DIAGRAMS AND MOD p REPRESENTATIONS OF p-ADIC GROUPS

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Abstract. This is an expository article explaining the theory of diagrams of Breuil and Paskunas and its recent application to the construction of non-admissible irreducible mod p representations of GL_2 over unramified extensions of $\mathbb{Q}_p$.

Contents

1. Introduction

Smooth complex representations of reductive p-adic groups play a pivotal role in the global Langlands program as they appear as local factors of automorphic representations. These local representations are admissible. Recall that a representation is smooth if every vector has an open subgroup fixing that vector, and it is admissible if the subspace fixed by any open subgroup is finite-dimensional. The mod p analogue of the local Langlands correspondence makes it necessary to understand smooth mod p representations of reductive p-adic groups. Unlike complex representations, one does not have analytic methods at one's disposal to study smooth mod p representations of p-adic groups because they do not admit a non-zero $\overline{\mathbb{F}_p}$-valued Haar measure. Diagrams give a powerful tool to construct interesting smooth mod p representations of reductive p-adic groups.

Breuil and Paskunas used diagrams attached to certain Galois representations to construct irreducible admissible supercuspidal mod p representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ where $\mathbb{Q}_{p^f}$ is the degree f unramified extension of $\mathbb{Q}_p$ ([BP12]). The universal supercuspidal representations, i.e., the compact inductions of weights modulo the image of the Hecke operator, classify all irreducible admissible supercuspidal mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, while in general, their irreducible admissible quotients exhaust all admissible supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ for $f > 1$ ([Bre07], Proposition 4.6). The theory of diagrams can be used to show that, for $f > 1$, the universal supercuspidal representation is not of finite length and is also not admissible ([Bre10], Theorem 3.3). This also follows from [Sch15],

2010 *Mathematics Subject Classification.* 22E50, 11S37.

Corollary 2.21 and [Wu21], Corollary 4.5. This indicates that the mod p representation theory of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ is more involved than that of $\mathrm{GL}_2(\mathbb{Q}_p)$ ([Hu10], [BP12]). For some work on the mod p representation theory of GL_2 over a totally ramified extension of $\mathbb{Q}_p$, see, for example, [Sch04], and for a general finite extension of $\mathbb{Q}_p$, see [Hen19].[1]

By the work of many mathematicians such as Harish-Chandra, Jacquet, Vignéras, it is known that all smooth irreducible representations of connected reductive[2] p-adic groups over algebraically closed fields of characteristic not equal to p are admissible ([Vig96], II §2.8). The main point is to show that all irreducible supercuspidal representations are admissible, since a general smooth irreducible representation is a subrepresentation of the parabolic induction of an irreducible supercuspidal representation and parabolic induction preserves admissibility. However, it is no longer true that irreducible supercuspidal representations over characteristic p fields are admissible. Recently, Daniel Le constructed non-admissible irreducible (supercuspidal) $\overline{\mathbb{F}_p}$-linear representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ using infinite-dimensional diagrams for all $f > 2$, although only the case $f = 3$ is presented in his paper for simplicity ([Le19]). Applying Le's method to a diagram attached to a split reducible Galois representation, the authors have constructed non-admissible irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ ([GS20]).

This article gives an expository treatment of the theory of diagrams of Breuil and Paskunas, and provides a proof of Le's construction of non-admissible irreducible mod p representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ for all $f > 2$. It is organized as follows. In Section 2, we introduce (finite dimensional) diagrams and describe how they give rise to smooth admissible representations of GL_2 over p-adic fields. Section 3 focuses on diagrams attached to Galois representations and on the irreducible admissible supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_{p^f})$ that they give rise to. Finally, we prove Le's theorem for all $f > 2$ in Section 4.

1.1. **Acknowledgments.** We acknowledge support of the Department of Atomic Energy, Government of India, under project number 12-R&D-TFR-5.01-0500. We thank the organizers of the ICTS (International Centre for Theoretical Sciences) program on "Perfectoid spaces" (Code: ICTS/perfectoid2019/09) for the discussion meeting on "p-adic automorphic forms and perfectoid spaces". The research talks on mod p representation theory of p-adic groups during the meeting provided an impetus to our joint work mentioned in this article.

1.2. **Notation.** Let $p > 2$ be a prime number and $\mathbb{Q}_p$ be the field of p-adic numbers. Let $\mathbb{Q}_{p^f}$ denote the unramified extension of $\mathbb{Q}_p$ of degree f with ring of integers $\mathbb{Z}_{p^f}$. The residue field of $\mathbb{Q}_{p^f}$ is the finite field $\mathbb{F}_{p^f}$ with p^f elements. Fix an algebraic closure $\overline{\mathbb{F}_p}$ of $\mathbb{F}_p$ and an embedding $\mathbb{F}_{p^f} \hookrightarrow \overline{\mathbb{F}_p}$.

For an arbitrary but fixed f, let $G = \mathrm{GL}_2(\mathbb{Q}_{p^f})$, $K = \mathrm{GL}_2(\mathbb{Z}_{p^f})$, and $\Gamma = \mathrm{GL}_2(\mathbb{F}_{p^f})$. Let B and U be the subgroups of Γ consisting of the upper triangular matrices and the upper triangular unipotent matrices respectively. Let I and I_1 be the preimages of B

[1]The words *supersingular* and *supercuspidal* are used interchangeably in the literature for mod p representations. These two *a priori* different notions are now known to be equivalent ([AHHV17]).

[2]The reductive hypothesis is necessary, see [Hel90].

and U respectively under the natural surjection $K \twoheadrightarrow \Gamma$. The subgroups I and I_1 of K are called the Iwahori and the pro-p Iwahori subgroup of K respectively. Let K_n denote the n-th principal congruence subgroup of K, i.e., the kernel of the reduction map $K \longrightarrow \mathrm{GL}_2(\mathbb{Z}_{p^f}/p^n\mathbb{Z}_{p^f})$ modulo p^n for $n \geq 1$. Write N for the normalizer of I (and of I_1) in G. Then N is generated by I, the center Z of G and by the element $\Pi = \left(\begin{smallmatrix} 0 & 1 \\ p & 0 \end{smallmatrix}\right)$.

Unless stated otherwise, all representations considered in this paper are on $\overline{\mathbb{F}_p}$-vector spaces and are sometimes referred to as mod p representations. A weight is a smooth irreducible representation of K. The K-action on such a representation factors through Γ and thus a weight is an irreducible representation of Γ ([Bre07], Lemma 2.14). For a character χ of I, χ^s denotes its Π-conjugate sending g in I to $\chi(\Pi g \Pi^{-1})$. Given a weight σ, the subspace σ^{I_1} of its I_1-invariants has dimension 1. We denote the corresponding smooth character of I afforded by the space σ^{I_1} by χ_σ. If $\chi_\sigma \neq \chi_\sigma^s$, then there exists a unique weight σ^s such that $\chi_{\sigma^s} = \chi_\sigma^s$ ([Pas04], Theorem 3.1.1). For an I-representation V and an I-character χ, we write V^χ for the χ-isotypic part of V.

2. Diagrams and the existence theorem

Diagrams were introduced by Paskunas in [Pas04] to construct smooth admissible representations of G.

Definition 2.1. A diagram is a triple (D_0, D_1, r) where D_0 is a smooth representation of KZ, D_1 is a smooth representation of N, and $r : D_1 \longrightarrow D_0$ is an IZ-equivariant map. A diagram (D_0, D_1, r) is called a basic diagram if p acts trivially on D_0 and D_1, and r induces an isomorphism $D_1 \xrightarrow{\sim} D_0^{I_1}$ of IZ-representations.

The idea is to use the data of a basic diagram to construct a space Ω admitting actions of both KZ and N which agree on $IZ = KZ \cap N$. Let G^0 be the subgroup of G consisting of matrices whose determinant is a p-adic unit. Since G^0 is an amalgamated product of K and $\Pi K \Pi^{-1}$, and $G = G^0 \rtimes \Pi^{\mathbb{Z}}$, the actions of KZ and N on Ω glue together to give a G-action on Ω. This G-action is unique because KZ and N generate the group G ([Bre07], Theorem 3.3 and Corollary 3.4).

A way to construct Ω is to use injective envelopes of finite-dimensional representations of finite groups. An injective envelope of a representation is the “smallest” injective object containing the representation ([Bre07], Definition 5.12). If the subspace $D_0^{K_1}$ of K_1-invariants of D_0 is finite-dimensional, then the K-socle $\mathrm{soc}_K D_0$ of D_0, i.e., the maximal semi-simple K-subrepresentation of D_0, is finite-dimensional, and therefore the direct limit $\varinjlim_n \mathrm{inj}_{K/K_n}(\mathrm{soc}_K D_0)$ of finite-dimensional injective envelopes exists in the category of smooth K-representations. By [Bre07], Proposition 5.17, this direct limit is the smooth injective envelope $\mathrm{inj}_K(\mathrm{soc}_K D_0)$ of D_0.

Let $\Omega := \mathrm{inj}_K(\mathrm{soc}_K D_0)$ be equipped with the KZ-action such that p acts trivially. The smooth injective I-envelope $\mathrm{inj}_I D_1$ of D_1 appears as an I-direct summand of Ω via the IZ-equivariant map r. There is a unique N-action on $\mathrm{inj}_I D_1$ compatible with that of I and compatible with the action of N on D_1 ([Bre07], Corollary 6.7). Let e denote the projection of Ω onto $\mathrm{inj}_I D_1$. By [BP12], Lemma 9.6, there is a non-canonical N-action on

$(1-e)(\Omega)$ extending the given I-action. This gives an N-action on Ω whose restriction to IZ is compatible with the action coming from KZ on Ω. Consequently, there is a G-action on Ω as discussed two paragraphs above.

Let π be the G-representation generated by D_0 inside Ω. Then, we see that

$$\mathrm{soc}_K D_0 \subseteq \mathrm{soc}_K \pi \subseteq \mathrm{soc}_K(\mathrm{inj}_K(\mathrm{soc}_K D_0)) = \mathrm{soc}_K D_0$$

so that equality holds throughout.

We summarize the above discussion in the following theorem ([Bre07], Theorem 5.10).

Theorem 2.2 (The existence theorem). *Let (D_0, D_1, r) be a basic diagram such that D_0 is finite-dimensional and K_1 acts trivially on D_0. Then there exists a smooth admissible representation π of G such that*

(1) $(\pi^{K_1}, \pi^{I_1}, \mathrm{can})$ *contains (D_0, D_1, r), where* can *is the canonical inclusion,*
(2) π is generated by D_0 as a G-representation,
(3) $\mathrm{soc}_K \pi = \mathrm{soc}_K D_0$.

Note that the representation π in the theorem above is admissible because $\pi^{K_n} \subseteq (\mathrm{inj}_K(\mathrm{soc}_K D_0))^{K_n} = \mathrm{inj}_{K/K_n}(\mathrm{soc}_K D_0)$ which is finite-dimensional (cf. [Pas04], Lemma 6.2.4).

We remark that the discussion in this section, i.e., the notion of a basic diagram and the existence theorem, works for $G = \mathrm{GL}_2(F)$ for any finite extension F of $\mathbb{Q}_p$.

Example 2.3. Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and σ be a weight. Take $D_0 = \sigma \oplus \sigma^s$ and $D_1 = D_0^{I_1} = \chi_\sigma \oplus \chi_\sigma^s$. Let Π map a basis vector of the underlying vector space of χ_σ to that of χ_σ^s. By letting p act trivially on D_0 and D_1, we get a basic diagram (D_0, D_1, can) where can is the canonical injection. The existence theorem applied to this diagram gives rise to a G-representation π that is irreducible and supercuspidal, and is *uniquely* determined by the diagram (D_0, D_1, can) ([Bre07], Lemma 5.2). In fact, one obtains all irreducible admissible supercuspidal representations of G up to a smooth twist in this way as σ varies. Under the mod p local Langlands correspondence for GL_2 over $\mathbb{Q}_p$, π is mapped to a continuous 2-dimensional irreducible representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ whose restriction to the inertia subgroup contains the information of $\mathrm{soc}_K \pi = \mathrm{soc}_K D_0 = \sigma \oplus \sigma^s$.

3. Diagrams attached to Galois representations

Let $f > 2$ for the rest of the article.

3.1. **Diamond diagrams.** Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p^f}) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{F}_p})$ be a continuous irreducible generic Galois representation ([BP12], Definition 11.7). In [BDJ10], Buzzard, Diamond and Jarvis associate to ρ a finite set $\mathcal{D}(\rho)$ of distinct weights anticipating that it would describe the K-socle of the supercuspidal representation of G corresponding to ρ under the conjectural mod p local Langlands correspondence for GL_2 over $\mathbb{Q}_{p^f}$.[3] As we shall see, the set $\mathcal{D}(\rho)$ can indeed be used to construct irreducible supercuspidal representations

[3]They associate a finite set of weights to any continuous semi-simple generic Galois representation ρ. We stick to irreducible ρ in this exposition. However, see Remark 4.5.

with K-socle described by $\mathcal{D}(\rho)$. However, it turns out that there are infinitely many such representations up to isomorphism. The mod p local Langlands correspondence for GL_2 over finite extensions of $\mathbb{Q}_p$ thus still remains puzzling.

The set $\mathcal{D}(\rho)$ has cardinality 2^f. By elementary representation theoretic arguments, there exists a unique finite dimensional $\overline{\mathbb{F}_p}$-linear representation $D_0(\rho)$ of Γ whose Γ-socle equals $\bigoplus_{\sigma\in\mathcal{D}(\rho)}\sigma$, and is maximal with respect to the property that each $\sigma\in\mathcal{D}(\rho)$ occurs exactly once in $D_0(\rho)$ as a Jordan–Hölder factor. Further, there is an isomorphism of Γ-representations

$$D_0(\rho)\cong\bigoplus_{\sigma\in\mathcal{D}(\rho)} D_{0,\sigma}(\rho)$$

with $\mathrm{soc}_\Gamma D_{0,\sigma}(\rho)=\sigma$ ([BP12], Proposition 13.1). Viewing $D_0(\rho)$ as a K-representation, let us denote by $D_1(\rho)$ the I-representation $D_0(\rho)^{I_1}$ and by $D_{1,\sigma}(\rho)$ the I-representation $D_{0,\sigma}(\rho)^{I_1}$. If an I-character χ appears in $D_1(\rho)$ then so does χ^s.

While for any finite set of weights, there exists a finite-dimensional Γ-representation D_0 satisfying the same properties listed above, the properties of $D_0(\rho)$ and $D_1(\rho)$ specific to the set of weights $\mathcal{D}(\rho)$ are summarized below.

Proposition 3.1.
1. *The Jordan–Hölder factors of $D_0(\rho)$ are multiplicity free.*
2. *$D_1(\rho)$ is a multiplicity free semi-simple I-representation of dimension 3^f-1 and thus*
$$D_1(\rho)=\bigoplus_{I-\text{character }\chi}\chi\oplus\chi^s.$$

Proof. See [BP12], Corollary 13.5, Corollary 13.6, Lemma 14.1 and Proposition 14.7. □

Proposition 3.1 allows us to define an action of Π on $D_1(\rho)$ by mapping I-characters to their Π-conjugates, thereby giving a family of basic diagrams $D(\rho,r):=(D_0(\rho),D_1(\rho),r)$ parameterized by IZ-equivariant injections $r:D_1(\rho)\hookrightarrow D_0(\rho)$. The diagrams $D(\rho,r)$ attached to Galois representations ρ in this way are called *Diamond diagrams* in [BP12].

3.2. **The map δ.** We now introduce the map $\delta:\mathcal{D}(\rho)\longrightarrow\mathcal{D}(\rho)$ which governs the dynamics of the Π-action on $D(\rho,r)$ and plays an important role in proving the irreducibility of representations of G coming from $D(\rho,r)$. There is a natural identification of the set $\mathcal{D}(\rho)$ of weights with the set of subsets of $\mathbb{Z}/f\mathbb{Z}=\{0,1,\dots,f-1\}$ ([BP12], §11). Under this identification, the map δ is defined as follows:

Definition 3.2. For $J\subseteq\mathbb{Z}/f\mathbb{Z}$,

$$\delta(J):=\begin{cases}\{j-1\mid j\in J\}\cup\{0\} & \text{if } 1\notin J\\ \{j-1\mid j\in J\}\setminus\{0\} & \text{if } 1\in J\end{cases}$$

with the convention $-1=f-1$.

Note that δ is a bijection and partitions the set $\mathcal{D}(\rho)$ into δ-orbits.

Example 3.3. We list the four δ-orbits for $f = 5$.

$$\Delta_1 = \big\{\phi \mapsto \{0\} \mapsto \{0,4\} \mapsto \{0,3,4\} \mapsto \{0,2,3,4\} \mapsto \{0,1,2,3,4\} \mapsto \{1,2,3,4\} \mapsto \{1,2,3\} \mapsto \{1,2\} \mapsto \{1\}\big\},$$
$$\Delta_2 = \big\{\{2\} \mapsto \{0,1\} \mapsto \{4\} \mapsto \{0,3\} \mapsto \{0,2,4\} \mapsto \{0,1,3,4\} \mapsto \{2,3,4\} \mapsto \{0,1,2,3\} \mapsto \{1,2,4\} \mapsto \{1,3\}\big\},$$
$$\Delta_3 = \big\{\{3\} \mapsto \{0,2\} \mapsto \{0,1,4\} \mapsto \{3,4\} \mapsto \{0,2,3\} \mapsto \{0,1,2,4\} \mapsto \{1,3,4\} \mapsto \{2,3\} \mapsto \{0,1,2\} \mapsto \{1,4\}\big\},$$
$$\Delta_4 = \big\{\{2,4\} \mapsto \{0,1,3\}\big\}.$$

The map δ has a nice reinterpretation. Identify the set of subsets of $\mathbb{Z}/f\mathbb{Z}$ as the set of binary numbers (sequences of 0s and 1s) of length f. The subset $J \subseteq \mathbb{Z}/f\mathbb{Z}$ corresponds to the binary number $a_1a_2\ldots a_f$ under the rule $a_j = 1$ if and only if $j \in J$, where we make the identification $f = 0$. Under this identification, δ is the map that moves the first digit of a binary number to the end and changes its parity:

$$\delta(a_1a_2\ldots a_f) = a_2a_3\ldots a_f(a_1+1) \text{ with the convention } 2 = 0.$$

Example 3.4. Let $f = 5$. The subset $\{0,1,3\}$ corresponds to the binary number 10101 and $\delta(10101) = 01010$ which corresponds to $\{2,4\}$.

It follows from the definition that

$$\delta^{2f}(a_1a_2\ldots a_f) = (a_1+2)(a_2+2)\ldots(a_f+2) = a_1a_2\ldots a_f.$$

Hence the order of δ is at most $2f$. In fact the order of δ equals $2f$ as one easily sees by considering the δ-orbit of the empty set ($= f$ zeros). It follows that the size of any δ-orbit divides $2f$. Observe that δ changes the size of a subset J by ± 1. So any δ-orbit contains an even number of subsets. Therefore, the size of a δ-orbit is $2f'$ for some f' dividing f. Using the reinterpretation of δ, we can prove the following result which is of independent interest.

Lemma 3.5. *The set $\mathcal{D}(\rho)$ has a δ-orbit of size $2f'$ if and only if $\frac{f}{f'}$ is odd.*

Proof. ($\Rightarrow$) Suppose $d := \frac{f}{f'}$ is even. Let $a = a_1a_2\ldots a_f$ belongs to a δ-orbit of size $2f'$. We write

$$a = a_1a_2\ldots a_{f'}a_{f'+1}a_{f'+2}\ldots a_{2f'}\ldots a_{(d-1)f'+1}a_{(d-1)f'+2}\ldots a_f.$$

Then

$$\delta^{f'}(a) = a_{f'+1}a_{f'+2}\ldots a_{2f'}a_{2f'+1}a_{2f'+2}\ldots a_{3f'}\ldots(a_1+1)(a_2+1)\ldots(a_{f'}+1).$$

Since a_j and $\delta^{f'}(a)_j$ must have opposite parity for all $1 \le j \le f$ and d is even by assumption, we get $a_1 = a_{(d-2)f'+1}$. Comparing the parity of the last block of f' digits in a and $\delta^{f'}(a)$, we also have $a_{(d-1)f'+1} = a_1$. This implies that the first digit $a_{(d-2)f'+1}$ of the second last block of f' digits in a is equal to the first digit $a_{(d-1)f'+1}$ of the second last block of f' digits in $\delta^{f'}(a)$, a contradiction.

($\Leftarrow$) Let a be the f-digit binary number starting with f' 0s, followed by f' 1s, followed by f' 0s, and so on. The number a ends with f' 0s as $\frac{f}{f'}$ is odd. Clearly $\delta^{f'}(a)$ flips the parity of the digits of a, showing that the δ-orbit of a has size $2f'$.

□

If $\sigma \in \mathcal{D}(\rho)$ corresponds to a subset J, let $\delta(\sigma)$ denote the weight corresponding to the subset $\delta(J)$. The map δ is characterized by the following property.

Lemma 3.6. *For $\sigma \in \mathcal{D}(\rho)$, $\delta(\sigma) \in \mathcal{D}(\rho)$ is the unique weight such that σ^s is a Jordan–Hölder factor of $D_{0,\delta(\sigma)}(\rho)$.*

Proof. See [BP12], Lemma 15.2. □

Using the combinatorics of the Π-action dynamics on $D(\rho, r)$ described by δ, one obtains the following theorem.

Theorem 3.7. *The basic diagram $D(\rho, r)$ is indecomposable, i.e., the KZ-representation $D_0(\rho)$ does not have a proper non-zero KZ-direct summand X such that X^{I_1} is stable under the action of* Π.

Proof. See [BP12], Theorem 15.4. □

3.3. **Irreducible admissible supercuspidal representations.** Let $\tau(\rho, r)$ be a smooth admissible representation of G given by the existence theorem applied to a Diamond diagram $D(\rho, r)$. We briefly sketch the argument of the irreducibility of $\tau(\rho, r)$ using Theorem 3.7. Let $\tau' \subseteq \tau(\rho, r)$ be a non-zero subrepresentation. Since $0 \neq \mathrm{soc}_K\tau' \subseteq \mathrm{soc}_K\tau(\rho, r) = \mathrm{soc}_K D_0(\rho)$, we have $\sigma \in \mathrm{soc}_K\tau'$ for some $\sigma \in \mathcal{D}(\rho)$. Thus, $D_1(\rho)^{\chi_\sigma} \subseteq \tau'$. As τ' is stable under the Π-action, we have $D_1(\rho)^{\chi_\sigma^s} \subseteq \tau'$. By Lemma 3.6, we see that $D_{1,\delta(\sigma)}(\rho)^{\chi_\sigma^s} \subseteq \tau'$. As τ' is clearly a K-representation, it follows that τ' contains the unique K-subrepresentation $I(\delta(\sigma), \sigma^s)$ of $D_{0,\delta(\sigma)}(\rho)$ with quotient σ^s. It is a non-trivial fact that the embedding $I(\delta(\sigma), \sigma^s) \hookrightarrow \tau'$ extends uniquely to an embedding $D_{0,\delta(\sigma)}(\rho) \hookrightarrow \tau'$. This requires delicate analysis of non-split extensions between weights (cf. [BP12], §17 and 18). Repeating the argument for $\delta(\sigma)$, we get $D_{0,\delta^2(\sigma)}(\rho) \subseteq \tau'$ and so on. Since the map δ has finite order, we get $D_{0,\sigma}(\rho) \subseteq \tau'$. It then follows that

$$\bigoplus_{\sigma \in \mathrm{soc}_K\tau'} D_{0,\sigma}(\rho) = \tau' \cap D_0(\rho).$$

Since the space of I_1-invariants of the right hand side in the above is stable under the action of Π, the same is true for the left hand side which is a non-zero direct summand of $D_0(\rho)$. This contradicts Theorem 3.7 unless $\tau' = \tau(\rho, r)$. Hence $\tau(\rho, r)$ is irreducible.

As $\mathrm{soc}_K\tau(\rho, r) = \mathrm{soc}_K D_0(\rho)$, the number of weights in the K-socle of $\tau(\rho, r)$ is equal to the size of $\mathcal{D}(\rho)$ which is $2^f > 2$. Any subquotient of a principal series representation of G has at most two weights in its K-socle ([Bre07], Remark 4.9). It follows that $\tau(\rho, r)$ is supercuspidal.

Finally, we remark that if $D(\rho, r)$ and $D(\rho, r')$ are two non-isomorphic basic diagrams, then any two smooth admissible G-representations $\tau(\rho, r)$ and $\tau(\rho, r')$ are non-isomorphic

([BP12], Theorem 19.8 (ii)). In fact, even the representation $\tau(\rho, r)$ is not uniquely determined by $D(\rho, r)$ ([Hu10]).

3.4. **Extra characters.** Let us now fix a diagram $D = (D_0, D_1, r)$ in the family $\{D(\rho, r)\}_r$ for the rest of the article. We have $D_0 = \bigoplus_{\sigma\in\mathcal{D}(\rho)} D_{0,\sigma}$. Write $D_{1,\sigma} = (D_{0,\sigma})^{I_1}$. For any δ-orbit Δ, we write

$$D_{0,\Delta} := \bigoplus_{\sigma\in\Delta} D_{0,\sigma} \text{ and } D_{1,\Delta} := (D_{0,\Delta})^{I_1}.$$

We call an I-character $\chi \subset D_1$ *extra* if $\chi \neq \chi_\sigma$ and $\chi \neq \chi_\sigma^s$ for any $\sigma \in \mathcal{D}(\rho)$. There are a total of $3^f - 1$ characters in D_1 (Proposition 3.1). Of these at most 2^{f+1} characters correspond to the socle weights and their Π-conjugates. Therefore, the set of extra characters is non-empty because $3^f - 1 > 2^{f+1}$ as $f > 2$. We remark that Lemma 3.6 together with Theorem 3.7 imply that for a given δ-orbit Δ, there is an extra character χ such that $(D_{1,\Delta})^\chi \neq 0$.

Let n be the number of δ-orbits of $\mathcal{D}(\rho)$. As the set $\mathcal{D}(\rho)$ has cardinality 2^f and $f > 2$, we have $n > 1$. The existence of the set of extra characters established in the following lemma is used crucially by Le in his construction of non-admissible irreducible G-representations.

Lemma 3.8. *There exists a set S of $2(n-1)$ extra characters closed under Π-conjugation such that given a δ-orbit Δ, there is a $\chi \in S$ satisfying $(D_{1,\Delta})^\chi \neq 0$.*

Proof. Choose any δ-orbit, call it Δ_1, and pick an extra character, say χ_1, such that

$$(D_{1,\Delta_1})^{\chi_1} \neq 0 \text{ and } \Big(\bigoplus_{\sigma\in\mathcal{D}(\rho)\setminus\Delta_1} D_{1,\sigma}\Big)^{\chi_1^s} \neq 0.$$

The existence of such a χ_1 is guaranteed by Theorem 3.7. Call the orbit Δ_2 for which $(D_{1,\Delta_2})^{\chi_1^s} \neq 0$. Using Theorem 3.7 again, there is an extra character χ_2 such that

$$\big(D_{1,\Delta_1} \bigoplus D_{1,\Delta_2}\big)^{\chi_2} \neq 0 \text{ and } \Big(\bigoplus_{\sigma\in\mathcal{D}(\rho)\setminus(\Delta_1\sqcup\Delta_2)} D_{1,\sigma}\Big)^{\chi_2^s} \neq 0.$$

Note that $\chi_2 \notin \{\chi_1, \chi_1^s\}$. Call the orbit Δ_3 for which $(D_{1,\Delta_3})^{\chi_2^s} \neq 0$. Proceeding in this way, we find n δ-orbits $\Delta_1, \Delta_2, \ldots, \Delta_n$ of $\mathcal{D}(\rho)$ and $(n-1)$ extra characters $\chi_1, \chi_2, \ldots, \chi_{n-1}$ such that $(D_{1,\Delta_{j+1}})^{\chi_j^s} \neq 0$ for all $1 \leq j \leq n-1$. Take $S = \{\chi_1, \chi_1^s, \chi_2, \chi_2^s, \ldots, \chi_{n-1}, \chi_{n-1}^s\}$. ☐

4. Infinite-dimensional diagrams and non-admissible representations

We now explain Le's method of constructing infinite-dimensional diagrams from Diamond diagrams to produce non-admissible irreducible representations. Let $D_0(\infty) := \bigoplus_{i\in\mathbb{Z}} D_0(i)$ be the smooth KZ-representation with componentwise KZ-action, where there is a fixed isomorphism $D_0(i) \cong D_0$ of KZ-representations for every $i \in \mathbb{Z}$. Denote the natural inclusion $D_0 \xrightarrow{\sim} D_0(i) \hookrightarrow D_0(\infty)$ by ι_i, and write $v_i := \iota_i(v)$ for $v \in D_0$ for every $i \in \mathbb{Z}$. Let $D_1(\infty) := D_0(\infty)^{I_1}$.

We make use of the δ-orbits and the set S of extra characters from the proof of Lemma 3.8 to define a Π-action on $D_1(\infty)$ which is different from the componentwise Π-action.

Pick a pair of extra characters $\{\psi, \psi^s\}$ not belonging to the set S. To justify the existence of such a pair, note that it is enough to show the inequality $2(n-1) < 3^f - 1 - 2^{f+1}$ for all $f > 2$. Since the size of any δ-orbit is even, we have $n \leq 2^{f-1}$. Thus $2(n-1) \leq 2^f - 2$. It is now easy to check that $2^f - 2 < 3^f - 1 - 2^{f+1}$ for all $f > 2$.

Let us choose a weight $\sigma_k \in \Delta_k$ for all $1 \leq k \leq n$ and let $\lambda = (\lambda_i) \in \prod_{i\in\mathbb{Z}} \overline{\mathbb{F}_p}^{\times}$. For all integers $i \in \mathbb{Z}$, define

$$\Pi v_i := \begin{cases} (\Pi v)_i & \text{if } v \in D_1^{\chi} \text{ for } \chi \notin \{\chi_{\sigma_1}, \chi_{\sigma_1}^s, \ldots, \chi_{\sigma_n}, \chi_{\sigma_n}^s, \psi, \psi^s\}, \\ (\Pi v)_{i+1} & \text{if } v \in D_1^{\chi} \text{ for } \chi \in \{\chi_{\sigma_1}, \ldots, \chi_{\sigma_{n-1}}\}, \\ (\Pi v)_{i-1} & \text{if } v \in D_1^{\chi_{\sigma_n}}, \\ \lambda_i (\Pi v)_i & \text{if } v \in D_1^{\psi}. \end{cases}$$

This uniquely determines a smooth N-action on $D_1(\infty)$ such that $p = \Pi^2$ acts trivially on it. Thus we get a basic diagram $D(\lambda) := (D_0(\infty), D_1(\infty), \mathrm{can})$ with the above actions where can is the canonical inclusion $D_1(\infty) \hookrightarrow D_0(\infty)$.

Theorem 4.1 (Le). *There exists a smooth representation π of G such that*

(1) $(\pi|_{KZ}, \pi|_N, \mathrm{id})$ *contains* $D(\lambda)$*,*
(2) π *is generated by* $D_0(\infty)$ *as a* G*-representation, and*
(3) $\mathrm{soc}_K \pi = \mathrm{soc}_K D_0(\infty)$*.*

Proof. The idea is to consider the infinite direct sum $\bigoplus_{i\in\mathbb{Z}} \Omega(i)$ where each $\Omega(i)$ is isomorphic to the smooth injective K-envelope Ω of D_0, and equip this direct sum with an N-action extending the N-action on $D_1(\infty)$ defined above. The proof is same as that of [Le19], Theorem 3.2, presented for $f = 3$. □

Theorem 4.2 (Le). *If $\lambda_i \neq \lambda_0$ for all $i \neq 0$, then any smooth representation π of G satisfying the properties (1), (2), and (3) of Theorem 4.1 is irreducible and non-admissible.*

Proof. Let $\pi' \subseteq \pi$ be a non-zero subrepresentation of G. By property (3), we have $\mathrm{Hom}_K(\sigma, \pi') \neq 0$ for some $\sigma \in \mathrm{soc}_K D_0$. Considering that σ could be embedded diagonally in π', there exists a non-zero $(c_i) \in \bigoplus_{i\in\mathbb{Z}} \overline{\mathbb{F}_p}$ such that

$$\Big(\sum_i c_i \iota_i\Big)(\sigma) \subset \pi',$$

or equivalently

$$\Big(\sum_i c_i \iota_i\Big)(D_{0,\sigma}) \cap \pi' \neq 0,$$

because the K-socle of $\big(\sum_i c_i \iota_i\big)(D_{0,\sigma})$ is $\big(\sum_i c_i \iota_i\big)(\sigma)$, which is irreducible.

We claim that

$$\Big(\sum_i c_i \iota_{i+j}\Big)(D_0) \subset \pi' \quad \text{for all } j \in \mathbb{Z}. \tag{4.3}$$

We prove the claim (4.3) assuming $\sigma \in \Delta_n$. The cases where σ is in an orbit other than Δ_n are proved similarly. If $\sigma \in \Delta_n$, then σ is in the same δ-orbit Δ_n as σ_n is. So it follows from the discussion in Subsection 3.3 that

$$\Big(\sum_i c_i\iota_i\Big)(\sigma) \subset \pi' \Longrightarrow \Big(\sum_i c_i\iota_i\Big)(\sigma_n) \subset \pi'.$$

Note that the indices i are unchanged since the action of Π on $\iota_i(D_1^{\chi_\sigma})$ fixes the index i of the embedding ι_i for all $\sigma \in \Delta_n$ except σ_n. Since the Π-action takes $\iota_i(D_1^{\chi_{\sigma_n}})$ to $\iota_{i-1}(D_1^{\chi^s_{\sigma_n}})$, we have

$$\Big(\sum_i c_i\iota_{i-1}\Big)(D_{0,\delta(\sigma_n)}) \subset \pi'.$$

Therefore, again from the discussion in Subsection 3.3, we get that

$$\Big(\sum_i c_i\iota_{i-1}\Big)(D_{0,\Delta_n}) \subset \pi'.$$

Continuing in this fashion, we obtain

$$\Big(\sum_i c_i\iota_{i+j}\Big)(D_{0,\Delta_n}) \subset \pi' \quad \text{for all } j < 0.$$

Making use of the extra character χ^s_{n-1} in the proof of Lemma 3.8, we have in particular,

$$\Big(\sum_i c_i\iota_{i+j}\Big)(D_1^{\chi^s_{n-1}}) \subset \pi' \quad \text{for all } j < 0.$$

Therefore,

$$\Big(\sum_i c_i\iota_{i+j}\Big)(D_1^{\chi_{n-1}}) \subset \pi' \quad \text{for all } j < 0.$$

We know from the proof of Lemma 3.8 that $(D_{1,\Delta_k})^{\chi_{n-1}} \neq 0$ for some $1 \leq k < n$. Since the Π-action takes $\iota_i(D_1^{\chi_{\sigma_k}})$ to $\iota_{i+1}(D_1^{\chi^s_{\sigma_k}})$, we obtain

$$\Big(\sum_i c_i\iota_{i+j}\Big)(D_{0,\Delta_k}) \subset \pi' \quad \text{for all } j \in \mathbb{Z} \quad \text{for some } 1 \leq k < n.$$

Making use of the extra character χ^s_{k-1}, by the same arguments as above, we obtain

$$\Big(\sum_i c_i\iota_{i+j}\Big)(D_{0,\Delta_{k'}}) \subset \pi' \quad \text{for all } j \in \mathbb{Z} \quad \text{for some } 1 \leq k' < k.$$

Continuing in this fashion, we finally get that

$$\Big(\sum_i c_i\iota_{i+j}\Big)(D_{0,\Delta_1}) \subset \pi' \quad \text{for all } j \in \mathbb{Z}.$$

Recall from the proof of Lemma 3.8 that

$$\Big(\bigoplus_{m=1}^{l} D_{1,\Delta_m}\Big)^{\chi_l} \neq 0 \ \text{ and } \ (D_{1,\Delta_{l+1}})^{\chi^s_l} \neq 0 \ \text{ for all } \ 1 \leq l \leq n-1. \tag{4.4}$$

Using (4.4) with $l = 1$ we get

$$\Big(\sum_i c_i\iota_{i+j}\Big)((D_{1,\Delta_1})^{\chi_1}) \subset \pi' \quad \text{and} \quad \Big(\sum_i c_i\iota_{i+j}\Big)((D_{1,\Delta_2})^{\chi^s_1}) \subset \pi' \quad \text{for all } j \in \mathbb{Z}.$$

This implies

$$\big(\sum_i c_i\iota_{i+j}\big)(D_{0,\Delta_2}) \subset \pi \quad \text{for all } j \in \mathbb{Z}.$$

Similarly, using (4.4) successively for $l = 2, \dots, n-1$, we obtain

$$\big(\sum_i c_i\iota_{i+j}\big)(D_{0,\Delta_r}) \subset \pi' \quad \text{for all } j \in \mathbb{Z} \quad \text{and for all } 1 \le r \le n.$$

Hence $\big(\sum_i c_i\iota_{i+j}\big)(D_0) \subset \pi'$ for all $j \in \mathbb{Z}$ as desired.

For $(d_i) \in \bigoplus_{i\in\mathbb{Z}} \overline{\mathbb{F}_p}$, let $\#(d_i)$ denote the number of non-zero d_i's. Among all the non-zero elements (c_i) of $\bigoplus_{i\in\mathbb{Z}} \overline{\mathbb{F}_p}$ for which $\big(\sum_i c_i\iota_i\big)(D_0) \subset \pi'$, we pick one with $\#(c_i)$ minimal. We may also assume that $c_0 \neq 0$ using (4.3). We now show that $\#(c_i) = 1$. Assume to the contrary that $\#(c_i) > 1$. Since $\big(\sum_i c_i\iota_i\big)(D_1^{\psi}) \subset \pi'$ and π' is stable under the Π-action, we have

$$\big(\sum_i \lambda_i c_i\iota_i\big)(D_1^{\psi^s}) \subset \pi'.$$

Since $\big(\sum_i \lambda_0 c_i\iota_i\big)(D_1^{\psi^s})$ is also clearly in π', subtracting it from the above, we get

$$\big(\sum_i (\lambda_i - \lambda_0) c_i\iota_i\big)(D_1^{\psi^s}) \subset \pi'.$$

Let $\nu \in \mathcal{D}(\rho)$ be the weight for which $D_{1,\nu}^{\psi^s} \neq 0$. Writing $(c_i') := ((\lambda_i - \lambda_0)c_i)$, we see that

$$\big(\sum_i c_i'\iota_i\big)(D_{0,\nu}) \cap \pi' \neq 0.$$

Following the same arguments as in the previous paragraph proving the claim (4.3), we get that $\big(\sum_i c_i'\iota_i\big)(D_0) \subset \pi'$. However, the hypothesis $\lambda_i \neq \lambda_0$ for all $i \neq 0$, and the assumption $\#(c_i) > 1$ imply that (c_i') is non-zero and $\#(c_i') = \#(c_i) - 1$ contradicting the minimality of $\#(c_i)$. Therefore, we have $c_0\iota_0(D_0) \subset \pi'$. So $\iota_0(D_0) \subset \pi'$. Using (4.3) again, we get that $\bigoplus_{j\in\mathbb{Z}} \iota_j(D_0) = D_0(\infty) \subset \pi'$. By property (2) of Theorem 4.1, we have $\pi' = \pi$.

The non-admissibility of π is clear because $\pi^{K_1} \supseteq \mathrm{soc}_K\pi$ and $\mathrm{soc}_K\pi$ is not finite-dimensional by property (3) of Theorem 4.1. □

Remark 4.5. The strategy to construct non-admissible irreducible representations explained above fails for the group $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ because of the absence of extra characters in $D_1(\rho)$ when $f = 2$. However, it turns out that a Diamond diagram attached to a *reducible split* mod p Galois representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_{p^2})$ does have enough extra characters to employ Le's strategy to produce non-admissible irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_{p^2})$ (cf. [GS20]).

Remark 4.6. Note that the smooth irreducible non-admissible representations π in Theorem 4.2 and in [GS20], Theorem 3.2 have a central character because the action of p on π is trivial. By [BL94], Theorem 33 (1), π is a quotient of $\text{c-Ind}_{KZ}^G\sigma/(T-\lambda)(\text{c-Ind}_{KZ}^G\sigma)$ for some $\sigma \in \mathrm{soc}_K\pi$ and $\lambda \in \overline{\mathbb{F}_p}$. If $\lambda \neq 0$, by [BL94], Corollary 31, π is the unique irreducible quotient and by [BL94], Lemma 28 (1) and Theorem 33, all such quotients are admissible.

It follows that $\lambda = 0$ and π is a quotient of c-$\mathrm{Ind}_{KZ}^{G}\sigma/T(\text{c-}\mathrm{Ind}_{KZ}^{G}\sigma)$, i.e., π is supercuspidal. Since quotients of admissible representations are admissible, by [Hen09], Theorem 1, we deduce that the universal supercuspidal representation c-$\mathrm{Ind}_{KZ}^{G}\sigma/T(\text{c-}\mathrm{Ind}_{KZ}^{G}\sigma)$ is not admissible. This was already known, as mentioned in the introduction.

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