Local behavior of automorphic Galois representations

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of these local Galois representations.

Some classical results

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a primitive classical cusp form of

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attached to f is **irreducible**. However, the **local** representation

$$\rho_{f,p}\big|_{G_p}$$

obtained by restricting $\rho_{f,p}$ to a decomposition subgroup G_p at p may be **reducible**.

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- $\lambda(\alpha)$ is the unramified character of G_p taking Frob_p to α, with

$$\alpha = \begin{cases} \text{unit root of} & \\ \quad x^2 - a_p x + p^{k-1}\psi(p), & \text{if } p \nmid N, \\ a_p, & \text{if } p || N, p \nmid \mathrm{cond}(\psi) \ \& \ k = 2, \\ a_p, & \text{if } v_p(N) = v_p(\mathrm{cond}(\psi)) \geq 1. \end{cases}$$

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- $\beta = \psi'(p) p^{k-1}/\alpha$.

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Furthermore, even when $n = 2$, can we specify when the local reducible representation above is semi-simple?

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We show that the local Galois representations coming from automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ have a particularly simple behaviour, if the underlying Weil-Deligne representation is **indecomposable**.

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We work under a technical assumption from p-adic Hodge theory.

Motivation from GSp_4

Let π be a cuspidal automorphic form on $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ with π_∞ in the discrete series of weight $(a, b; a+b)$ with $a \geq b \geq 0$.

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Towards GL_n

Let π be a cuspidal automorphic representation on $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with infinitesimal character

$$\begin{aligned} \chi_H : \mathcal{Z}_n = \mathbb{C}[X_1, \ldots, X_n] &\rightarrow \mathbb{C} \\ X_i &\mapsto x_i \end{aligned}$$

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Conjecture

If H consists of n distinct integers $-\beta_1 > \cdots > -\beta_n$, then there exists a strictly compatible system of Galois representations

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Much progress had been made on this by Clozel, Harris, Taylor, ...

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Remark: The general WD representation for $d \geq 1$ arises as the unique irreducible quotient (Langlands quotient) of the double induction: $\mathrm{Ind}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_d))$.

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   - *If π is not p-ordinary, then $\rho_{\pi,p}|_{G_p}$ is irreducible.*
2. *If $m \geq 2$, then $\rho_{\pi,p}|_{G_p}$ is always irreducible.*

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Jumps in the Hodge filtration $= -H = \{\beta_1 < \cdots < \beta_n\}$.

Assume: the Hodge filtration on D is in general position with respect to the Newton filtration.

Proof continued

Lemma

Let $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$ be two increasing sequences of integers s.t.

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So equality holds: all $b_n - b_i = a_n - a_i$, and $a_i = b_i$ for all i.

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If two consecutive D_i and D_{i+1} are admissible, then all D_r are admissible, and the β_j are necessarily consecutive integers.

Proof: Let $\alpha_j^{-1} = p^{j-1}/\alpha$. Then:

$$t_H(D_i) = \sum_{j=1}^{i} \beta_j = -\sum_{j=1}^{i} v_p(\alpha_j) = t_N(D_i), \tag{1}$$

$$t_H(D_{i+1}) = \sum_{j=1}^{i+1} \beta_j = -\sum_{j=1}^{i+1} v_p(\alpha_j) = t_N(D_{i+1}). \tag{2}$$

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Lemma again $\implies$ $a_j = b_j$, for $i+1 \le j \le n$. This shows $\beta_j = -v_p(\alpha_j)$ for all j, and all D_r $(1 \le r \le n)$ are admissible.

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Theorem

The filtered module $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ is either irreducible or reducible, in which case all the D_r $(1 \leq r \leq n)$ are admissible.

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2. Semisimplicty

Recall: if f is p-ordinary, then the local Galois representation

$$\rho_{f,p}\big|_{G_p} \sim \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

is **reducible**.

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We will, in fact, prove results more generally for Hilbert modular cusp forms.

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Let

$$\rho_{f,p} : G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$$

be the Galois representation attached to f by Wiles, Taylor, Ohta, Carayol, Blasius-Rogawksi.

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- level $\mathfrak{N} \subset \mathcal{O}_F$, and,
- character $\psi : \mathrm{Cl}_{F,+}(\mathfrak{N}) \to \mathbb{C}^\times$.

Let

$$\rho_{f,p} : G_F \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$$

be the Galois representation attached to f by Wiles, Taylor, Ohta, Carayol, Blasius-Rogawksi. Thus, for all primes $\mathfrak{q} \nmid \mathfrak{N}p$,

$$\mathrm{tr}(\rho_{f,p}(\mathrm{Frob}_\mathfrak{q})) = c(\mathfrak{q}, f) \quad \text{and} \quad \det(\rho_{f,p}(\mathrm{Frob}_\mathfrak{q})) = \psi(\mathfrak{q})\mathrm{N}(\mathfrak{q})^{k-1}.$$

Split and CM

Wiles: If f is p-ordinary (i.e., $v_p(c(\mathfrak{p}, f)) = 0$, for all $\mathfrak{p}|p$), then

$$\rho_{f,p}\big|_{G_{\mathfrak{p}}} \sim \begin{pmatrix} \delta_{\mathfrak{p}} & u_{\mathfrak{p}} \\ 0 & \epsilon_{\mathfrak{p}} \end{pmatrix}$$

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Then $f \in S_k(\mathrm{N}_{K/F}(M) \cdot D_{K/F}, \lambda|_{\mathbb{A}_F^\times} \cdot \omega_{K/F})$.

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Today: We show that 'most' forms in 'most' non-CM Hilbert modular Hida families are NOT p-split.

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Then except for a 'Zariski small' subset of $S(\mathfrak{n}_0)$

$$f \text{ is } p\text{-spilt} \iff f \text{ has CM.}$$

Remarks

1. The set $S(\mathfrak{n}_0)$ is parametrized by a power series ring

$$\Lambda = \mathbb{Z}_p[[X_0, X_1, \ldots, X_\delta]]$$

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4. When $k = 2$, see also B. Zhao's forthcoming UCLA thesis.

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Hida families

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Define primitive forms $\mathcal{F}$ (eigen + new + normalized forms) appropriately.

Theorem (Hida)

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Local semisimplicity for families

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Theorem Λ implies the main Theorem, by descent to the classical world.

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Then, there is a Hilbert cusp form f of weight 1 such that $\rho \sim \rho_f$, the Rogawski-Tunnel representation attached to f.

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As ϵ varies, we see $\mathcal{F}$ has a Zariski dense set of classical weight 1 specializations.

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Thus $\mathcal{F}$ is a CM form, and we are done.

Aside: Weight 1 forms

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E.g.: Greenberg-Vatsal have remarked that if there is Steinberg-type prime in the prime-to-p level N_0 of $\mathcal{F}$, then $\mathcal{F}$ has **no** classical weight 1 specializations.

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where h_K is the class number of K and ϵ_K is a fundamental unit of K.

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- On the other hand, there are examples of non-CM and residually CM families $\mathcal{F}$ with classical weight 1 CM points, with $p | D_K$.

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Thus Hida's Hecke algebra is not étale at weight 1 points, but is there still a chance that uniqueness (up to conjugacy) holds?

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$$\begin{aligned} \mathcal{F} &= \mathcal{F} \otimes \chi_{D_2} = (\mathcal{G} \otimes \chi_{D_2})^\sigma = (\mathcal{G} \otimes \chi_{D_3}\chi_{D_1})^\sigma = (\mathcal{G} \otimes \chi_{D_1})^\sigma \\ &= \mathcal{F} \otimes \chi_{D_1}, \end{aligned}$$

a contradiction, since $\mathcal{F}$ cannot have RM forms in wts ≥ 2.

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Answers to these questions have implications for the geometry of the eigencurve at classical weight 1 points.

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Let $S_{k,w}(\mathfrak{n}, \mathbb{C})$ be the space of Hilbert modular forms of weight (k, w).

For $\mathfrak{a} \subset \mathcal{O}_F$, let

$$T_0(\mathfrak{a}) = \{\mathfrak{a}^v\}^{-1} T(\mathfrak{a})$$

be Hida's modified Hecke operator.

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Hida: An eigenform $f \in S_{k,w}(\mathfrak{n}, \mathbb{C})$ is **nearly** p-**ordinary** if it's $T_0(\mathfrak{p})$-eigenvalue is a p-adic unit for all $\mathfrak{p}|p$. In this case

$$\rho_{f,p}|_{G_\mathfrak{p}} \sim \begin{pmatrix} \delta_\mathfrak{p} & u_\mathfrak{p} \\ 0 & \epsilon_\mathfrak{p} \end{pmatrix}$$

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This may come out of methods from P. Allen's recent UCLA thesis.

Thank you!