 - 1, p - 4 - r_1) \otimes \text{det}^{(r_1+2)p}.
\end{align*}
$$

We have from [5], Corollary 14.10 that

$$
D_1 = \chi_\sigma \oplus \chi_\tau \oplus \chi_\sigma^s \oplus \chi_\tau^s \oplus \chi_\sigma^r \oplus \chi_\tau^r.
$$

For an $\mathbb{I} \mathbb{Z}$-representation $V$ and an $\mathbb{I} \mathbb{Z}$-character $\chi$, we write $V^\chi$ for the $\chi$-isotypic part of $V$. 

3. An infinite dimensional diagram and the construction

Let $D_0(\infty) := \bigoplus_{i \in \mathbb{Z}} D_0(i)$ be the smooth $KZ$-representation with component-wise $KZ$-action, where there is a fixed isomorphism $D_0(i) \cong D_0$ of $KZ$-representations for every $i \in \mathbb{Z}$. Following [9], we denote the natural inclusion $D_0 \hookrightarrow D_0(i) \hookrightarrow D_0(\infty)$ by $t_i$, and write $v_i := t_i(v)$ for $v \in D_0$ for every $i \in \mathbb{Z}$. Let $D_1(\infty) := D_0(\infty)^I$. We define a $\Pi$-action on $D_1(\infty)$ as follows. Let $\lambda = (\lambda_i) \in \prod_{i \in \mathbb{Z}} \mathbb{F}_p^\infty$. For all integers $i < 0$, define

$$\Pi v_i := \begin{cases} (\Pi v)_i & \text{if } v \in D_1^{\lambda_i}, \\ (\Pi v)_{i+1} & \text{if } v \in D_1^{\lambda_i}, \\ \lambda_i(\Pi v)_i & \text{if } v \in D_1^{\lambda_i}. \end{cases}$$

This uniquely determines a smooth $N$-action on $D_1(\infty)$ such that $p = \Pi^2$ acts trivially on it. Thus we get a basic $0$-diagram $D(\lambda) := (D_0(\infty), D_1(\infty), \text{can})$ with the above actions where can is the canonical inclusion $D_1(\infty) \hookrightarrow D_0(\infty)$.

**Theorem 3.1.** There exists a smooth representation $\pi$ of $G$ such that

1. $(\pi|_{KZ}, \pi|_{N;} \text{id})$ contains $D(\lambda)$,
2. $\pi$ is generated by $D_0(\infty)$ as a $G$-representation, and
3. $\text{soc}_K \pi = \text{soc}_K D_0(\infty)$.

**Proof.** Let $\Omega$ be the smooth injective $K$-envelope of $D_0$ equipped with the $KZ$-action such that $p$ acts trivially. The smooth injective $I$-envelope $\text{inj}_j D_1$ of $D_1$ appears as an $I$-direct summand of $\Omega$. Let $e$ denote the projection of $\Omega$ onto $\text{inj}_j D_1$. There is a unique $N$-action on $\text{inj}_j D_1$ compatible with that of $I$ and compatible with the action of $N$ on $D_1$. By [5], Lemma 9.6, there is a non-canonical $N$-action on $(1 - e)(\Omega)$ extending the given $I$-action. This gives an $N$-action on $\Omega$ whose restriction to $IZ$ is compatible with the action coming from $KZ$ on $\Omega$.

Now let $\Omega(\infty) := \bigoplus_{i \in \mathbb{Z}} \Omega(i)$ with component-wise $KZ$-action where there is a fixed isomorphism $\Omega(i) \cong \Omega$ of $KZ$-representations for every $i \in \mathbb{Z}$. We wish to define a compatible $N$-action on $\Omega(\infty)$. As before, denote the natural inclusion $\Omega \hookrightarrow \Omega(i) \hookrightarrow \Omega(\infty)$ by $t_i$, and write $v_i := t_i(v)$ for $v \in \Omega$. Let $\Omega(\pi)$ denote the smooth injective $I$-envelope of an $I$-character $\chi$. Thus, from [2,1], we have $e(\Omega) = \text{inj}_j D_1 = \Omega_{\chi^t} \oplus \Omega_{\chi^t} \oplus \Omega_{\chi^t} \oplus \Omega_{\chi^t}$. If $v \in (1 - e)(\Omega)$, we define $\Pi v_i := (\Pi v)_i$ for all integers $i$. Otherwise, we define $\Pi v_i := (\Pi v)_i$ if $v \in \Omega_{\chi^t}$, $\Pi v_i := (\Pi v)_{i+1}$ if $v \in \Omega_{\chi^t}$, and $\Pi v_i := \lambda_i(\Pi v)_i$ if $v \in \Omega_{\chi^t}$. By demanding that $\Pi^2$ acts trivially, this defines a smooth $N$-action on $\Omega(\infty)$ which is compatible with the $N$-action on $D_1(\infty)$, and whose restriction to $IZ$ is compatible with the action coming from $KZ$ on $\Omega(\infty)$. By [10], Corollary 5.5.5, we have a smooth $G$-action on $\Omega(\infty)$. We then take $\pi$ to be the $G$-representation generated by $D_0(\infty)$ inside $\Omega(\infty)$. It follows easily from the construction that $\pi$ satisfies the properties (1), (2) and (3). \hfill \Box

**Theorem 3.2.** If $\lambda_i \neq \lambda_0$ for all $i \neq 0$, then any smooth representation $\pi$ of $G$ satisfying the properties (1), (2), and (3) of Theorem 3.1 is irreducible and non-admissible.

**Proof.** Let $\pi' \subseteq \pi$ be a non-zero subrepresentation of $G$. By property (3), we have either $\text{Hom}_K(\sigma, \pi') \neq 0$ or $\text{Hom}_K(\sigma^*, \pi') \neq 0$. We consider the case $\text{Hom}_K(\sigma, \pi') \neq 0$; the other
case is treated analogously. There exists a non-zero \( (c_i) \in \bigoplus_{i \in \mathbb{Z}} \mathbb{F}_p \) such that

\[
\left( \sum_i c_i \tau_i \right)(D_{0,\sigma}(\rho)) \cap \pi' \neq 0.
\]

We claim that

\[
(3.3) \quad \left( \sum_i c_i \tau_{i+j} \right)(D_0) \subset \pi' \quad \text{for all } j \in \mathbb{Z}.
\]

We first show that \( \left( \sum_i c_i \tau_i \right)(D_{0,\sigma^*}(\rho)) \subset \pi' \). Note that \( \left( \sum_i c_i \tau_i \right)(D_{0,\sigma}(\rho)) \cap \pi' \neq 0 \) is equivalent to \( \left( \sum_i c_i \tau_i \right)(\sigma) \subset \pi' \). Since \( \left( \sum_i c_i \tau_i \right)(D_{1,\sigma^*}) \subset \pi' \) and \( \pi' \) is stable under the \( \Pi \)-action, we have \( \left( \sum_i c_i \tau_i \right)(D_{1,\sigma^*}) \subset \pi' \). By Frobenius reciprocity, we have a non-zero \( K \)-equivariant map

\[
(3.4) \quad \text{Ind}_I^K \left( \left( \sum_i c_i \tau_i \right)(D_{1,\sigma^*}) \right) \rightarrow \pi'
\]

whose image is \( \left( \sum_i c_i \tau_i \right)(I(\delta(\sigma),\sigma^*)) \), where \( \delta \) is the bijection on the set of Diamond weights \( D(\rho) \) defined in [5], Section 15, and \( I(\delta(\sigma),\sigma^*) \) is the \( K \)-subrepresentation of \( D_{0,\delta(\sigma)}(\rho) \) with cosocle \( \sigma^* \) (and socle \( \delta(\sigma) \)). In our setting, \( \delta \) maps \( \sigma \) to \( \sigma^* \) and vice versa ([5], Lemma 15.2). Thus \( I(\delta(\sigma),\sigma^*) = \sigma^* \) and so \( \left( \sum_i c_i \tau_i \right)(\sigma^*) \subset \pi' \). Let \( R \left( \sum_i c_i \tau_i \right)(\sigma) \) be the \( K \)-subrepresentation of the compact induction \( c\text{-Ind}_{KZ}^G \left( \sum_i c_i \tau_i \right)(\sigma) \) defined in [5], Section 17. By [5], Lemmas 17.1, 17.4 and 17.8 we have

\[
\text{Ind}_I^K \left( \left( \sum_i c_i \tau_i \right)(D_{1,\sigma^*}) \right) \subset R \left( \sum_i c_i \tau_i \right)(\sigma),
\]

and by Frobenius reciprocity, there is a non-zero map

\[
(3.5) \quad c\text{-Ind}_{KZ}^G \left( \sum_i c_i \tau_i \right)(\sigma) \rightarrow \pi'
\]

which restricts to the map \( (3.4) \). So the image \( Q \) of \( R \left( \sum_i c_i \tau_i \right)(\sigma) \) in \( \pi' \) under the map \( (3.5) \) contains \( \left( \sum_i c_i \tau_i \right)(\sigma^*) \). Since \( \text{soc}_K Q \subset \text{soc}_K \pi = \text{soc}_K D_0(\infty) \) and the Jordan-Hölder factors of \( R \left( \sum_i c_i \tau_i \right)(\sigma) \) are multiplicity free ([5], Lemma 17.1), \( \text{soc}_K Q \) is isomorphic to a subrepresentation of the direct sum of the weights in \( D(\rho) \). Therefore by [5], Lemma 19.5, \( \text{soc}_K Q = \left( \sum_i c_i \tau_i \right)(\sigma^*) \), and by [5], Lemma 19.7, \( Q \) contains a copy of the \( K \)-representation \( D_{0,\sigma^*}(\rho) \). But \( \left( \sum_i c_i \tau_i \right)(D_{0,\sigma^*}(\rho)) \) is the unique \( K \)-subrepresentation of \( \pi \) isomorphic to \( D_{0,\sigma^*}(\rho) \) and with \( K \)-socle \( \left( \sum_i c_i \tau_i \right)(\sigma^*) \). Thus \( \left( \sum_i c_i \tau_i \right)(D_{0,\sigma^*}(\rho)) = Q \subset \pi' \).

Now, since \( \left( \sum_i c_i \tau_i \right)(\sigma^*) \subset \pi' \), a symmetric argument shows that \( \left( \sum_i c_i \tau_i \right)(D_{0,\sigma}(\rho)) \subset \pi' \). Thus

\[
\left( \sum_i c_i \tau_i \right)(D_0) \subset \pi' .
\]

Therefore

\[
\left( \sum_i c_i \tau_i \right)(D_{1,\sigma^*}) \subset \pi' \quad \text{and} \quad \left( \sum_i c_i \tau_i \right)(D_{1,\sigma^*}) \subset \pi' .
\]
In particular,

\[
(\sum_i c_i t_{i+1})(D_{0,\sigma}(\rho)) \cap \pi' \neq 0 \quad \text{and} \quad (\sum_i c_i t_{i-1})(D_{0,\sigma}(\rho)) \cap \pi' \neq 0.
\]

By the same arguments as above, we find that

\[
(\sum_i c_i t_{i+1})(D_0) \subset \pi' \quad \text{and} \quad (\sum_i c_i t_{i-1})(D_0) \subset \pi'.
\]

The claim (3.3) is now proved by repeatedly using the II-action.

For \((d_i) \in \bigoplus_{i \in \mathbb{Z}} \mathbb{F}_p^2\), let \(#(d_i)\) denote the number of non-zero \(d_i\)'s. Among all the non-zero elements \((c_i)\) of \(\bigoplus_{i \in \mathbb{Z}} \mathbb{F}_p^2\) for which \((\sum_i c_i t_{i+1})(D_0) \subset \pi'\), we pick one with \(#(c_i)\) minimal.

We may also assume that \(c_0 \neq 0\) using (3.3). We now show that \(#(c_i) = 1\) if the contrary that \(#(c_i) > 1\). Since \((\sum_i c_i t_{i+1})(D_1^{\lambda c_i}) \subset \pi'\) and \(\pi'\) is stable under the II-action, we have

\[
(\sum_i \lambda_i c_i t_i)(D_1^{\lambda c_i}) \subset \pi'.
\]

Since \((\sum \lambda_0 c_i t_i)(D_1^{\lambda c_i})\) is also clearly in \(\pi'\), subtracting it from the above, we get

\[
(\sum_i (\lambda_i - \lambda_0) c_i t_i)(D_1^{\lambda c_i}) \subset \pi'.
\]

Writing \((c'_i) := ((\lambda_i - \lambda_0) c_i)\), we see that

\[
(\sum_i c'_i t_i)(D_{0,\sigma}(\rho)) \cap \pi' \neq 0.
\]

Following the same arguments as in the previous paragraphs, we get that \((\sum_i c'_i t_i)(D_0) \subset \pi'\). However, the hypothesis \(\lambda_i \neq \lambda_0\) for all \(i \neq 0\), and the assumption \(#(c_i) > 1\) imply that \((c'_i)\) is non-zero and \(#(c'_i) = #(c_i) - 1\) contradicting the minimality of \(#(c_i)\). Therefore, we have \(c_0 t_0(D_0) \subset \pi'\). So \(\mu_0(D_0) \subset \pi'\). Using (3.3) again, we get that \(\bigoplus_{j \in \mathbb{Z}} t_j(D_0) = D_0(\infty) \subset \pi'\). By property (2), we have \(\pi' = \pi\).

The non-admissibility of \(\pi\) is clear because \(\pi^K \supseteq \text{soc}_K \pi\) and \(\text{soc}_K \pi\) is not finite dimensional by property (3). \(\square\)

**Remark 3.6.** If the diagram \((D_0(\rho), D_1(\rho), r)\) is defined over \(\mathbb{F}_p^2\) and \((\lambda_i) \in \prod_{i \in \mathbb{Z}} \mathbb{F}_p^\infty\), then the representation \(\pi\) in Theorem 3.1 has a model \(\pi_0\) over \(\mathbb{F}_p^2\). Furthermore, \(\pi_0\) is absolutely irreducible and non-admissible if the \((\lambda_i)\) satisfy the hypothesis of Theorem 3.2.

In fact, for any field \(C\) containing \(\mathbb{F}_p^2\), the methods of this paper produce an absolutely irreducible non-admissible smooth \(C\)-representation \(C \otimes_{\mathbb{F}_p^2} \pi_0\) of \(G\).

Now let \(C\) be an arbitrary field of characteristic \(p\) with algebraic closure \(\overline{C}\). From the discussion in the previous paragraph, the representation \(\overline{C} \otimes_{\mathbb{F}_p^2} \pi_0\) is a smooth absolutely
irreducible $\overline{C}$-representation which has a model $C' \otimes_{F_p^2} \pi_0$ over $C'$, where $C' = C \mathbb{F}_{p^2} \subset \overline{C}$. By [7], Lemma II.5, there exists a smooth irreducible $C$-representation $\pi_C$ such that $\overline{C} \otimes_{F_p^2} \pi_0$ is a $\overline{C}$-subrepresentation of $\overline{C} \otimes_C \pi_C$. Since $\overline{C} \otimes_{F_p^2} \pi_0$ is non-admissible, $\overline{C} \otimes_C \pi_C$ is non-admissible. It follows from [7], Lemma III.1 (ii) that $\pi_C$ is also non-admissible. Thus we obtain a smooth irreducible non-admissible representation of $G$ over any field $C$ of characteristic $p$.

References


