Ordinary forms and their local Galois representations

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Abstract. We describe what is known about the local splitting behaviour of Galois representations attached to ordinary cuspidal eigenforms. We relate this to a question of Coleman concerning the existence of non-CM eigenforms of weight $k \geq 2$ in the image of the $(k-1)$-st power of the theta derivation.

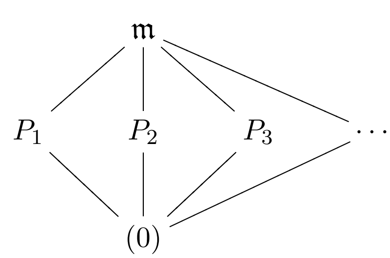
1. Introduction

Let p be a prime and let $\Lambda = \mathbb{Z}_p[[X]]$. Consider the following three kinds of cusp forms:

- mod p cusp forms,
- elliptic modular cusp forms (of weight ≥ 2), and,
- Λ-adic cusp forms.

These are intimately related. For instance a single Λ-adic form specializes to infinitely many different classical elliptic modular forms, and these characteristic 0 forms give rise to the same mod p form. In terms of the lattice of prime ideals in the dimension two local domain (e.g. Λ) which is the coefficient ring of the Λ-adic form:

(1.1)



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the Λ-adic form corresponds to the minimal prime ideal (0), the elliptic modular specializations correspond to certain height one prime ideals P_i, and the underlying mod p form corresponds to the maximal ideal $\mathfrak{m}$.

Assume that we have fixed a form of one of the three types above, and assume it is an eigenform. Then there is a Galois representation

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K)$$

attached to the form (in the case of Λ-adic forms one needs an assumption of ordinariness which we impose shortly). Here K is respectively

- a finite field $\mathbb{F}$ of characteristic p,
- a finite extension of $\mathbb{Q}_p$,
- a finite extension of the quotient field of Λ.

The representation ρ encapsulates important information about the form - for instance the Fourier coefficients of the form occur as traces of the images of Frobenius elements under ρ, at least outside a finite set of primes.

In recent years the local properties of such modular Galois representations have come under intensive study. This paper deals with one specific aspect of this study - we wish to describe what is known about the local splitting behaviour of such representations in the ordinary case. Let us elaborate. Recall that an eigenform as above is said to be ordinary at p if

- the eigenvalue at p is non-zero,
- the eigenvalue at p is a p-adic unit,
- the eigenvalue at p of every classical specialization of weight at least 2 is a p-adic unit,

respectively. The notion of ordinariness depends on choices of embeddings of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ and $\mathbb{C}$ which we assume have been implicitly fixed once and for all. It is a fundamental fact, due to

- Deligne [**6**],
- Mazur-Wiles [**15**] and Wiles [**19**],
- Hida [**11**], [**12**],

respectively, that if the eigenform is ordinary at p then the corresponding representation ρ is upper-triangular when restricted to a decomposition group G_p at p. More precisely, there are characters $\delta : G_p \to K^\times$ and $\epsilon : G_p \to K^\times$ and a map $u : G_p \to K$ such that the representation $\rho|_{G_p}$ has the shape

$$\rho|_{G_p} \sim \begin{pmatrix} \delta & u \\ 0 & \epsilon \end{pmatrix}. \tag{1.2}$$

Moreover the character ϵ is known to be unramified and its value on a Frobenius element at p can be specified completely.

Let us say that the representation ρ splits at p if u is the zero map (after a possible change of basis). More succinctly, the representation ρ is said to be split at p if the local representation $\rho|_{G_p}$ is semi-simple. This paper addresses the following natural question:

QUESTION 1. When is ρ split at p?

Interestingly, we shall see there are now almost complete answers to this question in the first and third cases. These cases correspond to the extremities in the diamond in figure (1.1) above. On the other hand the second case, which corresponds to the points in the center of the diamond, is still open. It is however possible to get quite a bit of information in this case by either 'going up' or 'going down' from the previous cases.

2. Mod p cusp forms

Let us begin with the case of mod p forms. There is a striking and almost complete answer to Question 1 in this case due to Gross and Coleman-Voloch. To recall it, let $f = \sum a_n q^n$ be a p-ordinary mod p cuspidal eigenform which has level N, prime to p, Serre weight $2 \leq k \leq p$, and nebentypus character ψ.

DEFINITION 2. Say that the mod p eigen cusp form $g = \sum b_n q^n$ of Serre weight $p+1-k$ is a companion form for f if the Fourier coefficients of f and g are related as follows:

$$n \cdot a_n = n^k \cdot b_n \tag{2.1}$$

for all $n = 1, 2, \ldots$

In terms of the derivation $\theta = q \cdot d/dq$, condition (2.1) says that $\theta f = \theta^k g$, whereas in terms of Galois representations it says $\rho_f \sim \rho_g \otimes \omega^{k-1}$ where ω is the mod p cyclotomic character. The condition of being a companion form is symmetric in f and g.

THEOREM 3 ([**10**], [**4**]). Assume either $k \neq 2$ or $k \neq p$. Then the representation

$$\rho_f \text{ is tamely ramified at } p \iff f \text{ has a companion form.}$$

Coleman-Voloch refer to the case $2 = k = p$ as the exceptional case.

In order to relate the theorem to Question 1 we need some terminology.

DEFINITION 4. Say that f is p-distinguished if the characters δ and ϵ are distinct.

More concretely, let $\lambda(x)$ denote the unramified character of G_p which takes the arithmetic Frobenius, Frob_p, to $x \in \mathbb{F}^\times$. Then $\epsilon = \lambda(a_p)$ and $\delta = \omega^{k-1} \cdot \lambda(\psi(p)/a_p)$. Thus f is p-distinguished if and only if $k < p$ or $a_p^2 \neq \psi(p)$.

Lemma 5. Let f be a mod p form as above.

(1) If ρ_f splits at p then ρ_f is tamely ramified at p.
(2) Assume f is p-distinguished. Then the converse is also true.

Proof. Let I_w denote the wild inertia group. Any character $G_p \to \mathbb{F}^\times$ has to be trivial on I_w since I_w is a pro-p group whereas $\mathbb{F}^\times$ has order prime to p. Hence if ρ_f splits at p then ρ_f is tamely ramified.

For the converse, assume that f is p-distinguished. By the inflation-restriction sequence, the restriction map

$$\mathrm{H}^1(G_p, \mathbb{F}(\delta\epsilon^{-1})) \to \mathrm{H}^1(I_p, \mathbb{F}(\delta\epsilon^{-1}))$$

has kernel $\mathrm{H}^1(G_p/I_p, \mathbb{F}(\delta\epsilon^{-1})^{I_p})$. We claim that this group vanishes so that the above map is injective. Indeed if $k < p$ then $\mathbb{F}(\delta\epsilon^{-1})^{I_p} = \mathbb{F}(\omega^{k-1})^{I_p} = 0$. If $k = p$ then $\mathbb{F}(\delta\epsilon^{-1})^{I_p} = \mathbb{F}$ and we must prove that $\mathrm{H}^1(G_p/I_p, \mathbb{F}(\chi))$ vanishes, for $\chi = \delta\epsilon^{-1}$. However, the condition $\delta \neq \epsilon$ (i.e. $a_p^2 \neq \psi(p)$) implies $\chi \neq 1$. Using the fact that $G_p/I_p = \hat{\mathbb{Z}}$ is topologically cyclic, we see that this group does indeed vanish.

We now further restrict to I_w. We claim that the restriction map

$$\mathrm{H}^1(I_p, \mathbb{F}(\chi)) \to \mathrm{H}^1(I_w, \mathbb{F}(\chi))$$

is also injective. By the inflation-restriction sequence again the kernel of this map is $\mathrm{H}^1(I_p/I_w, \mathbb{F}(\chi)^{I_w})$. This group vanishes since the tame inertia group is $I_p/I_w \xrightarrow{\sim} \prod_{\ell \neq p} \mathbb{Z}_\ell$ whereas $\mathbb{F}$ has exponent p.

Composing the two restriction maps above, we obtain an injective map

$$\mathrm{H}^1(G_p, \mathbb{F}(\chi)) \hookrightarrow \mathrm{H}^1(I_w, \mathbb{F}(\chi)). \tag{2.2}$$

Let $c \in \mathrm{Z}^1(G_p, \mathbb{F}(\chi))$ be the 1-cocycle defined by $c = \epsilon^{-1} \cdot u$. Recall ρ_f splits at p if and only if the cohomology class $[c] \in \mathrm{H}^1(G_p, \mathbb{F}(\chi))$ vanishes. Now assume $\rho_f(I_w) = 1$. Then the restriction of $[c]$ to I_w vanishes. By the injectivity of (2.2) we see that $[c]$ itself vanishes and ρ_f splits at p. □

It follows from the theorem and the lemma above that if f is p-distinguished then ρ_f splits at p if and only if f has a companion form, giving a complete answer to Question 1 in this case.

Note that if f is not p-distinguished (i.e. $\delta = \epsilon$ on G_p) then it is in principle possible to have ρ_f tamely ramified at p but ρ_f not split at p. Indeed any non-zero class [c] in the kernel of the map (2.2) will define a representation $\rho_f|_{G_p} \sim \left(\begin{smallmatrix} \delta & u \\ 0 & \delta \end{smallmatrix}\right)$ which is not split by assumption, but which splits on I_w, and hence is trivial on I_w. As Coleman-Voloch have remarked there does not yet appear to be any criterion to predict whether ρ_f is split at p if f is not p-distinguished (and not exceptional). Finally, in the exceptional case, nothing appears to be known.

Let $\Delta = \sum \tau(n) q^n$ be the unique normalized cusp form of weight 12 and level 1. It is known that Δ is ordinary for all primes $p < 10^6$, except $p = 2, 3, 5, 7$ and 2411. Atkin and Elkies have checked (see the end of Gross' paper [**10**]) that the mod p form associated to Δ does not have a companion form for $p < 3,500$ except $p = 23$ and $p = 691$. In the former case Δ is its own mod p companion form (note $\tau(n) \equiv n^{11}\tau(n)$ mod 23 for all n coprime to 23), and in the latter case $\bar{\rho}_\Delta$ is (globally) completely reducible. It follows that $\bar{\rho}_\Delta$ does not split at p for all primes $p < 3,500$ except for $p = 23$ and 691 where it does split.

Further examples of companion forms due to Atkin and Elkies may be found at the end of Gross' paper [**10**]. For instance the mod p form attached to the unique cusp form Δ_{16} of weight 16 and level 1 has a companion form for $p = 397$ so $\bar{\rho}_{\Delta_{16}}$ is split at p in this case.

3. Going up: consequences for elliptic modular forms

Let f be a primitive p-ordinary cusp form of weight at least 2. It is well known that ρ_f splits at p if f is of CM type. Some fairly general statements in the converse direction are available for forms of weight 2 thanks to ideas of Serre and Tate (see [**17**] and [**8**]).

Here we show that it is sometimes possible to lift the mod p results described in the previous section, to obtain some information for forms f of characteristic 0 of arbitrary weight $k \geq 2$. In particular if f is not of CM type, it is sometimes possible to deduce that ρ_f is not split at p by studying the mod p reduction $\bar{\rho}_f$ of ρ_f.

As in the mod p case, let us say that f is p-distinguished if the mod p reductions of δ and ϵ are distinct.

Proposition 6. Assume that

(1) f is p-distinguished,
(2) $\bar{\rho}_f$ is absolutely irreducible on $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

If $\bar{\rho}_f$ does not split at p then ρ_f does not split at p.

Proof. We first recall why it makes sense to speak of whether $\bar{\rho}_f$ is split or not split at p. The representation ρ_f takes values in $\mathrm{GL}_2(K)$, where K is a p-adic field, with ring of integers, say $\mathcal{O}$, and uniformizing element, say π. Let $V \cong K^2$ be a model for ρ_f. Let $L \cong \mathcal{O}^2 \subset V$ be a Galois stable lattice. Consider the reduced representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

on $L/\pi L$. In general this representation depends on the choice of L. However, under condition (2), it is an easy exercise to check that any two Galois stable lattices are homothetic, so that the representation $L/\pi L$ does not depend on L. In particular if the action of G_p on $L/\pi L$ is diagonal for one lattice L then it is diagonal for any lattice L.

We now claim that there is a Galois stable lattice $L \subset V$ and an $\mathcal{O}$-basis v_1, v_2 of L in which $\rho_f|_{G_p}$ has the usual upper-triangular form $\left(\begin{smallmatrix} \delta & u \\ 0 & \epsilon \end{smallmatrix}\right)$. To see this, note that by the ordinariness assumption there is a K-basis w_1, w_2 of V in which $\rho_f|_{G_p}$ is upper-triangular. Consider the lattice L_0 spanned by w_1 and w_2. Note δ and ϵ take values in $\mathcal{O}^\times$ and by replacing u by a scalar multiple of itself (i.e. modifying w_2 by a scalar) we may assume that u is $\mathcal{O}$-valued as well. Thus we may assume that L_0 is stable under G_p. Of course L_0 need not be stable under the action of the full group $G = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. But we may choose a lattice L containing L_0 which is Galois stable. This can be done as follows. Say that L_0 is obtained from the standard lattice in V via an element $x \in \mathrm{GL}_2(K)$. Then L_0 is stabilized by the compact open subgroup $K = x\,\mathrm{GL}_2(\mathcal{O})\,x^{-1}$. Now $\rho_f(G) \cap K$ is an open subgroup of the compact group $\rho_f(G)$, and so must have finite index in it. Choosing a set of coset representatives g_i, we see that the lattice $L = \sum g_i L_0$ is stable under $\rho_f(G)$. Now choose $n \geq 0$ such that $w_1/\pi^n \in L$, but $w_1/\pi^{n+1} \notin L$. Set $v_1 = w_1/\pi^n$ and choose v_2 such that v_1, v_2 is an $\mathcal{O}$-basis for L. Then L is a Galois stable lattice, and the basis v_1, v_2 has the desired upper-triangular property.

Now set $\chi = \delta\epsilon^{-1}$. Then there is a natural map

$$\mathrm{H}^1(G_p, \mathcal{O}(\chi)) \to \mathrm{H}^1(G_p, K(\chi)) \tag{3.1}$$

whose kernel is $\mathrm{H}^0(G_p, K/\mathcal{O}(\chi))$. By condition (1) we have $\bar{\chi} \neq 1$. An easy check shows that this is equivalent to $\mathrm{H}^0(G_p, K/\mathcal{O}(\chi)) = 0$. So the map (3.1) is injective. Now use the lattice L above to define the usual cohomology class $[c]$. Assume ρ_f splits at p. Then $[c]$ vanishes as an element of the second cohomology group in (3.1) above. Since the map (3.1) is injective $[c]$ vanishes integrally as well. In particular, L has a basis in which the action of G_p is diagonal. Thus we may assume $u = 0$ and $g \cdot v_1 = \delta(g)v_1$ and $g \cdot v_2 = \epsilon(g)v_2$ for all $g \in G_p$. Going mod π we have $g \cdot \bar{v}_1 = \bar{\delta}(g)\bar{v}_1$ and $g \cdot \bar{v}_2 = \bar{\epsilon}(g)\bar{v}_2$ for all such g. Since $\bar{\delta} \neq \bar{\epsilon}$, we see $\bar{v}_1$ and $\bar{v}_2$ are linearly independent, so $\bar{\rho}_f$ also splits at p. This proves the proposition. □

The p-distinguished assumption (1) is necessary in order to ensure the validity of the conclusion of the proposition. The following toy example, shown to us by K. Buzzard, illustrates what may go wrong. Start with a faithful representation $\rho : \mathrm{SL}_2(\mathbb{F}_5) \hookrightarrow \mathrm{GL}_2(\mathcal{O})$ where $\mathcal{O}$

is the ring of integers of a 5-adic field K. This representation may be constructed as follows. Take a faithful representation of $\mathrm{SL}_2(\mathbb{F}_5)$ in $\mathrm{GL}_2(\mathbb{C})$ (recall that $\mathrm{PSL}_2(\mathbb{F}_5) \cong A_5$). The image may be taken to lie in a number field, and by completing at a prime lying above 5, we may assume that the image is in a 5-adic field K. Finally, after conjugation, we may assume that the image lies in $\mathrm{GL}_2(\mathcal{O})$. Consider the reduction of ρ to $\mathrm{GL}_2(\mathbb{F})$ where $\mathbb{F}$ is the residue field of $\mathcal{O}$. Then the reduced representation $\bar{\rho}$ must be the inclusion $\mathrm{SL}_2(\mathbb{F}_5) \hookrightarrow \mathrm{GL}_2(\mathbb{F})$. In particular it is absolutely irreducible. Now the subgroup $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{F}_5)$ lifts to a subgroup of $\mathrm{GL}_2(K)$ of order 5, and so is certainly diagonalizable, but it is not diagonalizable itself.

Let us now show how the proposition can be used to study the splitting question in characteristic 0. Consider the Δ function. Then condition (1) is automatically satisfied. Condition (2) holds for all ordinary primes except $p = 691$ (cf. Serre's 1971/72 Bourbaki talk). By the work of Atkin and Elkies mentioned above, there is no mod p companion form in the range $p < 3,500$, $p \neq 691$ except for $p = 23$. It follows that $\rho_\Delta|_{G_p}$ is not split for all ordinary primes $p < 3,500$ with $p \neq 691$, except possibly for $p = 23$. The argument used here can't be used in the case $p = 23$ since $\bar{\rho}_\Delta|_{G_{23}}$ splits. However, Vatsal has shown that, even so, $\rho_\Delta|_{G_{23}}$ does not split (see [**18**]).

4. Λ-adic cusp forms

We now turn to the case of Λ-adic forms.

Let F be a p-ordinary primitive Λ-adic eigen cusp form and let ρ_F be the associated Galois representation. As in the elliptic modular case F may or may not be of CM type. Recall here that a Λ-adic form F is said to be of CM type if F has at least one classical specialization of weight $k \geq 2$ which is of CM type. In turns out that if there is one such specialization then every classical specialization of weight $k \geq 2$ is of CM type.

As in the elliptic modular case, it is not difficult to show that if F is of CM type then ρ_F splits at p. Interestingly, the converse was recently shown to hold in the Λ-adic setting in joint work with V. Vatsal, under some technical conditions. To state the precise result let us introduce the following notation. Assume in this section that p is an odd prime and let $M = \mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$ be the associated quadratic field. As usual we shall say that the form F is p-distinguished if $\bar{\delta} \neq \bar{\epsilon}$. Write $\bar{\rho}_F$ for the mod p representation attached to ρ_F. Then we have:

Theorem 7 ([**9**]). Say $p > 2$. Assume that

(1) F is p-distinguished, and,

(2) $\bar{\rho}_F$ is absolutely irreducible when restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

Then ρ_F splits at p if and only if F has CM.

The theorem completely answers Question 1 when p is odd, under the hypotheses (1) and (2) above. The proof is, somewhat surprisingly, based on a study of the weight one specializations of F. In particular it turns out that a form F as above is of CM type if and only if F has infinitely many weight 1 specializations which are classical (and not just p-adic) cusp forms. This is proved using recent work of Buzzard [**?**] which gives conditions under which a p-adic Galois representation arises from a classical weight 1 form.

5. Going down: consequences for elliptic modular forms

As already mentioned, for ordinary elliptic modular eigenforms f of CM type, the associated Galois representation ρ_f splits at p. Much less is known about the converse except for some results in weight 2.

However, it is possible to obtain some information about elliptic modular forms of higher weight, by descent, from Theorem 7. We have:

Theorem 8 ([**9**]). Let p be an odd prime. Let S be the set of p-ordinary elliptic modular forms of weight $k \geq 2$ such that

- f is p-distinguished,
- $\bar{\rho}_f$ is absolutely irreducible when restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$,

and such that the $f \in S$ are specializations of the same primitive p-ordinary Λ-adic form F. Then, for all but finitely many $f \in S$, the representation

$$\rho_f \text{ splits at } p \text{ if and only if } f \text{ is of CM type.}$$

It is a result of Hida that every (p-stabilized) p-ordinary cusp form f of weight at least 2 is the specialization of some primitive p-ordinary Λ-adic form F (for the CM case see below). In view of this and the theorem above, we see for odd p, the naive guess to Question 1, namely ρ_f splits at p exactly when f is of CM type, holds generically, at least for forms f satisfying the condition (1) and (2) above.

Let us give an example to show how the theorem gives new information. Let $p = 397$ and let Δ_{16} denote the unique p-ordinary cusp form of level 1 and weight 16, discussed at the end of section 2. Consider the Λ-adic form F which contains Δ_{16}. As mentioned at the end of section 2, Δ_{16} has a mod p companion form and $\bar{\rho}_{\Delta_{16}}$ splits at p. So one cannot hope to 'go up' and obtain some information about the local splitting behaviour of $\rho_{\Delta_{16}}$, or for that matter, any of the characteristic 0 Galois representations attached to the members in S. However the theorem above shows that, even so, for all but finitely many specializations $f \in S$, ρ_f does not split at p, as is to be expected of the members of the non-CM family F. Incidentally, note that the members $f \in S$ automatically satisfy condition (1) in the theorem. Condition (2) holds since $\bar{\rho}_{\Delta_{16}}$ has image containing $\mathrm{SL}_2(\mathbb{F}_p)$. (We did not consider the first example at the end of section 2 here, namely $f = \Delta$ and $p = 23$, since $\bar{\rho}_\Delta$ is reducible on $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$).

Though it is slightly tangential to our exposition here, for the convenience of the reader we recall the explicit construction of the CM form F which contains a given CM form f. Our exposition is based on [**11**, Theorem 7.1] and [**13**, pages 235-6].

So assume that p is an odd prime, and that N_0 is an integer prime to p. Let ψ be a character mod $N_0 p$. For a primitive p^{r_0-1}-th root of unity ζ_0, with $r_0 \geq 1$, let χ_{ζ_0} denote the wild character of $(\mathbb{Z}/p^{r_0})^\times$ obtained by mapping $1+p$ to ζ_0. Assume f is a p-stabilized p-ordinary cusp form of weight $k \geq 2$, level $N = N_0 p^{r_0}$ and character $\chi = \psi\omega^{-k}\chi_{\zeta_0}$. Note that ψ and the root of unity ζ_0 are uniquely determined by χ.

Assume f is of CM type. Let K be the associated imaginary quadratic field. An easy check, using the p-ordinariness of f, shows that p must split in K. Thus if $\mathfrak{p}$ is a prime of K lying over p then $p = \mathfrak{p}\bar{\mathfrak{p}}$ where $\bar{\mathfrak{p}}$ is the complex conjugate of $\mathfrak{p}$.

Assume f is p-new. Since f has CM, there is a primitive Hecke character ϕ of K of infinity type $(k-1, 0)$ such that

$$f = \sum \phi(\mathfrak{a}) q^{\mathrm{N}(\mathfrak{a})} \tag{5.1}$$

where $\mathfrak{a}$ varies over all integral ideals of K and $\mathrm{N}(\mathfrak{a})$ is the norm from K to $\mathbb{Q}$ of $\mathfrak{a}$. Let $\mathfrak{c}$ denote the conductor of ϕ. An easy check shows that exactly one of $\mathfrak{p}$ or $\bar{\mathfrak{p}}$ divides $\mathfrak{c}$. Assume that $\mathfrak{p}$ divides $\mathfrak{c}$ and $\bar{\mathfrak{p}} \nmid \mathfrak{c}$, so $\phi(\mathfrak{p}) = 0$ and $\phi(\bar{\mathfrak{p}}) = a(p, f)$ is a p-adic unit.

Now assume f is p-old. Then f is in fact the p-stabilization of a series similar to that occurring in the right hand side of (5.1) for some primitive Hecke character ϕ_0 of prime-to-p conductor $\mathfrak{c}_0$. However if ϕ is defined by $\phi(\mathfrak{q}) = \phi_0(\mathfrak{q})$ for all primes $\mathfrak{q}$ different from $\mathfrak{p}$ and $\phi(\mathfrak{p}) = 0$, then (5.1) continues to hold for f and ϕ. In this case we set $\mathfrak{c} = \mathfrak{c}_0\mathfrak{p}$. In either case the prime-to-p part $\mathfrak{c}_0$ of $\mathfrak{c}$ satisfies $\mathrm{N}(\mathfrak{c}_0)|D| = N_0$ where D is the discriminant of K. As for the p-part of $\mathfrak{c}$, note $\mathfrak{p}|\mathfrak{c}$ but $\bar{\mathfrak{p}} \nmid \mathfrak{c}$.

We now construct the CM form F containing f as follows. Fix any $\mathbb{C}^\times$-valued Hecke character λ of K of infinity type $(1, 0)$ and conductor $\mathfrak{p}$. Let $\mathbb{Q}(\lambda)$ be the number field generated by the values of λ. Let E be the completion of $\mathbb{Q}(\lambda)$ at the prime determined by the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ which has been implicitly fixed throughout this paper. Write $\mathcal{O}_E$ for the ring of integers of E. Decompose $\mathcal{O}_E^\times = \mu_E \times W_E$ where μ_E is a finite abelian group and W_E is $\mathbb{Z}_p$-free. Write the projection of $x \in \mathcal{O}_E^\times$ onto W_E as $\langle x \rangle$. Let W_K denote the subgroup of W_E topologically generated by the $\langle \lambda(\mathfrak{a}) \rangle$ as $\mathfrak{a}$ varies through the integral ideals of K; it is isomorphic to $\mathbb{Z}_p$. Moreover it naturally contains $W = 1+p\mathbb{Z}_p$. Let $\gamma \geq 0$ be defined by $[W_K : W] = p^\gamma$. Fix a generator w of W_K so that $w^{p^\gamma} = 1+p$. Define the character s by

$$s(\mathfrak{a}) = \frac{\log \langle \lambda(\mathfrak{a}) \rangle}{\log w} \in \mathbb{Z}_p.$$

Let $\mathcal{O}$ be the ring of integers in $E(\phi, \zeta_0)$. Recall that $\Lambda = \mathbb{Z}_p[[X]]$. Consider the extension $L = \mathcal{O}[[Y]]$ of Λ defined by

$$\zeta_0(1+Y)^{p^\gamma} = (1+X). \tag{5.2}$$

Define the formal q-expansion F in $L[[q]]$ by

$$F = \sum_{\mathfrak{a}} \phi(\mathfrak{a}) \langle \lambda(\mathfrak{a}) \rangle^{-k} (1+Y)^{s(\mathfrak{a})} q^{\mathrm{N}(\mathfrak{a})}.$$

This q-expansion does not depend on the choice of λ since any two Hecke characters λ and λ' of infinity type $(1,0)$ and conductor $\mathfrak{p}$ differ by a finite order character, so $\langle \lambda \rangle = \langle \lambda' \rangle$.

In view of (5.2) every arithmetic point of L is given by $Y = \zeta w^l - 1$ where $l \geq 2$ and ζ is a p^{r-1}-th root of 1, for $r \geq 1$. This point extends the evaluation $X = \zeta_0 \zeta^{p^\gamma}(1+p)^l - 1$ of Λ. Set $\delta_\zeta(\mathfrak{a}) = \zeta^{s(\mathfrak{a})}$ and set

$$\phi_{l,\zeta}(\mathfrak{a}) = \phi(\mathfrak{a}) \langle \lambda(\mathfrak{a}) \rangle^{l-k} \delta_\zeta(\mathfrak{a}).$$

Then δ_ζ is a finite order character and $\phi_{l,\zeta}$ is a Hecke character of infinity type $(l-1,0)$. Specializing at the point $Y = \zeta w^l - 1$ we get $(1+Y)^{s(\mathfrak{a})} = \delta_\zeta(\mathfrak{a}) \cdot \langle \lambda(\mathfrak{a}) \rangle^l$. So F specializes to the q-expansion

$$f_{l,\zeta} = \sum_{\mathfrak{a}} \phi_{l,\zeta}(\mathfrak{a}) q^{\mathrm{N}\mathfrak{a}}$$

which is clearly a CM cusp form of weight l. For a Hecke character θ of infinity type $(t,0)$ write $\theta|_{\mathbb{Q}}$ for the induced finite order Dirichlet character defined by $m \mapsto \theta((m))/m^t$. Then $\langle \lambda \rangle|_{\mathbb{Q}} = \omega^{-1}$ and $\delta_\zeta|_{\mathbb{Q}} = \chi_{\zeta^{p^\gamma}}$. By the standard formula for the nebentypus of the CM form f we have $\chi = \phi|_{\mathbb{Q}} \cdot \chi_{K/\mathbb{Q}} = \psi \omega^{-k} \chi_{\zeta_0}$ where $\chi_{K/\mathbb{Q}}$ is the quadratic character corresponding to K. Hence the character of $f_{l,\zeta}$ is given by

$$\phi_{l,\zeta}|_{\mathbb{Q}} \cdot \chi_{K/\mathbb{Q}} \;=\; \phi|_{\mathbb{Q}} \cdot \chi_{K/\mathbb{Q}} \cdot \langle \lambda \rangle|_{\mathbb{Q}}^{l-k} \cdot \delta_\zeta|_{\mathbb{Q}} \;=\; \psi \omega^{-l} \chi_{\zeta_0 \zeta^{p^\gamma}}.$$

Finally one may check that $f_{l,\zeta}$ has level $N_0 p^{r'}$ where $p^{r'-1}$ is the exact order of $\zeta_0 \zeta^{p^\gamma}$.

It follows that F is a Λ-adic form of CM type. Furthermore, F is clearly an eigenform of level N_0. It is even a newform of level N_0, since the conductor of $\phi_{l,\zeta}$ is divisible by $\mathfrak{c}_0$, so the level of the primitive form attached to each $f_{l,\zeta}$ is divisible by $N_0 = \mathrm{N}(\mathfrak{c}_0)|D|$. Moreover, F is p-ordinary since $a(p, f_{l,\zeta}) = \phi_{l,\zeta}(\bar{\mathfrak{p}})$ has the same p-adic valuation as $\phi(\bar{\mathfrak{p}})$. In sum F is a primitive Λ-adic form of level N_0.

Finally when $l = k$ and $\zeta = 1$, then $\phi_{l,\zeta} = \phi$ and $f_{k,1} = f$. Thus F contains f as a specialization.

6. Λ-adic companion forms and a question of Coleman

We wish to end this note by explaining how Question 1 is related to a question of Coleman on the image of a power of the θ derivation acting on p-adic modular forms, and to the work of Buzzard-Taylor on Λ-adic companion forms.

Let p be a prime and let N_0 be an integer prime to p. Let $S_k(N_0)$ denote the space of overconvergent p-adic modular cusp forms of tame level N_0 and weight $k \in \mathbb{Z}$. Recall that $S_k(N_0)$ is an infinite dimensional p-adic Banach space. Classical primitive cusp forms f of weight $k \geq 2$ and level N_0p^r, for $r \geq 0$, may be thought of as overconvergent p-adic modular forms of tame level N_0. In the case $r = 0$ this may be done in two ways.

Let $\theta = q \cdot d/dq$ be the usual derivation which takes p-adic modular cusp forms of weight $k \in \mathbb{Z}$ and tame level N_0 to such forms of weight $k + 2$. While in general the operator θ destroys overconvergence, it is a result of Coleman [**5**, thm. 2] that for $k \geq 2$ the operator θ^{k-1} induces a linear map

$$\theta^{k-1} : S_{2-k}(N_0) \quad \rightarrow \quad S_k(N_0).$$

Proposition 9 ([**3**], prop. 7.1). *Every classical CM eigenform of weight $k \geq 2$ and slope $k - 1$ is in the image of θ^{k-1}.*

Note that the images of eigenforms under θ^{k-1} have slope at least $k - 1$. This is because eigenforms of any weight have slope at least 0 and θ^{k-1} knocks the slope up by $k-1$. Coleman has asked the following question ([**3**], rem. 2, pg. 232):

Question 10. *Are there any classical non-CM forms f of weight $k \geq 2$ in the image of θ^{k-1}?*

As one might expect from the results about mod p cusp forms discussed in section 2, Question 10 is closely connected to Question 1. Let us explain this in more detail.

Say $f = \sum a_n q^n$ is a classical primitive p-ordinary form of weight $k \geq 2$ and level N_0p^r, for $r \geq 0$. We will define a form f' of slope $k - 1$ and show that if the answer to Coleman's question for the form f' is 'yes', then ρ_f splits at p. Conversely, we show (under some technical conditions) that if ρ_f is split at p then f' is in the image of θ^{k-1}. To define f' we consider two cases.

Case $r = 0$. Let χ be the character of f and let α and β be the roots of the polynomial

$$x^2 - a_p x + p^{k-1}\chi(p)$$

with p-adic valuations 0 and $k-1$ respectively. Let $f_\alpha = f(z) - \beta f(pz)$ and $f_\beta = f(z) - \alpha f(pz)$ be the corresponding p-stabilized forms of level $N = N_0 p$ with U_p-eigenvalue α and β respectively. Both f_α and f_β may be considered as overconvergent eigenforms of tame level N_0, of slopes 0 and $k-1$ respectively. Moreover the p-adic Galois representations associated to f_α and f_β coincide with ρ_f. We set $f' := f_\beta$.

Case $r > 0$. In this case f is p-new. By our assumption of ordinariness we have $a_p \neq 0$ so that the local factor at p of the automorphic representation corresponding to f is either

- in the ramified principal series, or,
- a twist of the Steinberg representation.

By ordinariness again, the latter case occurs only if $k = 2$. It is known by work of Ribet that in this case ρ_f does not split at p and that f is not of CM type (see, for instance, [**8**]). For this reason we will disregard such forms in the discussion below. So assume we are in the first case. Then $a_p \bar{a}_p = p^{k-1}$ by [**16**, thm. 4.6.17(1)]. Now let $\chi = \chi_0 \chi_p$ be the factorization of χ into its N_0 and p^r components. Let η_p be the matrix defined in equation (4.6.21) of [**16**] (it is called η_q there). Then, by [**16**, thm. 4.6.16(4)], we see that $f|_{\eta_p}$ is, up to a constant, equal to a primitive form with the same weight and level as f, but with character equal to $\chi_0 \bar{\chi}_p$. We take f' to be this primitive form. One deduces from [**16**, thm. 4.6.16(4)] that

$$a_m(f') = \begin{cases} \bar{\chi}_p(m) a_m(f) & \text{if } m \text{ is an integer coprime to } p. \\ \chi_0(m) \bar{a}_m(f) & \text{if } m = p. \end{cases}$$

Thus the form f' is close to (but not exactly the same as) the forms $f \otimes \bar{\chi}_p$ and $\bar{f} \otimes \chi_0$ where $\bar{f} = \sum \bar{a}_n q^n$ is the form obtained by taking the complex conjugates of the Fourier coefficients of f.

Note that in both the $r = 0$ case and the $r > 0$ case f' has slope $k-1$ and is an eigenvector for U_p.

The following proposition is the first step towards our goal. The case $r = 0$ was proved by Emerton [**7**, prop. 1.2]. The idea of the proof is essentially the same as that used in the original mod p setting [**10**], [**4**] and in the Λ-adic setting [**2**].

PROPOSITION 11. If f' is in the image of θ^{k-1} then ρ_f splits at p.

PROOF. By the inflation-restriction sequence the restriction map

$$\mathrm{H}^1(G_p, K(\delta\epsilon^{-1})) \to \mathrm{H}^1(I_p, K(\delta\epsilon^{-1}))$$

is injective. It therefore suffices to show that ρ_f splits on the inertia subgroup I_p. Let ν denote the p-adic cyclotomic character. We show that on I_p the representation ρ_f is simultaneously both an extension of 1 by $\chi_p \cdot \nu^{k-1}$, as well as an extension of $\chi_p \cdot \nu^{k-1}$ by 1, where χ_p is the p-part of the nebentypus χ (and so is trivial in the case $r = 0$). Since $k \geq 2$, we have $\nu^{k-1} \neq 1$ and it follows that ρ_f must split on I_p.

The first extension follows immediately from the ordinariness of f and (1.2), and the fact that $\det \rho_f = \chi\nu^{k-1}$. To get the second extension we use the hypothesis that f' is in the image of θ^{k-1} along with some results of Kisin [**14**] (see especially the arguments surrounding the proof of his Theorem 6.6(2)). Indeed, suppose that $f' = \theta^{k-1}(g)$ where g is an overconvergent p-adic modular form of tame level N_0 and weight $2-k$ and slope 0. Then by Kisin [**14**, thm. 6.3] applied to the cuspidal overconvergent eigenform g there is a non-zero G_p-equivariant map

$$V_{g,E} \to (B^+_{\mathrm{cris}} \hat{\otimes}_{\mathbb{Q}_p} E)^{\phi = a_p(g)}$$

where E is a finite extension of $\mathbb{Q}_p$ over which g is defined and ϕ is the crystalline Frobenius. Since the p-adic valuation of $a_p(g)$ is 0, the right hand side of the above map is of the form $\alpha \cdot E$ where $\alpha \in (\hat{\mathbb{Q}}_p^{\mathrm{un}} \otimes_{\mathbb{Q}_p} E)^\times$ and $\phi(\alpha) = a_p(g)\alpha$. On the other hand since the kernel of the above map has all its $[E : \mathbb{Q}_p]$ Hodge-Tate weights equal to $1-k$ it must be a twist of $\mathbb{Q}_p(1-k)$ by a character ψ_0 which has finite image on inertia. So $V_{g,E}$ is an extension of an unramified character by $\nu^{1-k} \cdot \psi_0$. Noting that $\rho_f \otimes \bar{\chi}_p \sim \rho_{f'} \sim \rho_g \otimes \nu^{k-1}$ we see that on inertia $\psi_0 = \bar{\chi}_p$, and that ρ_f is an extension of $\chi_p \cdot \nu^{1-k}$ by 1. This finishes the proof.

Alternatively one may note, following Kisin, that since ρ_g is potentially semi-stable on G_p (being a twist of the potentially semi-stable representation ρ_f) the extension $V_{g,E}$ must split on G_p. So ρ_f also splits on G_p. $\square$

Here is a partial converse to the above result.

PROPOSITION 12. Assume that

(1) $p \geq 5$,
(2) $a_q = 0$ for all $q | N_0$,
(3) f is p-distinguished, and,
(4) $\bar{\rho}_f$ is absolutely irreducible on $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

If ρ_f splits at p then f' is in the image of the θ^{k-1}.

PROOF. The form f (or its p-stabilization f_α, in the case $r = 0$) lives in a Λ-adic family F which by condition (2) has the property that $U_q F = 0$ for all primes $q | N_0$. By hypothesis ρ_f splits at p. By [**2**, theorem 2] (for which we need the other conditions (1), (3) and (4) above) we see that F has a Λ-adic companion form G. Assume that $r = 0$. Then this means that there is a p-ordinary overconvergent eigenform g of tame level N_0 and weight $2 - k$ which satisfies:

- $a_m(g) = a_m(f)m^{1-k}$ for all integers m coprime to p, and,
- $a_p(g) = \chi(p)/a_p(f_\alpha) = \chi(p)/\alpha = \beta/p^{k-1}$.

Here we have used the conditions 1 and 2 on page 909 of [**2**] which are part of the definition of a Λ-adic companion form. We also remark that the paper [**2**] makes the blanket assumption that $N_0 \geq 5$ but this is not needed in the first section of that paper where their theorem 2 is proved. In any case, taken together the above formulas show that $m^{k-1}a_m(g) = a_m(f_\beta)$ for all integers m, since $a_m(f) = a_m(f_\beta)$ for m coprime to p, and $a_p(f_\beta) = \beta$. It follows that the forms $\theta^{k-1}(g)$ and f_β have the same q-expansion, so that they are equal.

Now assume that $r > 0$. This time g is seen to satisfy:

- $a_m(g) = a_m(f)m^{1-k}\bar{\chi}_p(m)$ for all integers m coprime to p, and,
- $a_p(g) = \chi_0(p)/a_p(f) = \chi_0(p)\bar{a}_p(f)/p^{k-1}$.

It follows that $\theta^{k-1}(g)$ and f' have the same q-expansion, and are hence equal. We note in passing that under condition (2) in the statement of the proposition the form f' is actually equal to the form $\bar{f} \otimes \chi_0$. This can be seen by noting that $a_m(f) = \bar{a}_m(f)\chi(m)$ for all integers m coprime to p. In fact this relation usually holds for all m coprime to $N_0 p$, but under (2) it also holds if $(m, N_0) \neq 1$ since both $a_m(f)$ and $\bar{a}_m(f)$ vanish.

We have shown that $\theta^{k-1}(g) = f'$ in both cases and the proposition follows. $\square$

The above propositions show that

$$f' = \theta^{k-1}(g) \iff \text{the representation } \rho_f|_{G_p} \text{ splits,} \tag{6.1}$$

at least under some technical conditions. Let us now show how (6.1) may be used to

(a) recover some of our earlier examples of non-CM forms f with non-split local Galois representations, and,
(b) give some information towards Coleman's question.

We start with (a). We will assume $N_0 = 1$ and consider $f = \Delta$. Recall that Δ has Fourier coefficients $a_n = \tau(n) \in \mathbb{Z}$ and that Δ is ordinary for all primes $< 10^6$ except for $2, 3, 5, 7$ and 2411. So we may assume that $p \geq 11$. The first three conditions in proposition 12 are then automatically satisfied. As for the last condition, we will need to assume $p \neq 23$ or 691.

Now, suppose towards a contradiction, that ρ_Δ is split at p. Then by (6.1) we see that $\Delta_\beta = \theta^{11} g$ for some g of weight -10. Let

$$\Delta_k \in S_k(p, \omega^{12-k})$$

for $k \in \mathbb{Z}$ be the members of the p-ordinary family containing Δ. (So $\Delta_{12} = \Delta_\alpha$ in the notation introduced above.) Similarly let g_l for $l \in \mathbb{Z}$ be the members of the family containing g. Comparing Galois representations we see that Δ and g_{-10} must have the same nebentypus, from which it follows that

$$g_l \in S_l(p, \omega^{-10-l}).$$

Now specialize to weight 2. Then $\Delta_2 \in S_2(p, \omega^{10})$ and $g_2 \in S_2(p, \omega^{-12})$.

Let us now discuss the first few cases:

- $p = 11$: One sees that $S_2(11, \omega^{-12}) = 0$ so there is no g_2, a contradiction!
- $p = 13$: Again $S_2(13) = 0$ and we get a contradiction.
- $p = 17$: Again $S_2(17, \omega^4) = 0$.
- $p = 19$: Again $S_2(19, \omega^6) = 0$.

In all the above cases there is no form g_2. We conclude that ρ_Δ is non-split at p for the first few ordinary primes $11 \leq p \leq 19$. Of course this is compatible with what we obtained earlier, and could have been deduced directly, since for such p there is no mod p companion form.

The next case is $p = 23$ which we skip since the absolute irreducibility condition fails. In any case mod 23 there is a companion form and $S_2(23, \omega^{-12})$ has dimension one, so there is a potential candidate for g_2, namely Δ_2 itself. However, as mentioned at the end of section 3, it is shown in [**18**] that ρ_Δ does not split at $p = 23$.

The remaining ordinary primes $p > 23$ with $p \neq 691$ can be similarly treated, at least in principle. The dimension of the space in which $g_2 = \sum b_n q^n$ lies starts to grow. However, whenever one shows that the congruence $\tau(n) \equiv n^{11} b_n \mod p$ fails for each candidate $g_2 = \sum b_n q^n$ for some small value of n (coprime to p), we may conclude that ρ_Δ is not split at p!

The above algorithm extends to give information about the other members Δ_k of the p-ordinary family containing Δ. Suppose that Δ_k is split at p for some $k \geq 2$. We have $\Delta'_k = \theta^{k-1} g_{2-k}$. Again one checks that $g_2 = \sum b_n q^n \in S_2(p, \omega^{-12})$. Now, for example, if $11 \leq p \leq 19$, one sees as before that $g_2 = 0$ since $S_2(p, \omega^{-12}) = 0$. This contradiction shows that each of the members Δ_k of the p-adic family containing

Δ have locally non-split Galois representation. Again, we could have deduced this directly, since there is no mod p companion form.

Let us now turn to (b). Again let us assume $N_0 = 1$. The following partial answer towards Coleman's question (Question 10) is an immediate consequence of Theorem 7 and (6.1).

Corollary 13. Let $p \geq 5$ and let F be a primitive p-ordinary Λ-adic form F of tame level $N_0 = 1$. Assume

(1) F is p-distinguished,
(2) $\bar{\rho}_F$ is absolutely irreducible when restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$, and,
(3) F is not of CM type.

Let S be the set of ordinary (slope $= 0$) classical specializations f of F of weight $k \geq 2$, and let f' be the associated forms of slope $k-1$ defined above. Then, for all but finitely many $f \in S$, the forms

$$f' \text{ are not in the image of } \theta^{k-1}.$$

We remind the reader that since F does not have CM, the forms f' are not of CM type.

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