ON LOCAL GALOIS REPRESENTATIONS ASSOCIATED TO ORDINARY HILBERT MODULAR FORMS

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Abstract. Let F be a totally real field and p be an odd prime which splits completely in F. We show that a *generic* p-ordinary non-CM primitive Hilbert modular cuspidal eigenform over F of parallel weight two or more must have a locally non-split p-adic Galois representation, at at least one of the primes of F lying above p. This is proved under some technical assumptions on the global residual Galois representation. We also indicate how to extend our results to nearly ordinary families and forms of non-parallel weight.

1. Introduction

The purpose of this article is to generalize a result of the latter two authors [1] to the case of totally real fields. Thus let F be a totally real field and $\mathcal{O}_F$ its ring of integers. Let p be an odd integer prime. Let $k \geq 2$ be an integer and suppose that $f \in S_k(\mathfrak{n}, \psi)$ is a Hilbert modular form of (parallel) weight k, level $\mathfrak{n}$ and character $\psi : \mathrm{Cl}_{F,+}(\mathfrak{n}) \to \mathbb{C}^\times$, which is a primitive eigenform for all the Hecke operators. Denote by $c(\mathfrak{a}, f)$ the eigenvalue of f with respect to the $T(\mathfrak{a})$ operator. As $\mathfrak{a}$ varies over the integral ideals of F, the $c(\mathfrak{a}, f)$ generate a finite extension E of $\mathbb{Q}$. Let $\mathcal{P}$ be a prime ideal of $\bar{\mathbb{Q}}$, hence E, lying over p. Let

$$\rho_f : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(E_{\mathcal{P}})$$

be the Galois representation associated to f by Wiles, Taylor, Ohta, Carayol and Blasius-Rogawski, building on the work of Eichler-Shimura and Deligne in the case $F = \mathbb{Q}$. For all prime ideals $\mathfrak{q} \nmid \mathfrak{n}p$, we have

$$\mathrm{Tr}\ \rho_f(\mathrm{Frob}\ \mathfrak{q}) = c(\mathfrak{q}, f), \text{ and}$$

$$\det \rho_f(\mathrm{Frob}\ \mathfrak{q}) = \psi(\mathfrak{q})\,\mathrm{N}\mathfrak{q}^{k-1}.$$

We say that f is p-ordinary if for each prime $\mathfrak{p}|p$, $c(\mathfrak{p}, f)$ is a $\mathcal{P}$-adic unit. If f is p-ordinary, then by Wiles [18], for $\mathfrak{p}|p$, the local representation

$$\rho_f|_{D_\mathfrak{p}} \sim \begin{pmatrix} \epsilon_\mathfrak{p} & u_\mathfrak{p} \\ 0 & \delta_\mathfrak{p} \end{pmatrix} \tag{1}$$

is reducible, where $\epsilon_\mathfrak{p}$ and $\delta_\mathfrak{p}$ are characters on the decomposition group $D_\mathfrak{p}$ with values in $E_{\mathcal{P}}^\times$ and $\delta_\mathfrak{p}$ is unramified. We say that f is *p-distinguished* if $\epsilon_\mathfrak{p}$ and $\delta_\mathfrak{p}$ are distinct mod $\mathcal{P}$ for each $\mathfrak{p}|p$.

Definition 1. ρ_f is said to be split at $\mathfrak{p}$ if the local representation $\rho|_{D_\mathfrak{p}}$ is the direct sum of two 1-dimensional characters. ρ_f is split at p if it is split at all $\mathfrak{p}|p$.

In general, it is rare that the representation ρ_f is split at p, and in fact Greenberg asked (for $F = \mathbb{Q}$) whether for weight $k \geq 2$ this only happens for the so-called CM forms, which we now proceed to describe.

Let K be a totally imaginary quadratic extension of F and let c denote the nontrivial automorphism of K/F. Let I_K be the set of infinite places of K. Fix a set $\Sigma_\infty \subset I_K$ such that Σ_∞ and $\Sigma_\infty c$ are disjoint and $\Sigma_\infty \cup \Sigma_\infty c = I_K$. Let $I(M)$ be the fractional ideals of K coprime to a modulus $M \subset \mathcal{O}_K$ and let $\lambda : I(M) \to \overline{\mathbb{Q}}_p^\times$ be an arithmetic Hecke character of K of modulus M such that $\lambda((\alpha)) = \prod_{\sigma \in \Sigma_\infty} \sigma(\alpha)^{k-1}$ for all $\alpha \equiv 1 \bmod M$, where $k \geq 1$. Then by [2] there is a Hilbert modular form f of parallel weight k and level $\mathrm{N}_{K/F}(M)\,\mathfrak{d}_{K/F}$ (where $\mathfrak{d}_{K/F}$ is the relative discriminant of K/F) associated to λ. That is, for all but finitely many primes $\mathfrak{q}$ of F,

$$f|T(\mathfrak{q}) = c(\mathfrak{q}, f)f,$$

where

$$c(\mathfrak{q}, f) = \begin{cases} \lambda(Q) + \lambda(\bar{Q}) & \text{if } \mathfrak{q} = Q\bar{Q} \\ \lambda(Q) & \text{if } \mathfrak{q} = Q^2 \\ 0 & \text{if } \mathfrak{q} = Q. \end{cases}$$

A Hilbert modular form is called a CM form if it arises from a Hecke character in this way. The corresponding Galois representation of $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ is induced from the index two subgroup $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, and has projectively dihedral image. For a construction of these Galois representations we refer the reader to the final section of this paper, where a construction of Λ-adic CM-forms and their Galois representations is given in the ordinary and nearly ordinary cases.

Remarks.

(i) In weight 1 there exist non-CM forms f whose associated representations ρ_f have projectively dihedral image. Let K/F be an arbitrary quadratic extension. Let λ be a finite order Hecke character such that whenever a real place v of F splits into real places v_1 and v_2 in K, $\lambda_{v_1} = sgn$ and $\lambda_{v_2} = 1$, or vice versa. Let f be the form corresponding to λ, so that ρ_f is induced from G_K. The form f is not of CM type if K is not a CM field. Note that ρ_f satisfies

$$\rho_f(c_v) \sim \begin{pmatrix} \lambda_{v_1}(c_v) & \\ & \lambda_{v_2}(c_v) \end{pmatrix} = \begin{pmatrix} \mp 1 & \\ & \pm 1 \end{pmatrix}$$

for all split real places v of F as above, where c_v is complex conjugation at v, and so is totally odd, as it should be.

(ii) The non-CM holomorphic forms f with projectively dihedral image described in (i) do not occur for (parallel) weight $k \geq 2$. Let π be the automorphic representation associated to f and v be a real place of F. If v splits in K as above, then π_v will be a principal series representation (see, e.g., [15, §3.1.4]), but for $k \geq 2$ this is not possible since π_v is a discrete series representation.

(iii) CM forms of non-parallel weight can be obtained by taking powers of integers $k_\sigma \geq 1$ for each $\sigma \in \Sigma_\infty$ in the construction above.

If f is p-ordinary and CM, then it can easily be shown that ρ_f splits at p. We prove the following theorem which states that the converse is also true with a small set of exceptions.

Theorem 1. *Let F be a totally real field and p an odd prime that splits completely in F. Fix an integral ideal $\mathfrak{n}_0$ of F which is prime to p. Let $S(\mathfrak{n}_0)$ be the set of* *primitive, p-ordinary Hilbert modular cusp forms f of weight $k \geq 2$ and tame level $\mathfrak{n}_0$ such that*

(1) *f is p-distinguished, and*
(2) *$\bar{\rho}_f$ is absolutely irreducible when restricted to $G_{F(\zeta_p)} = \mathrm{Gal}(\overline{\mathbb{Q}}/F(\zeta_p))$.*

Then, except for a Zariski small subset of $S(\mathfrak{n}_0)$,

$$\rho_f \text{ is split at } p \text{ if and only if } f \text{ is CM.}$$

Moreover, if we assume that Leopoldt's conjecture is true for F and p, then there are only finitely many exceptions.

Remarks. (a) We assume in this theorem that the prime p splits completely in F. This is only necessary to apply the results of [17], specifically the result stated in Theorem 2 below.
(b) The forms in $S(\mathfrak{n}_0)$ are parameterized by certain coheight one primes in $\mathrm{Spec}(\Lambda)$, where Λ is a power-series ring in $1+\delta$ variables over a p-adic ring, where δ is the defect in Leopoldt's conjecture. When we say that the set of exceptions is Zariski small, we mean that the exceptional set of parameterization is a Zariski closed set of positive codimension in $\mathrm{Spec}(\Lambda)$. Thus the theorem holds for a Zariski dense set of parameterizations. If Leopoldt's conjecture holds and $\delta = 0$ (e.g., if $F/\mathbb{Q}$ is abelian), then Λ has only one (cyclotomic) variable, and the exceptional set is finite.

When $F = \mathbb{Q}$ this problem was studied in [1]. It is shown that as f varies over forms of weight $k \geq 2$ in a non-CM Hida family of tame level N_0, ρ_f is not split at p with finitely many exceptions. We extend the proof to the case of totally real fields. The key ingredient in the proof is a result by Sasaki [17, Theorem 2] which characterizes Galois representations associated to weight one cuspidal Hilbert modular forms. We state the relevant part of the theorem here.

Theorem 2. *Let p be an odd prime that splits completely in F. Suppose that $\rho : G_F \to GL_2(O_{\mathcal{P}})$, where $O_{\mathcal{P}}$ is the valuation ring in a finite extension $E_{\mathcal{P}}$ of $\mathbb{Q}_p$, is a continuous representation satisfying the following conditions:*

(1) *ρ ramifies only at finitely many primes*
(2) *ρ mod $\mathcal{P}$ is modular and absolutely irreducible when restricted to $G_{F(\zeta_p)}$*
(3) *For every prime $\mathfrak{p}|p$, $\rho|_{D_\mathfrak{p}}$ is the direct sum of two 1-dimensional characters $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p} : D_\mathfrak{p} \to O_{\mathcal{P}}^\times$ such that $\alpha(I_\mathfrak{p})$ and $\beta(I_\mathfrak{p})$ have finite cardinality, and $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p}$ are distinct mod $\mathcal{P}$.*

Then there is an embedding $i : E_{\mathcal{P}} \to \mathbb{C}$ and a classical weight 1 cuspform f such that $i \circ \rho$ is isomorphic to the representation associated to f by Rogawski and Tunnell in [16].

2. Ordinary families over totally real fields

In this section we recall basic properties of ordinary Λ-adic Hilbert modular cusp forms due to Hida and Wiles (for more details see [18, 3, 4]) and prove a Λ-adic version of our main theorem. Theorem 1 will follow from it.

Let M be the maximal abelian extension of F unramified outside p, and define $\mathbf{G}$ to be the torsion-free part of the group $\mathrm{Gal}(M/F)$. Then $\mathbf{G}$ is isomorphic to $1+\delta$ copies of $\mathbb{Z}_p$, where δ is the defect in Leopoldt's conjecture for F and p. Furthermore, $\mathbf{G}$ may be identified with the group $\mathrm{Gal}(M/F')$, where F' is a subfield of a (strict) ray class field of F of conductor p^α for some $\alpha \geq 1$. Thus $\mathrm{Gal}(M/F) \simeq \mathbf{G} \times \mathrm{Gal}(F'/F)$. Let $\mathscr{O}$ be a finite extension of $\mathbb{Z}_p$ and let $\Lambda = \mathscr{O}[\![\mathbf{G}]\!] \simeq \mathscr{O}[\![X_0, X_1, \ldots, X_\delta]\!]$, where the last isomorphism is given by sending generators γ_i of $\mathbf{G}$ to $X_i + 1$.

Let $N : G_F \to \mathbb{Z}_p^\times$ be the cyclotomic character of G_F. By restriction we may view N as a character of $\mathbf{G}$. For any integer k and finite order character $\epsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$, define $\kappa_{k,\epsilon} : \Lambda \to \overline{\mathbb{Q}}_p$ induced by the map $a \mapsto \epsilon(a)Na^{k-1}$ on $\mathbf{G}$. Let $P_{k,\epsilon}$ denote the kernel of $\kappa_{k,\epsilon}$.

Let L be a finite extension of the quotient field of Λ. We denote by Λ_L the integral closure of Λ in L. Let

$$\kappa : \Lambda_L \to \overline{\mathbb{Q}}_p$$

be a homomorphism that restricts to $\kappa_{k,\epsilon}$ on Λ and let P_κ denote the kernel of this map. When $k \geq 2$, the homomorphisms κ or the coheight 1 primes P_κ are called the arithmetic points of Λ_L. We remark that the points corresponding to $k = 1$ are precisely those which correspond to finite order characters of $\mathbf{G}$.

Let $\boldsymbol{\chi}$ denote the map

$$\boldsymbol{\chi} : G_F \twoheadrightarrow \mathrm{Gal}(M/F) = \mathbf{G} \times \mathrm{Gal}(F'/F) \twoheadrightarrow \mathbf{G} \hookrightarrow \Lambda^\times.$$

Let $\mathfrak{n}$ be an integral ideal in F and let $\mathrm{Cl}_{F,+}(\mathfrak{n})$ be the strict ray class group of F modulo $\mathfrak{n}$. Let ψ be a fixed finite order character of conductor dividing $\mathfrak{n}_0 p$ and define the Λ-adic character $\Psi = \psi\boldsymbol{\chi}$, so that

$$\Psi : G_F \twoheadrightarrow \varprojlim \mathrm{Cl}_{F,+}(\mathfrak{n}_0 p^r) \to \Lambda_L^\times.$$

Observe that Ψ modulo P_κ is the product of a finite order character with N^{k-1}. In particular, Ψ has finite order modulo primes of weight 1.

Now let $\mathbf{h}^{ord}(\mathfrak{n}_0)$ be Hida's universal ordinary Hecke algebra of tame level $\mathfrak{n}_0$.

Definition 2. A Λ-adic cusp form $\mathcal{F}$ of level $\mathfrak{n}_0$ and character Ψ is a Λ-algebra homomorphism $\mathcal{F} : \mathbf{h}^{ord}(\mathfrak{n}_0) \to \Lambda_L$ such that for any arithmetic point κ the composition $\kappa \circ \mathcal{F}$ gives the Hecke eigenvalues of a classical ordinary Hilbert cusp form of weight k, prime-to-p level $\mathfrak{n}_0$ and (finite order) character $\psi_\kappa = (N^{1-k} \cdot \Psi) \bmod P_\kappa$. We denote these forms by f_κ.

One may define the notion of a primitive Λ-adic cusp form by requiring a primitivity condition on the specializations. If f_0 is a p-ordinary p-stabilized primitive Hilbert modular cusp form of weight k, there is a primitive Λ-adic form $\mathcal{F}$ interpolating f_0 (i.e., there is an arithmetic point κ such that $\kappa \circ \mathcal{F}$ gives rise to f_0).

We can further say that $\mathcal{F}$ is unique. Let Q be the kernel of $\kappa \circ \lambda$; it lies above the prime $P = P_\kappa$ of Λ. In order to show that $\mathcal{F}$ is unique we need to show that there is a unique minimal prime ideal of $\mathbf{h} = \mathbf{h}^{ord}(\mathfrak{n}_0)$ contained in Q. This is true because the localization $\mathbf{h}_Q$ over Λ_P is an étale extension of regular local rings. We know that h_Q is flat over Λ_P. By Hida's control theorem [4] and [12, Lemma 12.7.6 (i)] we know that Q is unramified over P. Since Λ is isomorphic to a power-series ring it is regular and hence Λ_P is also a regular local ring. The generators of $Q\mathbf{h}_Q = P\mathbf{h}_Q$ form an $\mathbf{h}_Q$-regular sequence since they form a Λ_P-regular sequence and $\mathbf{h}_Q$ is flat. Indeed, if $x_0, \cdots, x_m$ are generators of $P\Lambda_P$ giving a regular sequence, then x_i is not a zero-divisor in $\Lambda_P/(x_0, \cdots, x_{i-1})$. So multiplication by x_i is injective on this ring. Tensoring with $\mathbf{h}_Q$ and using flatness, we see that $x_0, \cdots, x_m$ continues to be a regular sequence in $\mathbf{h}_Q$. This proof also works in the nearly ordinary case which will be dealt with later. If f_0 is also CM by some field K, then there is a CM form $\mathcal{F}$ passing through it (see section 4 below for the definition of Λ-adic CM forms; here we only note that all arithmetic specializations of such CM forms have CM) and by the uniqueness above this is the only form passing through it. Hence primitive Λ-adic forms can be divided into CM and non-CM families.

To each primitive Hida family $\mathcal{F}$ with character Ψ there is a Galois representation

$$\rho_{\mathcal{F}} : G_F \to \mathrm{GL}_2(L)$$

such that

$$\mathrm{Tr}\ \rho_{\mathcal{F}}(\mathrm{Frob}\ \mathfrak{q}) = c(\mathfrak{q}, \mathcal{F}),\ \text{and}$$
$$\det \rho_{\mathcal{F}}(\mathrm{Frob}\ \mathfrak{q}) = \Psi(\mathfrak{q})$$

for all $\mathfrak{q} \nmid \mathfrak{n}p$. We also know that for each $\mathfrak{p}|p$ the restriction of $\rho_{\mathcal{F}}$ to the decomposition group at $\mathfrak{p}$ looks like

$$\rho_{\mathcal{F}}|_{D_\mathfrak{p}} \sim \begin{pmatrix} \mathcal{E}_\mathfrak{p} & * \\ 0 & \mathcal{D}_\mathfrak{p} \end{pmatrix}$$

where $\mathcal{E}_\mathfrak{p}$ and $\mathcal{D}_\mathfrak{p}$ are characters $D_\mathfrak{p} \to \Lambda_L^\times$ with $\mathcal{D}_\mathfrak{p}$ unramified. If $\mathcal{F}$ is CM, then $\rho_{\mathcal{F}}$ splits at all primes $\mathfrak{p}$ of F lying over p which are split in the corresponding quadratic extension K/F.

Theorem 3. *Let p be an odd prime that splits completely in F and let $\mathcal{F}$ be a primitive Λ-adic form such that*

- *$\mathcal{F}$ is p-distinguished (i.e., $\bar{\mathcal{E}}_\mathfrak{p} \neq \bar{\mathcal{D}}_\mathfrak{p}$, for all $\mathfrak{p}|p$), and*
- *$\bar{\rho}_{\mathcal{F}}$ is absolutely irreducible when restricted to $G_{F(\zeta_p)}$.*

Then $\rho_{\mathcal{F}}|_{D_\mathfrak{p}}$ splits for each prime $\mathfrak{p}|p$ if and only if $\mathcal{F}$ is of CM type.

Remarks. As we have already remarked, we need to assume that p splits completely in F in order to apply Theorem 2. Kassaei, Pilloni, Sasaki, Stroh and Tian are working on removing this assumption. Their methods involve gluing overconvergent eigenforms using analytic continuation techniques on rigid analytic Hilbert-Blumenthal varieties. For instance, the papers [10] and [11] assume that p is unramified in F and the paper [13] that p is mildly ramified in F. However, these works also assume that the (quotient of) certain diagonal characters occurring on inertia are at worst tamely ramified; their main results are therefore not directly applicable in our method. Further work of Pilloni and Stroh [14] is expected to remove the assumption of tame ramification, and to allow p to have arbitrary ramification in F as well. Thus, it is expected that Theorems 2 and 3, and hence Theorem 1, will become true without the splitting assumption on p in F in the near future.

Before we prove this theorem we need the following lemma.

Lemma 1. *Let $\mathbf{G}$ be as above. Let $\epsilon : \mathbf{G} \to \overline{\mathbb{Q}}_p^\times$ be any finite order character and let $\kappa_\epsilon = \kappa_{1,\epsilon}$ be the induced map from $\Lambda = \mathscr{O}[\![\mathbf{G}]\!]$ to $\overline{\mathbb{Q}}_p$ and let $P_\epsilon = P_{1,\epsilon}$ denote the kernel of this map. Then the ideals P_ϵ form a dense subset of* $\mathrm{Spec}(\Lambda)$.

Proof. If $I = \cap P_\epsilon$, then the closure of P_ϵ in $\mathrm{Spec}(\Lambda)$ is $\mathrm{Spec}(\Lambda/I)$. The collection P_ϵ is dense if and only if $I = 0$. Suppose that $w \in I \subset \mathscr{O}[[\mathbf{G}]] \cong \mathscr{O}[[X_0, \dots, X_\delta]]$, with $\mathbf{G} \cong \mathbb{Z}_p^{1+\delta}$. We may identify $\mathscr{O}[[\mathbf{G}]]$ with the space of bounded $\mathscr{O}$-valued measures μ on the compact group $\mathbf{G}$, as in Section 3.7 of [9]. If ϵ is a finite order character of $\mathbf{G}$ we may view it as a locally constant function, and then $\int_\mathbf{G} \epsilon\, d\mu = \mu \pmod{P_\epsilon}$. It follows that if $w \in \cap P_\epsilon$ and μ is the corresponding measure, then $\int_\mathbf{G} \epsilon\, d\mu = 0$

for all ϵ, which implies that μ and w are zero (since the finite order characters are dense in the space of locally constant functions on the compact group $\mathbf{G}$). □

Proof of Theorem 3. This result is a generalization of [1, Proposition 14]. We sketch the proof here.

We know that if $\mathcal{F}$ is CM, then $\rho_{\mathcal{F}}$ is split at each $\mathfrak{p}|p$. We now prove the converse. We prove that any weight 1 specialization of $\rho_{\mathcal{F}}$ satisfies the conditions of Theorem 2 and thus has to come from a classical weight 1 form. Let $P_{1,\epsilon}(\mathcal{F})$ be a weight one specialization of $\mathcal{F}$. If $\rho_{\mathcal{F}}$ is split at each $\mathfrak{p}|p$, then the specialization $\rho_{\mathcal{F}}|_{D_{\mathfrak{p}}}$ to $P_{1,\epsilon}$ is a direct sum of two characters, say $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$. To apply Theorem 2, the only condition that needs to be verified is that $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ have finite images on inertia. We know that $\beta_{\mathfrak{p}}$ is unramified, so $\beta_{\mathfrak{p}}(I_{\mathfrak{p}}) = 1$. Since $\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}} = \Psi \pmod{P_{1,\epsilon}}$ has finite order, the claim follows.

Let $\mathscr{X}$ be the set of all finite order characters ϵ such that the weight 1 specialization $f_{1,\epsilon}$ is a *dihedral* form induced from a character of a quadratic extension of F. The same group theoretic argument as in [1] shows that all but finitely many characters ϵ lie in $\mathscr{X}$ and this implies that the set of primes $\{P_{1,\epsilon}\}_{\epsilon\in\mathscr{X}}$ is dense in $\mathrm{Spec}(\Lambda)$. There is a bound on the discriminant of the quadratic extension that can occur in this family and hence there are only finitely many of them. Hence there exist quadratic extensions $K_1, ..., K_h$ over F such that for every $\epsilon \in \mathscr{X}$, $f_{1,\epsilon}$ is dihedral by one of the K_i. This gives a finite partition of $\mathscr{X}$ into the $\mathscr{X}_i$. We claim that one of the sets $\{P_{1,\epsilon}\}_{\epsilon\in\mathscr{X}_i}$ is dense in $\mathrm{Spec}(\Lambda)$. If not, the closure of each of these sets will be of the form $\mathrm{Spec}(\Lambda/Q_i)$ for some ideal $Q_i \neq 0$. We also know that the dimension of $\mathrm{Spec}(\Lambda/Q_i)$ is less than that of $\mathrm{Spec}(\Lambda)$. The density of $\{P_{1,\epsilon}\}_{\epsilon\in\mathscr{X}}$ (see the lemma above) implies that $\mathrm{Spec}(\Lambda) = \cup_1^h \mathrm{Spec}(\Lambda/Q_i)$ and this is impossible by comparing dimensions. We denote by K the quadratic extension corresponding to this dense subset. We also let $\mathscr{X}_K$ denote the corresponding set of characters.

Suppose that κ is arithmetic of weight 1 such that the specialization f_κ is dihedral by K, then $\rho_\kappa \sim \rho_\kappa \otimes \phi$ where ϕ is the character of the extension K/F. We need to show that $\rho_{\mathcal{F}} \sim \rho_{\mathcal{F}} \otimes \phi$ and we do this by proving that $\mathrm{Tr}(\rho_{\mathcal{F}}(\mathrm{Frob}\ \mathfrak{l})) = \mathrm{Tr}((\rho_{\mathcal{F}} \otimes \phi)(\mathrm{Frob}\ \mathfrak{l}))$ for all but finitely many prime ideals $\mathfrak{l}$ in F. This is trivially true if $\mathfrak{l}$ splits in K (since $\phi(\mathrm{Frob}\ \mathfrak{l}) = 1$ for split ideals) and when $\mathfrak{l}$ is inert we claim both sides are zero. This is because for a κ of the form $(1, \epsilon)$ with $\epsilon \in \mathscr{X}_K$ the specialization $\mathrm{Tr}(\rho_\kappa(\mathrm{Frob}\ \mathfrak{l})) = 0$. Since their kernels P_κ are dense in $\mathrm{Spec}(\Lambda)$ we know that $\mathrm{Tr}(\rho_{\mathcal{F}}(\mathrm{Frob}\ \mathfrak{l})) = 0$. This shows that the family $\mathcal{F}$ is dihedral by K/F. In particular, all its weight 2 or more specializations are dihedral by K. But by remark (ii) in section 1, these specializations must come from a CM field K, as desired. □

Theorem 1 now follows in exactly the same manner as the main theorem in [1].

3. Nearly ordinary case

We would now like to consider the behavior of local Galois representations in the case of modular forms of non-parallel weight. It is necessary to consider nearly ordinary families of Hilbert modular forms in this case. For details see [3], [4] and [6, §2,3].

Let I be the set of infinite places of F and by a weight vector we mean an element of $\mathbb{Z}[I]$. Let $(k_\sigma) \in \mathbb{Z}[I]$ be such that all the k_σ are congruent mod 2 and all $k_\sigma \geq 2$.

Let $t = (1, ..., 1)$ and set $n = k - 2t$. Choose a weight vector $v \geq 0$ such that at least one $v_\sigma = 0$ and $n + 2v = \mu t$ is parallel (i.e., $\mu \in \mathbb{Z}$), and set $w = v + k - t$. Let $S_{k,w}(\mathfrak{n}, \mathbb{C})$ denote the set of cusp forms of level $\mathfrak{n}$ and weight (k, w). The classical automorphy factor for such forms is given in [3], equation (0.1), and the cusp forms of parallel weight considered earlier correspond to the case $v = 0$. We note also that the pair (k, w) of weight vectors is determined by the pair (k, μ), where k is a weight vector and μ is an ordinary integer satisfying the condition that $k - (2 + \mu)t = -2v$ is a weight vector with all entries even.

There is an action of the Hecke operators $T(\mathfrak{a})$ on this space for integral ideals $\mathfrak{a}$ of F. Hida also defines the modified Hecke operators $T_0(\mathfrak{a})$ which we describe now. For details, see [3] and [4]. For each $0 \leq v \in \mathbb{Z}[I]$ we define a character $F^\times \to \bar{\mathbb{Q}}^\times$ by sending $x \mapsto \prod x^{v_\sigma \sigma}$. The image of this map is in Φ, the Galois closure of F. By continuity we extend the map above to $F_{\mathbb{A}}^\times \to \Phi_{\mathbb{A}}^\times$. We let A be the ring of integers in the Hilbert class field of Φ. Then for every finite idele $x \in F_{\mathbb{A}}^\times$ the ideal $x^v A$ is a principal ideal in A. For each prime ideal $\mathfrak{l}$ of $\mathcal{O}_F$ let $x \in F_{\mathbb{A}}^\times$ be an idele such that $\mathfrak{l} = x\mathcal{O}_F$. We fix an element $\{x^v\} = \{\mathfrak{l}^v\} \in A$ which is a generator for the ideal $x^v A$ for every weight vector v. We extend this definition for any ideal $\mathfrak{a}$ of F multiplicatively. We then define the modified Hecke operator by

$$T_0(\mathfrak{a}) = \{\mathfrak{a}^v\}^{-1} T(\mathfrak{a}).$$

If Φ'/Φ is a field extension inside $\mathbb{C}$, then it can be shown that the space of Φ'-rational cusp forms $S_{k,w}(\mathfrak{n}, \Phi')$ is stable under $T(\mathfrak{a})$ and $T_0(\mathfrak{a})$ (cf. [3, Theorem 4.8]). Furthermore, if A is as above, then the A-integral cusp forms $S_{k,w}(\mathfrak{n}, A)$ are stable under $T_0(\mathfrak{a})$ for all integral ideals $\mathfrak{a}$ (cf. [3, Theorem 4.11]). If $v = 0$, we can take all the $\{\mathfrak{l}^v\}$ to be 1 and this is the usual Hecke operator.

Definition 3. An eigenform $f \in S_{k,w}(\mathfrak{n}, \mathbb{C})$ is said to be nearly ordinary at p if its $T_0(\mathfrak{p})$ eigenvalue is a $\mathcal{P}$-adic unit, for all $\mathfrak{p}|p$.

If f is nearly ordinary at p, and E is the number field generated by the $T_0(\mathfrak{a})$-eigenvalues of f, then the associated Galois representation

$$\rho_f : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(E_\mathcal{P})$$

has the following local behavior. At every $\mathfrak{p}|p$,

$$\rho_f|_{D_\mathfrak{p}} \sim \begin{pmatrix} \epsilon_\mathfrak{p} & u_\mathfrak{p} \\ 0 & \delta_\mathfrak{p} \end{pmatrix}$$

where $\epsilon_\mathfrak{p}$ and $\delta_\mathfrak{p}$ are again characters on the decomposition group $D_\mathfrak{p}$, but unlike the ordinary case, $\delta_\mathfrak{p}$ need not be unramified.

We now discuss the Λ-adic analogues of nearly ordinary forms. Let M/F denote the maximal abelian extension of F unramified outside p, and let $\mathbf{G}'$ denote the torsion-free part of $\mathrm{Gal}(M/F)$. This is the group denoted by $\mathbf{G}$ in the ordinary case and corresponds to the weight space there. In the nearly ordinary case at hand, the weight space $\mathbf{G}$ is given by

$$\mathbf{G} = U_{1,F} \times \mathbf{G}'$$

where $U_{1,F}$ is the torsion-free part of $\mathcal{O}_{F,p}^\times$, with $\mathcal{O}_{F,p} = \mathcal{O}_F \otimes \mathbb{Z}_p$. Thus $\mathbf{G}$ is isomorphic to $[F : \mathbb{Q}] + 1 + \delta$ copies of $\mathbb{Z}_p$. Furthermore, $\mathbf{G}'$ contains the subgroup $\mathbf{H}'$ = torsion-free part of $U_{1,F}/\bar{\mathcal{O}}_{F,1}^\times$ as a subgroup of finite index, where $\mathcal{O}_{F,1}^\times$ is the subset of $\mathcal{O}_F^\times$ of principal global units. Let $\mathbf{H} = U_{1,F} \times \mathbf{H}'$. There is an action of

$\mathbf{H}$ on $S_{k,w}(\mathfrak{n}_0 p^\alpha; \mathscr{O})$ denoted by $\langle a, d\rangle$. Following [6, §3] we modify this action for each n, v above. Define $\langle a, d\rangle_{n,v} = d^{-\mu t}\langle a, d\rangle$. It is known that this action factors through a finite quotient of $\mathbf{H}$. Given finite order characters $\epsilon_1 : U_{1,F} \to \mathscr{O}^\times$ and $\epsilon_2 : \mathbf{H}' \to \mathscr{O}^\times$, we define the subset of cusp forms

$$S_{k,w}(\mathfrak{n}_0 p^\alpha, \epsilon_1, \epsilon_2; \mathscr{O}) = \{f \mid f|_{\langle a,d\rangle_{n,v}} = \epsilon_1(a)\epsilon_2(d) f\}.$$

Let $\mathbf{h}^{n.ord}(\mathfrak{n}_0)$ be the universal nearly ordinary Hecke algebra of tame level $\mathfrak{n}_0$ over the Iwasawa algebra $\Lambda = \Lambda_{n.ord} = \mathscr{O}[\![\mathbf{G}]\!]$. Let L be a finite extension of the fraction field of Λ and let Λ_L be the integral closure of Λ in L. Let $\kappa_{n,v,\epsilon_1,\epsilon_2}$ denote the homomorphism $\mathscr{O}[[\mathbf{H}]] \to \overline{\mathbb{Q}}_p$ induced by $(a, d) \mapsto \epsilon_1(a)\epsilon_2(d) d^{\mu t} a^v$; a homomorphism $\kappa : \Lambda_L \to \overline{\mathbb{Q}}_p$ is called an arithmetic point if it extends some $\kappa_{n,v,\epsilon_1,\epsilon_2}$ with $k \geq 2t$. Here note that d^t is essentially the cyclotomic character. A nearly ordinary Λ-adic cusp form is a Λ-algebra homomorphism $\mathcal{F} : \mathbf{h}^{n.ord}(\mathfrak{n}_0) \to \Lambda_L$ such that for every arithmetic point κ the map $\kappa \circ \mathcal{F}$ corresponds to a classical nearly ordinary Hilbert modular cusp form in $S_{k,w}(\mathfrak{n}_0 p^\alpha, \epsilon_1, \epsilon_2; \mathscr{O})$. This normalization is slightly different than the one given above for ordinary forms; here the points corresponding to finite order characters of $\mathbf{G}$ are given by $v = 0$ and $k = 2t$.

We know that every primitive nearly ordinary form f of weight k with $k_\sigma \geq 2$ lives in a unique primitive nearly ordinary family $\mathcal{F}$, i.e., there exists a primitive Λ-adic form $\mathcal{F}$ and an arithmetic point κ such that f corresponds to the specialization $\kappa \circ \mathcal{F}$.

By [5], there is a Galois representation $\rho_{\mathcal{F}} : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_2(\Lambda_L)$ unramified outside $\mathfrak{n}_0 p$ such that for each $\mathfrak{p}|p$

$$\rho_{\mathcal{F}|_{D_\mathfrak{p}}} \sim \begin{pmatrix} \mathcal{E}_\mathfrak{p} & * \\ 0 & \mathcal{D}_\mathfrak{p} \end{pmatrix}$$

where $\mathcal{E}_\mathfrak{p}$ and $\mathcal{D}_\mathfrak{p}$ are Λ-adic characters, but unlike the ordinary case $\mathcal{D}_\mathfrak{p}$ is not necessarily unramified. We will construct Λ-adic forms with CM from a quadratic extension K/F and the corresponding representations in section 4 below, and we will see that if the primes above p split in K, then the Λ-adic representation is split at p.

Now suppose that $\mathcal{F}$ is arbitrary, and that $\rho_{\mathcal{F}}$ is split at p. We can show using Sasaki's theorem that for any specialization of the form $\kappa = \kappa_{-1,0,\epsilon_1,\epsilon_2}$ the Galois representation ρ_κ comes from a classical weight 1 form. Let $\alpha_\mathfrak{p} = \kappa \circ \mathcal{E}_\mathfrak{p}$ and $\beta_\mathfrak{p} = \kappa \circ \mathcal{D}_\mathfrak{p}$. The only non-trivial part is to check that $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p}$ have finite image on the inertia subgroup. We know by [7, Theorem 2.43] that $\beta_\mathfrak{p}([u, F_\mathfrak{p}]) = \epsilon_{1,\mathfrak{p}}(u) u^{-v}$, where $\epsilon_{1,\mathfrak{p}}$ is the $\mathfrak{p}$-part of ϵ_1, and $u \in \mathfrak{p}$-part of $U_{1,F}$. Also the units map onto the inertia subgroup via the local Artin map. It follows that when $v = 0$ the restriction to $I_\mathfrak{p}$ of $\beta_\mathfrak{p}$ has finite image. Let $\widehat{\mathcal{O}}_F = \mathcal{O}_F \otimes \widehat{\mathbb{Z}}$. For κ extending $\kappa_{n,v,\epsilon_1,\epsilon_2}$, there is an arithmetic Hecke character $\epsilon_+(\kappa)$ with infinity type $-(n + 2v)$ such that $\epsilon_+(\kappa)(z) = \epsilon_1(z)\epsilon_2(z)$ for $z \in \widehat{\mathcal{O}}_F^\times$. We know that on $\mathbf{H}'$, $\det(\rho_\kappa) = \epsilon_+(\kappa) \cdot N$ which is a character with infinity type $-(t + n + 2v)$ (see *loc. cit.* and errata on Hida's webpage). When k is of parallel weight 1, so $n = -t$ and $v = 0$, the determinant is a finite order character, hence $\alpha_\mathfrak{p}$ has finite order on $I_\mathfrak{p}$ as well.

Using the same arguments as before we can show that a dense set of these weight 1 specializations are CM forms for a fixed CM-field K/F and that $\mathcal{F}$ is CM by K. This proves a nearly ordinary version of Theorem 3. We leave it to the reader to conclude the obvious nearly ordinary version of Theorem 1 from this.

4. Construction of Λ-adic CM forms and representations

Let F be a totally real field, and K/F a totally imaginary quadratic extension. Let c denote the nontrivial automorphism of K/F, so that c is induced by complex conjugation under any embedding $\sigma : K \to \mathbb{C}$. Let p denote a rational prime, and assume that every prime above p in F splits in K. We fix once and for all an isomorphism $\iota : \mathbb{C} \mapsto \mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$.

Fix a set Σ_∞ of embeddings of K into $\mathbb{C}$ as in section 1. Then if $\sigma \in \Sigma_\infty$, the composite $\iota \circ \sigma$ gives an embedding $K \to \mathbb{C}_p$, and thus determines a place σ_p of K above p. Set $\Sigma_p = \{\sigma_p : \sigma \in \Sigma_\infty\}$. Then the set Σ_p is disjoint from $\Sigma_p c$ and gives 'half' the places of K lying above p; it is called a p-adic CM type in [8]. We may view σ_p as a map $K \otimes \mathbb{Q}_p \to \mathbb{C}_p$ by extension of scalars. Similarly, we view σ as a map $K \otimes \mathbb{R} \to \mathbb{C}$.

Now consider an arithmetic Hecke character $\lambda : K^\times \backslash K_{\mathbb{A}}^\times \to \mathbb{C}^\times$ such that the restriction of λ to $(K \otimes \mathbb{R})^\times$ is of the form $x \mapsto \prod_{\sigma \in \Sigma_\infty} \sigma(x)^{k_\sigma}$, with $k_\sigma \geq 1$. Then consider the map $\lambda_p : K_{\mathbb{A}}^\times \to \mathbb{C}_p^\times$ defined by

$$\lambda_p(x) = \iota(\lambda(x)) \cdot \prod_{\sigma \in \Sigma_\infty} \iota(\sigma(x_\infty))^{-k_\sigma} \cdot \prod_{\sigma \in \Sigma_p} \sigma_p(x_p)^{k_\sigma}.$$

Then λ_p is a continuous homomorphism, and λ_p is trivial on $K^\times$ and on $(K \otimes \mathbb{R})^\times$ by construction. (Note however, that λ_p and λ take the same values at the ideles $(1, 1, \cdots, \pi_Q, \cdots, 1)$ where π_Q is a uniformizer at a prime Q not lying over p.) By class field theory, we may view λ_p as a character $\mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathbb{C}_p^\times$. Since $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ is a subgroup of index 2 in $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, we may induce the character λ_p to obtain a Galois representation

$$\rho_\lambda : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\mathbb{C}_p).$$

Since by hypothesis all primes above p in F are split in K, the decomposition groups of such primes are subgroups of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Since ρ_λ is a direct sum of λ_p and its conjugate character on $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$, we see ρ_λ is locally split at all primes above p.

We now wish to construct Λ-adic representations interpolating the representations described above. Let K/F be as above, and let $\mathbb{G}$ denote the torsion-free part of the Galois group of the maximal abelian pro-p extension of K which is unramified outside p. Then by class field theory $\mathbb{G}$ fits into an exact sequence

$$0 \to \overline{U} \to \prod_{P|p} U_{1,P} \to \mathbb{G} \to H \to 0$$

where H is finite, $U_{1,P}$ denotes the $\mathbb{Z}_p$-free summand of the units of K_P, and U denotes the set of global units of K which are contained in $U_{1,P}$ for each P. The bar on top of the U indicates that we take the closure of U in $\prod_{P|p} U_{1,P}$.

Observe that U is a subgroup of finite index in the group of all units of K. Furthermore, by Dirichlet's unit theorem, the units of F form a subgroup of finite index in the units of K. Thus U has rank $d-1$ as an abelian group, with $d = [F : \mathbb{Q}]$. We now claim that $\overline{U}$ has $\mathbb{Z}_p$-rank $d - 1 - \delta$, where δ is the defect in Leopoldt's conjecture for F. To see this, let $U' \subset U$ denote $U \cap F$. Then U/U' is finite, and by definition of δ, the closure of U' in $\prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}}$ has $\mathbb{Z}_p$-rank $d - 1 - \delta$, where $\mathfrak{p}$ runs over the primes of F over p. Furthermore, since p is odd, we may decompose $\prod_{P|p} U_{1,P}$ as the direct sum of $\pm$-eigenspaces for the action of the order 2 group

$\mathrm{Gal}(K/F) \cong \mathbb{Z}/2\mathbb{Z}$. Under this identification, it is clear that $\prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}}$ is precisely the eigenspace where the action is trivial, from which the claim follows.

Since $\prod_{P|p} U_{1,P}$ has $\mathbb{Z}_p$-rank $2d$, we may conclude that the $\mathbb{Z}_p$-rank of the group $\mathbb{G}$ is $d+1+\delta$. Thus if we let $\mathscr{A}$ denote the completed group algebra $\mathscr{O}[[\mathbb{G}]]$, then we get a tautological character

$$\tilde{\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathbb{G} \to \mathscr{A}^\times,$$

and a representation $\tilde{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_2(\mathscr{A})$ by induction from K to F. The representation $\tilde{\rho}$ is locally split at primes above p, since all such primes split from F to K.

Now we return to the question of Λ-adic forms, as described in the previous section. First of all, we need to give $\mathscr{A}$ the structure of a Λ-algebra of finite type, and to this end, we start by constructing a map $\mathbf{G} \hookrightarrow \mathbb{G}$. Recall that $\mathbf{G} = \mathbf{G}' \times U_{1,F}$. Our fixed CM-type Σ_p determines a prime P^* dividing $\mathfrak{p}$ in K, for each prime $\mathfrak{p}$ of F over p. Since $\mathfrak{p}$ splits, there is an isomorphism between U_{1,P^*} and $U_{1,\mathfrak{p}}$. Thus we get a map

$$\prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}} \cong \prod_{\mathfrak{p}|p} U_{1,P^*} \subset \prod_{P|p} U_{1,P},$$

and a map $\prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}} \to \mathbb{G}$. This map is injective, since the kernel is the intersection of $\prod_{\mathfrak{p}|p} U_{1,P^*} \subset \prod_{P|p} U_{1,P}$ with the global units of K, and the global units have nonzero projection to every completion K_P. Thus we get a map $\prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}} \hookrightarrow \mathbb{G}$. On the other hand, it is clear that there is a map $\mathbf{G}' \to \mathbb{G}$, induced by base extension of the maximal abelian p-unramified extension of F to the analogous object for K. This map is injective, since p is odd and K/F is quadratic. Thus we get a map $\mathbf{G} = \mathbf{G}' \times \prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}} \to \mathbb{G}$. This map is injective since the image of $\mathbf{G}'$ is fixed by $\mathrm{Gal}(K/F)$, whereas the image of the other factor is clearly not invariant. Consideration of $\mathbb{Z}_p$-ranks shows that $\mathbf{G}$ has finite index in $\mathbb{G}$, so that $\mathscr{A}$ is in fact an $\mathscr{O}[[\mathbf{G}]]$-algebra of finite type.

Let λ denote a grössencharacter of K such that the restriction of λ to $(K \otimes \mathbb{R})^\times$ is given by $x \mapsto \prod_{\sigma \in \Sigma_\infty} \sigma(x)^{k_\sigma}$, as before. We form the p-adic avatar λ_p as explained above, and consider the character

$$\lambda_p \tilde{\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathscr{A}$$

where we understand that the ring $\mathscr{O}$ may have to be enlarged to include the values of λ_p. Then we observe that $\lambda_p \cdot \tilde{\lambda}$ is an $\mathscr{A}$-valued character that interpolates λ_p, since we may specialize to the trivial character of $\mathbb{G}$ to recover λ_p. By inducing as before, we obtain an $\mathscr{A}$-valued Galois representation which interpolates ρ_λ, and which is locally split at all primes above p. Finally, one verifies readily that the numbers $\lambda_p \cdot \tilde{\lambda}(Q) + \lambda_p \cdot \tilde{\lambda}(\bar{Q})$, for prime $\mathfrak{q} = Q\bar{Q}$ split in K, etc., give rise after appropriate renormalization to an $\mathscr{A}$-valued Λ-adic form over F, and that after specialization they coincide with the eigenvalues of a Hilbert modular form of CM type.

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