(p,p)-GALOIS REPRESENTATIONS ATTACHED TO AUTOMORPHIC FORMS ON GL_n

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Abstract. We study the local reducibility at p of the p-adic Galois representation attached to a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. In the case that the underlying Weil-Deligne representation is Frobenius semisimple and indecomposable, we analyze the reducibility completely. We use methods from p-adic Hodge theory, and work under a transversality assumption on the Hodge and Newton filtrations in the corresponding filtered module.

1. Introduction

Let $f = \sum_{n=1}^{\infty} a_n(f)q^n$ be a primitive elliptic modular cusp form of weight $k \geq 2$, level $N \geq 1$, and nebentypus $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. Let K_f denote the number field generated by the Fourier coefficients of f. Fix an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$, and let $\wp$ be the prime of $\bar{\mathbb{Q}}$ determined by this embedding. Let $\wp$ also denote the induced prime of K_f, and let $K_{f,\wp}$ be the completion of K_f at $\wp$. For a global or local field F of characteristic 0, let G_F denote the absolute Galois group of F. There is a global Galois representation

$$\rho_{f,\wp} : G_{\mathbb{Q}} \to \mathrm{GL}_2(K_{f,\wp}) \tag{1.1}$$

associated to f (and $\wp$) by Deligne which has the property that for all primes $\ell \nmid Np$,

$$\mathrm{trace}(\rho_{f,\wp}(\mathrm{Frob}_\ell)) = a_\ell(f) \quad \text{and} \quad \det(\rho_{f,\wp}(\mathrm{Frob}_\ell)) = \chi(\ell)\ell^{k-1}.$$

Thus $\det(\rho_{f,\wp}) = \chi\chi_{\mathrm{cyc},p}^{k-1}$, where $\chi_{\mathrm{cyc},p}$ is the p-adic cyclotomic character.

It is a well-known result of Ribet that the global representation $\rho_{f,\wp}$ is irreducible. However, if f is ordinary at $\wp$, i.e., $a_p(f)$ is a $\wp$-adic unit, then an important theorem of Wiles, valid more generally for Hilbert modular forms, says that the corresponding local representation is reducible.

Theorem 1.1 ([W88]). *Let f be a $\wp$-ordinary primitive form as above. Then the restriction of $\rho_{f,\wp}$ to the decomposition subgroup $G_{\mathbb{Q}_p}$ is reducible. More precisely, there exists a basis in which*

$$\rho_{f,\wp}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \chi_p \cdot \lambda(\beta/p^{k-1}) \cdot \chi_{\mathrm{cyc},p}^{k-1} & u \\ 0 & \lambda(\alpha) \end{pmatrix}, \tag{1.2}$$

where $\chi = \chi_p\chi'$ is the decomposition of χ into its p and prime-to-p-parts, $\lambda(x) : G_{\mathbb{Q}_p} \to K_{f,\wp}^\times$ is the unramified character which takes arithmetic Frobenius to x, and $u : G_{\mathbb{Q}_p} \to K_{f,\wp}$ is a continuous function. Here α is (i) the unit root of $X^2 - a_p(f)X + p^{k-1}\chi(p)$ if $p \nmid N$ (ii) the unit $a_p(f)$ if $p||N$, $p \nmid \mathrm{cond}(\chi)$, $k = 2$ (iii) the unit $a_p(f)$ if $p|N$, $v_p(N) = v_p(\mathrm{cond}(\chi))$. In all cases $\alpha\beta = \chi'(p)p^{k-1}$.

The numbering used in this paper differs slightly from that used in the published version.

Moreover, in case *(ii)*, $a_p(f)$ is a unit if and only if $k = 2$, and one can easily show that $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is irreducible when $k > 2$.

Urban has generalized Theorem 1.1 to the case of primitive Siegel modular cusp forms of genus 2. We briefly recall this result here. Let π be a cuspidal automorphic representation on $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ whose archimedean component π_∞ belongs to the discrete series, with cohomological weights $(a, b; a + b)$ with $a \geq b \geq 0$. For each prime p, Laumon, Taylor and Weissauer have defined a four-dimensional Galois representation

$$\rho_{\pi,p} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_4(\bar{\mathbb{Q}}_p)$$

with standard properties. Let p be an unramified prime for π. Then Tilouine and Urban have generalized the notion of ordinariness for such primes p in three ways to what they call Borel ordinary, Siegel ordinary, and Klingen ordinary (these terms come from the underlying parabolic subgroups of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$). In the Borel case, the p-ordinariness of π implies that the Hecke polynomial of π_p, namely

$$(X - \alpha)(X - \beta)(X - \gamma)(X - \delta),$$

has the property that the p-adic valuations of α, β, γ and δ are 0, $b + 1$, $a + 2$ and $a + b + 3$, respectively.

Theorem 1.2 ([U05], [TU99])**.** *Say π is a Borel p-ordinary cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$ which is stable at ∞ with cohomological weights $(a, b; a + b)$. Then the restriction of $\rho_{\pi,p}$ to the decomposition subgroup $G_{\mathbb{Q}_p}$ is upper-triangular. More precisely, there is a basis in which $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$*

$$\sim \begin{pmatrix} \lambda(\delta/p^{a+b+3}) \cdot \chi_{\mathrm{cyc},p}^{a+b+3} & * & * & * \\ 0 & \lambda(\gamma/p^{a+2}) \cdot \chi_{\mathrm{cyc},p}^{a+2} & * & * \\ 0 & 0 & \lambda(\beta/p^{b+1}) \cdot \chi_{\mathrm{cyc},p}^{b+1} & * \\ 0 & 0 & 0 & \lambda(\alpha) \end{pmatrix},$$

where $\lambda(x)$ is the unramified character which takes arithmetic Frobenius to x.

We remark that $\rho_{\pi,p}$ above is the contragredient of the one used in [U05] (we also use the arithmetic Frobenius in defining our unramified characters), so the theorem matches exactly with [U05, Cor. 1 (iii)]. Similar results in the Siegel and Klingen cases can be found in the other parts of [U05, Cor. 1].

The local Galois representations appearing in Theorems 1.1 and 1.2 are sometimes referred to as (p, p)-Galois representations. The goal of this paper is to prove structure theorems for the local (p, p)-Galois representations attached to automorphic forms on $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$.

In the first part of this paper (cf. Section 3) we reprove Theorem 1.1 using the celebrated work of Colmez-Fontaine [CF00] establishing an equivalence of categories between potentially semistable representations and filtered (φ, N)-modules with coefficients and descent data. Our proof is quite simple and serves to illustrate the techniques that will be used in the rest of the paper.

In the second and main part of this work (Section 4 onwards), we generalize Theorem 1.1 to the local (p, p)-Galois representations attached to an automorphic form π on $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. We assume that the global p-adic Galois representation $\rho_{\pi,p}$ attached to π exists, and that it satisfies several natural properties, e.g., it lives in a strictly compatible system of Galois representations, and satisfies Local-Global compatibility. Recently, much progress has been made on this front: such Galois representations have been attached to what are referred to as RAESDC (regular, algebraic, essentially self dual, cuspidal) automorphic forms on $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ by Clozel, Harris, Kottwitz and Taylor, and for conjugate self-dual automorphic forms over CM fields these representations were shown to satisfy Local-Global compatibility away from p by Taylor-Yoshida. Under some standard hypotheses (e.g., that the Hodge and Newton filtrations are in general position in the corresponding crystal, cf. Assumption 4.6 in the text), we show that in several cases the corresponding local (p,p)-representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ has an 'upper-triangular' form, and completely determine the 'diagonal' characters. In other cases, and perhaps more interestingly, we give conditions under which this local representation is irreducible. For instance, we directly generalize the comment about irreducibility made just after the statement of Theorem 1.1. As a sample of our results, let us state the following theorem (which is a collation of Theorems 6.6, 6.7 and 7.17 in the text).

Theorem 1.3 (Indecomposable Case)**.** *Say π is a cuspidal automorphic form on $\mathrm{GL}_{mn}(\mathbb{A}_\mathbb{Q})$ with infinitesimal character given by the integers $-\beta_1 > \cdots > -\beta_{mn}$. Suppose that the Weil-Deligne representation attached to π_p is Frobenius semisimple and indecomposable, i.e.,*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \tau_m \otimes \mathrm{Sp}(n),$$

where τ_m is an irreducible representation of $W_{\mathbb{Q}_p}$ of dimension $m \geq 1$, and $\mathrm{Sp}(n)$ *is the special representation, for $n \geq 1$. Assume that Assumption 4.6 holds.*

(i) Suppose $m = 1$ and $\tau_1 = \chi_0 \cdot \chi'$ is a character, where χ_0 is the ramified part, and χ' is an unramified character mapping arithmetic Frobenius to α.

a) If π is ordinary at p (i.e., $v_p(\alpha) = -\beta_1$), then the β_i are necessarily consecutive integers, and $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim$

$$\begin{pmatrix} \chi_0 \cdot \lambda(\frac{\alpha}{p^{v_p(\alpha)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1} & * & \cdots & * \\ 0 & \chi_0 \cdot \lambda(\frac{\alpha}{p^{v_p(\alpha)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-1} & \cdots & * \\ 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \chi_0 \cdot \lambda(\frac{\alpha}{p^{v_p(\alpha)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-(n-1)} \end{pmatrix},$$

where $\lambda(x)$ is the unramified character taking arithmetic Frobenius to x.

b) If π is not p-ordinary, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.

(ii) Suppose $m \geq 2$. Then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.

The theorem gives complete information about the reducibility of the (p,p)-representation in the indecomposable case (under Assumption 4.6). In particular, the image of the (p,p)-representation tends to be either in a minimal parabolic subgroup or a maximal parabolic subgroup of GL_n. While this is forced in the GL_2 setting, it is somewhat surprising that the image does not lie in any 'intermediate' parabolic subgroups even in the GL_n setting. Finally we point out that parts (i) $b)$ and (ii) of the theorem imply that the global representation $\rho_{\pi,p}$ is irreducible (see also [TY07, Cor. B] for the case of conjugate self-dual representations over CM fields).

The theorem is proved in Sections 6 and 7, using methods from p-adic Hodge theory. It is well-known that the category of Weil-Deligne representations is equivalent to the category of (φ, N)-modules [BS07, Prop. 4.1]. In Section 7, we classify the (φ, N)-submodules of the (φ, N)-module associated to the indecomposable Weil-Deligne representation in the theorem. This classification plays a key role in analyzing the (p,p)-representation once the Hodge filtration is introduced. Along the way, we take a slight detour to write down explicitly the filtered (φ, N)-module attached to an m-dimensional 'unramified supercuspidal' representation, since this might be a useful addition to the literature (cf. [GM09] for the two-dimensional case).

The terminology 'indecomposable case' in the discussion above refers to the standard fact that every Frobenius semisimple indecomposable Weil-Deligne representation has the form stated in the theorem. Some results in the decomposable case (where the Weil-Deligne representation is a direct sum of indecomposables) are given in Section 8, though the principal series case is treated completely a bit earlier, in Section 5 (in the spherical case our results overlap with those in D. Geraghty's recent thesis, and we thank T. Gee for pointing this out to us). We refer the reader to these sections for explicit statements of results. Of the remaining sections, Section 2 recalls some useful facts from p-adic Hodge theory, whereas Section 4 recalls some general facts and conjectures about Galois representations associated to automorphic forms on $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$.

2. p-adic Hodge theory

We start by recalling some results we need from p-adic Hodge theory. For the basic definitions in the subject, e.g., of Fontaine's ring $\mathbf{B}_{\mathrm{st}}$, filtered (φ, N)-modules with coefficients and descent data, Newton and Hodge numbers, see [F94], [FO], [GM09, §2].

2.1. **Newton and Hodge Numbers.** We start by stating some facts about Newton and Hodge numbers, which do not seem to be in the literature when the coefficients are not necessarily $\mathbb{Q}_p$.

Let F and E be two finite field extensions of $\mathbb{Q}_p$ and assume that all the conjugates of F are contained in E. Suppose D is a free of finite rank module over $F \otimes_{\mathbb{Q}_p} E$. Then clearly

$$\dim_F D = [E : \mathbb{Q}_p] \cdot \mathrm{rank}_{F \otimes_{\mathbb{Q}_p} E} D. \tag{2.1}$$

Lemma 2.1. *Suppose $D_2 \subseteq D_1$ are two free of finite rank modules over $F \otimes_{\mathbb{Q}_p} E$. Then D_1/D_2 is also free of finite rank* $= \mathrm{rank}(D_1) - \mathrm{rank}(D_2)$.

Proof. It suffices to show any basis of D_2 can be extended to a basis of D_1. For any $F \otimes_{\mathbb{Q}_p} E$-module D, we have:

$$D \ \simeq \prod_{\sigma : F \hookrightarrow E} D_\sigma, \tag{2.2}$$

where $D_\sigma = D \otimes_{F \otimes E, \sigma} E$. Apply this isomorphism to D_1. Look at the image of $D_{2\sigma}$ of D_2 in each projection $D_{1\sigma}$. Since the $D_{1\sigma}$ are vector spaces over E, we can extend to the basis of the E-subspaces $D_{2\sigma}$ to a basis of the $D_{1\sigma}$. Now pulling back the extended basis vectors in each $D_{1\sigma}$, we get a basis of D_1 which extends the basis of D_2. □

Lemma 2.2 (Newton number)**.** *Suppose D is a filtered (φ, N, F, E)-module of rank n, such that the action of φ is E-semisimple, i.e., there exists a basis $\{e_1, \cdots, e_n\}$ of D such that $\varphi(e_i) = \alpha_i e_i$, for some $\alpha_i \in E^\times$. Then*

$$t_N(D) = [E : \mathbb{Q}_p] \cdot \sum_{i=1}^{n} v_p(\alpha_i).$$

Proof. The proof is standard from the definition of the Newton number. □

Lemma 2.3 (Hodge number)**.** *Suppose D is a filtered (φ, N, F, E)-module of rank n. Then*

$$t_H(D) = [E : \mathbb{Q}_p] \cdot \sum_{i \in \mathbb{Z}} i \cdot \mathrm{rank}_{F \otimes_{\mathbb{Q}_p} E}\ \mathrm{gr}^i D_F. \tag{2.3}$$

Proof. Since D is a filtered (φ, N, F, E)-module, there exists a separated, exhausted and decreasing filtration $(\mathrm{Fil}^i D_F)_{i \in \mathbb{Z}}$ of $(F \otimes_{\mathbb{Q}_p} E)$-submodules of $D_F := F \otimes_{F_0 \otimes_{\mathbb{Q}_p} E} D$. Then, by forgetting the E-module structure, D is also a filtered $(\varphi, N, F, \mathbb{Q}_p)$-module and each term of the filtration $\{\mathrm{Fil}^i D_F\}$ can also thought of as an F-module. By a standard formula (see, e.g., [FO, Prop. 6.45]), we have:

$$t_H(D) = \sum_{i \in \mathbb{Z}} i \cdot \dim_F\ \mathrm{gr}^i D_F,$$

with $\mathrm{gr}^i D_F = \mathrm{Fil}^i D_F / \mathrm{Fil}^{i+1} D_F$, with $\mathrm{gr}^i D_F$ thought of as an F-module. Savitt has observed [S05] that each $\mathrm{Fil}^i D_F$ is free of finite rank over $F \otimes_{\mathbb{Q}_p} E$. By Lemma 2.1, we have $\mathrm{gr}^i D_F$ is also a free of finite rank module over $F \otimes_{\mathbb{Q}_p} E$. Applying (2.1), we obtain the lemma. □

Remark 1. By the last two lemmas, one can drop the common factor of $[E : \mathbb{Q}_p]$ when checking the admissibility of a filtered (φ, N, F, E)-module.

Corollary 2.4. *Suppose D is a filtered (φ, N, F, E)-module of rank* 1*. Then*

$$t_H(D) = [E : \mathbb{Q}_p] \cdot \beta, \tag{2.4}$$

where β is the unique integer such that $0 = \mathrm{Fil}^{\beta+1} D_F \subsetneq \mathrm{Fil}^{\beta} D_F = D_F$.

Lemma 2.5. *Let D_1, D_2 be two filtered (φ, N, F, E)-modules, of rank r_1, r_2, respectively. Assume that the action of φ on D_1, D_2 is semisimple. Then*

$$t_N(D_1 \otimes D_2) = \mathrm{rank}(D_1)\ t_N(D_2) + \mathrm{rank}(D_2)\ t_N(D_1), \tag{2.5a}$$

$$t_H(D_1 \otimes D_2) = \mathrm{rank}(D_1)\ t_H(D_2) + \mathrm{rank}(D_2)\ t_H(D_1). \tag{2.5b}$$

Remark 2. The above formulas are well-known if $E = \mathbb{Q}_p$.

Proof. The proof of (2.5a) is an easy check. It is enough to prove (2.5b) when D_1, D_2 are of rank 1, since

$$\wedge^{r_1 r_2}(D_1 \otimes D_2) = \underbrace{\wedge^{r_2}(D_2) \otimes \cdots \otimes \wedge^{r_2}(D_2)}_{r_1\text{-times}} \otimes \underbrace{\wedge^{r_1}(D_1) \otimes \cdots \otimes \wedge^{r_1}(D_1)}_{r_2\text{-times}},$$

where the tensor products are taken over $F \otimes_{\mathbb{Q}_p} E$. In this case (2.5b) follows from Corollary 2.4. □

2.2. **Potentially semistable representations.** Let E and F be two finite extensions of $\mathbb{Q}_p$, and let V be a finite dimensional vector space over E.

Definition 1. A representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(V)$ is said to be *semistable* over F, or F-*semistable*, if $\dim_{F_0} D_{\mathrm{st},F}(V) = \dim_{F_0}(\mathbf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_F} = \dim_{\mathbb{Q}_p} V$ where $F_0 = \mathbf{B}_{\mathrm{st}}^{G_F}$. If such an F exists, ρ is said to be a potentially semistable representation. If $F = \mathbb{Q}_p$, we say that ρ is semistable.

Remark 3. If ρ is F-semistable, then ρ is F'-semistable for any finite extension of F'/F. Hence we may and do assume that F is Galois over $\mathbb{Q}_p$.

The following fundamental theorem plays a key role in subsequent arguments.

Theorem 2.6 ([CF00])**.** *There is an equivalence of categories between F-semistable representations $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ with Hodge-Tate weights $-\beta_n \leq \cdots \leq -\beta_1$ and admissible filtered (φ, N, F, E)-modules D of rank n over $F_0 \otimes_{\mathbb{Q}_p} E$ such that the jumps in the Hodge filtration $\mathrm{Fil}^i D_F$ on $D_F := F \otimes_{F_0} D$ are at $\beta_1 \leq \cdots \leq \beta_n$.*

The jumps in the filtration on $D_F = F \otimes_{F_0} D_{\mathrm{st},F}(\rho)$ are the negatives of the Hodge-Tate weights of ρ. That is, if h is a Hodge-Tate weight, then $\mathrm{Fil}^{-h+1}(D_F) \subsetneq \mathrm{Fil}^{-h}(D_F)$. The equivalence of categories in the theorem is induced by Fontaine's functor $D_{\mathrm{st},F}$. The Frobenius φ, monodromy N, and filtration on $\mathbf{B}_{\mathrm{st}}$ induce the corresponding structures on $D_{\mathrm{st},F}(V)$. There is also an induced action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ on $D_{\mathrm{st},F}(V)$. As an illustration of the power of the theorem we recall the following useful (and well-known) fact:

Corollary 2.7. *Every potentially semistable character $\chi : G_{\mathbb{Q}_p} \to E^\times$ is of the form $\chi = \chi_0 \cdot \lambda(a_0) \cdot \chi^i_{\mathrm{cyc},p}$, where χ_0 is a finite order character of $\mathrm{Gal}(F/\mathbb{Q}_p)$, for a cyclotomic extension F of $\mathbb{Q}_p$, $-i \in \mathbb{Z}$ is the Newton number of $D_{\mathrm{st},F}(\chi)$, and $\lambda(a_0)$ is the unramified character that takes arithmetic Frobenius to the unit $a_0 = p^{-i}/a \in \mathcal{O}_E^\times$, where $a = \varphi(v)/v$ for any non-zero vector v in $D_{\mathrm{st},F}(\chi)$.*

Proof. Every potentially semistable $\chi : G_{\mathbb{Q}} \to E^\times$ is F-semistable for a sufficiently large cyclotomic extension of $\mathbb{Q}_p$. Let $D_{\mathrm{st},F}(\chi)$ be the corresponding filtered (φ, N)-module with coefficients and descent data. Suppose that the induced $\mathrm{Gal}(F/\mathbb{Q}_p)$-action on $D_{\mathrm{st},F}(\chi)$ is given the character χ_0. Now consider the F-semistable E-valued character $\chi' := \chi_0 \cdot \lambda(a_0) \cdot \chi^i_{\mathrm{cyc},p}$. One easily checks that $D_{\mathrm{st},F}(\chi') = D_{\mathrm{st},F}(\chi)$. By Theorem 2.6, we have $\chi = \chi' = \chi_0 \cdot \lambda(a_0) \cdot \chi^i_{\mathrm{cyc},p}$. □

2.3. **Weil-Deligne representations.** We now recall the definition of the Weil-Deligne representation associated to an F-semistable representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$, due to Fontaine. We assume that $F/\mathbb{Q}_p$ is Galois and $F \subseteq E$. Let W_F denote the Weil group of F. For any (φ, N, F, E)-module D, we have the decomposition

$$D \simeq \prod_{i=1}^{[F_0:\mathbb{Q}_p]} D_i \tag{2.6}$$

where $D_i = D \otimes_{(F_0 \otimes_{\mathbb{Q}_p} E, \sigma^i)} E$, and σ is the arithmetic Frobenius of $F_0/\mathbb{Q}_p$.

Definition 2 (Weil-Deligne representation)**.** Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ be an F-semistable representation. Let D be the corresponding filtered module. Noting $W_{\mathbb{Q}_p}/W_F = \mathrm{Gal}(F/\mathbb{Q}_p)$, we let

$$g \in W_{\mathbb{Q}_p} \text{ act on } D \text{ by } (g \bmod W_F) \circ \varphi^{-\alpha(g)}, \tag{2.7}$$

where the image of g in $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is the $\alpha(g)$-th power of the arithmetic Frobenius at p. We also have an action of N via the monodromy operator on D. These actions induce a Weil-Deligne action on each D_i in (2.6) and the resulting Weil-Deligne representations are all isomorphic. This isomorphism class is defined to be the Weil-Deligne representation $\mathrm{WD}(\rho)$ associated to ρ.

Remark 4. If $F/\mathbb{Q}_p$ is totally ramified and $\mathrm{Frob}_p \in W_{\mathbb{Q}_p}$ is the arithmetic Frobenius, then observe that $\mathrm{WD}(\rho)(\mathrm{Frob}_p)$ acts by φ^{-1}.

Lemma 2.8. *Let* $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_n(E)$ *be a potentially semistable representation. If* $\mathrm{WD}(\rho)$ *is irreducible, then so is* ρ.

Proof. Suppose the space V which affords ρ is reducible. By Theorem 2.6, the reducibility of V is equivalent to the existence of a non-trivial admissible filtered (φ, N, F, E)-submodule of $D_{\mathrm{st},F}(V)$. From the definition of the Weil-Deligne action given above, this submodule is both $W_{\mathbb{Q}_p}$ and N stable. Thus $\mathrm{WD}(\rho)$ is reducible, a contradiction. $\square$

3. The case of GL_2

Let f be a primitive cusp form, which is $\wp$-ordinary. Let π be the corresponding cuspidal automorphic representation on $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. Let $\rho_{f,\wp}$ be the associated Galois representation. In this section, we shall reprove Theorem 1.1 (reducibility of $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$).

Wiles' original proof in [W88] involves some amount of p-adic Hodge theory. More precisely it uses some Dieudonné theory for the abelian varieties associated by Shimura to cuspforms of weight 2. The result for forms of weight greater than 2 is then deduced by a clever use of certain auxiliary Λ-adic Galois representations attached to Hida families of ordinary forms [H86] (see [BGK10, §6] for a detailed exposition of the argument). The proof we give below avoids Hida theory completely, and as the expert will note, is a simple extension of Wiles' weight 2 argument. We remark that this proof could not have been given in [W88], since the equivalence of categories of Colmez and Fontaine (Theorem 2.6) was of course unavailable at the time.

Another key ingredient in our proof is the fact that Galois representations attached to elliptic modular eigenforms live in strictly compatible systems of Galois representations [S97]. The consequent ability to transfer information about the Weil-Deligne parameter between various members of the family has been used to great effect in recent times (e.g., in the Khare-Wintenberger proof of Serre's conjecture) and is important for us as well. We start by recalling the definition of such a system of Galois representations following [KW09, §5].

Let F be a number field, ℓ be a prime, and let $\rho : G_F \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ be a continuous global Galois representation.

Definition 3. Say that ρ is *geometric* if it is unramified outside a finite set of primes of F and its restrictions to the decomposition group at primes above ℓ are potentially semistable.

A geometric representation defines, for every prime q of F, a representation of the Weil-Deligne group at q, denoted by WD_q, with values in $\mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$, well defined up to conjugacy. For q of characteristic not ℓ, the definition is classical, and comes from the theory of Deligne-Grothendieck, and for q of characteristic ℓ, the definition comes from Fontaine theory (Definition 2).

Definition 4. For a number field L, we call an L-rational, n-dimensional *strictly compatible system* of geometric representations (ρ_ℓ) of G_F the data of:

1. For each prime ℓ and each embedding $i : L \hookrightarrow \bar{\mathbb{Q}}_\ell$, a continuous, semisimple representation $\rho_\ell : G_F \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ that is geometric.

(2) for each prime q of F, an F-semisimple (Frobenius semisimple) representation r_q of the Weil-Deligne group WD_q with values in $\mathrm{GL}_n(L)$ such that:
- r_q is unramified for all q outside a finite set.
- for each ℓ and each $i : L \hookrightarrow \bar{\mathbb{Q}}_\ell$, the Frobenius semisimple Weil-Deligne representation $\mathrm{WD}_q \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$ associated to $\rho_\ell|_{D_q}$ is conjugate to r_q (via the embedding $i : L \hookrightarrow \bar{\mathbb{Q}}_\ell$).
- There are n distinct integers $\beta_1 < \cdots < \beta_n$, such that ρ_ℓ has Hodge-Tate weights $\{-\beta_1, \ldots, -\beta_n\}$ (the minus signs arise since the weights are the negatives of the jumps in the Hodge filtration on the associated filtered module).

By work of Faltings, it is known that $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is a potentially semistable representation with Hodge-Tate weights $(0, k-1)$. Let D be the admissible filtered (φ, N, F, E)-module associated to this representation for a suitable choices of F, and $E = K_{f,\wp}$. By Theorem 2.6, the study of the structure of the (p,p)-Galois representation $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ reduces to that of the study of the filtered module D. In particular, $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is reducible if and only if D has a non-trivial admissible submodule.

As mentioned above, a key ingredient in our proof of Theorem 1.1 is that the representation $\rho_{f,\wp}$ lives in a strictly compatible system of Galois representations $(\rho_{f,\lambda})$, where λ varies over the primes of K_f. This result is the culmination of the work of several people, including Langlands, Deligne, Carayol, Katz-Messing and most recently Saito [S97]. In particular, one may read off the Weil-Deligne representation $\mathrm{WD}(\rho_{f,\wp}|_{G_{\mathbb{Q}_p}})$ on D (cf. Definition 2) by looking at the Weil-Deligne representation attached to $\rho_{f,\lambda}|_{G_{\mathbb{Q}_p}}$ for a place λ of K_f with $\lambda \nmid p$. As a consequence, one may read off, e.g., the characteristic polynomial of crystalline Frobenius φ purely in terms of a λ-adic member of the strictly compatible family for $\lambda \nmid p$. Now it is well-known that f is ordinary at $\wp$ only if the underlying local automorphic representation π_p is in the principal series or (an unramified twist of) the Steinberg representation. We obtain:

Theorem 3.1 (Carayol, Deligne, Langlands, Katz-Messing, Saito [S97])**.** *The characteristic polynomial $P(X)$ of the (inverse of) crystalline Frobenius φ on D coincides with that of $\rho_{f,\lambda}(\mathrm{Frob}_p)$ for a place λ of K_f with $\lambda \nmid p$. More precisely, in the cases we need, $P(X)$ is given by:*
- *(Unramified principal series) If $p \nmid N$, then $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is crystalline and $P(X) = X^2 - a_p(f)X + \chi(p)p^{k-1}$.*
- *(Steinberg) If $p||N$ and $v_p(cond(\chi)) = 0$, then $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is $\mathbb{Q}_p$-semistable and $P(X) = (X - a_p(f))(X - pa_p(f))$.*
- *(Ramified principal series) If $p|N$ with $v_p(N) = v_p(cond(\chi))$, then $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is potentially crystalline and $P(X) = (X - a_p(f))(X - \chi'(p)\bar{a}_p(f))$, where χ' denotes the prime-to-p part of the conductor χ.*

More generally, the complete rank 2 filtered (φ, N, F, E)-module D can be written down quite explicitly in all the above cases ([B01], [GM09]). We now give a proof of Theorem 1.1 using this structure of D, which we first recall below in various cases.

3.1. **Good reduction:** $p \nmid N$**.** In this case $F = \mathbb{Q}_p$ and $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is crystalline. Let $D = D_{\mathrm{st},\mathbb{Q}_p}(\rho_{f,\wp}|_{G_{\mathbb{Q}_p}})$ be the corresponding filtered (φ, N)-module. Let α and β be the two roots of $P(X) = X^2 - a_p(f)X + \chi(p)p^{k-1}$, with $v_p(\alpha) = 0$ and $v_p(\beta) = k-1$ (recall $v_p(a_p(f)) = 0$).

The structure of the filtered module D in this case is well-known (see [B01, p. 30-32], where the normalizations are a bit different). There are essentially two possibilities for D depending on whether D is decomposable or indecomposable. Assume that the crystal is decomposable (e.g., if f is a CM form). Let e_1, e_2 be the eigenvectors for φ with eigenvalues $1/\alpha$ and $1/\beta$ (cf. Remark 4). Then $D = Ee_1 \oplus Ee_2$, with

$$D = \begin{cases} \varphi(e_1) & = & \alpha^{-1}e_1 \\ \varphi(e_2) & = & \beta^{-1}e_2 \\ N & = & 0 \\ \mathrm{Fil}^i(D_F) & = & 0 \text{ if } i \geq 1,\ Ee_1 \text{ if } 2-k \leq i \leq 0,\ D \text{ if } i \leq 1-k. \end{cases}$$

If D is indecomposable, then $D = Ee_1 \oplus Ee_2$, and

$$D = \begin{cases} \varphi(e_1) & = & \alpha^{-1}e_1 + p^{1-k}e_2 \\ \varphi(e_2) & = & \beta^{-1}e_2 \\ N & = & 0 \\ \mathrm{Fil}^i(D_F) & = & 0 \text{ if } i \geq 1,\ Ee_1 \text{ if } 2-k \leq i \leq 0,\ D \text{ if } i \leq 1-k. \end{cases}$$

3.2. **Steinberg case: $p \parallel N$ and $p \nmid \mathbf{cond}(\chi)$.** In this case $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is semistable over $F = \mathbb{Q}_p$ but is not crystalline, and $v_p(a_p(f)) = \frac{k-2}{2}$. Note $v_p(a_p(f)) = 0$ if and only if $k = 2$. Let D be the corresponding filtered (φ, N)-module over $\mathbb{Q}_p$ (cf. [GM09, §3.1]). Set $\alpha = a_p(f)$ and $\beta = pa_p(f)$, so that the eigenvalues of crystalline Frobenius are $1/\alpha = p/\beta$ and $1/\beta$. Then $D = Ee_1 \oplus Ee_2$, and

$$D = \begin{cases} \varphi(e_1) & = & p\beta^{-1}e_1 \\ \varphi(e_2) & = & \beta^{-1}e_2 \\ N(e_1) & = & e_2 \\ N(e_2) & = & 0 \\ \mathrm{Fil}^i(D_F) & = & 0 \text{ if } i \geq 1,\ E(e_1 - \mathcal{L}e_2) \text{ if } 2-k \leq i \leq 0,\ D \text{ if } i \leq 1-k, \end{cases}$$

for some unique non-zero $\mathcal{L} \in E$.

3.3. **Ramified principal series: $v_p(\mathbf{cond}(\chi)) = v_p(N) \geq 1$.** In this case $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is potentially crystalline. If m is the exact power of p dividing N and $\mathrm{cond}(\chi)$, then $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ becomes crystalline over the totally ramified abelian extension $F = \mathbb{Q}_p(\mu_{p^m})$ of $\mathbb{Q}_p$. Decompose $\chi = \chi_p\chi'$ into its p-part and prime-to-p part.

Let D be the associated admissible filtered (φ, N, F, E)-module. An explicit description of this module was given in [GM09, §3.2]. Let $\alpha = a_p(f)$ and $\beta = \chi'(p)\bar{a}_p(f)$. Then $D = Ee_1 \oplus Ee_2$, with

$$D = \begin{cases} \varphi(e_1) & = & \alpha^{-1}e_1 \\ \varphi(e_2) & = & \beta^{-1}e_2 \\ N & = & 0 \\ g(e_1) & = & e_1 \\ g(e_2) & = & \chi_p(g)e_2 \end{cases}$$

for $g \in \mathrm{Gal}(F/\mathbb{Q}_p)$. Moreover, in [GM09, §3.2], the filtered module associated to $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ was further classified into three types. Since $v_p(a_p(f)) = 0$, this module is either $D_{\text{ord-split}}$ or $D_{\text{ord-non-split}}$ (cf. [GM09, 3.2]). The corresponding filtrations in these cases are given by

$$\begin{aligned}\mathrm{Fil}^i(D_F) &= 0 \text{ if } i \geq 1,\ (F \otimes E)e_1 \text{ if } 2-k \leq i \leq 0,\ D_F \text{ if } i \leq 1-k, \text{ and,}\\ \mathrm{Fil}^i(D_F) &= 0 \text{ if } i \geq 1,\ (F \otimes E)(xe_1 + ye_2) \text{ if } 2-k \leq i \leq 0,\ D_F \text{ if } i \leq 1-k,\end{aligned}$$

respectively, where x and y are explicit non-zero Gauss sum like quantities in $F \otimes E$ (see [GM09, §3.2] for their exact definitions, which we do not need here).

3.4. **Proof of Wiles' theorem.** Let D_n be the submodule of D generated by e_n, for $n = 1, 2$. Since, in all cases, the 'line' which determines the interesting step in the filtration on D_F, is transverse to $D_{2,F}$, the induced filtration on $D_{2,F}$ is given by

$$\cdots = \mathrm{Fil}^0(D_{2,F}) = \cdots = \mathrm{Fil}^{2-k}(D_{2,F}) = 0 \subsetneq \mathrm{Fil}^{1-k}(D_{2,F}) = D_{2,F} = \cdots \tag{3.1}$$

Thus $t_H(D_2) = 1-k$ in all cases.

In the principal series cases (i.e., the first and third cases above), we see that $t_N(D_2) = v_p(\beta^{-1}) = 1-k$, so that D_2 is an admissible φ-submodule of D (in the corresponding split subcases D_1 is also admissible but we do not need that here). In the second (Steinberg) case, D_2 is the only (φ, N)-submodule of rank 1, since N must act trivially on any such module. Moreover, $t_N(D_2) = v_p(\beta^{-1}) = -\frac{k}{2}$ which equals $1-k = t_H(D_2)$ if and only if $k = 2$. Thus D_2 is an admissible submodule if and only if $k = 2$, and D is irreducible if $k > 2$. Thus, in all (ordinary) cases, we have shown the existence of an admissible submodule D_2 of the filtered module D associated to $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ such that on the quotient D/D_2, crystalline Frobenius φ acts by an explicit element α^{-1} of valuation zero.

Assume that we are in the first two cases, so that $F = \mathbb{Q}_p$. By Theorem 2.6, the representation $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is clearly reducible, with a one-dimensional submodule given by the character $\lambda(\beta/p^{k-1})\chi_{\mathrm{cyc},p}^{k-1}$ (see Corollary 2.7), and quotient given by the unramified character $\lambda(\alpha)$ (again, by Corollary 2.7), proving the theorem in these cases. In the last case (when $F \neq \mathbb{Q}_p$) we may still use Theorem 2.6 to deduce that $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is reducible. Indeed, the one dimensional module D_2 is a filtered (φ, N, F, E)-module, with descent data given by the character χ_p of $\mathrm{Gal}(F/\mathbb{Q}_p)$, so by the one dimensional case of Theorem 2.6, D_2 corresponds to the character $\psi = \chi_p \lambda(\beta/p^{k-1})\chi_{\mathrm{cyc},p}^{k-1}$ of $G_{\mathbb{Q}_p}$. Since $D_2 \subset D$ as filtered modules with descent data, we see $V(\psi)$ is a one dimensional submodule of $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ with unramified quotient given by $\lambda(\alpha)$, proving the theorem in this case as well.

To recap, we have proved that if $v(a_p(f)) = 0$, then $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is reducible with quotient given by the unramified character which sends arithmetic Frobenius to

- the unique p-adic unit root of $X^2 - a_p(f)X + \chi(p)p^{k-1} = 0$ if $p \nmid N$, or,
- $a_p(f)$ if $p \parallel N$, $p \nmid \mathrm{cond}(\chi)$ and $k = 2$, or if $v_p(N) = \mathrm{cond}(\chi) \geq 1$,

and submodule completely determined by the condition that the determinant is $\chi\chi_{\mathrm{cyc},p}^{k-1}$. Moreover, $\rho_{f,\wp}|_{G_{\mathbb{Q}_p}}$ is irreducible if $p \parallel N$, $p \nmid \mathrm{cond}(\chi)$ and $k > 2$.

4. The case of GL_n

The goal of this paper is to prove various generalizations of Theorem 1.1 for the local (p,p)-Galois representations attached to automorphic forms on $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. In this section we collect together some facts about such automorphic forms and their Galois representations needed for the proof. The main results we need are the Local

Langlands correspondence (now a theorem of Henniart [H00] and Harris-Taylor [HT01]), and the existence of strictly compatible systems of Galois representations attached to cuspidal automorphic forms on GL_n (much progress has been made on this by Clozel, Harris, Kottwitz and Taylor [CHT08]).

4.1. **Local Langlands correspondence.** We state a few results concerning the Local Langlands correspondence, which we need later. We follow Kudla's article [K91], noting this article follows Rodier [R82], which in turn is based on the original work of Bernstein and Zelevinsky.

Let F be a complete non-Archimedean local field of residue characteristic p, let $n \geq 1$, and let $G = \mathrm{GL}_n(F)$. For a partition $n = n_1 + n_2 + \cdots + n_r$ of n, let P be the corresponding parabolic subgroup of G, M the Levi subgroup of P, and N the unipotent radical of P. Let δ_P denote the modulus character of the adjoint action of M on N. If $\sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_r$ is a smooth representation of M on V, we let

$$I_P^G(\sigma) = \{f : G \to V \mid f \text{ smooth on } G \text{ and } f(nmg) = \delta_P^{\frac{1}{2}}(m)(\sigma(m)(f(g))\},$$

for $n \in N$, $m \in M$, $g \in G$. G acts on functions in $I_P^G(\sigma)$ by right translation and $I_P^G(\sigma)$ is the usual induced representation of σ. It is an admissible representation of finite length.

A result of Bernstein-Zelevinsky says that if all the σ_i are supercuspidal, and σ is irreducible, smooth and admissible, then $I_P^G(\sigma)$ is reducible if and only if $n_i = n_j$ and $\sigma_i = \sigma_j(1)$ for some $i \neq j$. For the partition $n = m + m + \cdots + m$ (r times), and for a supercuspidal representation of σ of $\mathrm{GL}_m(F)$, call the data

$$(\sigma, \sigma(1), \cdots, \sigma(r-1)) = [\sigma, \sigma(r-1)] = \Delta \tag{4.1}$$

a segment. Clearly $I_P^G(\Delta)$ is reducible. By [K91, Thm. 1.2.2], the induced representation $I_P^G(\Delta)$ has a unique irreducible quotient $Q(\Delta)$ which is essentially square-integrable.

Two segments

$$\Delta_1 = [\sigma_1, \sigma_1(r_1 - 1)] \quad \text{and} \quad \Delta_2 = [\sigma_2, \sigma_2(r_2 - 1)]$$

are said to be linked if $\Delta_1 \not\subseteq \Delta_2$, $\Delta_2 \not\subseteq \Delta_1$, and $\Delta_1 \cup \Delta_2$ is a segment. We say that Δ_1 precedes Δ_2 if Δ_1 and Δ_2 are linked and if $\sigma_2 = \sigma_1(k)$, for some positive integer k.

Theorem 4.1 (Langlands classification)**.** *Given segments $\Delta_1, \cdots, \Delta_r$, assume that for $i < j$, Δ_i does not precede Δ_j. Then*

(1) *The induced representation $I_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r))$ admits a unique irreducible quotient $Q(\Delta_1, \cdots, \Delta_r)$, called the Langlands quotient. Moreover, r and the segments Δ_i up to permutation are uniquely determined by the Langlands quotient.*
(2) *Every irreducible admissible representation of $\mathrm{GL}_n(F)$ is isomorphic to some $Q(\Delta_1, \cdots, \Delta_r)$.*
(3) *The induced representation $I_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r))$ is irreducible if and only if no two of the segments Δ_i and Δ_j are linked.*

So much for the automorphic side. We now turn to the Galois side. Recall that a representation of W_F is said to be Frobenius semisimple if arithmetic Frobenius acts semisimply. An admissible representation of the Weil-Deligne group of F is one for which the action of W_F is Frobenius semisimple. Let $\mathrm{Sp}(r)$ denote the

Weil-Deligne representation of order r with the usual definition. When $F = \mathbb{Q}_p$ there is a basis $\{f_i\}$ of $\mathrm{Sp}(r)$ for which $\varphi f_i = p^{i-1} f_i$, and $N f_i = f_{i-1}$ for $i > 1$ and $N f_1 = 0$. It is well-known that every indecomposable admissible representation of the Weil-Deligne group of F is of the form $\tau \otimes \mathrm{Sp}(r)$ where τ is an irreducible admissible representation of W_F and $r \geq 1$. Moreover (cf. [R94, §5, Cor. 2]), every admissible representation of the Weil-Deligne group of F is of the form

$$\oplus_i \ \tau_i \otimes \mathrm{Sp}(r_i),$$

where the τ_i are irreducible admissible representations of W_F and the r_i are positive integers.

Theorem 4.2 (Local Langlands correspondence)**.** ([HT01, VII.2.20], [H00], [K80]). *There exists a bijection between isomorphism classes of irreducible admissible representations of* $\mathrm{GL}_n(F)$ *and isomorphism classes of admissible* n*-dimensional representations of the Weil-Deligne group of* F*.*

The correspondence is given as follows. The key point is to construct a bijection $\Phi_F : \sigma \mapsto \tau = \Phi_F(\sigma)$ between the set of isomorphism classes of supercuspidal representations of $\mathrm{GL}_n(F)$ and the set of isomorphism classes of irreducible admissible representations of W_F. This is due to Henniart [H00] and Harris-Taylor [HT01]. Then, to $Q(\Delta)$, for the segment $\Delta = [\sigma, \sigma(r-1)]$, one associates the indecomposable admissible representation $\Phi_F(\sigma) \otimes \mathrm{Sp}(r)$ of the Weil-Deligne group of F. More generally, to the Langlands quotient $Q(\Delta_1, \cdots, \Delta_r)$, where $\Delta_i = [\sigma_i, \sigma_i(r_i - 1)]$, for $i = 1$ to r, one associates the admissible representation $\oplus_i \Phi_F(\sigma_i) \otimes \mathrm{Sp}(r_i)$ of the Weil-Deligne group of F.

4.2. **Automorphic forms on** GL_n**.** The Harish-Chandra isomorphism identifies the center $\mathfrak{z}_n$ of the universal enveloping algebra of the complexified Lie algebra $\mathfrak{gl}_n$ of GL_n, with the algebra $\mathbb{C}[X_1, X_2, \cdots, X_n]^{S_n}$, where the symmetric group S_n acts by permuting the X_i. Given a multiset $H = \{x_1, x_2, \ldots, x_n\}$ of n complex numbers one obtains an infinitesimal character of $\mathfrak{z}_n$ given by $\chi_H : X_i \mapsto x_i$.

Cuspidal automorphic forms with infinitesimal character χ_H (or more simply just H) are smooth functions $f : \mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}) \to \mathbb{C}$ satisfying the usual finiteness condition under a maximal compact subgroup, a cuspidality condition, and a growth condition for which we refer the reader to [T04]. In addition, if $z \in \mathfrak{z}_n$ then $z \cdot f = \chi_H(z) f$. The space of such functions is denoted by $\mathcal{A}^\circ_H(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}))$. This space is a direct sum of irreducible admissible $\mathrm{GL}_n(\mathbb{A}^{(\infty)}_\mathbb{Q}) \times (\mathfrak{gl}_n, O(n))$-modules each occurring with multiplicity one, and these irreducible constituents are referred to as cuspidal automorphic representations on $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with infinitesimal character χ_H. Let π be such an automorphic representation (we will also refer to π as an automorphic form). By a result of Flath, π is a restricted tensor product $\pi = \otimes'_p \pi_p$ (cf. [B97, Thm. 3.3.3]) of local automorphic representations.

4.3. **Galois representations.** Let π be an automorphic form on $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with infinitesimal character χ_H, where H is a multiset of integers. The following very strong, but natural, conjecture seems to be part of the folklore.

Conjecture 4.3. *Let* H *consist of* n *distinct integers. There is a strictly compatible system of Galois representations* $(\rho_{\pi,\ell})$ *associated to* π*, with Hodge-Tate weights* H*, such that Local-Global compatibility holds.*

Here Local-Global compatibility means that the underlying semisimplified Weil-Deligne representation at p in the compatible system (which is independent of the residue characteristic ℓ of the coefficients by hypothesis) corresponds to π_p via the Local Langlands correspondence. Considerable evidence towards this conjecture is available for self-dual representations thanks to the work of Clozel, Kottwitz, Harris and Taylor. We quote the following theorem from Taylor's paper [T04], referring to that paper for the original references (e.g., [C91]).

Theorem 4.4 (cf. [T04], Thm. 3.6)**.** *Let H consist of n distinct integers. Suppose that the contragredient representation $\pi^\vee = \pi \otimes \psi$ for some character $\psi : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$, and suppose that for some prime q, the representation π_q is square-integrable. Then there is a continuous representation*

$$\rho_{\pi,\ell} : G_{\mathbb{Q}} \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$$

such that $\rho_{\pi,\ell}|_{G_{\mathbb{Q}_\ell}}$ is potentially semistable with Hodge-Tate weights given by H, and such that for any prime $p \neq \ell$, the semisimplification of the Weil-Deligne representation attached to $\rho_{\pi,\ell}|_{G_{\mathbb{Q}_p}}$ is the same as the Weil-Deligne representation associated by the Local Langlands correspondence to π_p, except possibly for the monodromy operator.

Subsequent work of Taylor and Yoshida [TY07] shows that the two Weil-Deligne representations in the theorem above are in fact the same (i.e., the monodromy operators also match).

In any case, for the rest of this paper we shall *assume* that Conjecture 4.3 holds. In particular, we assume that the Weil-Deligne representation at p associated to a p-adic member of the compatible system of Galois representations attached to π using Fontaine theory is the same as the Weil-Deligne representation at p attached to an ℓ-adic member of the family, for $\ell \neq p$.

4.4. **A variant, following** [CHT08]**.** A variant of the above result can be found in [CHT08]. We state this now using the notation and terminology from [CHT08, §4.3].

Say π is an RAESDC (regular, algebraic, essentially self dual, cuspidal) automorphic representation if π is a cuspidal automorphic representation such that

- $\pi^\vee = \pi \otimes \chi$ for some character $\chi : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$.
- π_∞ has the same infinitesimal character as some irreducible algebraic representation of GL_n.

Let $a \in \mathbb{Z}^n$ satisfy

$$a_1 \geq \cdots \geq a_n. \tag{4.2}$$

Let Ξ_a denote the irreducible algebraic representation of GL_n with highest weight a. We say that an RAESDC automorphic representation π has weight a if π_∞ has the same infinitesimal character as $\Xi_a^\vee$; in this case there is an integer w_a such that $a_i + a_{n+1-i} = w_a$ for all i.

Let S be a finite set of primes of $\mathbb{Q}$. For $v \in S$ let ρ_v be an irreducible square integrable representation of $\mathrm{GL}_n(\mathbb{Q}_v)$. Say that an RAESDC representation π has type $\{\rho_v\}_{v\in S}$ if for each $v \in S$, π_v is an unramified twist of $\rho_v^\vee$.

With this setup, Clozel, Harris, and Taylor attached a Galois representation to an RAESDC π.

Theorem 4.5 ([CHT08], Prop. 4.3.1)**.** *Let* $\iota : \bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$*. Let* π *be an* RAESDC *automorphic representation as above of weight* a *and type* $\{\rho_v\}_{v\in S}$*. Then there is a continuous semisimple Galois representation*

$$r_{\ell,\iota}(\pi) : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$$

with the following properties:

(1) *For every prime* $p \nmid \ell$*, we have*

$$r_{\ell,\iota}(\pi)|^{\mathrm{ss}}_{G_{\mathbb{Q}_p}} = (r_\ell(\iota^{-1}\pi_p)^\vee)(1-n)^{\mathrm{ss}},$$

where r_ℓ *is the reciprocity map defined in* [HT01]*.*

(2) *If* $\ell = p$*, then the restriction* $r_{\ell,\iota}(\pi)|_{G_{\mathbb{Q}_p}}$ *is potentially semistable and if* π_p *is unramified then it is crystalline, with Hodge-Tate weights* $-(a_j + n - j)$ *for* $j = 1, \ldots, n$*.*

4.5. **Newton and Hodge filtration.** Let $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ be the (p,p)-representation attached to an automorphic form π and D be the corresponding filtered (φ, N, F, E)-module (for suitable choices of F and E).

Note that there are are two natural filtrations on D_F, the Hodge filtration $\mathrm{Fil}^i D_F$ and the Newton filtration defined by ordering the slopes of the crystalline Frobenius (the valuations of the roots of φ.) To keep the analysis of the structure of the (p,p)-representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ within reasonable limits in this paper, we shall make the following assumption.

Assumption 4.6. *The Newton filtration on* D_F *is in general position with respect to the Hodge filtration* $\mathrm{Fil}^i D_F$*.*

Here, if V is a space and $\mathrm{Fil}^i_1 V$ and $\mathrm{Fil}^j_2 V$ are two filtrations on V then we say they are in general position if each $\mathrm{Fil}^i_1 V$ is as transverse as possible to each $\mathrm{Fil}^j_2 V$. We remark that the condition above is in some sense generic since two random filtrations on a space tend to be in general position.

4.6. **(Quasi-) Ordinary representations.** As mentioned earlier, our goal is to prove that the (p,p)-representation attached to π is 'upper-triangular' in several cases. To this end it is convenient to recall the following terminology (see, e.g., Greenberg [G94, p.152] or Ochiai [O01, Def. 3.1]).

Definition 5. Let F be a number field. A p-adic representation V of G_F is called *ordinary* (respectively *quasi-ordinary*), if the following conditions are satisfied:

(1) For each place v of F over p, there is a decreasing filtration of G_{F_v}-modules

$$\cdots \mathrm{Fil}^i_v V \supseteq \mathrm{Fil}^{i+1}_v V \supseteq \cdots$$

such that $\mathrm{Fil}^i_v V = V$ for $i \ll 0$ and $\mathrm{Fil}^i_v V = 0$ for $i \gg 0$.

(2) For each v and each i, I_v acts on $\mathrm{Fil}^i_v V/\mathrm{Fil}^{i+1}_v V$ via the character $\chi^i_{\mathrm{cyc},p}$, where $\chi_{\mathrm{cyc},p}$ is the p-adic cyclotomic character (respectively, there exists an open subgroup of I_v which acts on $\mathrm{Fil}^i_v V/\mathrm{Fil}^{i+1}_v V$ via $\chi^i_{\mathrm{cyc},p}$).

5. Principal series

Let π be an automorphic representation on $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with infinitesimal character H, for a set of distinct integers H. Let π_p denote the local automorphic representation of $\mathrm{GL}_n(\mathbb{Q}_p)$. In this section we study the behaviour of the (p,p)-Galois representation assuming that π_p is in the principal series.

5.1. **Spherical case.** Assume that π_p is an unramified principal series representation. Since π_p is a spherical representation of $\mathrm{GL}_n(\mathbb{Q}_p)$, there exist unramified characters $\chi_1,\ldots,\chi_n$ of $\mathbb{Q}_p^\times$ such that π_p is the Langlands quotient $Q(\chi_1,\ldots,\chi_n)$. We can parametrize the isomorphism class of this representation by the Satake parameters $\alpha_1,\ldots,\alpha_n$, for $\alpha_i=\chi_i(\omega)$, where ω is a uniformizer for $\mathbb{Q}_p$.

Note that $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is crystalline with Hodge-Tate weights H. Let D be the corresponding filtered φ-module. Let the jumps in the filtration on D be $\beta_1<\beta_2<\cdots<\beta_n$ (so that the Hodge-Tate weights H are $-\beta_1>\cdots>-\beta_n$.

Definition 6. Say that the automorphic representation π is *p-ordinary* if $\beta_i+v_p(\alpha_i)=0$ for all $i=1,\ldots,n$.

We remark that if π is p-ordinary, then the $v_p(\alpha_i)$ are integers.

Theorem 5.1 (Spherical case). *Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with infinitesimal character given by the integers $-\beta_1>\cdots>-\beta_n$ and such that π_p is in the unramified principal series with Satake parameters $\alpha_1,\ldots,\alpha_n$. If π is p-ordinary, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}\sim$*

$$\begin{pmatrix} \lambda(\frac{\alpha_1}{p^{v_p(\alpha_1)}})\cdot\chi_{\mathrm{cyc},p}^{-\beta_1} & * & \cdots & * \\ 0 & \lambda(\frac{\alpha_2}{p^{v_p(\alpha_2)}})\cdot\chi_{\mathrm{cyc},p}^{-\beta_2} & \cdots & * \\ 0 & 0 & \cdots & * \\ 0 & 0 & \lambda(\frac{\alpha_{n-1}}{p^{v_p(\alpha_{n-1})}})\cdot\chi_{\mathrm{cyc},p}^{-\beta_{n-1}} & * \\ 0 & 0 & 0 & \lambda(\frac{\alpha_n}{p^{v_p(\alpha_n)}}))\cdot\chi_{\mathrm{cyc},p}^{-\beta_n} \end{pmatrix}.$$

In particular, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is ordinary.

Proof. Since π_p is p-ordinary, we have that $v_p(\alpha_n)<v_p(\alpha_{n-1})<\cdots<v_p(\alpha_1)$. By strict compatibility, the characteristic polynomial of the inverse of crystalline Frobenius of D_n is equal to $\prod_i(X-\alpha_i)$.

Since the $v_p(\alpha_i)$ are distinct, there exists a basis of eigenvectors of D_n for the operator φ, say $\{e_i\}$, with corresponding eigenvalues $\{\alpha_i^{-1}\}$. For any integer $1\le i\le n$, let D_i be the φ-submodule generated by $\{e_1,\cdots,e_i\}$. Since D_n is admissible we know that $t_H(D_i)\le t_N(D_i)$ for all $i=1,\ldots,n$.

The filtration on D_n is

$$\cdots\subseteq 0\subsetneq \mathrm{Fil}^{\beta_n}(D_n)\subseteq\cdots\subsetneq\mathrm{Fil}^{\beta_1}(D_n)=D_n\subseteq\cdots \tag{5.1}$$

Since D_n is admissible, we have that

$$\sum_{i=1}^n\beta_i=-\sum_{i=1}^n v_p(\alpha_i). \tag{5.2}$$

By Assumption 4.6, we have that the jumps in the induced filtration on D_{n-1} are $\beta_1,\ldots,\beta_{n-1}$. By (5.2), we have

$$t_H(D_{n-1})=\sum_{i=1}^{n-1}\beta_i=-\sum_{i=1}^{n-1}v_p(\alpha_i)=t_N(D_{n-1}), \tag{5.3}$$

since $\beta_n=-v_p(\alpha_n)$. This implies that D_{n-1} is admissible. Moreover, D_n/D_{n-1} is also admissible since $t_H(D_n/D_{n-1})=\beta_n$ and $t_N(D_n/D_{n-1})=-v_p(\alpha_n)$, since φ

acts on D_n/D_{n-1} by α_n^{-1}. Therefore, the Galois representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ looks like

$$\rho \sim \begin{pmatrix} \rho_{n-1} & * \\ 0 & \lambda(\frac{\alpha_n}{p^{v_p(\alpha_n)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_n} \end{pmatrix}, \tag{5.4}$$

where ρ_{n-1} is the $n-1$-dimensional representation of $G_{\mathbb{Q}_p}$ which corresponds to D_{n-1}.

Repeating this argument successively for $D_{n-1}, D_{n-2}, \ldots, D_1$, we obtain the theorem. □

Corollary 5.2. *When $n = 2$, and π corresponds to f, we recover part (i) of Theorem 1.1 (at least when $k \geq 3$ is odd).*

Proof. Say $0 \leq v_p(\alpha) \leq v_p(\beta) \leq k-1$. It is well-known (cf. [B97]) that the Satake parameters at p satisfy the formulas $\alpha_1 = \beta \cdot p^{\frac{1-k}{2}}$ and $\alpha_2 = \alpha \cdot p^{\frac{k-1}{2}}$. In fact $L(s,\pi) = L(s + \frac{k-1}{2}, f)$ and $\rho_{\pi,p} = \rho_{f,\wp} \otimes \chi_{\mathrm{cyc},p}^{\frac{1-k}{2}}$, where $(k-1)/2 \in \mathbb{Z}$ if k is odd. In particular, the Hodge-Tate weights of $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ are $-\beta_2 = \frac{1-k}{2}$ and $-\beta_1 = \frac{k-1}{2}$, distinct integers if $k \geq 3$ is odd. By the p-ordinariness condition for π, we have that $v_p(\alpha_2) = \frac{1-k}{2}$ and $v_p(\alpha_1) = \frac{k-1}{2}$. By the theorem above, we obtain

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \lambda(\alpha_1/p^{(k-1)/2}) \cdot \chi_{\mathrm{cyc},p}^{\frac{k-1}{2}} & * \\ 0 & \lambda(\alpha_2/p^{(1-k)/2}) \cdot \chi_{\mathrm{cyc},p}^{\frac{1-k}{2}} \end{pmatrix}.$$

Twisting both sides by $\chi_{\mathrm{cyc},p}^{\frac{k-1}{2}}$, we recover part (i) of Theorem 1.1. □

5.1.1. *Variant, following* [CHT08]. Let π now be an RAESDC form of weight a as in Section 4.4 and let π_p denote the local p-adic automorphic representation associated to π. For any $i = 1, \ldots, n$, set $\beta'_{n+1-i} := a_i + n - i$, where a_i's are as in (4.2). We have that $\beta'_n > \beta'_{n-1} > \cdots > \beta'_1$, and the Hodge-Tate weights are $-\beta'_n < -\beta'_{n-1} < \cdots < -\beta'_1$.

Assume that π_p is in the unramified principal series, so $\pi_p = Q(\chi_1, \chi_2, \ldots, \chi_n)$, where χ_i's are unramified characters of $\mathbb{Q}_p^\times$. Set $\alpha'_i = \chi_i(\omega)p^{\frac{n-1}{2}}$. Let $t_p^{(j)}$ denote the eigenvalue of $T_p^{(j)}$ on $\pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$, where $T_p^{(j)}$ is the j-th Hecke operator as in [CHT08], and $\pi_p^{\mathrm{GL}_n(\mathbb{Z}_p)}$ is spanned by a $\mathrm{GL}_n(\mathbb{Z}_p)$-fixed vector, unique up to a constant. We would like to compute the right hand side in the display in part (1) of Theorem 4.5. By [CHT08, Cor. 3.1.2], in the spherical case, one has

$$\begin{aligned}(r_\ell(\iota^{-1}\pi_p)^\vee)(1-n)(\mathrm{Frob}_p^{-1}) &= \prod_i (X - \alpha'_i) \\ &= X^n - t_p^{(1)} X^{n-1} + \cdots + (-1)^j p^{\frac{j(j-1)}{2}} t_p^{(j)} X^{n-j} + \cdots + (-1)^n p^{\frac{n(n-1)}{2}} t_p^{(n)},\end{aligned}$$

where Frob_p^{-1} is geometric Frobenius. Let s_j denote the j-th elementary symmetric polynomial. Then from the equation above, for any $j = 1, \cdots, n$, we have $p^{\frac{j(j-1)}{2}} t_p^{(j)} = s_j(\alpha'_i) = p^{\frac{j(n-1)}{2}} s_j(\chi_i(p))$ and hence $t_p^{(j)} = s_j(\chi_i(p))p^{\frac{j(n-j)}{2}}$. In this setting we make:

Definition 7. Say that the automorphic representation π is *p-ordinary* if $\beta'_i + v_p(\alpha'_i) = 0$ for all $i = 1, \cdots, n$.

Again, if π is p-ordinary, then the $v_p(\alpha'_i)$ are integers.

Note that by strict compatibility, crystalline Frobenius has characteristic polynomial exactly that above. The following theorem follows in a manner identical to that used to prove Theorem 5.1.

Theorem 5.3 (Spherical case, variant). *Let π be a cuspidal automorphic form on $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$ of weight a, as is Section 4.4. Let $r_{p,\iota}(\pi)$ be the corresponding p-adic Galois representation, with Hodge Tate weights $-\beta'_{n+1-i} := a_i + n - i$, for $i = 1, \ldots, n$. Suppose π_p is in the principal series with Satake parameters $\alpha_1, \ldots, \alpha_n$, and set $\alpha'_i = \alpha_i p^{\frac{n-1}{2}}$. If π is p-ordinary, then $r_{p,\iota}(\pi)|_{G_{\mathbb{Q}_p}} \sim$*

$$\begin{pmatrix} \lambda(\frac{\alpha'_1}{p^{v_p(\alpha'_1)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta'_1} & * & \cdots & * \\ 0 & \lambda(\frac{\alpha'_2}{p^{v_p(\alpha'_2)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta'_2} & \cdots & * \\ 0 & 0 & \cdots & * \\ 0 & 0 & \lambda(\frac{\alpha'_{n-1}}{p^{v_p(\alpha'_{n-1})}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta'_{n-1}} & * \\ 0 & 0 & 0 & \lambda(\frac{\alpha'_n}{p^{v_p(\alpha'_n)}})) \cdot \chi_{\mathrm{cyc},p}^{-\beta'_n} \end{pmatrix}.$$

In particular, $r_{p,\iota}(\pi)|_{G_{\mathbb{Q}_p}}$ is ordinary.

The result above was also obtained by D. Geraghty in the course of proving modularity lifting theorems for GL_n (see Lem. 2.7.7 and Cor. 2.7.8 of [G10]). We thank T. Gee for pointing this out to us.

5.2. **Ramified principal series case.** Returning to the case where π is an automorphic form with infinitesimal character H, we assume now that the automorphic representation $\pi_p = Q(\chi_1, \ldots, \chi_n)$, where χ_i are possibly ramified characters of $\mathbb{Q}_p^\times$.

By the local Langlands correspondence, we think of the χ_i as characters of the Weil group $W_{\mathbb{Q}_p}$. In particular χ_i restricted to the inertia group have finite image. By strict compatibility, $\mathrm{WD}(\rho)|_{I_p} \simeq \bigoplus_i \chi_i|_{I_p}$. The characters $\chi_i|_{I_p}$ factor through $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}(\zeta_{p^m})/\mathbb{Q}_p^{\mathrm{nr}}) \simeq \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^m})/\mathbb{Q}_p)$ for some $m \geq 1$. Denote $\mathbb{Q}_p(\zeta_{p^m})$ by F. Observe that F is a finite abelian totally ramified extension of $\mathbb{Q}_p$. Let $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(E)$ be the corresponding (p,p)-representation. Note that $\rho_{\pi,p}|_{G_F}$ is crystalline.

Let D_n be the corresponding filtered module. Then $D_n = Ee_1 + \cdots + Ee_n$, where $g \in \mathrm{Gal}(F/\mathbb{Q}_p)$ acts by χ_i on e_i. A short computation shows that $\varphi(e_i) = \alpha_i^{-1} e_i$, where $\alpha_i = \chi_i(\omega_F)$, for ω_F a uniformizer of F.

Using Corollary 2.7, and following the proof of Theorem 5.1, we obtain:

Theorem 5.4 (Ramified principal series). *Say $\pi_p = Q(\chi_1, \ldots, \chi_n)$ is in the ramified principal series. If π is p-ordinary, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim$*

$$\begin{pmatrix} \chi_1 \cdot \lambda(\frac{\alpha_1}{p^{v_p(\alpha_1)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1} & * & \cdots & * \\ 0 & \chi_2 \cdot \lambda(\frac{\alpha_2}{p^{v_p(\alpha_2)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_2} & \cdots & * \\ 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \chi_n \cdot \lambda(\frac{\alpha_n}{p^{v_p(\alpha_n)}})) \cdot \chi_{\mathrm{cyc},p}^{-\beta_n} \end{pmatrix}.$$

In particular, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is quasi-ordinary.

6. Steinberg case

In this section we treat the case where the Weil-Deligne representation attached to π_p is a twist of the special representation $\mathrm{Sp}(n)$.

6.1. **Unramified twist of Steinberg.** We start with the case where the Weil-Deligne representation attached to π_p is of the form $\chi \otimes \mathrm{Sp}(n)$, where χ is an unramified character.

Let D be the filtered $(\varphi, N, \mathbb{Q}_p, E)$-module attached to $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$. Thus D is a vector space over E. Note $N^n = 0$ and $N^{n-1} \neq 0$ so that there is a basis $\{f_n, f_{n-1}, \ldots, f_1\}$ of D with $f_{i-1} := Nf_i$, for $1 < i \leq n$ and $Nf_1 = 0$, i.e.,

$$f_n \overset{N}{\mapsto} f_{n-1} \overset{N}{\mapsto} \cdots \overset{N}{\mapsto} f_1 \overset{N}{\mapsto} 0. \tag{6.1}$$

Say χ takes arithmetic Frobenius to α. Since $N\varphi = p\varphi N$, we may assume that $\varphi(f_i) = \alpha_i^{-1} f_i$, for all $i = 1, \ldots, n$, where $\alpha_i^{-1} = \frac{p^{i-1}}{\alpha}$. When $\alpha = 1$, D reduces to $\mathrm{Sp}(n)$ mentioned in Section 4.1.

For each $1 \leq i \leq n$, let D_i denote the subspace $\langle f_i, \cdots, f_1 \rangle$. Clearly, $\dim(D_i) = i$ and $D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_n$. We have:

Lemma 6.1. *For every integer $1 \leq r \leq n$, there is a unique N-submodule of D, of rank r, namely D_r.*

Proof. Let D' be a N-submodule of D, of rank r. Say the order of nilpotency of N on D' is i, i.e., $N^i = 0$ and $N^{i-1} \neq 0$. Then, $D' \subseteq \mathrm{Ker}(N^i)$. Observe that $\dim(\mathrm{Ker}(N^i)) = i$, because $\mathrm{Ker}(N^i)$ is generated by $\langle f_i, \cdots, f_1 \rangle$. Hence, we have $r \leq i$. Clearly, the order of nilpotency of N on D' is less than or equal to r. Hence $i = r$ and $D' = \mathrm{Ker}(N^r) = \langle f_r, \cdots, f_1 \rangle$. $\square$

Let $\beta_n > \cdots > \beta_1$ be the jumps in the Hodge filtration on D. We assume that the Hodge filtration is in general position with respect to the Newton filtration given by the D_i (cf. Assumption 4.6). An example of such a filtration is

$$\langle f_n \rangle \subsetneq \langle f_n, f_{n-1} \rangle \subsetneq \cdots \subsetneq \langle f_n, f_{n-1}, \cdots, f_2 \rangle \subsetneq \langle f_n, f_{n-1}, \cdots, f_1 \rangle. \tag{6.2}$$

The following elementary lemma plays an important role in later proofs.

Lemma 6.2. *Let m be a natural number. Let $\{a_i\}_{i=1}^n$ be an increasing (resp., decreasing) sequence of integers such that $|a_{i+1} - a_i| = m$. Let $\{b_i\}_{i=1}^n$ be another increasing (resp., decreasing) sequence of integers, such that $|b_{i+1} - b_i| \geq m$. Assume that $\sum_i a_i = \sum_i b_i$. If $a_n = b_n$ or $a_1 = b_1$, then $a_i = b_i, \forall\ i$.*

Proof. Let us prove the lemma when $a_n = b_n$ and the a_i are increasing. The proof in the other cases is similar. We have:

$$m(n-1+n-2+\cdots+1) \leq \sum_{i=1}^{n}(b_n - b_i) = \sum_{i=1}^{n}(a_n - a_i) = m(n-1+n-2+\cdots+1) \tag{6.3}$$

The first equality in the above expression follows from $a_n = b_n$. From the equation above, we see that $b_n - b_i = a_n - a_i$, for every $1 \leq i \leq n$. Since $a_n = b_n$, we have that $a_i = b_i$, for every $1 \leq i \leq n$. $\square$

By Lemma 6.1, the D_i are the only (φ, N)-submodules of D. The following proposition shows that if two 'consecutive' submodules D_i and D_{i+1} are admissible then all the D_i are admissible.

Proposition 6.3. *Suppose there exists an integer $1 \leq i \leq n$ such that both D_i and D_{i+1} are admissible. Then each D_r, for $1 \leq r \leq n$, is admissible. Moreover, the β_i are consecutive integers.*

Proof. Since D_i and D_{i+1} are admissible, we have the following equalities:

$$\begin{aligned} \beta_1 + \beta_2 + \cdots + \beta_i &= -\sum_{r=1}^{i} v_p(\alpha_r), \\ \beta_1 + \beta_2 + \cdots + \beta_{i+1} &= -\sum_{r=1}^{i+1} v_p(\alpha_r). \end{aligned} \tag{6.4}$$

From these expressions, we have that

$$-v_p(\alpha_{i+1}) = \beta_{i+1}. \tag{6.5}$$

Define $a_r = -v_p(\alpha_r)$ and $b_r = \beta_r$, for $1 \leq r \leq n$. Hence, we have

$$\begin{aligned} a_n > \cdots > a_{i+2} > a_{i+1} > a_i > \cdots > a_1, \\ b_n > \cdots > b_{i+2} > b_{i+1} > b_i > \cdots > b_1. \end{aligned} \tag{6.6}$$

By (6.5), $a_{i+1} = b_{i+1}$. By Lemma 6.2 and by (6.4), we have that $a_r = b_r$, for all $1 \leq r \leq i+1$.

Since D_n is admissible, we have

$$t_H(D_n) = \sum_{r=1}^{n} \beta_r = -\sum_{r=1}^{n} v_p(\alpha_r) = t_N(D_n). \tag{6.7}$$

From (6.4) and (6.7), we have that

$$\sum_{r=i+1}^{n} \beta_r = -\sum_{r=i+1}^{n} v_p(\alpha_r). \tag{6.8}$$

Again, by (6.6) and Lemma 6.2, we have that $a_r = b_r$, for all $i+1 \leq r \leq n$. Hence $\beta_r = -v_p(\alpha_r)$, for all $1 \leq r \leq n$. This shows that all the other D_i's are admissible. Also, the β_i are consecutive integers since the $v_p(\alpha_i)$ are consecutive integers. □

Corollary 6.4. *Keeping the notation as above, admissibility of D_1 or D_{n-1} implies the admissibility of all other D_i.*

Theorem 6.5. *Assume that the Hodge filtration on D is in general position with respect to the D_i (cf. Assumption 4.6). Then the crystal D is either irreducible or reducible, in which case each D_i, for $1 \leq i \leq n$ is admissible.*

Proof. If D is irreducible, then we are done. If not, there exists an i, such that D_i is admissible. If D_{i-1} or D_{i+1} is admissible, then by Proposition 6.3, all the D_r are admissible. So, it is enough to consider the case where neither D_{i-1} nor D_{i+1}

is admissible (and D_i is admissible). We have:

$$\beta_1 + \beta_2 + \cdots + \beta_{i-1} < -\sum_{r=1}^{r=i-1} v_p(\alpha_r), \tag{6.9a}$$

$$\beta_1 + \beta_2 + \cdots + \beta_i = -\sum_{r=1}^{r=i} v_p(\alpha_r), \tag{6.9b}$$

$$\beta_1 + \beta_2 + \cdots + \beta_{i+1} < -\sum_{r=1}^{r=i+1} v_p(\alpha_r). \tag{6.9c}$$

Subtracting (6.9b) from (6.9a), we get $-\beta_i < v_p(\alpha_i)$. Subtracting (6.9b) from (6.9c), we get $\beta_{i+1} < -v_p(\alpha_{i+1}) = -v_p(\alpha_i) + 1$. Adding these inequalities, we obtain $\beta_{i+1} - \beta_i < 1$. But this is a contradiction, since $\beta_{i+1} > \beta_i$. This proves the theorem. □

Definition 8. Say π is *p-ordinary* if $\beta_1 + v_p(\alpha) = 0$.

Note that if π is p-ordinary, then D_1 is admissible, so the flag $D_1 \subset D_2 \subset \cdots \subset D_n$ is an admissible flag by Theorem 6.5 (an easy check shows that if π is p-ordinary then Assumption 4.6 holds automatically).

Applying the above discussion to the local Galois representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$, we obtain:

Theorem 6.6 (Unramified twist of Steinberg)**.** *Say π is a cuspidal automorphic form with infinitesimal character given by the integers $-\beta_1 > \cdots > -\beta_n$. Suppose that π_p is an unramified twist of the Steinberg representation, i.e.,* $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \chi \otimes \mathrm{Sp}(n)$*, where χ is the unramified character mapping arithmetic Frobenius to α. If π is ordinary at p (i.e., $v_p(\alpha) = -\beta_1$), then the β_i are necessarily consecutive integers and $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim$*

$$\begin{pmatrix} \lambda(\frac{\alpha}{p^{v_p(\alpha)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1} & * & \cdots & * \\ 0 & \lambda(\frac{\alpha}{p^{v_p(\alpha)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1 - 1} & \cdots & * \\ 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \lambda(\frac{\alpha}{p^{v_p(\alpha)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-(n-1)} \end{pmatrix},$$

where $\lambda(\frac{\alpha}{p^{v_p(\alpha)}})$ is an unramified character taking arithmetic Frobenius to $\frac{\alpha}{p^{v_p(\alpha)}}$, and in particular, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is ordinary. If π is not p-ordinary, and Assumption 4.6 holds, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.

Proof. By strict compatibility, D is the filtered $(\varphi, N, \mathbb{Q}_p, E)$-module attached to $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$. If π is p-ordinary, then we are done and the characters on the diagonal are determined by corollary 2.7.

If π is not p-ordinary, we claim that D is irreducible. Indeed, if D is reducible, then by Theorem 6.5, all D_i, and in particular D_1, are admissible, so π is p-ordinary. □

6.2. **Ramified twist of Steinberg.**

Theorem 6.7 (Ramified twist of Steinberg)**.** *Let the notation and hypotheses be as in Theorem 6.6, except that this time assume that* $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \chi \otimes \mathrm{Sp}(n)$*, where χ is an arbitrary, possibly ramified, character. Write $\chi = \chi_0 \cdot \chi'$ where χ_0 is the* *ramified part of χ, and χ' is an unramified character taking arithmetic Frobenius to α. If π is p-ordinary ($\beta_1 = -v_p(\alpha)$), then the β_i are consecutive integers and $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim$*

$$\begin{pmatrix} \chi_0 \cdot \lambda(\frac{\alpha}{p^{v_p(\alpha)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1} & * & \cdots & * \\ 0 & \chi_0 \cdot \lambda(\frac{\alpha}{p^{v_p(\alpha)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-1} & \cdots & * \\ 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \chi_0 \cdot \lambda(\frac{\alpha}{p^{v_p(\alpha)}}) \cdot \chi_{\mathrm{cyc},p}^{-\beta_1-(n-1)} \end{pmatrix}.$$

If π is not p-ordinary, and Assumption 4.6 holds, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.

Proof. Let F be a totally ramified abelian (cyclotomic) extension of $\mathbb{Q}_p$ such that $\chi_0|_{I_F} = 1$. Then the reducibility of $\rho_{\pi,p}|_{G_F}$ over F can be shown exactly as in Theorem 6.6, and the theorem over $\mathbb{Q}_p$ follows using the descent data of the underlying filtered module. If π is not p-ordinary, then by arguments similar to those used in proving Theorem 6.6, $\rho_{\pi,p}|_{G_F}$ is irreducible, so that $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is also irreducible. □

7. Supercuspidal ⊗ Steinberg

We now turn to the case where the Weil-Deligne representation attached to π_p is indecomposable. Thus we assume that $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}})$ is Frobenius semisimple and is of the form $\tau \otimes \mathrm{Sp}(n)$, where τ is an irreducible m-dimensional representation corresponding to a supercuspidal representation of GL_m, for $m \geq 1$, and $\mathrm{Sp}(n)$ for $n \geq 1$ denotes the usual special representation.

We first classify the (φ, N, F, E)-submodules of D, the crystal attached to the local representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$, where $\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) = \tau \otimes \mathrm{Sp}(n)$, for $m \geq 1$ and $n \geq 1$. This classification will be used in the last subsection to study the structure of $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$, taking the filtration on D into account.

Recall that there is a equivalence of categories between (φ, N)-modules with coefficients and descent data, and Weil-Deligne representations [BS07, Prop. 4.1]. Write D_τ, respectively $D_{\mathrm{Sp}(n)}$, for the (φ, N)-modules corresponding to τ, respectively $\mathrm{Sp}(n)$, etc. The main result of the first few sections is:

Theorem 7.1. *All the (φ, N, F, E)-submodules of $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ are of the form $D_\tau \otimes D_{\mathrm{Sp}(r)}$, for some $1 \leq r \leq n$.*

We prove the theorem in stages, since this is how the theorem was discovered, and it also allows one to appreciate the general argument. However the impatient reader may turn straight to the Section 7.3 where the general case is treated. Note the case that $m = 1$ was treated in the previous section (twist of Steinberg), and the case $n = 1$ is vacuously true. The next simplest case is when $m = 2$ and $n = 2$, and we start with this case in the next section.

The following lemma will be useful in our analysis throughout.

Lemma 7.2. *The theory of Jordan canonical forms can be extended to nilpotent operators on free of finite rank $(F_0 \otimes E)$-modules, and we call the number of blocks in the Jordan decomposition of the monodromy operator N as the 'index' of N.*

Proof. One simply extends the usual theory of Jordan canonical forms on each projection under (2.6) to modules over $F_0 \otimes E$-modules. □

7.1. $m = 2$ **and** $n = 2$. We start with an exhaustive study of the case when τ corresponds to a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ whose Weil-Deligne representation is the 2-dimensional representation obtained by inducing a character from a quadratic extension K of $\mathbb{Q}_p$, and $n = 2$. The methods used in this case will be used to deal with more general cases in subsequent subsections.

We also assume, for simplicity, that the quadratic extension of $\mathbb{Q}_p$ mentioned above is $K = \mathbb{Q}_{p^2}$, the unramified quadratic extension of $\mathbb{Q}_p$. By abusing language a bit we shall say that τ corresponds to an unramified supercuspidal representation. The argument in the ramified supercuspidal case (i.e., $K/\mathbb{Q}_p$ ramified quadratic) was also worked out in detail, but is excluded here for the sake of brevity.

Thus we assume that there is a character χ of W_{p^2}, the Weil group of $\mathbb{Q}_{p^2}$, which does not extend to W_p, the Weil group of $\mathbb{Q}_p$ (i.e., $\chi \neq \chi^\sigma$ on W_{p^2}, equivalently $\chi \neq \chi^\sigma$ on W_{p^2}, where σ is the non-trivial automorphism of $\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$) such that

$$\tau|_{I_p} \simeq \mathrm{Ind}_{W_{p^2}}^{W_p} \chi|_{I_p} \simeq \chi|_{I_p} \oplus \chi^\sigma|_{I_p}.$$

The corresponding (φ, N)-module in this case can be written down quite explicitly and we refer the reader to [GM09, §3.3] for details. Briefly there is an abelian field F (depending on χ), generated by σ (a lift to F of the automorphism σ above) and elements g of the inertia group $I(F/K)$ of $\mathrm{Gal}(F/K)$, and an element $t \in E$ such that D_τ is free of rank 2 with basis e_1, e_2, and satisfies:

$$D_\tau = D_{\mathrm{unr-sc}}[a:b] \quad = \quad \begin{cases} \varphi(e_1) & = & \frac{1}{\sqrt{t}} e_1, \\ \varphi(e_2) & = & \frac{1}{\sqrt{t}} e_2, \\ N & = & 0, \\ \sigma(e_1) & = & e_2, \\ \sigma(e_2) & = & e_1, \\ g(e_1) & = & (1 \otimes \chi(g)) e_1, \quad g \in I(F/K) \\ g(e_2) & = & (1 \otimes \chi^\sigma(g)) e_2. \quad g \in I(F/K). \end{cases}$$

We do not write down the Hodge filtration since we do not need it here.

For the second factor, we recall that the module $D_{\mathrm{Sp}(2)}$ has a basis f_1, f_2 with properties described at the start of Section 6.1. Then $D = D_\tau \otimes D_{\mathrm{Sp}(2)}$ has a basis of 4 vectors $\{v_i\}_{i=1}^4$ defined as follows

$v_1 = e_1 \otimes f_2$	$v_2 = e_1 \otimes f_1$
$v_3 = e_2 \otimes f_2$	$v_4 = e_2 \otimes f_1$

Observe that on D, the monodromy operator $N = 1 \otimes N + N \otimes 1 = 1 \otimes N$ has kernel of rank 2, and this space is generated by the vectors v_2 and v_4.

For notational simplicity, write z for $1 \otimes z$, if $z \in E$ and note that for $z = \sum_i z_i \otimes e_i \in F_0 \otimes E$, $\sigma(z) = \sum_i \sigma(z_i) \otimes e_i$. Then the induced action on D, the tensor product of the two (φ, N)-modules D_τ and $D_{\mathrm{Sp}(2)}$, is summarized in the following table:

	$v_1 = e_1 \otimes f_2$	$v_2 = e_1 \otimes f_1$	$v_3 = e_2 \otimes f_2$	$v_4 = e_2 \otimes f_1$
φ	$\frac{p}{\sqrt{t}} v_1$	$\frac{1}{\sqrt{t}} v_2$	$\frac{p}{\sqrt{t}} v_3$	$\frac{1}{\sqrt{t}} v_4$
N	v_2	0	v_4	0
σ	v_3	v_4	v_1	v_2
g	$\chi(g) v_1$	$\chi(g) v_2$	$\chi^\sigma(g) v_3$	$\chi^\sigma(g) v_4$

7.1.1. *Rank* 1 *submodules of* D.

Lemma 7.3. *There are no rank* 1 (φ, N, F, E)*-submodules of* D.

Proof. Let $\langle v\rangle$ be a free module of rank-1 admissible (φ, N, F, E)-submodule of D. Write $v = av_1 + bv_2 + cv_3 + dv_4$ where $a, b, c, d \in (F_0 \otimes_{\mathbb{Q}_p} E)$. Since the rank of $\langle v\rangle$ is 1, we have Nv is zero. From the above table, it is easy to see that v has to be equal to $bv_2 + dv_4$, for some b, d. Assume that $\varphi(v) = \alpha_\varphi v$. Since φ is bijective, α_φ is a unit. We compute:

$$\alpha_\varphi(bv_2 + dv_4) = \alpha_\varphi v \overset{(1)}{=} \varphi v \overset{(2)}{=} \sigma(b)\frac{1}{\sqrt{t}}v_2 + \sigma(d)\frac{1}{\sqrt{t}}v_4$$

(equality (1) follows from the fact that $\langle v\rangle$ is φ-stable and equality (2) from the table). By comparing coefficients, we have

$$\alpha_\varphi\sqrt{t}b = \sigma(b), \quad \alpha_\varphi\sqrt{t}d = \sigma(d). \tag{7.1}$$

Lemma 7.4. *Suppose x is a non-zero and a non-unit element of $F_0 \otimes E$. Then $\sigma(x) \neq cx$, for every $c \in F_0 \otimes E$.*

From the above relations for b (resp., d) and by Lemma 7.4, we conclude that b (resp., d) is either zero or a unit.

We know that σ also acts on v. Suppose that $\sigma(v) = c_\sigma v$, for some $c_\sigma \in (F_0 \otimes E)$, with $c_\sigma^{2m} = 1$ (this m is the m of [GM09, §3.3], and not the m of this paper which is presently 2). Then

$$c_\sigma(bv_2 + dv_4) = c_\sigma v \overset{(1)}{=} \sigma v \overset{(2)}{=} \sigma(b)v_4 + \sigma(d)v_2$$

(equality (1) follows from the fact that $\langle v\rangle$ is σ-stable and equality (2) from the table). By comparing coefficients, we have

$$c_\sigma b = \sigma(d), \quad c_\sigma d = \sigma(b). \tag{7.2}$$

From these relations, we can conclude that b is zero if and only if d is zero. Since v is non-zero, we have that both b and d are both units.

Now, we shall use the fact that $\langle v\rangle$ is $I(F/K)$-invariant. For every $g \in I(F/K)$, we have

$$c_g(bv_2 + dv_4) = c_g v \overset{(1)}{=} g \cdot v \overset{(2)}{=} \chi(g)bv_2 + \chi^\sigma(g)dv_4$$

(equality (1) follows from the fact that $\langle v\rangle$ is $I(F/K)$-stable and equality (2) from the table). Now, by comparing coefficients, and using the fact that b and d are units, we have

$$c_g = \chi(g), \quad c_g = \chi^\sigma(g). \tag{7.3}$$

Hence $\chi(g) = \chi^\sigma(g)$, for every $g \in I(F/K)$. This is a contradiction. Hence, there are no 1-dimensional (φ, N, F, E)-submodules of D. □

Remark 5. The above argument shows that there are no $(\varphi, \mathrm{Gal}(F/\mathbb{Q}_p))$-stable submodules of rank 1 of either $\langle v_2, v_4\rangle$ or, as one can show similarly, $\langle v_1, v_3\rangle$. This observation will be used later. There is also a simpler proof of this Lemma which avoids working with the above explicit manipulations which works for general m and n (cf. the proof of Lemma 7.11).

7.1.2. *Rank 2 submodules of D.*

Lemma 7.5. *The only rank 2 (φ, N, F, E)-submodule of D is $\langle v_2, v_4\rangle = D_\tau \otimes D_{\mathrm{Sp}(1)}$.*

Proof. Let D' be a filtered (φ, N, F, E)-module of rank 2. Suppose the index of $N = 1$, i.e., there exists a basis, say $\langle w_1, N(w_1)\rangle$, of D' such that $N^2(w_1)$ is zero. It is easy to see that $\langle N(w_1)\rangle$ is a rank 1 (φ, N, F, E)-submodule of D. But, we know that there are no such submodules of D by the previous Lemma. Therefore, the index of N is not equal to 1.

Suppose the index of $N = 2$, i.e., there exists a basis, say $\langle w_1, w_2\rangle$, of D' such that $N(w_1)$ and $N(w_2)$ are both zero. We know that the kernel of N is $\langle v_2, v_4\rangle$. Therefore, we have that $\langle w_1, w_2\rangle \subseteq \langle v_2, v_4\rangle$. But both are free modules of the same rank, hence equality holds. □

7.1.3. *Rank 3 submodules of D.*

Lemma 7.6. *There are no rank 3 (φ, N, F, E)-submodules of D.*

Proof. Suppose D' is a (φ, N, F, E)-submodule of D of rank 3. Suppose the index of nilpotency of N is 3, i.e., there are 3 linearly independent vectors of D', such that N acts trivially on these vectors. But, this cannot happen, because the kernel of N has rank 2.

Suppose the index of nilpotency of N is 2, that is, there exists a basis, say $w_1, w_2, N(w_2)$, of D' such that $N(w_1)$, $N^2(w_2)$ are both zero. Clearly $\langle w_1, N(w_2)\rangle = \langle v_2, v_4\rangle$. Thus there is a vector $w_2' = a_1v_1 + a_3v_3$ such that $D' = \langle w_2', v_2, v_4\rangle$. But $\langle w_2'\rangle$ is stable by φ. Indeed φ acts by a scalar on w_2' since, *a priori*, $\varphi w_2'$ is a linear combination of w_2', v_2 and v_4, but the v_2 and v_4 components do not appear since φ preserves the space $\langle v_1, v_3\rangle$. Similarly $\langle w_2'\rangle$ is also $\mathrm{Gal}(F/\mathbb{Q}_p)$-stable. But then $\langle w_2'\rangle \subset \langle v_1, v_3\rangle$ is a rank 1 module stable by φ and $\mathrm{Gal}(F/\mathbb{Q}_p)$, which is not possible (cf. Remark 5).

Suppose the index of nilpotency of N is 1. But this case does not arise since the index of nilpotency of N on D is 2. □

7.1.4. *Proof of Theorem 7.1 when $m = 2$ and $n = 2$.* This follows immediately from Lemmas 7.3, 7.5, and 7.6, when τ is an unramified supercuspidal representation of dimension $m = 2$, and $n = 2$. As we remarked earlier, the case when τ is a ramified supercuspidal representation is proved in a very similar manner using the notation in [GM09, §3.4] (the only difference in the computations are the role of σ is now played by the automorphism ι there).

We mention some immediate corollaries of Theorem 7.1 in the present case. Let π be an automorphic form on $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ with infinitesimal character consisting of distinct integers $-\beta_4 < \cdots < -\beta_1$. Let $\rho = \rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ be the corresponding (p,p)-representation. Suppose that $\mathrm{WD}(\rho) \sim \tau \otimes \mathrm{Sp}(2)$, where τ is a supercuspidal representation of dimension 2, as above. Let $D = D(\rho)$ be the corresponding admissible filtered (φ, N, F, E)-module. Note that $\beta_4 > \beta_3 > \beta_2 > \beta_1$ are also the drops in the Hodge filtration on D_F.

Corollary 7.7. *With notation as above, the crystal D is irreducible if and only if $\langle v_2, v_4\rangle$ is not an admissible (φ, N, F, E)-submodule of D.*

Proof. We know that D does not have any rank 1 or rank 3 (φ, N, F, E)-submodules. By Lemma 7.5, there exists a unique rank 2 (φ, N, F, E)-submodule of D, which is $D' = \langle v_2, v_4\rangle$. Hence the admissibility of D' is equivalent to the reducibility of D. □

Corollary 7.8. *The crystal D is irreducible if no two of the four β_i add up to $-v_p(t)$. In particular ρ is irreducible in this case.*

Proof. By Corollary 7.7, D is irreducible if and only if $D' = \langle v_2, v_4\rangle$ is not an admissible submodule. The submodule D' is not admissible if no two of the four β_i add up to $t_N(D')$ which is $-v_p(t)$, from the table above. □

7.2. **τ unramified supercuspidal of dim $m \geq 2$ and $n \geq 2$.** We now prove Theorem 7.1 for general m and n, assuming τ is an 'unramified supercuspidal representation'. Let us explain this terminology. We assume that τ is an induced representation of dimension m, i.e., $\tau \simeq \mathrm{Ind}_{W_K}^{W_p}\chi$, where K is a p-adic field such that $[K:\mathbb{Q}_p] = m$, and χ is a character of W_K. This is known to always hold if $(p,m) = 1$ or $p > m$. For simplicity, we shall assume that K is the unique unramified extension of $\mathbb{Q}_p$, namely $\mathbb{Q}_{p^m}$. We refer to τ in this case as an unramified supercuspidal representation.

Following the methods of [GM09, §3.3], we first explicitly write down the crystal $D = D_\tau$ whose underlying Weil-Deligne representation is an unramified supercuspidal representation τ of dimension m. This is done in the next few subsections. The arguments are similar to those given in [GM09, §3.3], with some minor modifications. We outline the steps now.

Let σ be the generator of $\mathrm{Gal}(\mathbb{Q}_{p^m}/\mathbb{Q}_p)$ and let I_{p^m} denote the inertia subgroup of $\mathbb{Q}_{p^m}$. Then

$$\tau|_{I_p = I_{p^m}} \simeq (\mathrm{Ind}_{W_{p^m}}^{W_p}\chi)|_{I_{p^m}} \simeq \oplus_{i=1}^m \chi^{\sigma^i}|_{I_{p^m}}.$$

Since τ is irreducible, by Mackey's criterion, we have that $\chi \neq \chi^{\sigma^i}$, for all i, on W_{p^m} and also on I_{p^m}. Moreover, we have that $\chi^{\sigma^j} \neq \chi^{\sigma^i}$, for any $i \neq j$.

7.2.1. *Description of* $\mathrm{Gal}(F/\mathbb{Q}_p)$. First, we need to construct a finite extension F over K such that $\tau|_{I_F}$ is trivial and which simultaneously has the property that $F/\mathbb{Q}_p$ is Galois. The construction of an explicit such field F is given in [GM09, §3.3.1], in the case $K = \mathbb{Q}_{p^2}$ using Lubin-Tate theory. The structure of $\mathrm{Gal}(F/\mathbb{Q}_p)$ is described in the same place. The case $K = \mathbb{Q}_{p^m}$ may be treated in a very similar manner. Write d for what was called m in [GM09, §3.3.1], since m already has meaning here. Then F may be chosen such that $\mathrm{Gal}(F/\mathbb{Q}_p)$ is the semi-direct product of a cyclic group $\langle\sigma\rangle$ with $\sigma^{md} = 1$, with the product of the cyclic groups $\Delta = \langle\delta\rangle$, with $\delta^{p^m-1} = 1$, and $\Gamma = \prod_{i=1}^m \langle\gamma_i\rangle$, with each $\gamma_i^{p^n} = 1$, for some n. Moreover, the maximal unramified extension F_0 of F is $F_0 = \mathbb{Q}_p^{md}$ and $\mathrm{Gal}(F_0/\mathbb{Q}_p) = \langle\tilde{\sigma}\rangle = \mathbb{Z}/md$, such that $\tilde{\sigma}|_K$ is the generator of $\mathrm{Gal}(K/\mathbb{Q}_p)$. By abuse of notation, we denote $\tilde{\sigma}$ by σ itself.

7.2.2. *Description of the Galois action.* Recall D is a free $F_0 \otimes_{\mathbb{Q}_p} E$-module of rank m. Let $D_i = D \otimes_{F_0\otimes E, \sigma^i} E$, for $i = 0, 1, \cdots, md-1$, be the component of D corresponding to σ^i. Each D_i is a Weil-Deligne representation with an action of W_p.

By the definition of the Weil-Deligne representation, the action of I_p matches with the action of the inertia subgroup of $\mathrm{Gal}(F/\mathbb{Q}_p)$, namely $\Delta\times\Gamma$. The restriction of χ to I_p can be written as $\chi|_{I_p} = \omega_m^r \prod_{i=1}^m \chi_i$, where ω_m is the fundamental character of level m, $r \geq 1$, and χ_i is the character of Γ which takes γ_i to a p^n-th root of unity ζ_i, for $i = 1, \ldots, m$.

We see that each D_i has a basis $v_{i,1}, v_{i,2}, \ldots, v_{i,m}$ such that if $i \equiv k \pmod m$, for some $0 \leq k \leq m-1$, then

$$g \cdot v_{i,j} = \chi^{\sigma^{j-k-1}}(g) v_{i,j}, \tag{7.4}$$

for all $j = 1, \cdots, m$.

Since σ takes D_i to D_{i+1}, using (7.4), we may assume that

$$\sigma(v_{i,j}) = v_{i+1,j},$$

for all i, j.

7.2.3. *Description of action of φ.* The operator φ acts in a cyclic manner as well, taking D_i to D_{i+1}. Since φ commutes with the action of inertia, we see that

$$\varphi(v_{i,j}) = c_j v_{i+1,j+1},$$

for some c_j, for all $1 \leq j \leq m$. Observe that c_j's does not depend on i, since φ commutes with σ.

For all $1 \leq k \leq m$, define, $\frac{1}{t_k} := \prod_{j=1}^{k} c_j$ and $\frac{1}{t_0} = 1$. Replace the extension E with a finite extension, again denoted by E, so that it contains all m-th roots of all c_j. Let $\sqrt[m]{c_j}$ denote a particular m-th root, for each j.

We now write down a basis of D, say $\{e_i\}_{i=1}^m$, such that $\varphi(e_i) = \frac{1}{\sqrt[m]{t_m}} e_i$. First, we shall define e_1 and the other e_i's are defined by $e_i = \sigma^{i-1} e_1$. The vector e_1 is given by

$$e_1 = \sum_{\substack{j=0 \\ j \equiv j_0 \pmod m,\ 0 \leq j_0 \leq m-1}}^{md-1} \frac{(t_m)^{\frac{j_0}{m}}}{t_{j_0}} v_{j,j+1}.$$

Here we use the obvious convention that if j is such that $j \equiv j_0 \pmod m$, with $1 \leq j_0 \leq m$, then $v_{i,j} := v_{i,j_0}$. A small computation shows that $\varphi(e_1) = \frac{1}{\sqrt[m]{t_m}} e_1$. Since φ commutes with σ, we have $\varphi(e_i) = \frac{1}{\sqrt[m]{t_m}} e_i$, for all $1 \leq i \leq m$. We obtain that D_τ is a free rank m module over $F_0 \otimes E$ with basis e_i, $i = 1, \ldots, m$ such that

$$D_\tau = \begin{cases} \varphi(e_i) &= \frac{1}{\sqrt[m]{t_m}} e_i, \\ N(e_i) &= 0, \\ \sigma(e_i) &= e_{i+1}, \\ g(e_i) &= (1 \otimes \chi^{\sigma^{i-1}}(g))(e_i), \quad g \in I(F/K), \end{cases} \tag{7.5}$$

for all $1 \leq i \leq m$.

When $m = 2$, this (φ, N)-module is exactly the one given in [GM09, §3.3] (though the e_i used here differ by a scalar from the e_i used there).

7.2.4. *Description of the filtration.* For the sake of completeness, let us make some brief comments about the filtration on D_τ, even though we shall not need to use the filtration in this paper.

Let D be an arbitrary filtered (φ, N, F, E)-module and write $D_F = F \otimes D$, where the tensor product is taken over $F_0 \otimes E$. It is known that every Galois stable line $(F \otimes_{\mathbb{Q}_p} E) \cdot v$ in D_F is generated by a Galois stable vector v (cf. [GM09, Lemma

3.1]). The proof uses the fact $\mathrm{H}^1(\mathrm{Gal}(F/\mathbb{Q}_p), (F \otimes_{\mathbb{Q}_p} E)^\times) = 0$. In fact, for any $n \geq 1$:

$$\begin{aligned}\mathrm{H}^1(\mathrm{Gal}(F/\mathbb{Q}_p), \mathrm{GL}_n(F \otimes_{\mathbb{Q}_p} E)) &= \mathrm{H}^1(\mathrm{Gal}(F/\mathbb{Q}_p), \prod_{F \hookrightarrow E} \mathrm{GL}_n(E)),\\ &\underset{(1)}{=} \mathrm{H}^1(\mathrm{Gal}(F/\mathbb{Q}_p), \mathrm{Ind}_{\{e\}}^{\mathrm{Gal}(F/\mathbb{Q}_p)} \mathrm{GL}_n(E)),\\ &\underset{(2)}{=} \mathrm{H}^1(\{e\}, \mathrm{GL}_n(E)) = \{0\},\end{aligned}$$

(where (1) follows from the permutation action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ on $\prod_{F \hookrightarrow E} \mathrm{GL}_n(E)$ and (2) follows from Shapiro's lemma). Using this vanishing, we can prove the following general fact.

Lemma 7.9. *Every* $\mathrm{Gal}(F/\mathbb{Q}_p)$*-stable submodule of* D_F *has a basis consisting of Galois invariant vectors.*

Proof. Let D' be a Galois stable submodule of D_F. Savitt has observed that any $F \otimes_{\mathbb{Q}_p} E$-submodule of a filtered module with descent data (i.e., $\mathrm{Gal}(F/\mathbb{Q}_p)$-action) has to be free [S05, Lemma 2.1]. Hence D' is a free of finite rank, say, r. If $\{v_1, v_2, \cdots, v_r\}$ is a basis of D', then for every $g \in \mathrm{Gal}(F/\mathbb{Q}_p)$, we have $g \cdot (v_1, v_2, \cdots, v_n)^t = c_g(v_1, v_2, \cdots, v_r)^t$, for some $c_g \in \mathrm{GL}_n(F \otimes_{\mathbb{Q}_p} E)$. Moreover, c_g is 1-cocycle, i.e., $c_g \in Z^1(\mathrm{Gal}(F/\mathbb{Q}_p), \mathrm{GL}_n(F \otimes_{\mathbb{Q}_p} E))$. By the vanishing result above, c_g is coboundary, hence $c_g = cg(c)^{-1}$, for some $c \in \mathrm{GL}_n(F \otimes_{\mathbb{Q}_p} E)$. Replacing the basis $(v_1, v_2, \cdots, v_r)^t$ with $c \cdot (v_1, v_2, \cdots, v_r)^t$, we may assume that $c_g = 1$ and that each vector in $\{v_1, v_2, \ldots, v_r\}$ is invariant under $\mathrm{Gal}(F/\mathbb{Q}_p)$. □

In particular each step $\mathrm{Fil}^i(D_F)$ in the filtration on D_F is spanned by $\mathrm{Gal}(F/\mathbb{Q}_p)$-invariant vectors. In [GM09, §3.3.4] the Hodge filtration on D_F was written down explicitly when $D = D_\tau$ and τ is a 2-dimensional unramified supercuspidal representation. Presumably this can be done also when τ has dimension $m \geq 2$, but we refrain from pursuing this here.

7.2.5. *Proof of Theorem 7.1.* We now turn to the proof of Theorem 7.1, when τ is an unramified supercuspidal representation of dimension m.

Now D_τ has basis $e_1, \ldots, e_m$ with properties described above. Recall that $D_{\mathrm{Sp}(n)}$ has basis $f_n, f_{n-1}, \ldots, f_1$, with properties described at the start of Section 6.1. Then $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ has a basis of mn vectors $\{v_i\}_{i=1}^{mn}$ defined by the table:

$v_1 = e_1 \otimes f_n$	$v_2 = e_1 \otimes f_{n-1}$	$\cdots$	$v_n = e_1 \otimes f_1$
$v_{n+1} = e_2 \otimes f_n$	$v_{n+2} = e_2 \otimes f_{n-1}$	$\cdots$	$v_{2n} = e_2 \otimes f_1$
$\vdots$	$\vdots$	$\cdots$	$\vdots$
$v_{in+1} = e_{i+1} \otimes f_n$	$v_{in+2} = e_{i+1} \otimes f_{n-1}$	$\cdots$	$v_{in} = e_{i+1} \otimes f_1$
$\vdots$	$\vdots$	$\cdots$	$\vdots$
$v_{(m-1)n+1} = e_m \otimes f_n$	$v_{(m-1)n+2} = e_m \otimes f_{n-1}$	$\cdots$	$v_{mn} = e_m \otimes f_1$

The action of φ, N, $\sigma \in \mathrm{Gal}(F/\mathbb{Q}_p)$ and $g \in I(F/K)$, on the vectors in the table above can be written down explicitly, as in the table in Section 7.1. We only note that φ acts by a scalar, N shifts one column to the right, σ shifts one row down, and each g acts by a scalar. In particular, the span of the vectors in each *column* in the table above is stable under the actions of φ and $\mathrm{Gal}(F/\mathbb{Q}_p)$. The following lemma concerning this submodule is useful (cf. Remark 5 for the case $m = 2$ and $n = 2$).

Lemma 7.10. *Let $1 \leq i \leq n$. There are no rank r $(\varphi, \mathrm{Gal}(F/\mathbb{Q}_p))$-sub-modules of*
$$\langle v_i, v_{n+i}, \ldots, v_{(m-1)n+i}\rangle \subset D_\tau \otimes D_{\mathrm{Sp(n)}},$$
for $1 \leq r \leq m-1$.

Proof. Suppose there exists such a (free) module D' with basis $w_1, w_2, \ldots, w_r$. Project D' to a fixed component say j-th under (2.6), and write $w_1^j, \ldots, w_r^j$ for the corresponding basis elements of the projection. Since $\{w_l^j\}_{l=1}^r$'s are linearly independent, up to an ordering of $\{i, n+i, \ldots, (m-1)n+i\}$, we may write:

$$\begin{pmatrix} w_1^j \\ w_2^j \\ \cdots \\ w_r^j \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & * & * \\ 0 & 1 & \cdots & 0 & * & * \\ \cdots & \cdots & \cdots & \cdots & * & * \\ 0 & 0 & \cdots & 1 & * & * \end{pmatrix}_{r\times m} \begin{pmatrix} v_i^j \\ v_{n+i}^j \\ \vdots \\ v_{(m-1)n+i}^j \end{pmatrix}. \tag{7.6}$$

Assume one of the $*$'s in the above matrix is non-zero. Without loss of generality, assume that a $*$ in the first row is non-zero, and denote it by a. Then
$$w_1^j = v_i^j + a v_{kn+i}^j + \cdots,$$
for some $r+1 \leq k \leq m$. Using the action of $g \in I(F/K)$, we see from the above matrix that
$$g \cdot w_1^j = c_g w_1^j,$$
for some $c_g \in E$. Now (7.5) and by comparing the coefficients of v_i^j, v_{kn+i}^j, in the above equality we see that
$$\chi(g) = \chi^{\sigma^k}(g),$$
for every $g \in I(F/K)$, a contradiction.

Thus, we may assume that all the $*$'s in the above matrix are zero. So, locally $(D')^j$, the image of D' in the j-th projection, is generated by r-vectors from the set $\{v_i^j, v_{n+i}^j, \ldots, v_{(m-1)n+i}^j\}$, say $(D')^j = \langle v_{j_1 n+i}^j, v_{j_2 n+i}^j, \ldots, v_{j_r n+i}^j\rangle$, where $0 \leq j_1 < j_2 < \ldots < j_r \leq m-1$.

Let $\tilde{w}_1$ be the element of D corresponding to $(v_{j_1 n+i}^j)$, as j varies through all projections, under (2.6). Similarly, define $\tilde{w}_2, \ldots, \tilde{w}_r$. If any $\tilde{w}_j$ is in the set $\{v_i, v_{n+i}, \ldots, v_{(m-1)n+i}\}$, then using the action of $\sigma \in \mathrm{Gal}(F/\mathbb{Q}_p)$, we see that $D' = \langle v_i, v_{n+i}, \ldots, v_{(m-1)n+i}\rangle$, a contradiction, since the rank of D' is at most $m-1$.

Therefore, none of the $\tilde{w}_i$ are basis vectors. Hence, we may write:
$$\tilde{w}_1 = a_k v_{kn+i} + \sum_{j=k+1}^{m-1} a_j v_{jn+i},$$
where the $a_j \in F_0 \otimes E$ and a_k is neither zero nor a unit in $F_0 \otimes E$, for some $1 \leq k \leq m-1$. The other basis vectors $\tilde{w}_2, \ldots, \tilde{w}_r$ are contained in the span of $\{v_{(k+1)n+i}, \ldots, v_{(m-1)n+i}\}$. Moreover, we have
$$D' = \langle w_1, w_2, \cdots, w_r\rangle = \langle \tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_r\rangle \tag{7.7}$$

Write $\varphi(\tilde{w}_1) = \sum_i c_i \tilde{w}_i$, for some c_i. Now, by comparing the coefficients of v_{kn+i} on both sides of the above equality, we see that

$$\sigma(a_k) = c_1 \cdot a_k.$$

But, by Lemma 7.4, this cannot happen. □

Now, we prove Theorem 7.1 for τ an unramified supercuspidal representation of dimension n.

Let D' be a rank r (φ, N, F, E)-submodule of D. By the lemma above (applied when $i = n$, i.e., to the span of the vectors in the last column of the table above), the index of N on D' cannot be smaller than m, since otherwise the kernel of N on D' would be a $(\varphi, \mathrm{Gal}(F/\mathbb{Q}_p))$-submodule of $\langle v_n, v_{2n}, \ldots, v_{mn}\rangle$ of rank smaller than m. The index of D' can also clearly not be bigger than m, since the index of N on D is exactly m. We conclude that the index of N on D' has to be m. Hence, there are positive integers $r_1 \le r_2 \le \ldots \le r_m$ such that $r = \sum_i^m r_i$ and vectors $w_1, w_2, \ldots, w_m$ in D' such that D' is a free module generated by the basis elements $N^j w_i$. We arrange these basis elements as follows:

$$\begin{array}{rrl}
 & w_1,\ Nw_1, & \ldots,\ N^{r_1-1}w_1, \\
 & w_2,\ Nw_2, & \ldots,\ N^{r_2-1}w_2, \\
 & & \quad\vdots \\
 & w_k,\ Nw_k, & \ldots,\ N^{r_k-1}w_k, \\
w_{k+1},\ Nw_{k+1}, \ldots, N^{r_{k+1}-r_1-1}w_{k+1}, & N^{r_{k+1}-r_1}w_{k+1}, & \ldots,\ N^{r_{k+1}-1}w_{k+1}, \\
 & & \quad\vdots \\
w_m,\ Nw_m,\ N^2w_m, \ldots, N^{r_m-r_1-1}w_m, & N^{r_m-r_1}w_m, & \ldots,\ N^{r_m-1}w_m.
\end{array} \tag{7.8}$$

Suppose, towards a contradiction, that there exists $1 \le k \le m-1$ such that $r_1 = r_2 = \cdots = r_{k-1} = r_k$ but $r_k < r_{k+1}$. It is easy to check that the span of the vectors in the last r_1 columns in the table above is nothing but $D'' = \langle v_{n-(r_1-1)}, \ldots, v_n, v_{2n-(r_1-1)}, \ldots, v_{2n}, \ldots, v_{mn-(r_1-1)}, \ldots, v_{mn}\rangle$, the kernel of N^{r_1} on D. Indeed, both spaces are of the same rank and the former is contained in the latter since N^{r_1} kills the former space. Thus $D' =$

$$\langle w_{k+1}, Nw_{k+1}, \ldots, N^{r_{k+1}-r_1-1}w_{k+1}, \ldots, w_m, Nw_m, \ldots, N^{r_m-r_1-1}w_m\rangle \oplus D''.$$

We now study the span of the basis vectors of D' contained in the (r_k+1)-th column from the right in (7.8) (this is the first, from the right,'short' column of vectors). Since, e.g., $N(N^{r_{k+1}-r_1-1}w_{k+1}) \in D''$, the top-most vector in this 'short' column

$$N^{r_{k+1}-r_1-1}w_{k+1} \in \langle v_{n-r_1}, v_{2n-r_1}, \ldots, v_{mn-r_1}\rangle \oplus D''.$$

Thus as a basis element of D', we may replace $N^{r_{k+1}-r_1-1}w_{k+1}$ by a linear combination $w'_{k+1} := a_1 v_{n-r_1} + a_2 v_{2n-r_1} + \cdots + a_m v_{mn-r_1}$. The same applies to the other basis vectors of D' in this column and we may replace the second through last vector in this column by similar linear combinations $w'_{k+2}, \ldots, w'_m$.

We now claim that the module $\langle w'_{k+1}, \ldots, w'_m\rangle$ is $(\varphi, \mathrm{Gal}(F/\mathbb{Q}_p))$-invariant. Indeed, $\varphi w'_{k+1} \in D'$ has order of nilpotency r_1+1 (since w'_{k+1} does), hence is a linear combination of the w'_i and the vectors

$$v_{n-(r_1-1)}, \ldots, v_n, \ldots, v_{mn-(r_1-1)}, \ldots, v_{mn}.$$

However the last vectors do not appear since $\langle v_{n-r_1}, v_{2n-r_1}, \cdots, v_{mn-r_1}\rangle$ is φ-stable, so $\varphi w'_{k+1} \in \langle w'_{k+1}, \ldots, w'_m\rangle$. An identical argument applies to the other vectors $\varphi w'_{k+2}, \cdots, \varphi w'_m$. In a very similar fashion one can show that $\langle w'_{k+1}, \cdots w'_m\rangle$ is $\mathrm{Gal}(F/\mathbb{Q}_p)$-stable (this time $g \in I(F/K)$ acts by a scalar on each of the vectors $v_{n-r_1}, v_{2n-r_1}, \cdots, v_{mn-r_1}$ and $\sigma \in \mathrm{Gal}(F/\mathbb{Q}_p)$ permutes them, but the same argument applies). But now

$$\langle w'_{k+1}, \ldots, w'_m\rangle \subset \langle v_{n-r_1}, v_{2n-r_1}, \ldots v_{mn-r_1}\rangle$$

violates Lemma 7.10, since it a $(\varphi, \mathrm{Gal}(F/\mathbb{Q}_p))$-submodule of rank $1 \leq m-k \leq m-1$.

The upshot is that such a k does not exist, all the r_i's are equal, r is divisible by m, and $D' = D''$ is spanned only by vectors in the last r_1 columns in (7.8) above. That is, D' is exactly $D_\tau \otimes D_{\mathrm{Sp}(r_1)}$, proving the theorem.

7.3. **General case: $m \geq 2$ and $n \geq 2$.** Now, we shall prove Theorem 7.1 in general. Thus, we show that the only (φ, N, F, E)-submodules of $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$, for any irreducible representation τ of $W_{\mathbb{Q}_p}$ dimension $m \geq 2$, and $n \geq 2$, are of the form $D_\tau \otimes D_{\mathrm{Sp}(r)}$, for some $1 \leq r \leq n$.

The proof uses ideas introduced for the special cases proved so far. Note that in the previous section many explicit formulas were used regarding the action of φ and $\mathrm{Gal}(F/\mathbb{Q}_p)$ which depended on the shape of the unramified supercuspidal τ. In fact it is possible to do away with these explicit formulas completely.

Recall that the module $D_{\mathrm{Sp}(n)}$ has a basis $\{f_n,\ f_{n-1}, \cdots, f_1\}$, with properties as in Section 6.1, and say that D_τ has a basis $\{e_1, e_2, \cdots, e_m\}$ over $F_0 \otimes E$. Let $\{v_i\}_{i=1}^{mn}$ denote the basis of $D = D_\tau \otimes D_{\mathrm{Sp}(r)}$ defined exactly as in the table at the start of Section 7.2.5.

Lemma 7.11. *There are no rank r (φ, N, F, E)-submodules of D on which N acts trivially, for $1 \leq r \leq m-1$.*

Proof. Suppose there exists such a module, say $\tilde{D}$, of rank $r < m$. Since N acts trivially on $\tilde{D}$, we have $\tilde{D} \subseteq \langle v_n, v_{2n}, \cdots, v_{mn}\rangle = D_\tau \otimes D_{\mathrm{Sp}(1)} \simeq D_\tau$. But τ is irreducible, so D_τ is irreducible by Lemma 2.8, a contradiction. □

Corollary 7.12. *The index of N on a (φ, N, F, E)-submodule of D is m.*

Proof of Theorem 7.1. Let D' be a (φ, N, F, E)-submodule of $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$. From the corollary above, there are m blocks in the Jordan canonical form of N on D'. Without loss of generality assume that the blocks have sizes $r_1 \leq r_2 \leq \cdots \leq r_m$ with $\sum_{i=1}^m r_i = \mathrm{rank}\ D'$. Suppose $w_1, \cdots, w_m$ are the corresponding basis vectors in D' such that the order of nilpotency of N on w_i is r_i, so that the $N^j(w_i)$ form a basis of D'. If all the r_i are equal to say r, then the usual argument shows $D' = D_\tau \otimes D_{\mathrm{Sp}(r)}$. We show that this is indeed the case.

Suppose towards a contradiction that $r_i \neq r_{i+1}$ for some $1 \leq i < m$. For $1 \leq i \leq n$, let D_i be span of the vectors in the last i columns in the table at the start of Section 7.2.5. Observe that $D_i = D_\tau \otimes \mathrm{Ker}(N^i) = D_\tau \otimes D_{\mathrm{Sp}(i)}$.

Now, arrange the basis vectors of D', i.e., the $N^j w_k$, as in (7.8). With respect to this arrangement, denote the span of the vectors in the last i columns as A_i. Since $r_i \neq r_{i+1}$, the rank of the space A_{r_i+1}/A_{r_i} is less than m. Moreover, A_{r_i+1}/A_{r_i} is a subspace of D_{r_i+1}/D_{r_i}, i.e., there is an inclusion of (φ, N, F, E)-modules

$$A_{r_i+1}/A_{r_i} \hookrightarrow D_{r_i+1}/D_{r_i}.$$

Now

$$
\begin{aligned}
D_{r_i+1}/D_{r_i} &= (D_\tau \otimes D_{\mathrm{Sp}(r_i+1)})/(D_\tau \otimes D_{\mathrm{Sp}(r_i)}) \\
&\simeq D_\tau \otimes (D_{\mathrm{Sp}(r_i+1)}/D_{\mathrm{Sp}(r_i)}) \\
&\simeq D_\tau \otimes D_{\mathrm{Sp}(1)} \\
&\simeq D_\tau.
\end{aligned} \tag{7.9}
$$

All the isomorphisms above are isomorphisms of (φ, N, F, E)-modules over $F_0 \otimes E$. By Lemma 7.11, the above inclusion is not possible! Hence all the r_i are indeed equal. This finishes the proof of Theorem 7.1. □

7.4. **Filtration on $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$.** We finally can apply the discussion above to write down the structure of the (p,p)-representation attached to an automorphic form on $\mathrm{GL}_{mn}(\mathbb{A}_\mathbb{Q})$.

We start with some remarks. Suppose D_1 and D_2 are two admissible filtered modules. It is well-known (cf. [T96]) that the tensor product $D_1 \otimes D_2$ is also admissible. The difficulty in proving this lies in the fact that one does not have much information about the structure of the (φ, N)-submodules of the tensor product. If they are of the form $D' \otimes D''$, where D' and D'' are admissible (φ, N)-submodules of D_1 and D_2 respectively, then one could use Lemma 2.5 to prove that $D' \otimes D''$ is also admissible. But not all the submodules of $D_1 \otimes D_2$ are of this form.

However in the previous section we have just shown (cf. Theorem 7.1), that for $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$, all the (φ, N, F, E)-submodules of D are of the form $D_\tau \otimes D_{\mathrm{Sp}(r)}$, for some $1 \le r \le n$. This fact allows us to study the crystal D and its submodules, once we introduce the Hodge filtration.

7.4.1. *Filtration in general position.* Assume that the Hodge filtration on D is in general position with respect to the Newton filtration (cf. Assumption 4.6). Let m be the rank of D_τ. Let $\{\beta_{i,j}\}_{i=1,j=1}^{i=n,j=m}$ be the jumps in the Hodge filtration with $\beta_{i_1,j_1} > \beta_{i_2,j_2}$, if $i_1 > i_2$, or if $i_1 = i_2$ and $j_1 > j_2$. Thus

$$\beta_{n,m} > \beta_{n,m-1} > \cdots > \beta_{n,1} > \beta_{n-1,m} > \cdots > \beta_{1,m} > \cdots > \beta_{1,1}.$$

Define, for every $1 \le k \le n$,

$$b_k = \sum_{j=1}^{j=m} \beta_{k,j},$$

and

$$a_k = t_N(D_\tau \otimes D_{\mathrm{Sp}(k)}) - t_N(D_\tau \otimes D_{\mathrm{Sp}(k-1)}) = t_N(D_\tau) + m(k-1),$$

where the last equality follows from Lemma 2.5. Clearly, we have that

$$
\begin{aligned}
b_n > b_{n-1} > \cdots > b_2 > b_1, \\
a_n > a_{n-1} > \cdots > a_2 > a_1.
\end{aligned}
$$

Observe that $b_{i+1} - b_i \ge m^2$ and $a_{i+1} - a_i = m$, for every $1 \le i \le n$. Since D is admissible, the submodule $D_\tau \otimes D_{\mathrm{Sp}(i)}$ of D is admissible if and only if $\sum_{k=1}^{i} b_k = \sum_{k=1}^{i} a_k$.

The arguments below are similar to the ones used when analyzing the Steinberg case. We start with an analog of Lemma 6.2.

Lemma 7.13. *Let $\{a_i\}_{i=1}^n$ be an increasing sequence of integers, such that $a_{i+1} - a_i = m$, for every i and for some fixed natural number m. Let $\{b_i\}_{i=1}^n$ be an increasing sequence of integers, such that $b_{i+1} - b_i \geq m^2$, for every i. Suppose that $\sum_i a_i = \sum_i b_i$. If $a_n = b_n$ or $a_1 = b_1$, then $m = 1$ and hence $a_i = b_i$, for all i.*

Proof. Let us prove the lemma when $a_n = b_n$. We have:

$$m^2(n-1+n-2+\cdots+1) \leq \sum_{i=1}^{n}(b_n - b_i) = \sum_{i=1}^{n}(a_n - a_i) = m(n-1+n-2+\cdots+1).$$

The first equality in the above expression follows from $a_n = b_n$. From the above inequality, we see $m = 1$. Now, the rest of the proof follows from Lemma 6.2. The proof when $a_1 = b_1$ is similar. □

Theorem 7.14. *If $D_\tau \otimes D_{\mathrm{Sp}(i)}$ and $D_\tau \otimes D_{\mathrm{Sp}(i+1)}$ are admissible submodules of D, then $m = 1$, in which case all the $D_\tau \otimes D_{\mathrm{Sp}(i)}$, for $1 \leq i \leq n$, are admissible.*

Proof. Since $D_\tau \otimes D_{\mathrm{Sp}(i)}$ and $D_\tau \otimes D_{\mathrm{Sp}(i+1)}$ are admissible, we have:

$$\begin{aligned} b_1 + b_2 + \cdots + b_i &= \sum_{r=1}^{i} a_r, \\ b_1 + b_2 + \cdots + b_{i+1} &= \sum_{r=1}^{i+1} a_r. \end{aligned} \tag{7.10}$$

From these expressions, we have $b_{i+1} = a_{i+1}$. As recalled above:

$$b_n > \cdots > b_{i+2} > b_{i+1} > b_i > \cdots > b_1,$$
$$a_n > \cdots > a_{i+2} > a_{i+1} > a_i > \cdots > a_1.$$

Since $a_{i+1} = b_{i+1}$ and (7.10) holds, by Lemma 7.13 we have $m = 1$ and $a_i = b_i$, for all $1 \leq i \leq n$. This shows that all the $D_\tau \otimes D_{\mathrm{Sp}(i)}$ are admissible. □

Theorem 7.15. *Let $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ and assume that the Hodge filtration on D is in general position (cf. Assumption 4.6). Then either D is irreducible or D is reducible, in which case $m = 1$ and the (φ, N, F, E)-submodules $D_\tau \otimes D_{\mathrm{Sp}(i)}$, for $1 \leq i \leq n$ are all admissible.*

Proof. Let $D_i = D_\tau \otimes D_{\mathrm{Sp}(i)}$, for $1 \leq i \leq n$. If D is irreducible then we are done. If not, by Theorem 7.1, there exists an $1 \leq i \leq n$ such that D_i is admissible. If D_{i-1} or D_{i+1} is also admissible, then by the theorem above, $m = 1$ and hence all the (φ, N, F, E)-submodules of D are admissible. So, assume D_{i-1} and D_{i+1} are not admissible, but D_i is admissible. We shall show that this is not possible. Indeed, we have:

$$b_1 + b_2 + \cdots + b_{i-1} < \sum_{r=1}^{r=i-1} a_r, \tag{7.11a}$$

$$b_1 + b_2 + \cdots + b_i = \sum_{r=1}^{r=i} a_r, \tag{7.11b}$$

$$b_1 + b_2 + \cdots + b_{i+1} < \sum_{r=1}^{r=i+1} a_r. \tag{7.11c}$$

Subtracting (7.11b) from (7.11a), we get $-b_i < -a_i$. Subtracting (7.11b) from (7.11c), we get $b_{i+1} < a_{i+1}$. Adding these two inequalities, we get $b_{i+1} - b_i < a_{i+1} - a_i = m$. But this is a contradiction, since $b_{i+1} - b_i \geq m$. □

For emphasis we state separately the following corollary:

Corollary 7.16. *With assumptions as above, for any $m \geq 2$, the crystal $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ is irreducible.*

Definition 9. Say π is *ordinary* at p if $a_1 = b_1$, i.e., $t_N(D_\tau) = \sum_{j=1}^{m} \beta_{1,j}$.

This condition implies $m = 1$, and this definition then coincides with Definition 8.

Applying the above discussion to the local (p,p)-representation in a strictly compatible system, we obtain:

Theorem 7.17 (Indecomposable case)**.** *Say π is a cuspidal automorphic form with infinitesimal character consisting of distinct integers. Suppose that*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \tau_m \otimes \mathrm{Sp}(n),$$

where τ_m is an irreducible representation of $W_{\mathbb{Q}_p}$ of dimension $m \geq 1$, and $n \geq 1$. Assume that Assumption 4.6 holds.

- *If π is ordinary at p, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is reducible, in which case $m = 1$ and τ_1 is a character and $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is (quasi)-ordinary as in Theorems 6.6 and 6.7.*
- *If π is not ordinary at p, then $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is irreducible.*

7.4.2. *Tensor product filtration.* One might wonder what happens if the filtration on D is not necessarily in general position. As an example, here we consider just one case arising from the so called tensor product filtration.

Assume that D_τ and $D_{\mathrm{Sp}(n)}$ are the usual filtered (φ, N, F, E)-modules and equip $D_\tau \otimes D_{\mathrm{Sp}(n)}$ with the tensor product filtration. By the formulas in Lemma 2.5 one can prove:

Lemma 7.18. *Suppose that $D = D_\tau \otimes D_{\mathrm{Sp}(n)}$ has the tensor product filtration. Fix $1 \leq r \leq n$. Then $D_\tau \otimes D_{\mathrm{Sp}(r)}$ is an admissible submodule of D if and only if $D_{\mathrm{Sp}(r)}$ is an admissible submodule of $D_{\mathrm{Sp}(n)}$.*

We recall that if the filtration on $D_{\mathrm{Sp}(n)}$ is in general position (as in Assumption 4.6), then we have shown that furthermore $D_{\mathrm{Sp}(r)}$ is an admissible submodule of $D_{\mathrm{Sp}(n)}$ if and only if $D_{\mathrm{Sp}(1)}$ is an admissible submodule.

Remark 6. The lemma can be used to give an example where the tensor product filtration on D is not in general position (i.e., does not satisfy Assumption 4.6). Suppose τ is an irreducible representation of dimension $m = 2$ and $D_{\mathrm{Sp}(2)}$ has weight 2 (cf. [GM09, §3.1]). Note $\langle f_1 \rangle$ is an admissible submodule of $D_{\mathrm{Sp}(2)}$. Hence, by the lemma, $D_\tau \otimes \langle f_1 \rangle$ is an admissible submodule of $D_\tau \otimes D_{\mathrm{Sp}(2)}$. If the tensor product filtration satisfies Assumption 4.6, then the admissibility of $D_\tau \otimes \langle f_1 \rangle$ would contradict Theorem 7.15, since $m = 2$.

In any case, we have the following application to local Galois representations.

Proposition 7.19. *Suppose that $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \rho_\tau \otimes \rho_{\mathrm{Sp}(n)}$ is a tensor product of two (p,p)-representations, with underlying Weil-Deligne representations τ and $\mathrm{Sp}(n)$ respectively. If $\rho_{\mathrm{Sp}(n)}$ is irreducible, then so is $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$.*

8. General Weil-Deligne representations

So far, we have studied the (p,p)-representation attached to π_p when the underlying Weil-Deligne representation is indecomposable. We now make some remarks in the general setting where the Weil-Deligne representation can be decomposed into a direct sum of indecomposable representations.

8.1. **Sum of twisted Steinberg.** For simplicity we start with the case where the indecomposable pieces are twists of the Steinberg representation by an unramified character. Thus we assume the underlying Weil-Deligne representation is

$$D_{\chi_1} \otimes D_{\mathrm{Sp}(n_1)} \oplus D_{\chi_2} \otimes D_{\mathrm{Sp}(n_2)} \oplus \cdots \oplus D_{\chi_r} \otimes D_{\mathrm{Sp}(n_r)},$$

where $n_i \geq 1$ and χ_i are unramified characters taking arithmetic Frobenius to α_i. Let $\chi_i(\omega) = \alpha_i$ where ω is a uniformizer of $\mathbb{Q}_p^\times$. Without loss of generality we may assume that $v_p(\alpha_1) \geq v_p(\alpha_2) \geq \cdots \geq v_p(\alpha_r)$.

Let $n = \sum_i^r n_i$. Let $\{\beta_{i,j}\}_{i=1,j=1}^{r,n_i}$ be the jumps in the Hodge filtration such that $\beta_{i_1,j_1} > \beta_{i_2,j_2}$, if $i_1 > i_2$ or $i_1 = i_2$ and $j_1 > j_2$. Thus

$$\begin{array}{rcr} \beta_{r,n_r} > \cdots > & \beta_{r,1} > & \\ \beta_{r-1,n_{r-1}} > \cdots > & \beta_{r-1,1} > \cdots > & \\ \beta_{2,n_2} > \cdots > & \beta_{2,1} > & \\ \beta_{1,n_1} > \cdots > & \beta_{1,1}. & \end{array} \tag{8.1}$$

Let D be a filtered $(\varphi, N, \mathbb{Q}_p, E)$-module with associated Weil-Deligne representation as above. We now define a flag inside D as follows.

$$D_i = \begin{cases} D_{\chi_1} \otimes D_{\mathrm{Sp}(i)} & \text{if } 1 \leq i \leq n_1, \\ D_{\chi_1} \otimes D_{\mathrm{Sp}(n_1)} \oplus D_{\chi_2} \otimes D_{\mathrm{Sp}(i-n_1)} & \text{if } n_1+1 \leq i \leq n_1+n_2, \\ \vdots & \\ \oplus_{k=1}^{r-1} D_{\chi_i} \otimes D_{\mathrm{Sp}(n_i)} \oplus D_{\chi_r} \otimes D_{\mathrm{Sp}(i-(n-n_r))} & \text{if } n - n_r + 1 \leq i \leq n. \end{cases}$$

Clearly, D_n is the full (φ, N)-module D. We now show that the above flag is admissible if and only if π is ordinary at p in the following sense:

Definition 10. Say π is ordinary at p if $\beta_{i,1} = -v_p(\alpha_i)$ for all $1 \leq i \leq r$.

We remark that the notion of ordinariness extends the previous definitions given in Definition 6, when all the $n_i = 1$ and Definition 8, when $r = 1$ and $m = 1$. We have:

Theorem 8.1. *Assume that Assumption 4.6 holds. Then the flag $\{D_i\}$ is an admissible flag in D (i.e., each D_i is an admissible submodule of D) if and only if π is ordinary at p.*

Proof. The 'only if' part is clear. Indeed if the Hodge filtration is in general position then the jump in the filtration on D_1 will be the last number in (8.1), i.e., $\beta_{1,1}$, and the admissibility of D_1 shows that $\beta_{1,1} = -v_p(\alpha_1)$. Similarly, the jumps in the filtration on D_{n_1} (respectively D_{n_1+1}) are the last n_1 numbers (respectively the last n_1 numbers along with $\beta_{2,1}$) in (8.1) above, and clearly $t_N(D_{n_1+1}) = t_N(D_{n_1}) - v_p(\alpha_2)$, so the admissibility of D_{n_1} and D_{n_1+1} together shows that $\beta_{2,1} = -v_p(\alpha_2)$, etc.

Let us prove the 'if' part. Since $\beta_{1,1} = -v_p(\alpha_1)$, D_1 is admissible. Since D_2 is a $(\varphi, N, \mathbb{Q}_p, E)$-submodule of D we have that

$$\beta_{1,1} + \beta_{1,2} \leq (1 - v_p(\alpha_1)) + (-v_p(\alpha_1))$$

and hence $\beta_{1,2} \leq 1 - v_p(\alpha_1) = 1 + \beta_{1,1}$. Thus $\beta_{1,2} - \beta_{1,1} \leq 1$. But $\beta_{1,2} - \beta_{1,1} \geq 1$, by (8.1), hence equality holds, i.e.,

$$\beta_{1,2} = (1 - v_p(\alpha_1)).$$

Therefore,

$$\beta_{1,1} + \beta_{1,2} = (1 - v_p(\alpha_1)) + (-v_p(\alpha_1)).$$

By a similar argument, we see that

$$\sum_{j=1}^{n_1} \beta_{1,j} = \sum_{j=1}^{n_1} (j - 1 - v_p(\alpha_1)). \tag{8.2}$$

This shows that D_{n_1} is admissible.

Since D_{n_1+1} is an (φ, N, F, E)-submodule of D, we have that

$$\sum_{j=1}^{n_1} \beta_{1,j} + \beta_{2,1} \leq \sum_{j=1}^{n_1} (j - 1 - v_p(\alpha_1)) + (-v_p(\alpha_2)),$$

but this inequality is actually an equality, by (8.2), and since $\beta_{2,1} = -v_p(\alpha_2)$ by assumption. This shows that D_{n_1+1} is also admissible. The admissibility of the other D_i is proved in a similar manner. $\square$

8.2. **General ordinary case.** We now assume that as a (φ, N, F, E)-module,

$$D = D_{\tau_1} \otimes D_{\mathrm{Sp}(n_1)} \oplus D_{\tau_2} \otimes D_{\mathrm{Sp}(n_2)} \oplus \cdots \oplus D_{\tau_r} \otimes D_{\mathrm{Sp}(n_r)},$$

where $n_i \in \mathbb{N}$ and τ_i's are irreducible representations of $W_{\mathbb{Q}_p}$ of degree m_i. Without loss of generality we may assume $t_N(D_{\tau_1}) \leq t_N(D_{\tau_2}) \leq \cdots \leq t_N(D_{\tau_r})$.

We now define a flag inside $D_{\tau_1} \otimes D_{\mathrm{Sp}(n_1)} \oplus \cdots \oplus D_{\tau_r} \otimes D_{\mathrm{Sp}(n_r)}$, and show that this flag is admissible if and only if there is a relation between some numbers a_i (depending on Newton numbers) and b_i (depending on Hodge numbers). More precisely, define the flag $\{D_i\}$ in D by

$$D_i = \begin{cases} D_{\tau_1} \otimes D_{\mathrm{Sp}(i)} & \text{if } 1 \leq i \leq n_1, \\ D_{\tau_1} \otimes D_{\mathrm{Sp}(n_1)} \oplus D_{\tau_2} \otimes D_{\mathrm{Sp}(i-n_1)} & \text{if } n_1 + 1 \leq i \leq n_1 + n_2, \\ \qquad \vdots & \\ \oplus_{k=1}^{r-1} D_{\tau_k} \otimes D_{\mathrm{Sp}(n_k)} \oplus D_{\tau_r} \otimes D_{\mathrm{Sp}(i-(n-n_r))} & \text{if } n - n_r + 1 \leq i \leq n. \end{cases}$$

Clearly, $D_n = D$.

We now define the numbers a_i and b_i. Let $\{\beta_{i,j}\}$ be the jumps in the Hodge filtration associated to D such that $\beta_{i_1,j_1} > \beta_{i_2,j_2}$, if $i_1 > i_2$ or if $i_1 = i_2$ but $j_1 > j_2$. Thus, in the case $r = 2$, the jumps in the filtration are:

$$\begin{array}{llll}
\beta_{m_1+m_2,n_2} > \beta_{m_1+m_2-1,n_2} & > \cdots > \beta_{m_1+1,n_2} > & \\
\beta_{m_1+m_2,n_2-1} > \beta_{m_1+m_2-1,n_2-1} & > \cdots > \beta_{m_1+1,n_2-1} & > \cdots > \\
\beta_{m_1+m_2,1} > \beta_{m_1+m_2-1,1} & > \cdots > \beta_{m_1+1,1} > & \\
\beta_{m_1,n_1} > \beta_{m_1-1,n_1} & > \cdots > \beta_{1,n_1} > & \\
\beta_{m_1,n_1-1} > \beta_{m_1-1,n_1-1} & > \cdots > \beta_{1,n_1-1} & > \cdots > \\
\beta_{m_1,1} > \beta_{m_1-1,1} & > \cdots > \beta_{1,1}. &
\end{array}$$

Define, for every $1 \leq k \leq n_1$,

$$b_k = \sum_{i=1}^{i=m_1} \beta_{i,k},$$

$$a_k = t_N(D_{\tau_1} \otimes D_{\mathrm{Sp}(k)}) - t_N(D_{\tau_1} \otimes D_{\mathrm{Sp}(k-1)}) = t_N(D_{\tau_1}) + m_1(k-1).$$

Clearly, we have that

$$b_{n_1} > b_{n_1-1} > \cdots > b_2 > b_1,$$

$$a_{n_1} > a_{n_1-1} > \cdots > a_2 > a_1.$$

Observe that $b_{i+1} - b_i \geq m_1^2$ and $a_{i+1} - a_i = m_1$, for $1 \leq i < n_1$. Under Assumption 4.6, the jumps in the induced Hodge filtration on $D_{\tau_1} \otimes D_{\mathrm{Sp}(j)}$ are $\beta_{m_1,j} > \cdots > \beta_{1,1}$, so that $D_{\tau_1} \otimes D_{\mathrm{Sp}(j)}$ is an admissible submodule of D if and only if $\sum_{k=1}^{j} b_k = \sum_{k=1}^{j} a_k$.

Similarly define b_k and a_k for all $1 \leq k \leq n = \sum_i^r n_i$. For example, if $n_1 + 1 \leq n_1 + k \leq n_1 + n_2$, define

$$b_{n_1+k} = \sum_{i=1}^{i=m_2} \beta_{m_1+i,k},$$

$$a_{n_1+k} = t_N(D_{\tau_2} \otimes D_{\mathrm{Sp}(k)}) - t_N(D_{\tau_2} \otimes D_{\mathrm{Sp}(k-1)}) = t_N(D_{\tau_2}) + m_2(k-1).$$

Again, we have

$$b_{n_1+n_2} > b_{n_1+n_2-1} > \cdots > b_{n_1+2} > b_{n_1+1},$$

$$a_{n_1+n_2} > a_{n_1+n_2-1} > \cdots > a_{n_1+2} > a_{n_1+1},$$

and $b_{i+1} - b_i \geq m_2^2$ and $a_{i+1} - a_i = m_2$, for every $n_1 + 1 \leq i < n_1 + n_2$, etc.

Definition 11. Say π is ordinary at p if $a_{\sum_{j=1}^{i-1} n_j + 1} = b_{\sum_{j=1}^{i-1} n_j + 1}$, for $1 \leq i \leq r$.

Note that this definition of ordinariness reduces to Definition 9 when $r = 1$, but also to Definition 10 since it implies $m_i = 1$, for $1 \leq i \leq r$.

Theorem 8.2. *Assume that Assumption 4.6 holds. Then, the flag $\{D_i\}$ is admissible (i.e., each D_i is an admissible submodule of D) if and only if π is ordinary at p.*

Proof. We prove the 'only if' direction for $r = 2$, since the general case is similar, and only notationally more cumbersome. Thus we have to show that if the flag $\{D_i\}$ is admissible, then $a_1 = b_1$ and $a_{n_1+1} = b_{n_1+1}$. The proof is an easy application of Lemma 7.13. Indeed

- The admissibility of D_1 shows that $a_1 = b_1$.
- The admissibility of D_{n_1} and $a_1 = b_1$ shows $m_1 = 1$ and $a_i = b_i$ for $1 \leq i \leq n_1$ (by Lemma 7.13).
- The admissibility of D_{n_1+1} and D_{n_1} together shows $a_{n_1+1} = b_{n_1+1}$.

- The admissibility of $D_{n_1+n_2}$ and D_{n_1} and $a_{n_1+1} = b_{n_1+1}$ shows $m_2 = 1$ and $a_{n_1+i} = b_{n_1+i}$ for $1 \le i \le n_2$ (by Lemma 7.13).

Since all $m_i = 1$, the proof of the 'if' part of the theorem is exactly the same as the 'if' part of the proof of Theorem 8.1, noting $b_{\sum_{j=1}^{i-1} n_j+1} = \beta_{i,1}$ and and $a_{\sum_{j=1}^{i-1} n_j+1} = -v_p(\alpha_i)$, for $1 \le i \le r$. □

Remark 7. The theorem does not tell us when D is irreducible, since there are a large number of (φ, N, F, E)-submodules of D which are not part of the flag considered above. For instance, it seems hard to determine the admissibility of the submodule $D_{\tau_i} \otimes D_{\mathrm{Sp}(n_i)}$, except when $i = 1$.

Translating the theorem above in terms of the (p,p)-representation, we obtain:

Theorem 8.3 (Decomposable case)**.** *Say π is a cuspidal automorphic form on $\mathrm{GL}_N(\mathbb{A}_{\mathbb{Q}})$ with infinitesimal character given by the integers $-\beta_1 > \cdots > -\beta_N$. Suppose that $N = \sum_{i=1}^r m_i n_i$ and*

$$\mathrm{WD}(\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}) \sim \oplus_{i=1}^r \tau_i \otimes \mathrm{Sp}(n_i),$$

where τ_i is an irreducible representation of dimensions $m_i \ge 1$, and $n_i \ge 1$. If π is ordinary at p, then $m_i = 1$ for all i, the β_i occur in r blocks of consecutive integers, of lengths n_i, for $1 \le i \le r$, and

$$\rho_{\pi,p}|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \rho_{n_1} & * & \cdots & * \\ 0 & \rho_{n_2} & \cdots & * \\ 0 & 0 & \cdots & * \\ 0 & 0 & 0 & \rho_{n_r} \end{pmatrix},$$

where each ρ_{n_i} is an n_i-dimensional representation with shape similar to that in Theorem 6.7. In particular, $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ is quasi-ordinary.

8.3. **Further remarks.** Theorem 8.3 treats the general ordinary case. In the general non-ordinary case, the behaviour of the (p,p)-representation is more complex and we do not have complete information about reducibility (compare with the indecomposable case treated in Theorem 7.17). In this section we content ourselves with a few concluding remarks about new issues that arise.

To fix ideas we assume that the Weil-Deligne representation associated to π_p is of the form

$$\chi_1 \otimes \mathrm{Sp}(2) \oplus \chi_2 \otimes \mathrm{Sp}(2),$$

where χ_1 and χ_2 are unramified characters of $W_{\mathbb{Q}_p}$ taking arithmetic Frobenius to α_1 and α_2, respectively. Let D be the associated $(\varphi, N, \mathbb{Q}_p, E)$-module. Let $\beta_4 > \beta_3 > \beta_2 > \beta_1$ be the jumps in the Hodge filtration on D. We continue to assume that Assumption 4.6 holds.

8.3.1. *Classification of (φ, N)-submodules of D.* Just as in previous sections, we ignore the filtration, and first classify the $(\varphi, N, \mathbb{Q}_p, E)$-submodules of D.

Let e_1, $e_2 = N(e_1)$ be a basis of $\chi_1 \otimes \mathrm{Sp}(2)$ and f_1, $f_2 = N(f_1)$ be a basis of $\chi_2 \otimes \mathrm{Sp}(2)$. Sometimes we write $\langle e_2 \rangle$ for $\chi_1 \otimes \mathrm{Sp}(1)$, etc.

If $\alpha_1 = \alpha_2$, then any 1-dimensional subspace of $\langle e_2, f_2 \rangle$ is a (φ, N)-submodule of D, with Newton number $-v_p(\alpha_1)$. In particular, there exists infinitely many 1-dimensional submodules of D. This is already in striking contrast with the statement of Theorem 7.1, which says that in the indecomposable case there are only finitely many (φ, N)-submodules. So already we can expect that the analysis in the decomposable case might be much more complex. We remark, however, that if $\alpha_1 \neq \alpha_2$, then $\langle e_2 \rangle$ and $\langle f_2 \rangle$ are the only 1-dimensional submodules of D.

Again, if $\alpha_1 = \alpha_2$, then the 2-dimensional (φ, N)-submodule of D are of the form $\langle ae_1 + cf_1, ae_2 + cf_2 \rangle$, for some a, $c \in E$, with Newton number $1 - 2v_p(\alpha_1)$. If $\alpha_1 \neq \alpha_2$, but $p/\alpha_1 = 1/\alpha_2$, then again there are again infinitely many 2-dimensional (φ, N)-submodules and they are given by $\langle e_1 + bf_2, e_2 \rangle$ or $\langle f_1 + be_2, f_2 \rangle$, for some $b \in E$. If $\alpha_1 \neq \alpha_2$ and $p/\alpha_1 \neq 1/\alpha_2$, then there are only finitely many 2-dimensional (φ, N)-submodules: they are $\chi_1 \otimes \mathrm{Sp}(2)$, $\chi_2 \otimes \mathrm{Sp}(2)$ and the diagonal one $\chi_1 \otimes \mathrm{Sp}(1) \oplus \chi_2 \otimes \mathrm{Sp}(1)$.

Finally, like the 1-dimensional case, if $\alpha_1 = \alpha_2$, then all the 3-dimensional (φ, N)-submodules of D are of form $\langle ae_1 + bf_1, e_2, f_2 \rangle$, for any $a, b \in E$, with Newton number $1 - 3v_p(\alpha_1)$. If $\alpha_1 \neq \alpha_2$, there are exactly two 3-dimensional submodules, namely $\chi_1 \otimes \mathrm{Sp}(2) \oplus \chi_2 \otimes \mathrm{Sp}(1)$ and $\chi_1 \otimes \mathrm{Sp}(1) \oplus \chi_2 \otimes \mathrm{Sp}(2)$.

Hence, if we choose α_1 and α_2 generically (i.e., $\alpha_1 \neq \alpha_2$ and $\alpha_1 \neq p\alpha_2$), then there are only finitely many (φ, N)-submodules of D, otherwise there are infinitely many (φ, N)-submodules of D. The following table contains the possible Newton numbers of the (φ, N)-submodules D' of D.

$\dim_E D'$	$t_N(D')$ when $\alpha_1 \neq \alpha_2$	$t_N(D')$ when $\alpha_1 = \alpha_2$
1	$-v_p(\alpha_1)$, $-v_p(\alpha_2)$	$-v_p(\alpha_1)$
2	$-v_p(\alpha_1) - v_p(\alpha_2)$, $1 - 2v_p(\alpha_1)$, $1 - 2v_p(\alpha_2)$	$1 - 2v_p(\alpha_1)$
3	$1 - 2v_p(\alpha_1) - v_p(\alpha_2)$, $1 - v_p(\alpha_1) - 2v_p(\alpha_2)$	$1 - 3v_p(\alpha_1)$
4	$1 - 2v_p(\alpha_1) + 1 - 2v_p(\alpha_2)$	$2 - 4v_p(\alpha_1)$

8.3.2. *An irreducible example.* We can now easily construct examples such that the crystal D is irreducible. For instance, choose any $\alpha_1 \in E$ with $v_p(\alpha_1) = 0$ and take $\alpha_2 = \alpha_1$. Take $(\beta_4, \beta_3, \beta_2, \beta_1) = (2, 1, 0, -1)$. Using the table above, one can easily check that there are no admissible (φ, N)-submodules of D, except for D itself. We note that since $\alpha_1 = \alpha_2$, there are infinitely many (φ, N)-submodules of D, but only finitely many conditions to check for non-admissibility.

8.3.3. *All complete flags cannot be reducible.* In proving Theorem 8.3 we showed that ordinariness implies that a particular complete flag is admissible. We now wish to point out that not all complete flags in D are necessarily admissible, even under the ordinariness assumption. Indeed, in the setting of the example of this section, if we choose α_1 and α_2 such that $v_p(\alpha_1) \neq v_p(\alpha_2)$, then any two complete flags whose 1-dimensional subspaces are $\langle e_2 \rangle$ and $\langle f_2 \rangle$, respectively, cannot be admissible simultaneously, since (under Assumption 4.6) we have both $\beta_1 = -v_p(\alpha_1)$ and $\beta_1 = -v_p(\alpha_2)$.

8.3.4. *Intermediate cases.* Finally, in the general decomposable case, the regularity (distinct Hodge-Tate weights) of the (p, p)-representation $\rho_{\pi,p}|_{G_{\mathbb{Q}_p}}$ does not imply that it is either (quasi)-ordinary in the sense of Definition 5 or irreducible (compare with Theorem 7.17). We now give an example of such an 'intermediate case', i.e., an example for which the (p, p)-representation is reducible, but such that there is no complete flag of reducible submodules.

Let D be as above. Choose α_1 and α_2 such that $v_p(\alpha_1) = 1$ and $v_p(\alpha_2) = -10$. Take $(\beta_4, \beta_3, \beta_2, \beta_1) = (17, 4, 0, -1)$. From the table above, we see that $\chi_1 \otimes \mathrm{Sp}(1)$, $\chi_1 \otimes \mathrm{Sp}(2)$, and D, are admissible and all the other (φ, N)-submodules satisfy the condition that their Hodge numbers are less than or equal to their Newton numbers.

So D is reducible. However, since $\alpha_1 \neq \alpha_2$ and $\alpha_1 \neq p\alpha_2$, there are only two 3-dimensional (φ, N)-submodules of D, and again from the table we see that neither is admissible. Hence, there is no admissible complete flag of (φ, N)-submodules of D.

Errata. We end this paper by correcting some errors in [GM09].

- p. 2254, lines 6 and 7: $\mathbf{Q}$ should be $\mathbf{Q}_p$
- p. 2257, first two lines should be $\varphi(e_1) = \frac{1}{\sqrt{t}} e_1$ and $\varphi(e_2) = \frac{1}{\sqrt{t}} e_2$
- p. 2260: first three lines in the middle display should be $\varphi(e_1) = \frac{1}{\sqrt{t}} e_1$, $\varphi(e_2) = \frac{1}{\sqrt{t}} e_2$, and $t \in \mathcal{O}_E$, $\mathrm{val}_p(t) = k - 1$. Moreover, t is to be chosen in §3.4.3 satisfying $t^2 = 1/c$ (we may take $c = d$, since ι commutes with φ, and s is no longer required).

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