REDUCTIONS OF GALOIS REPRESENTATIONS VIA THE MOD p LOCAL LANGLANDS CORRESPONDENCE

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Abstract. We describe the semisimplification of the mod p reduction of certain crystalline two dimensional local Galois representations of weights bounded by $p^2 - p$ and slopes in $(1, 2)$. This builds on previous results, for weights bounded by $2p + 1$, for large slopes, and for slopes in (0,1). In proving our results we give a complete description of the submodules generated by the top two monomials in the mod p symmetric power representation of $\mathrm{GL}_2(\mathbb{F}_p)$ in the above range.

1. Introduction

About ten years ago, Breuil initiated the study of the mod p Local Langlands program in the papers [B03a] and [B03b]. One of the key applications was that he was able to study the local behaviour of the reductions of two dimensional crystalline p-adic Galois representations of the local Galois group $G_{\mathbb{Q}_p}$, beyond the classical range of weights $2 \leq k \leq p+1$. Since then several authors (in particular, Colmez and Berger) have clarified various parts of the theory, and obtained new results. Most recently Buzzard and Gee have harnessed the mod p Local Langlands correspondence to study the case where the slope satisfies $0 < v(a_p) < 1$.

We recall their results. Let E be a finite extension field of $\mathbb{Q}_p$ and let v be the valuation normalized so that $v(p) = 1$. Let $a_p \in E$ with $v(a_p) > 0$ and let $k \geq 2$. Let V_{k,a_p} be the irreducible crystalline representation of $G_{\mathbb{Q}_p}$ such that $D_{\mathrm{cris}}(V_{k,a_p}^*) = D_{k,a_p}$, where $D_{k,a_p} = Ee_1 \oplus Ee_2$ is the filtered φ-module given by

$$\begin{cases} \varphi(e_1) = p^{k-1}e_2 \\ \varphi(e_2) = -e_1 + a_p e_2, \end{cases} \qquad \text{and} \quad \mathrm{Fil}^i D_{k,a_p} = \begin{cases} D_{k,a_p} & \text{if } i \leq 0 \\ Ee_1 & \text{if } 1 \leq i \leq k-1 \\ 0 & \text{if } i \geq k. \end{cases}$$

Here φ is crystalline Frobenius. We remark that if $f \in S_k(\Gamma_0(N))$ is a primitive form and $p \nmid N$, then in many cases one knows that $\rho_f|_{G_{\mathbb{Q}_p}} \sim V_{k,a_p}$. Let $\omega = \omega_1$ and ω_2 denote the fundamental characters of level 1 and 2, respectively, and let $\mathrm{ind}(\omega_2^a)$ denote the irreducible representation of $G_{\mathbb{Q}_p}$ obtained by inducing the character ω_2^a from $G_{\mathbb{Q}_{p^2}}$ to $G_{\mathbb{Q}_p}$; on the inertia subgroup $I_p \subset G_{\mathbb{Q}_p}$, the representation $\mathrm{ind}(\omega_2^a) \sim \omega_2^a \oplus \omega_2^{ap}$ is semisimple. Here is a summary of what is known about the behaviour of $\bar{V}_{k,a_p}$, the semisimplification of the reduction of V_{k,a_p}:

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Theorem 1 (Fontaine, Edixhoven, Breuil, Berger, Li, Zhu, Buzzard, Gee, ...). *The reduction $\bar{V}_{k,a_p}$ has the following shape on $G_{\mathbb{Q}_p}$ (on I_p, in the reducible cases):*

(1) $2 \leq k \leq p+1 \implies \mathrm{ind}(\omega_2^{k-1})$.

(2) $k = p+2 \implies \begin{cases} \mathrm{ind}(\omega_2^2) & \text{if } 0 < v(a_p) < 1, \\ \omega \oplus \omega & \text{if } v(a_p) \geq 1. \end{cases}$

(3) $p+3 \leq k \leq 2p \implies \begin{cases} \mathrm{ind}(\omega_2^{k-p}) & \text{if } 0 < v(a_p) < 1, \\ \omega^{k-2} \oplus \omega & \text{if } v(a_p) = 1, \\ \mathrm{ind}(\omega_2^{k-1}) & \text{if } v(a_p) > 1. \end{cases}$

(4) $k = 2p+1$ *(and $p \neq 2$)* $\implies \begin{cases} \mathrm{ind}(\omega_2^2) & \text{if } v(a_p^2+p) < \frac{3}{2}, \\ \omega \oplus \omega & \text{if } v(a_p^2+p) \geq \frac{3}{2}. \end{cases}$

(5) $k \geq 2p+2$ *and* $v(a_p) > \lfloor \frac{k-2}{p-1} \rfloor \implies \begin{cases} \mathrm{ind}(\omega_2^{k-1}) & \text{if } p+1 \nmid k-1, \\ \omega^{\frac{k-1}{p+1}} \oplus \omega^{\frac{k-1}{p+1}} & \text{if } p+1 \mid k-1. \end{cases}$

(6) $k \geq 2p+2$ *and* $0 < v(a_p) < 1 \implies \begin{cases} \mathrm{ind}(\omega_2^t) & \text{with } \{1, \ldots, p-1\} \ni t \equiv k-1 \mod p-1, \\ \omega \oplus \omega. \end{cases}$

The classical case $2 \leq k \leq p+1$ is due to Fontaine and Edixhoven [E92]. Breuil proved parts (2) and (3) of the theorem in [B03a], [B03b], as well as part (4) (unpublished, but later stated in [B10]), thereby extending the range of weights treated to $k \leq 2p+1$. The results in parts (5) and (6) are for all larger weights, but with some restrictions on the slope. On the one hand, Berger-Li-Zhu [BLZ04] treated the case of 'large' slopes $v(a_p) > \lfloor \frac{k-2}{p-1} \rfloor$, reducing to the case $a_p = 0$ and $v(a_p) = \infty$. On the other hand, Buzzard-Gee [BG09] described the reduction for 'small' slopes $0 < v(a_p) < 1$ (they have since refined their result in [BG13] to separate out the two subcases that occur in this range). An alternative approach to the case $0 < v(a_p) < 1/2$ can be found in [G10].

The goal of this paper is to make some progress beyond the results above. In view of the results above, we will restrict our attention to weights $k \geq 2p+2$, as in parts (5) and (6) above. We will also assume that the weight k is essentially bounded by p^2. This bound is an artefact of our method which uses a structure theorem of Glover [G78, (6.4)] to describe the projective and non-projective parts of the mod p symmetric power representations of $\mathrm{GL}_2(\mathbb{F}_p)$ for weights $k \leq p^2 - p$. We will further restrict our attention to a range of slopes just beyond what was treated in [BG09], namely, we shall assume throughout this paper that $1 < v(a_p) < 2$. This is already an interesting case to handle. We remark here that the case $v(a_p) = 1$ could present some technical difficulties, so we do not consider it. Here is our result:

Theorem 2. *Assume $1 < v(a_p) < 2$ and that $2p+2 \leq k \leq p^2 - p$. Then, the possibilities for the reduction $\bar{V}_{k,a_p}$ can be written down as follows. Let $r = k-2 \equiv a \mod p-1$, with $1 \leq a \leq p-1$. $\bar{V}_{k,a_p}$ has the following shape on $G_{\mathbb{Q}_p}$ (and in the reducible cases, on I_p):*

(7) $a = 1$ *and*[1] $3p-2 \leq r \leq p^2-2p+2 \implies \mathrm{ind}(\omega_2^2)$.

(8) $a = 2$ *and* $2p \leq r \leq p^2-2p+3 \implies \begin{cases} \mathrm{ind}(\omega_2^{p+2}) & \text{if } r = 2p, \\ \mathrm{ind}(\omega_2^3) & \text{if } 3p-1 \leq r \leq p^2-2p+3. \end{cases}$

(9) $a = 3$ *and* $2p+1 \leq r \leq 3p \implies \begin{cases} \mathrm{ind}(\omega_2^{3p+1}), \\ \omega^2 \oplus \omega^2. \end{cases}$

(9)' $a = 3$ *and* $4p-1 \leq r \leq p^2-2p+4 \implies \begin{cases} \mathrm{ind}(\omega_2^4), \\ \mathrm{ind}(\omega_2^{3p+1}), \\ \omega^2 \oplus \omega^2. \end{cases}$

(10) $4 \leq a \leq p-1$ *and* $2p+2 \leq r \leq ap$ $(3p-3 \leq r < ap,$ *if* $a = p-1) \implies \mathrm{ind}(\omega_2^{ap+1})$.

(10)' $4 \leq a \leq p-3$ *and* $(a+1)p-1 \leq r \leq p^2-p-2 \implies \begin{cases} \mathrm{ind}(\omega_2^{a+1}), \\ \mathrm{ind}(\omega_2^{ap+1}). \end{cases}$

We remark that Theorem 2 evidently holds for all primes $p \geq 5$ (and $p \geq 7$ in parts (9)' and (10)'; the cases $p = 2$ and 3 for weights $k \leq p^2 - p$ are covered by Theorem 1). We also remark that in parts (7), (8) and (10) we specify a unique answer. In this paper we also provide several new results about the structure of the submodules of the mod p symmetric power representations of $\mathrm{GL}_2(\mathbb{F}_p)$ generated by the top two monomials in the usual model by homogeneous polynomials. These results are of independent interest, and we refer the reader to the text for precise details.

The proof of Theorem 2 uses the p-adic and mod p Local Langlands Correspondences due to Breuil, Berger and Colmez, and an important compatibility between them with respect to the process of reduction. The main ideas here develop those that have gone into proving several parts of Theorem 1 and that are explained in the papers of Breuil and Buzzard-Gee. To orient the reader and to mention the key innovations we introduce, we briefly recall the method now.

There is a p-adic Local Langlands Correspondence, which involves associating to a crystalline irreducible local p-adic representation V of $G_{\mathbb{Q}_p}$ a certain p-adic Banach space $B(V)$ with unitary $\mathrm{GL}_2(\mathbb{Q}_p)$ action (see, e.g., [C10]). For $V = V_{k,a_p}$ we obtain the first row in the square below.

$$\begin{tikzcd} V_{k,a_p} \arrow[rr, mapsto, "p-\text{adic}", "\text{LLC}"'] \arrow[d, mapsto] & & B(V_{k,a_p}) \arrow[d, mapsto] \\ \bar{V}_{k,a_p} \arrow[r, mapsto, "\text{mod } p", "\text{LLC}"'] & LL(\bar{V}_{k,a_p}) \arrow[r, "\sim", no head] & \bar{B}(V_{k,a_p}). \end{tikzcd}$$

As mentioned earlier, there is also a (semisimple) mod p Local Langlands Correspondence due to Breuil, given by the first map in the bottom row. It uses the classification of the irreducible smooth admissible mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ due to Barthel-Livné [BL94] and Breuil [B03a], to

[1] We exclude the case $r = 2p-1$ in part (7), though one can easily show that the possibilities are $\mathrm{ind}(\omega_2^2)$ or $\omega \oplus \omega$ when $v(a_p) > 0$. This is slightly weaker than the result for $k = 2p+1$ in part (4) which in addition specifies that the latter possibility only occurs when $v(a_p) = 1/2$.

associate to a semisimple mod p representation $\bar{V}$ of $G_{\mathbb{Q}_p}$ a smooth admissible semisimple mod p representation $LL(\bar{V})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. The vertical maps are (semisimplifications of) 'reduction' maps and it is a result of Berger [B10, Thm. A] that the square above 'commutes'. Moreover, there is a locally algebraic representation of $\mathrm{G} = \mathrm{GL}_2(\mathbb{Q}_p)$ given by compact induction:

$$\Pi_{k,a_p} = \frac{\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} \mathrm{Sym}^r(\bar{\mathbb{Q}}_p^2)}{T - a_p},$$

where $\mathrm{K} = \mathrm{GL}_2(\mathbb{Z}_p)$ is the standard compact subgroup, $\mathrm{Z} = \mathbb{Q}_p^*$ is the center of G and $T = T_p$ is the Hecke operator at p, and a lattice in Π_{k,a_p}, namely

$$\Theta_{k,a_p} := \mathrm{image}\left(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} \mathrm{Sym}^r(\bar{\mathbb{Z}}_p^2) \to \Pi_{k,a_p}\right) \simeq \frac{\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} \mathrm{Sym}^r(\bar{\mathbb{Z}}_p^2)}{(T - a_p)(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} \mathrm{Sym}^r(\bar{\mathbb{Q}}_p^2)) \cap \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} \mathrm{Sym}^r(\bar{\mathbb{Z}}_p^2)},$$

such that the semisimplification of the reduction of this lattice satisfies $\bar{\Theta}_{k,a_p}^{\mathrm{ss}} \simeq LL(\bar{V}_{k,a_p})$. (One requires here the conditions $a_p^2 \neq 4p^{k-1}$ and $a_p \neq \pm(1+p)p^{(k-2)/2}$, but these clearly hold if $k \geq 2p+2$ and $v(a_p) < 2$, see [BB10].) By the injectivity of the mod p correspondence, $\bar{\Theta}_{k,a_p}^{\mathrm{ss}}$ determines $\bar{V}_{k,a_p}$ completely.

One must therefore compute $\bar{\Theta}_{k,a_p}$. For $r = k-2 \geq 0$, let $V_r = \mathrm{Sym}^r(\bar{\mathbb{F}}_p^2)$ be the usual symmetric power representation of $\mathrm{GL}_2(\mathbb{F}_p)$ (hence of KZ, with $p \in \mathrm{Z}$ acting trivially). Clearly there is a surjective map

$$\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} V_r \twoheadrightarrow \bar{\Theta}_{k,a_p}. \tag{1.1}$$

Write $X(k, a_p)$ for the kernel of the map in (1.1). A model for V_r is the space of all homogeneous polynomials in the two variables X and Y over $\bar{\mathbb{F}}_p$ with the standard action of $\mathrm{GL}_2(\mathbb{F}_p)$. Let $X_{r-1} \subset V_r$ be the $\mathrm{GL}_2(\mathbb{F}_p)$- (whence KZ-) submodule generated by XY^{r-1}. Let $\theta := X^pY - XY^p \in V_{p+1}$ and let $V_r^{**} = V_{r-2p-2} \otimes D^2$ be the image of the 'multiplication by θ^2' map from V_{r-2p-2} to V_r, where $\otimes D^i$ means tensor with the i-th power of the determinant character D. Then, if $r \geq 2p+1$, a fundamental observation due to Buzzard-Gee (cf. [BG09, Remark 4.4]) shows:

- $v(a_p) > 1 \implies \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}}\, X_{r-1} \subset X(k, a_p)$,
- $v(a_p) < 2 \implies \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}}\, V_r^{**} \subset X(k, a_p)$.

By exactness (of the induction functor), it follows that when $1 < v(a_p) < 2$, the map (1.1) induces a surjective map $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}}\, Q \twoheadrightarrow \bar{\Theta}_{k,a_p}$, where

$$Q := \frac{V_r}{X_{r-1} + V_r^{**}}.$$

To proceed further, one needs to understand the denominator of the 'final quotient' Q. In particular, one must understand the submodule X_{r-1}, and to what extent it intersects with V_r^{**}. In this paper, we give a complete description of X_{r-1}, for weights $r \leq p^2 - p - 2$. That this is at all possible is somewhat surprising and is done as follows. On the one hand, we are able to give precise formulas for the dimension of X_{r-1} (this dimension is bounded by $2p+2$). On the other hand, if X_r denotes the submodule of V_r generated by the 'top' monomial X^r, we observe that X_{r-1} is a submodule of a homomorphic image of the tensor product $X_{r'} \otimes V_1$, with $X_{r'} \subset V_{r'}$ for $r' = r-1$, which can itself then be analyzed using Clebsch-Gordan style formulas. These two ingredients taken together allow us to describe X_{r-1} (and X_r) completely. Further, a socle analysis involving Glover's structure theorem applied to V_r^{**} tells us how large $X_{r-1}^{**} := X_{r-1} \cap V_r^{**}$ is. This gives the structure of the 'final quotient' Q.

When $a = 1$, 2, the structure of X_{r-1} (given in Propositions 6, 11 and Corollaries 8, 14) shows that the 'final quotient' Q has only one Jordan-Hölder factor as a $\mathrm{GL}_2(\mathbb{F}_p)$-module (see Theorems 9, 15), and then one can conclude by [BG09, Proposition 3.3], which treats exactly such a situation. This yields parts (7) and (8) of the Theorem, with $r = 2p$ needing a slightly different argument.

However, when $a \geq 3$ (and $2p+1 \leq r \leq p^2 - p - 2$), a complete analysis of the module X_{r-1} (for which see Propositions 19, 21, 24) shows that Q has two Jordan-Hölder factors and is a direct sum (see Theorems 20, 23, 25):

$$Q \sim V_{p-a+1} \otimes D^{a-1} \oplus V_{p-a-1} \otimes D^a,$$

independent of the complexity of X_{r-1}. We remark that when $0 < v(a_p) < 1$, the relevant 'final quotient' is always irreducible (cf. [BG09]). This brings us to the next stage of the argument, which is more number theoretic rather than algebraic in nature. We must now use the Hecke operator $T = T_p$ to 'eliminate' one of the two Jordan-Hölder factors in Q. This has been carried out for weights in the range $2p + 1 \leq r \leq ap$ (and strictly less than ap when $a = p - 1$), appealing to a periodicity isomorphism of Glover to deal with large weights, and some ingenuity in dealing with binomial coefficients mod p (cf. Section 5.4). In fact, we eliminate the *second* Jordan-Hölder factor above in this 'low weight' range to obtain parts (9) and (10) of the Theorem. In the case of 'higher weights' $ap < r \leq p^2 - p - 2$, we are able to show that *exactly one* of the two Jordan-Hölder factors in Q dies, giving parts (9)' and (10)' of the theorem.

We end this introduction by noting that this work owes a great debt to the works [B03b], [BG09], and the work [G78] on symmetric powers mod p, which we believe can still be profitably mined for further applications to number theory. We also note that after the results in this paper were obtained, we learned that G. Yamashita and S. Yasuda have also obtained results for weights in the range $k < p^2$, though we are not aware of the precise statements.

2. Preliminaries

2.1. **The mod p Local Langlands Correspondence.** Recall $\mathrm{G} = \mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathrm{KZ} = \mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^*$ is the standard compact mod center subgroup of G. Let V be a weight, i.e., an irreducible representation of $\mathrm{GL}_2(\mathbb{F}_p)$ thought of as a representation of KZ by inflating to $\mathrm{GL}_2(\mathbb{Z}_p)$ and making $p \in \mathbb{Q}_p^*$ act trivially. Let $V_r = \mathrm{Sym}^r(\bar{\mathbb{F}}_p^2)$ be the r-th symmetric power of the standard two-dimensional representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on $\bar{\mathbb{F}}_p^2$. The set of weights V is exactly the set of modules $V_r \otimes D^i$, for $0 \leq r \leq p-1$ and $0 \leq i \leq p-2$. For $0 \leq r \leq p-1$, $\lambda \in \bar{\mathbb{F}}_p$ and $\eta : \mathbb{Q}_p^* \to \bar{\mathbb{F}}_p^*$ a character, let

$$\pi(r, \lambda, \eta) \quad := \quad \frac{\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} V_r}{T - \lambda} \otimes \eta$$

be the smooth admissible representation of G, where ind stands for compact induction, and where $T = T_p$ is the Hecke operator coming from the fact that the Hecke algebra

$$\mathrm{End}_{\mathrm{G}}(\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} V_r) = \bar{\mathbb{F}}_p[T_p]$$

is a commutative polynomial algebra generated by the operator T_p. Barthel-Livné [BL94] and Breuil [B03a] have classified all smooth admissible irreducible representations of G. Briefly, they are:

- One-dimensional: $\eta \circ \det$,
- Principal Series: $\pi(r, \lambda, \eta)$, with $\lambda \neq 0$ and $(r, \lambda) \neq (0, \pm 1), (p-1, \pm 1)$,
- Steinberg: $\ker(\pi(0, 1, 1) \to 1) \otimes \eta$, or
- Supercuspidal: $\pi(r, 0, \eta)$,

there being some overlap among the members of this list. Breuil's semisimple mod p Local Langlands Correspondence (see, e.g., [B03b, Def. 1.1]) is given by:

- $\lambda = 0$: $\quad \mathrm{ind}(\omega_2^{r+1}) \otimes \eta \longleftrightarrow \pi(r, 0, \eta)$,
- $\lambda \neq 0$: $\quad \big(\omega^{r+1}\mathrm{unr}(\lambda) \oplus \mathrm{unr}(\lambda^{-1})\big) \otimes \eta \longleftrightarrow \pi(r, \lambda, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], \lambda^{-1}, \eta\omega^{r+1})^{\mathrm{ss}}$, where $\{0, 1, \ldots, p-2\} \ni [p-3-r] \equiv p-3-r \mod p-1$.

2.2. **Hecke operator T.** We need to work explicitly with the Hecke operator $T = T_p$. We recall some well-known formulas involving T from [B03b]. For $m = 0$, set $I_0 = \{0\}$, and for $m > 0$, let

$$I_m = \{[\lambda_0] + [\lambda_1]p + \cdots + p^{m-1}[\lambda_{m-1}] \mid \lambda_i \in \mathbb{F}_p\} \subset \mathbb{Z}_p,$$

where the square brackets denote Teichmüller representatives. For $m \geq 1$, there is a truncation map $[\quad]_{m-1} : I_m \to I_{m-1}$ given by taking the first $m-1$ terms in the p-adic expansion above; for $m = 1$, $[\quad]_{m-1}$ is the 0-map. Let $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. For $m \geq 0$ and $\lambda \in I_m$, if

$$g^0_{m,\lambda} = \begin{pmatrix} p^m & \lambda \\ 0 & 1 \end{pmatrix} \qquad \text{and} \quad g^1_{m,\lambda} = \begin{pmatrix} 1 & 0 \\ p\lambda & p^{m+1} \end{pmatrix},$$

then

$$\mathrm{G} \;=\; \coprod_{m, \lambda \in I_m} \mathrm{KZ}(g^0_{m,\lambda})^{-1} \coprod \coprod_{m, \lambda \in I_m} \mathrm{KZ}(g^1_{m,\lambda})^{-1}.$$

Let R be a $\bar{\mathbb{Z}}_p$-algebra and let $V = \mathrm{Sym}^r R^2$ be the usual symmetric power representation of KZ, modelled on homogeneous polynomials of degree r in the variables X, Y. For $g \in \mathrm{G}$, $v \in V$, let $[g, v] \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} V$ be the function with support in $\mathrm{KZ}g^{-1}$ given by

$$g' \mapsto \begin{cases} g'g \cdot v & \text{if } g' \in \mathrm{KZ}g^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Then the action of T on $[g, v]$ can be given explicitly as follows, when $g = g^0_{n,\mu}$ with $n \geq 0$ and $\mu \in I_n$. Write $v = \sum_{i=0}^{r} c_i X^{r-i} Y^i$, with $c_i \in R$. One has:

$$T = T^+ + T^-,$$

where

$$
\begin{aligned}
T^{+}([g^0_{n,\mu},v]) &= \sum_{\lambda\in I_1}\left[g^0_{n+1,\mu+p^n\lambda},\sum_{j=0}^{r}\left(p^j\sum_{i=j}^{r}c_i\binom{i}{j}(-\lambda)^{i-j}\right)X^{r-j}Y^j\right],\\
T^{-}([g^0_{n,\mu},v]) &= \left[g^0_{n-1,[\mu]_{n-1}},\sum_{j=0}^{r}\left(\sum_{i=j}^{r}p^{r-i}c_i\binom{i}{j}\left(\frac{\mu-[\mu]_{n-1}}{p^{n-1}}\right)^{i-j}\right)X^{r-j}Y^j\right] \quad (n>0),\\
T^{-}([g^0_{n,\mu},v]) &= [\alpha,\sum_{j=0}^{r}p^{r-j}c_jX^{r-j}Y^j] \quad (n=0).
\end{aligned}
$$

2.3. **Some $\mathrm{GL}_2(\mathbb{F}_p)$-modules.** We concentrate on what is necessary for the case $1<v(a_p)<2$. Let us set some notation. Let $\Gamma:=\mathrm{GL}_2(\mathbb{F}_p)$. Recall that the Γ-module $V_r=\mathrm{Sym}^r(\bar{\mathbb{F}}_p^2)$ has dimension $r+1$ over $\bar{\mathbb{F}}_p$ and is thought of as all homogeneous polynomials in two variables X, Y of degree r over $\bar{\mathbb{F}}_p$.

Let $X_t:=\langle X^{r-t}Y^t\rangle$ be the Γ-submodule of V_r generated by $X^{r-t}Y^t\in V_r$, for $t=0,1,\cdots,r$. Note that $X_t=X_{r-t}$, since $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\in\Gamma$. Of particular interest to us are the Γ-submodules X_{r-1} and X_r. We have the following easy lemma.

Lemma 3. *We have:*

(1) $X_r\subset X_{r-1}$
(2) $\dim X_{r-1}\le 2p+2$
(3) $\dim X_r\le p+1$.

Proof. Part (1) follows from the fact that $X^r=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\cdot X^{r-1}Y-X^{r-1}Y$. For part (2) note that if $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ and $ac\neq 0$, then $g\cdot X^{r-1}Y=(aX+cY)^{r-1}(bX+dY)=bc^{r-1}X(\frac{a}{c}X+Y)^{r-1}+a^{r-1}d(X+\frac{c}{a}Y)^{r-1}Y$. If $a=0$, then $g\cdot X^{r-1}Y=bc^{r-1}XY^{r-1}+c^{r-1}dY^r$, and similarly if $c=0$. This shows that the $2p+2$ elements X^r, $X(kX+Y)^{r-1}(k\in\mathbb{F}_p)$, $(X+lY)^{r-1}Y(l\in\mathbb{F}_p)$, Y^r span X_{r-1}. Part (3) is well-known and proved similarly. $\square$

For $1\le r\le p-1$ in the irreducible range, $X_r=X_{r-1}$. We claim that the inclusion in (1) above is strict thereafter, at least if $r\le p^2-p+2$. More precisely, one can prove the stronger result (in this range):

Proposition 4. *For $p\le r\le p^2-p+2$, we have $X_{r-1}\not\subset X_r+V_r^{**}$.*

Proof. We show that $XY^{r-1}\notin X_r+V_r^{**}$. If not, we have a relation of the form

$$XY^{r-1}=\sum c_i(a_iX+b_iY)^r+\theta^2\cdot\sum_{j=0}^{r-2p-2}d_jX^{r-2p-2-j}Y^j,$$

with $\theta^2=X^{2p}Y^2-2X^{p+1}Y^{p+1}+X^2Y^{2p}$. Recall that $\dim V_r^{**}=(r+1)-(2p+2)$ is non-zero only if $r\ge 2p+2$, and so the d_j only 'occur' in this range. Letting $x=\sum_i c_ia_ib_i^{r-1}$, and comparing coefficients of XY^{r-1}, X^pY^{r-p}, $X^{2p-1}Y^{r-2p+1}$, $X^{3p-2}Y^{r-3p+2}$, $X^{4p-3}Y^{r-4p+3}$ etc. on both sides, we obtain a system of equations in x and the d_j:

$$\begin{aligned}
1 &= \binom{r}{1}x \\
0 &= \binom{r}{p}x + d_{r-3p} \\
0 &= \binom{r}{2p-1}x - 2d_{r-3p} + d_{r-4p+1} \\
0 &= \binom{r}{3p-2}x + d_{r-3p} - 2d_{r-4p+1} + d_{r-5p+2} \\
0 &= \binom{r}{4p-3}x + d_{r-4p+1} - 2d_{r-5p+2} + d_{r-6p+3}, \quad \text{etc.}
\end{aligned}$$

Now if $p \leq r \leq 2p+1$, then $\binom{r}{p} = 1$ (or 2, for $r = 2p, 2p+1$) is non-zero, so the second equation shows $x = 0$, which contradicts the first equation. Actually, for $r = p$, $2p$, the first equation already leads to the contradiction $1 = 0$. So we may assume that $r \geq 2p+2$. Write $r = (m+1)p - a$, with $3 \leq m+1 \leq p$, $1 \leq a \leq p$. We consider three cases (in the third case, we assume $m+1 \neq p$).

(1) $mp \leq r \leq (m+1)p - (m+1)$. By 6.1.1 in the Appendix, the binomial coefficients $\binom{r}{bp-(b-1)}$ above vanish, except possibly the first two, for which $\binom{r}{1} = r$ and $\binom{r}{p} = m$. Now all the d_j vanish (working from the last equation up), so the above system reduces to solving the first two equations $1 = rx$ and $0 = mx$. But $p > m \implies p \nmid m$, showing that these two equations are inconsistent.

(2) $r = (m+1)p - m$ or $r = (m+1)p - m + 1$. These cases are similar to the previous case, except that the last binomial coefficient $\binom{r}{(m+1)p-m} = 1$ or $-(m-1)$, respectively, and so is also non-zero (in the latter case note $p > m$, so $1 \leq m-1 < p-1$). Since for these values of r, the d_j in the last equation are 0 (since all $j < 0$), we have $x = 0$, which contradicts the first equation.

(3) $r = (m+1)p - a$, with $m \neq p-1$ and $1 \leq a \leq m-2$. By 6.1.2 in the Appendix, the binomial coefficients satisfy $\binom{r}{1} = r$, $\binom{r}{p} = m$, and $\binom{r}{bp-(b-1)} = (-1)^{a+b-3}\binom{m}{b-1}\binom{b-2}{a-1}$, for $b = 2, 3, \ldots, m+1$. Note $j = r - (m+1)p + m - 2 \geq 0 \iff a \leq m-2$ and this is the last d_j in the equations above since for the next j, we have $j = r - (m+2)p + m - 1 \geq 0 \iff a \leq -2$. In fact, the last equation has $b = m+1$ and is $0 = (-1)^{a+m-2}\binom{m}{m}\binom{m-1}{a-1}x + d_{r-(m+1)p+m-2}$, which says that $d_{r-(m+1)p+m-2} = (-1)^{a+m-1}\binom{m}{m}\binom{m-1}{a-1}x$. The second last equation then gives

$$d_{r-mp+(m-3)} = (-1)^{a+m-1}\left[2\binom{m}{m}\binom{m-1}{a-1} - 1\binom{m}{m-1}\binom{m-2}{a-1}\right]x.$$

Similarly, the third last gives (after some simplification)

$$d_{r-(m-1)p+m-4} = (-1)^{a+m-1}\left[3\binom{m}{m}\binom{m-1}{a-1} - 2\binom{m}{m-1}\binom{m-2}{a-1} + 1\binom{m}{m-2}\binom{m-3}{a-1}\right]x.$$

Continuing 'upwards' we obtain $d_{r-3p} =$

$$(-1)^{a+m-1}\left[(m-1)\binom{m}{m}\binom{m-1}{a-1} - (m-2)\binom{m}{m-1}\binom{m-2}{a-1} + (m-3)\binom{m}{m-2}\binom{m-3}{a-1} - \cdots + (-1)^m\binom{m}{2}\binom{1}{a-1}\right]x,$$

which, by 6.1.3 in the Appendix, turns out to be just x. Thus the first two equations become $1 = -ax$ and $0 = (m+1)x$. Since $p > m+1$, we see $x = 0$, contradicting the first equation. $\square$

As explained in the introduction, we need to compute the Jordan-Hölder factors (from now often abbreviated to JH factors) of $Q = \frac{V_r}{X_{r-1}+V_r^{**}}$, which is nothing but the quotient of $\frac{V_r}{X_r+V_r^{**}}$ by the image of X_{r-1}. By the proposition, this image is non-zero in the range $p \leq r \leq p^2-p+2$. To get a feeling for the number of JH factors involved, we need to introduce a bit more notation. Write $V_r^* = V_{r-p-1} \otimes D$ for the image of the 'multiplication by θ' map from V_{r-p-1} to V_r. Set

$$X_r^* = X_r \cap V_r^*, \qquad X_r^{**} = X_r \cap V_r^{**} \qquad \text{and} \qquad X_{r-1}^{**} = X_{r-1} \cap V_r^{**}.$$

We consider three cases, depending on the size of the quotient $\frac{X_r^*}{X_r^{**}}$. We use the well-known fact (see, e.g., [G78, (4.2) and Note (2), p. 455] or [G10, Thm. 3.3.1]) that $\frac{V_r}{V_r^*}$ (and hence $\frac{V_r^*}{V_r^{**}}$) has 2 JH factors, for r sufficiently large.

(1) The 'full case': $X_r^* = X_r^{**}$. In this case, we have

$$0 \to \frac{V_r^*}{V_r^{**}} \to \frac{V_r}{X_r+V_r^{**}} \to \frac{V_r}{X_r+V_r^*} \to 0$$

since $V_r^* \cap (X_r + V_r^{**}) = V_r^{**}$, so that for sufficiently large r the middle term has 3 Jordan-Hölder factors (it was shown in [BG09] that the last term has 1 JH factor).

(2) The 'mixed case': $0 \subsetneq \frac{X_r^*}{X_r^{**}} \subsetneq \frac{V_r^*}{V_r^{**}}$. In this case, we have the following diagram:

(2.1)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \frac{X_r^*}{X_r^{**}} \arrow[r] \arrow[d] & \frac{X_r}{X_r^{**}} \arrow[r] \arrow[d] & \frac{X_r}{X_r^*} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{V_r^*}{V_r^{**}} \arrow[r] \arrow[d] & \frac{V_r}{V_r^{**}} \arrow[r] \arrow[d] & \frac{V_r}{V_r^*} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{X_r+V_r^*}{X_r+V_r^{**}} \arrow[r] \arrow[d] & \frac{V_r}{X_r+V_r^{**}} \arrow[r] \arrow[d] & \frac{V_r}{X_r+V_r^*} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}$$

so $V_r/(X_r + V_r^{**})$, given by the last row, has 2 Jordan-Hölder factors for sufficiently large r.

(3) The 'uncase': $\frac{X_r^*}{X_r^{**}} = \frac{V_r^*}{V_r^{**}}$. We have:

$$\frac{V_r}{X_r+V_r^{**}} \simeq \frac{V_r}{X_r+V_r^*}$$

has 1 Jordan-Hölder factor, for sufficiently large r. This seems to make life simpler, but this case does not occur when $r \geq 2p+2$. Indeed, by part (3) of Lemma 3, $\dim V_r/(X_r + V_r^{**}) \geq 2p+2-(p+1) = p+1$, whereas it is known that $\dim V_r/(X_r+V_r^{*}) < p+1$, since $X_r/X_r^* \neq 0$.

We now specify the Jordan-Hölder factors in each of the cases above in some detail. For the 'right column' in the 3×3 diagram above, we have (by [G78, Note (2), p. 455]) that, for $r \geq 2p+1$, there is a non-split exact sequence of $\mathrm{M}_2(\mathbb{F}_p)$-modules

$$0 \to V_{i'} \to \frac{V_r}{V_r^*} \to V_{p-1-i'} \otimes D^{i'} \to 0,$$

where $r+1 = r' + j'(p-1)$, with $r' = p+i'$, $1 \leq i' \leq p-1$, for some j'.

For the 'left column' in the 3×3 diagram above, note that multiplication by θ induces the isomorphism

$$\frac{V_r^*}{V_r^{**}} \simeq \frac{V_{r-p-1}}{V_{r-p-1}^*} \otimes D,$$

so the JH factors of the left column can be computed from the JH factors of the right column (for a smaller weight). When $r \geq 3p$, we obtain

$$0 \to V_{i''} \otimes D \to \frac{V_r^*}{V_r^{**}} \to V_{p-1-i''} \otimes D^{i''+1} \to 0,$$

where $r-(p+1)+1 = r'' + j''(p-1)$, with $r'' = p+i''$, $1 \leq i'' \leq p-1$, for some j'', whereas when $2p+1 \leq r \leq 3p-1$, we obtain

$$0 \to V_{r-2p} \otimes D \to \frac{V_r^*}{V_r^{**}} \to V_{3p-1-r} \otimes D^{r-1} \to 0.$$

These sequences may split.

Thus the JH factors of $\frac{V_r}{X_r+V_r^{**}}$ in the mixed and full cases are as follows. There is

- always $V_{p-1-i'} \otimes D^{i'}$,
- plus one of $V_{p-1-i''} \otimes D^{i''+1}$ or $V_{i''} \otimes D$, if $r \geq 3p$ (or one of $V_{3p-1-r} \otimes D^{r-1}$ or $V_{r-2p} \otimes D$, if $2p+1 \leq r \leq 3p-1$) in the mixed case,
- plus the other JH factor, in the full case.

We now do some dimension analysis. We have:

$$\dim V_r/(X_r + V_r^{**}) = \dim V_r - \dim X_r - \dim V_r^{**} + \dim X_r^{**}.$$

If follows that, for $r \geq 2p+2$,

- Mixed case: $\dim X_r - \dim X_r^{**} = 2+i'+i''$ or $p+1+i'-i''$, if $r \geq 3p$ (and equals $2+i'+r-2p$ or $3p+1+i'-r$, if $2p+2 \leq r \leq 3p-1$).
- Full case: $\dim X_r - \dim X_r^{**} = 1+i'$.

Since

$$(i', i'') = (1, p-2), (2, p-1), (3, 1), (4, 2), \ldots, (p-2, p-4), (p-1, p-3),$$

we see that the full case *must* occur for $p \geq 5$ if $(p+3)/2 \leq i' \leq p-1$ by part (3) of the Lemma above, if $r \geq 3p$ (and also if $(5p-1)/2 \leq r \leq 3p-3$, for r is the range $2p+2 \leq r \leq 3p-1$).

2.4. **The structure of V_r.** We recall some results on the structure of V_r, particularly for $r \leq p^2 - p - 2$, which we shall loosely refer to as the Glover range. The main result is the structure theorem (Theorem 5) below. We remark that while the full detail provided by the theorem may not be completely necessary for some of our subsequent arguments, the theorem does provide a convenient and natural context for our results. We also remark that Glover's symmetric powers representations are $\mathbb{F}_p$-vector spaces (whereas so far they have been $\bar{\mathbb{F}}_p$-vector spaces). This will not affect the main results. In particular, while some of the auxiliary results we prove in Sections 3-5 are *a priori* over $\mathbb{F}_p$, the structure of Q given in parts (4) of Theorems 9, 15, 20, 23 and 25 also holds over $\bar{\mathbb{F}}_p$.

In [G78], Glover writes V_r as V_{r+1} (so that the subscript is the dimension). This has the virtue of making the numerology in his formulas for the structure of V_r simpler. In this section, we shall follow this convention. In later sections, we distinguish the two notations by adding a superscript 'G' to Glover's V_r. Thus $V_r^G = V_{r-1}$. Also the local notation k and r below are Glover's and are independent of the global $r = k - 2$ used elsewhere in the paper. This will not cause any confusion.

Theorem 5 (Glover [G78], (6.4))**.** *For $0 \leq k \leq p-2$ and $0 \leq r \leq p-1$, one can write*

$$V_{kp+r}^G = \bar{V}_{kp+r} \oplus \bar{\bar{V}}_{kp+r}$$

where $\bar{V}_{kp+r}$ is a non-projective indecomposable module and $\bar{\bar{V}}_{kp+r}$ is a projective module. Moreover, one has:

$$\begin{aligned}
\operatorname{soc} \bar{V}_{kp+r} &= \begin{cases} \bigoplus_{i=1}^{p-\max(k+1,r)} V_{2p-k-r-2i}^G \otimes D^{k+r+i-1} & \text{if } k+r \geq p \\ \bigoplus_{i=0}^{\min(k+1,r)-1} V_{k+r-2i}^G \otimes D^{i} & \text{if } k+r < p, \end{cases} \\
\bar{V}_{kp+r}/(\operatorname{soc} \bar{V}_{kp+r}) &= \begin{cases} \bigoplus_{i=0}^{p-1-\max(p-k,r)} V_{p+k-r-1-2i}^G \otimes D^{r+i} & \text{if } k < r \\ \bigoplus_{i=0}^{\min(p-k,r)-1} V_{p-k+r-1-2i}^G \otimes D^{k+i} & \text{if } k \geq r, \end{cases}
\end{aligned}$$

and

$$\bar{\bar{V}}_{kp+r} = \begin{cases} P & \text{if } k < r \\ P \oplus V_{(k-r)p}^G \otimes D^r & \text{if } k \geq r, \end{cases} \quad \textit{where} \quad P = \begin{cases} V_{p(k+r+1-p)}^G & \text{if } k+r \geq p \\ 0 & \text{if } k+r < p. \end{cases}$$

Note that we have corrected two typos in the original statement of (6.4) in [G78]. The first is significant: the '$k+r+i$' has been replaced by '$k+r+i-1$' in the power of D in the first line of the formula for the socle of the non-projective part. Secondly, an incorrect '$k \leq r$' has been made into a '$k \geq r$' in the description of the projective part.

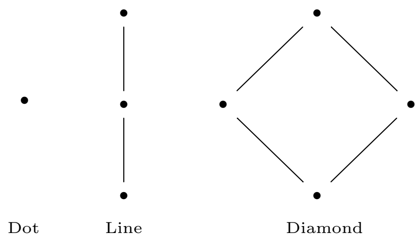
In [G78], Glover also shows that the non-projective part is eventually periodic, and that the projective part can be described in terms of certain principal indecomposable modules (PIMs), at least in small weights. Let $1 \leq m \leq p$ and $0 \leq n \leq p-2$, and let $P[m,n]$ be the principal indecomposable $\bar{\mathbb{F}}_p[\Gamma]$-module occurring in the regular representation of $\Gamma = \mathrm{GL}_2(\mathbb{F}_p)$, which

- for $m = p$, is just the projective module $V_p^G \otimes D^n$, and

- for $1 \leq m < p$, has top and bottom JH factors (cosocle and socle) $V_m^G \otimes D^n$, and middle JH factor:

$$\begin{cases} V_{p-2}^G \otimes D^{n+1} & \text{if } m = 1, \\ (V_{p-1-m}^G \otimes D^{m+n}) \oplus (V_{p+1-m}^G \otimes D^{m+n-1}) & \text{if } 1 < m < p. \end{cases}$$

See [G78, Sec. 6]. The lattice of submodules of $P[m,n]$ can be considered as a 'dot', a 'line' or a 'diamond', in the three cases above. Pictorially, we have:



where the bottom most dot is the socle and the top most dot the cosocle. By [G78, (6.3)], we have, for $0 \leq k \leq p-2$, the projective module

$$V_{(k+1)p}^G = (V_{(k-1)p}^G \otimes D) \oplus P[p-k,k] \tag{2.2}$$

$$= \oplus_{m=0}^{\lfloor k/2 \rfloor} P[p-k+2m, k-m], \tag{2.3}$$

where $\lfloor a \rfloor$ is the greatest integer less than or equal to a.

Finally, we also recall here a Clebsch-Gordan style identity for the tensor products of various V_r^G's [G78, (5.2)]:

$$V_r^G \otimes V_2^G \simeq (V_{r-1}^G \otimes D) \oplus V_{r+1}^G, \quad \text{if } p \nmid r. \tag{2.4}$$

3. The case $r \equiv 1 \mod p-1$

We treat this opening case completely, in the Glover range, for $r \geq 3p-2$. We start with the following key result.

Proposition 6. *For $2p-1 \leq r \leq p^2-p+1$ and $r \equiv 1 \mod p-1$, we have*

$$X_{r-1} \simeq V_{2p-1}.$$

Proof. Set $r' = 2p-1$ and think of $V_{r'}$ as homogeneous polynomials of degree r' in the variables S, T. We define an explicit Γ-map:

$$\begin{aligned} \eta : X_{r'-1} &\longrightarrow X_{r-1} \\ g \cdot S^{r'-1}T &\mapsto g \cdot X^{r-1}Y, \end{aligned}$$

for $g \in \Gamma$. The map η is compatible with multiplication by scalars. Indeed, if $g = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in \Gamma$, then $g \cdot S^{r'-1}T = a^{r'-1}dS^{r'-1}T = dS^{r'-1}T$ and similarly $g \cdot X^{r-1}Y = dX^{r-1}Y$. One may also check that, for the above values of r, η is well-defined (using the spanning set of $2p$ elements given below). We now claim that η is injective. If not, the kernel of η would contain $\mathrm{soc}\, V_{r'} \simeq V_{p-2} \otimes D \simeq V_{2p-1}^*$, which is generated by $\theta \cdot S^{p-2} = S^{2p-2}T - S^{p-1}T^p$, so that $S^{p-1}T^p \in X_{r'-1}$. By the proof of part (2) of Lemma 3, we may write

$$S^{p-1}T^p \quad = \quad \sum_{i=0}^{p-1} a_i S(iS+T)^{2p-2} + \sum_{i=0}^{p-1} b_i (S+iT)^{2p-2} T, \tag{3.1}$$

for some scalars a_i, b_i. Comparing coefficients of $S^{p-1}T^p$ and $S^{2p-2}T$, we get

$$1 = \sum_{i=0}^{p-1} a_i i^{p-2} \quad \text{and} \quad 2 = \sum_{i=0}^{p-1} b_i. \tag{3.2}$$

Since $\eta(S^{p-1}T^p) = \eta(S^{2p-2}T)$, applying η to (3.1) we get $X^{r-1}Y = \sum_{i=0}^{p-1} a_i X(iX+Y)^{r-1} + \sum_{i=0}^{p-1} b_i (X+iY)^{r-1}Y$. Comparing the coefficient of $X^{r-1}Y$, we get $1 = (\sum_{i=0}^{p-1} a_i i^{r-2})(r-1) + \sum_{i=0}^{p-1} b_i$. Since $r-2 \equiv p-2 \mod p-1$, the formulas (3.2) show that this simplifies to $r \equiv 0 \mod p$, which is a contradiction for the values of r considered here. Thus η is injective.

We now claim that $\dim X_{r-1} \leq 2p$ (cf. part (2) of Lemma 3). In view of the proof of the lemma, it suffices to show that X^r is in the span of the elements $X(kX+Y)^{r-1}$, for $k \in \mathbb{F}_p$ (and similarly Y^r is in the span of the $(X+lY)^{r-1}Y$, $l \in \mathbb{F}_p$). Note $X(k^{-1}X+Y)^{r-1}$ ($k \neq 0$) differs from $X(X+kY)^{r-1}$ by a scalar. We have

$$X(X+kY)^{r-1} \quad = \quad X^r + \binom{r-1}{1} kX^{r-1}Y + \cdots + \binom{r-1}{r-2} k^{r-2}X^2Y^{r-2} + k^{r-1}XY^{r-1}.$$

But $\sum_{k\neq 0} k^j \equiv 0 \mod p$, unless $j \equiv 0 \mod p-1$ (in which case the sum is $-1 \mod p$). On the other hand, when $j \equiv 0 \mod p-1$, we have $\binom{r-1}{j} \equiv 0 \mod p$, for $j \neq 0$, $r-1$, by 6.2 in the Appendix. Thus summing the above identity over all $0 \neq k \in \mathbb{F}_p$, only the first and last terms on the RHS survive, giving $(p-1)X^r + (p-1)XY^{r-1} = -X^r - XY^{r-1}$, showing X^r is in the desired span.

Finally, we claim $X_{r'-1} = V_{2p-1}$. Note $X_{r'-1}$ has the filtration $0 \subset X_{r'}^* \subset X_{r'} \subset X_{r'-1} \subset V_{2p-1}$. We showed in Proposition 4 that $X_{r'} \subsetneq X_{r'-1}$ (including $p=2$). Also by [G78, (4.5)], $X_{r'}/X_{r'}^* = V_2^G$. Finally $X_{r'}^* \neq 0$, since $\dim X_{r'} \geq 3$; indeed $X_{r'}$ contains $X^{r'}$, $Y^{r'}$ and mixed monomials as well (since $r' \not\equiv 0 \mod p$). Thus $X_{r'-1}$ has at least 3 JH factors, whereas $V_{2p-1} = V_{2p}^G = P[p-1,1]$ also does, so $X_{r'-1} = V_{2p-1}$.

We have constructed an injective map η from a space of dimension $2p$ to one of dimension at most $2p$, so it must be an isomorphism. □

Remark 7. The proof above breaks down for the next value $r = p^2 \equiv 1 \mod p-1$ since, for instance, $\binom{r-1}{p-1} \not\equiv 0 \mod p$.

Corollary 8. *For* $r \equiv 1 \mod p-1$ *as above, we are in the mixed case, and*

- $\dim X_{r-1} = 2p$
- $\operatorname{soc} X_{r-1} = V_{p-2} \otimes D = X_r^*$
- $X_r/X_r^* = V_1$
- $X_r^{**} = 0$, *and*

- $X_{r-1}/X_r \simeq V_{p-2} \otimes D$.

Proof. This follows from the Proposition, noting that $\dim X_r \geq 3$, so $0 \neq X_r^* \simeq V_{p-2} \otimes D$, so $\dim X_r = p+1$. We shall shortly see that $X_r^{**} \subset X_{r-1}^{**} = 0$, so we are in the mixed case (with $(i', i'') = (1, p-2)$). The last isomorphism follows, but can be proved directly noting that the surjective map $X_{r-1} \to V_{p-2} \otimes D$ induced by $X^{r-1}Y \mapsto S^{p-2}$ (where we model the space on the right by homogeneous polynomials in S and T of degree $p-2$) has kernel X_r. $\square$

Theorem 9. *Suppose $p \geq 5$ and $3p-2 \leq r \leq p^2-2p+2$ with $r \equiv 1 \mod p-1$. Write $V_r = \bar{V} \oplus \bar{\bar{V}}$. Then,*

(1) $X_{r-1} = \bar{\bar{V}}$
(2) $V_r^{**} \subset \bar{V}$, *for $p \gg 0$*
(3) $X_{r-1}^{**} = 0$, *so that $X_{r-1} + V_r^{**}$ is a direct sum*
(4) $V_r/(X_{r-1} + V_r^{**}) \simeq V_1 = V_2^G$ *has 1 JH factor.*

Proof. Write $r = (m+1)p - m$ for $2 \leq m \leq p-2$. Then $V_r = V_{(m+1)p-m+1}^G$. Write $(m+1)p-m+1 = m \cdot p + (p-m+1)$ so that "$k = m$" and "$r = p-m+1$" in the local notation of the structure theorem (Theorem 5). Note that $0 \leq k \leq p-2 \iff p \geq m+2$ and $0 \leq r \leq p-1$ is automatic. Note also that $k + r = p+1 \geq p$. Applying Theorem 5, we obtain $V_r = \bar{V} \oplus \bar{\bar{V}}$ (dropping the subscript from the notation), with

$$\begin{aligned}
\operatorname{soc} \bar{V} &= \begin{cases} \bigoplus_{i=1}^{m-1} V_{p-1-2i}^G \otimes D^{1+i} & \text{if } p \geq 2m \\ \bigoplus_{i=1}^{p-m-1} V_{p-1-2i}^G \otimes D^{1+i} & \text{if } m+2 \leq p < 2m, \end{cases} \\
\bar{V}/(\operatorname{soc} \bar{V}) &= \begin{cases} \bigoplus_{i=0}^{m-2} V_{2m-2-2i}^G \otimes D^{2-m+i} & \text{if } p \geq 2m \\ \bigoplus_{i=0}^{p-m-1} V_{2p-2m-2i}^G \otimes D^{m+i} & \text{if } m+2 \leq p < 2m, \end{cases}
\end{aligned}$$

and

$$\bar{\bar{V}} = \begin{cases} V_{2p}^G & \text{if } p \geq 2m \\ V_{2p}^G \oplus V_{(2m-p-1)p}^G \otimes D^{2-m} & \text{if } m+2 \leq p < 2m. \end{cases}$$

Let's turn our attention to $V_r^{**} = V_{(m-1)p-m-1}^G \otimes D^2$. Write $(m-1)p-m-1 = (m-2) \cdot p + (p-m-1)$ so that this time "$k = m-2$", "$r = p-m-1$" and $k + r = p-3 < p$ in the local notation of Theorem 5. Applying that result, we obtain $V_r^{**} = \bar{V}^{**} \oplus \bar{\bar{V}}^{**}$, with

$$\begin{aligned}
\operatorname{soc} \bar{V}^{**} &= \begin{cases} \bigoplus_{i=0}^{m-2} V_{p-3-2i}^G \otimes D^{2+i} & \text{if } p \geq 2m \\ \bigoplus_{i=0}^{p-m-2} V_{p-3-2i}^G \otimes D^{2+i} & \text{if } m+2 \leq p < 2m, \end{cases} \\
\bar{V}^{**}/(\operatorname{soc} \bar{V}^{**}) &= \begin{cases} \bigoplus_{i=0}^{m-3} V_{2m-2-2i}^G \otimes D^{2-m+i} & \text{if } p \geq 2m \\ \bigoplus_{i=0}^{p-m-2} V_{2p-2m-2i}^G \otimes D^{m+i} & \text{if } m+2 \leq p < 2m, \end{cases}
\end{aligned}$$

and

$$\bar{\bar{V}}^{**} = \begin{cases} 0 & \text{if } p \geq 2m \\ V^G_{(2m-p-1)p} \otimes D^{2-m} & \text{if } m+2 \leq p < 2m. \end{cases}$$

To prove (1), we note that if $X_{r-1} \cap \bar{V} \neq 0$, then the socles of these spaces would share a common JH factor (necessarily $V_{p-2} \otimes D$). But an inspection of the formulas above shows the weight of maximal dimension in the socle of $\bar{V}$ has dimension $p-3 < p-1$, a contradiction. Thus $X_{r-1} \subset \bar{\bar{V}}$. If $p \geq 2m$, this inclusion is necessarily an equality, by Proposition 6. Part (2) is proved similarly, at least for $p \geq 2m$. Indeed, for such p, the socle of (the non-projective part of) V_r^{**} does not have a JH factor in common with $\bar{\bar{V}}$. (3) now follows immediately from (1) and (2) since $\bar{V} \oplus \bar{\bar{V}}$ is a direct sum. We also see that $V_r/V_r^{**} = (\bar{V}/V_r^{**}) \oplus \bar{\bar{V}}$ has $1 \oplus 3 = 4$ Jordan-Hölder factors, as it should. In particular, taking a further quotient by X_{r-1} gives part (4) using (1) and noting that $\bar{V}/V_r^{**} = V_2^G$, from the formulas above. □

Remark 10. For smaller primes $m+2 \leq p < 2m$, the above argument becomes more complicated, since it seems the projective part of V_r^{**}, namely $\bar{\bar{V}}^{**} = V^G_{(2m-p-1)p} \otimes D^{2-m}$ is contained in $\bar{\bar{V}}$, and that (2) no longer holds. However, we still claim that (3) holds, and $V_r^{**} \cap X_{r-1} = 0$. Indeed, this is more or less obvious in view of the direct sum occurring in $\bar{\bar{V}}$. But it can also be seen directly using the fact that we can write $V^G_{(2m-p-1)p} \otimes D^{2-m}$ as a direct sum of PIMs, none of whose socles is $V^G_{p-1} \otimes D$. Indeed, the condition $m+2 \leq p \leq 2m-1$ implies $0 \leq 2m-p-1 \leq p-5$, so the formula to compute small dimensional projective symmetric powers (2.2) applies to give

$$V^G_{(2m-p-1)p} \otimes D^{2-m} = \oplus_{l=0}^{m-(p+3)/2} P[2p-2m+2+2l, m-p-l].$$

But $2p-2m+2+2l = p-1 \iff l = m-(p+3)/2$ is the top index, but then $m-p-l = (3-p)/2 \not\equiv 1 \mod p-1$. In any case, (3) holds, and so again does (4), for $p \geq m+2$.

The question remains of treating the 'small primes' (e.g., $p \leq m+1$, when $r \equiv 1 \mod p-1$). These primes are outside the Glover range where some things begin to fail, and as mentioned in the introduction, we do not consider such primes in this paper.

4. The case $r \equiv 2 \mod p-1$

We start by noting that X_{r-1} is as large as possible in (and just beyond) the Glover range.

Proposition 11. *Say $3p-1 \leq r \leq p^2-p+2$ and $r \equiv 2 \mod p-1$. Then*

$$\dim X_{r-1} = 2p+2.$$

Remark 12. The proposition is false for the two values of $r \equiv 2 \mod p-1$, namely $r = 2p$ and $r = p^2+1$, just outside the range of r treated above.

The proposition follows by hand for $p=3$, and from the following general proposition when $p \geq 5$ (this result will be used in the next section for weights in the 'upper triangle').

Proposition 13. *Let $2 \leq a \leq p-3$. If $ap < r \leq p^2-3$ with $r \equiv a \mod p-1$, then*

$$\dim X_{r-1} = 2p+2.$$

Proof. By part (2) of Lemma 3, it suffices to show that the standard spanning set of vectors X^r, $X(kX+Y)^{r-1}$ ($k \in \mathbb{F}_p$), $(X+lY)^{r-1}Y$ ($l \in \mathbb{F}_p$), Y^r are linearly independent. Suppose that there is a relation of the form:

$$AX^r + \sum_{k=0}^{p-1} c_k X(kX+r)^{r-1} + \sum_{l=0}^{p-1} d_l (X+lY)^{r-1}Y + BY^r = 0,$$

for A, B, c_k, $d_\ell \in \mathbb{F}_p$. We must show that all these constants are zero. Expanding the sums, and rearranging terms we get:

$$AX^r + \sum_{i=0}^{r-1} \binom{r-1}{i} \sum_{k=0}^{p-1} c_k k^i X^{1+i} Y^{r-1-i} + \sum_{j=0}^{r-1} \binom{r-1}{j} \sum_{l=0}^{p-1} d_l l^j X^{r-1-j} Y^{1+j} + BY^r = 0.$$

Comparing the coefficients of X^r and Y^r gives

$$A + \sum_{k=0}^{p-1} c_k k^{r-1} = 0 \quad \text{and} \quad B + \sum_{l=0}^{p-1} d_l l^{r-1} = 0,$$

so that it suffices to show that all the $c_k = d_j = 0$. Similarly comparing the coefficients of $X^{r-1}Y$ and XY^{r-1} yields

$$(r-1)\sum_{k=1}^{p-1} c_k k^{r-2} + \sum_{l=0}^{p-1} d_l = 0 \quad \text{and} \quad \sum_{k=0}^{p-1} c_k + (r-1)\sum_{l=1}^{p-1} d_l l^{r-2} = 0,$$

so that it further suffices to show that $c_k = d_l = 0$, for all $k, l \neq 0$.

Comparing the inner (but two) coefficients, i.e., those of $X^{r-t}Y^t$ for $2 \leq t \leq r-2$, yields the system of equations:

$$\binom{r-1}{t} \sum_{k=1}^{p-1} c_k k^{r-1-t} + \binom{r-1}{t-1} \sum_{l=1}^{p-1} d_l l^{t-1} = 0.$$

To solve these, it is convenient to define the 'power sums'

$$C_i = \sum_{k=1}^{p-1} c_k k^i \quad \text{and} \quad D_j = \sum_{l=1}^{p-1} d_l l^j,$$

for $1 \leq i, j \leq p-1$ (identifying C_i and D_j for other values of i and j with those in this range according to the congruence classes of i and j mod $p-1$; in particular, $C_0 = C_{p-1}$ and $D_0 = D_{p-1}$). The equations above then read

$$\binom{r-1}{t} C_{r-1-t} \quad + \quad \binom{r-1}{t-1} D_{t-1} = 0,$$

for $2 \leq t \leq r-2$. Write $r = (m+a)p - m$ for $1 \leq m \leq p-a$. By 6.3 in the Appendix, $\binom{r-1}{t} = 0$, for t in the successive ranges $p-m \leq t \leq p-1$; $2p-m \leq t \leq 2p-1$; $\ldots$; $(m+a-1)p-m \leq t \leq (m+a-1)p-1$, and is non-zero otherwise. Thus at the values of t in these ranges the equation above is useless, except at the beginning and ending (+1) values in each range, and when $a = 2$, those values give:

$$C_{p-m-1} = C_{p-m} = \cdots = C_{p-1} = 0 \quad \text{and} \quad D_{p-m-1} = D_{p-m} = \cdots = D_{p-1} = 0,$$

showing that the C_i and D_j vanish for 'large' values of i and j. When $3 \leq a \leq p-3$, the indices overshoot $p-1$ all the way to $p+a-3 \equiv a-2 \mod p-1$, so we see that C_i and D_j also vanish for 'small' values of i and j, namely for $1 \leq i, j \leq a-2$ (in particular, if $m = p-a$, then all the C_i and D_j vanish), but we do not make use of this here. To treat the 'small' values of i and j, namely $1 \leq i, j \leq p-m-2$, we consider instead the system of equations obtained by taking the t-th and $(t+p-1)$-th equations:

$$\begin{aligned} \binom{r-1}{t} C_{r-1-t} \quad &+ \quad \binom{r-1}{t-1} D_{t-1} = 0 \\ \binom{r-1}{t+p-1} C_{r-1-t} \quad &+ \quad \binom{r-1}{t+p-2} D_{t-1} = 0, \end{aligned}$$

for $2 \leq t \leq p-m-1$. Noting that $r \not\equiv 0 \mod p$, we see that the determinant of this system, after some simplification, is

$$\binom{r-1}{t}\binom{r-1}{t+p-2} - \binom{r-1}{t+p-1}\binom{r-1}{t-1} \quad = \quad \frac{p-1}{r} \cdot \binom{r}{t}\binom{r}{t+p-1},$$

which is easily seen to be non-zero mod p in the range $2 \leq t \leq p-m-1$. We obtain that $D_j = 0$ for $1 \leq t \leq p-m-2$, so that all the D_j vanish. One now checks (we omit the proof) that all the C_i vanish as well. But now an easy exercise involving Vandermonde determinants shows that $c_k = d_l = 0$, for $k, l \neq 0$, as desired. □

We return to the case $a = 2$. We investigate the quotient $\frac{V_r}{X_{r-1}+V_r^{**}}$ in the Glover range. We first treat the base case $r = 3p-1$ separately. The reason is that this case is a bit simpler from the case of other $r \equiv 2 \mod p-1$, since $V_{3p-1} = V_{3p}^G$ is projective.

In fact, by (2.2), we have for $p \geq 5$, that $V_{3p}^G = V_p^G \otimes D \oplus P[p-2, 2]$, where $P[p-2, 2]$ has socle filtration

$$V_{p-2}^G \otimes D^2 \quad — \quad V_3^G \ \oplus \ V_1^G \otimes D \quad — \quad V_{p-2}^G \otimes D^2.$$

By Proposition 11, we are forced to have

$$X_{r-1} = V_p^G \otimes D \ \oplus \ \left(V_{p-2}^G \otimes D^2 \quad — \quad V_3^G \ \oplus \ V_1^G \otimes D\right), \tag{4.1}$$

by dimension considerations. We will soon show that this holds for all $r \equiv 2 \mod p-1$ (in the range of the proposition). Accepting this we can now give the structure of X_r completely (in the above range). It is known, by [G78, (4.5)], that $X_r/X_r^* = V_3^G$, since $r^G := \dim V_r^G \equiv 3 \mod p-1$. If $X_r^* = 0$, then $V_3^G \in \operatorname{soc} X_r \subset \operatorname{soc} X_{r-1}$, which is clearly not the case. It follows that $0 \neq X_r^* = V_{p-2}^G \otimes D^2$, that

$$0 \to X_r^* = V_{p-2}^G \otimes D^2 \to X_r \to V_3^G \to 0 \tag{4.2}$$

is exact, and that $\dim X_r = p+1$. We obtain:

Corollary 14. *Let $3p-1 \le r \le p^2-p+2$, and $r \equiv 2 \mod p-1$. Then we are in the full case, X_{r-1} and X_r have the structure as in* (4.1) *and* (4.2) *above, and*

- $\dim X_r = p+1$
- $X_r/X_r^* = V_3^G$
- $X_r^* = X_r^{**} = V_{p-3} \otimes D^2$
- $X_{r-1}/X_r = V_1^G \otimes D \oplus V_p^G \otimes D.$

Proof. The claim about X_{r-1}/X_r follows from (4.1) and (4.2) (the structure of X_{r-1} for r not just $3p-1$ will be verified in the next theorem). We remark that this may also be proved directly by noting that there are surjective maps $X_{r-1} \to V_p^G \otimes D$ and $X_{r-1} \to V_1^G \otimes D$ induced by $X^{r-1}Y \mapsto S^{p-1}$ and $X^{r-1}Y \mapsto 1$ respectively (with the usual notation for the variables in the space on the right). Both these maps are compatible with the scalar action, are well-defined, and have X_r in the kernel. Thus $V_1^G \otimes D$ and $V_p^G \otimes D$ are quotients of X_{r-1}/X_r and since the second module is projective, it is also a submodule, whence X_{r-1}/X_r is a direct sum of these two modules. Unlike the case $r \equiv 1 \mod p-1$, we are not in the mixed case, since $(i', i'') = (2, p-1) \implies \dim X_r - \dim X_r^{**} = p+3$, a contradiction to part (3) of Lemma 3, or $= 4$, a contradiction to $X_r^{**} \subset X_r^*$, since $p \ge 5$. Thus, we are in the full case, with $X_r^{**} = X_r^*$. □

Theorem 15. *Suppose $p \ge 5$ and $3p-1 \le r \le p^2-2p+3$ with $r \equiv 2 \mod p-1$. Write $V_r = V_{r+1}^G = \bar{V} \oplus \bar{\bar{V}}$. Then,*

1. $X_{r-1} \subsetneq \bar{\bar{V}}$
2. $V_r^{**} \cap \bar{\bar{V}} = V_{p-2}^G \otimes D^2$, *for $p \gg 0$*
3. $X_{r-1}^{**} = V_{p-2}^G \otimes D^2$, *so that $X_{r-1} + V_r^{**}$ is not a direct sum*
4. $V_r/(X_{r-1} + V_r^{**}) \simeq V_{p-2}^G \otimes D^2$ *has 1 JH factor.*

Proof. Note that (1), (2), (3), (4) are trivial when $r = 3p-1$ ($p \ge 5$) since $V_r = \bar{\bar{V}}$, $\bar{V} = 0$ and $V_r^{**} = V_{p-3} \otimes D^2$ is in the irreducible range. So we assume $r \ge 4p-2$. Write $r = (m+2)p - m$ for $2 \le m \le p-3$. Then $V_r = V_{(m+2)p-m+1}^G$. Write $(m+2)p - m + 1 = (m+1)\cdot p + (p-m+1)$ so that "$k = m+1$" and "$r = p-m+1$" in the local notation of Glover's structure theorem. Again $0 \le k \le p-2 \iff p \ge m+3$ and $0 \le r \le p-1$ is automatic. Note also that $k + r = p+2 \ge p$. Applying Theorem 5, we obtain $V_r = \bar{V} \oplus \bar{\bar{V}}$, with

$$\operatorname{soc} \bar{V} = \begin{cases} \bigoplus_{i=1}^{m-1} V_{p-2-2i}^G \otimes D^{2+i} & \text{if } p \ge 2m+1 \\ \bigoplus_{i=1}^{p-m-2} V_{p-2-2i}^G \otimes D^{2+i} & \text{if } m+3 \le p < 2m+1, \end{cases}$$

$$\bar{V}/(\operatorname{soc} \bar{V}) = \begin{cases} \bigoplus_{i=0}^{m-2} V_{2m-1-2i}^G \otimes D^{2-m+i} & \text{if } p \ge 2m+1 \\ \bigoplus_{i=0}^{p-m-2} V_{2p-2m-1-2i}^G \otimes D^{m+1+i} & \text{if } m+3 \le p < 2m+1, \end{cases}$$

and

$$\bar{\bar{V}} = \begin{cases} V_{3p}^G & \text{if } p \geq 2m+1 \\ V_{3p}^G \oplus V_{(2m-p)p} \otimes D^{2-m} & \text{if } m+3 \leq p < 2m+1, \end{cases}$$

where $V_{3p}^G = P[p,1] \oplus P[p-2,2]$ has socle filtration:

$$V_p^G \otimes D \oplus V_{p-2}^G \otimes D^2 \quad \text{——} \quad V_3^G \oplus (V_1^G \otimes D) \quad \text{——} \quad V_{p-2}^G \otimes D^2.$$

Turning our attention to $V_r^{**} = V_{mp-m-1}^G \otimes D^2$, write $mp - m - 1 = (m-1) \cdot p + (p - m - 1)$ so that this time "$k = m-1$", "$r = p - m - 1$" and $k + r = p - 2 < p$ in the notation of Theorem 5. Applying that result, we obtain $V_r^{**} = \bar{V}^{**} \oplus \bar{\bar{V}}^{**}$, with

$$\begin{aligned} \operatorname{soc} \bar{V}^{**} \quad &= \quad \begin{cases} \bigoplus_{i=0}^{m-1} V_{p-2-2i}^G \otimes D^{2+i} & \text{if } p \geq 2m+1 \\ \bigoplus_{i=0}^{p-m-2} V_{p-2-2i}^G \otimes D^{2+i} & \text{if } m+3 \leq p < 2m+1, \end{cases} \\ \bar{V}^{**}/(\operatorname{soc} \bar{V}^{**}) \quad &= \quad \begin{cases} \bigoplus_{i=0}^{m-2} V_{2m-1-2i}^G \otimes D^{2-m+i} & \text{if } p \geq 2m+1 \\ \bigoplus_{i=0}^{p-m-2} V_{2p-2m-1-2i}^G \otimes D^{1+m+i} & \text{if } m+3 \leq p < 2m+1, \end{cases} \end{aligned}$$

and

$$\bar{\bar{V}}^{**} = \begin{cases} 0 & \text{if } p \geq 2m+1 \\ V_{(2m-p)p}^G \otimes D^{2-m} & \text{if } m+3 \leq p < 2m+1. \end{cases}$$

To prove (1), we use the multiplication or tensor product map (used to remarkable effect in [G78]):

$$\begin{aligned} \phi_{m,n} \quad : \quad & V_m \otimes V_n \longrightarrow V_{m+n} \\ & X^a Y^{m-a} \otimes X^b Y^{n-b} \mapsto X^{a+b} Y^{m+n-(a+b)}, \end{aligned}$$

for $m = r-1$, $n = 1$, to deduce the structure of X_{r-1} for $r \equiv 2 \mod p-1$ from that of X_r for $r \equiv 1 \mod p-1$ (which was written down in the previous section). This is one of the key new techniques used in this paper. Since we are now moving between various ambient symmetric powers and there can be some confusion regarding notation, let us set $r' = r - 1 \equiv 1 \mod p-1$, and let $\phi = \phi_{r',1}$. Now $X^{r-1}Y = \phi(X^{r'} \otimes Y)$, so

$$X_{r-1} \subset \phi(X_{r'} \otimes V_2^G), \tag{4.3}$$

where $X_{r'} \subset V_{r'}$, by the Γ-linearity of ϕ. By Corollary 8, we have $0 \to V_{p-1}^G \otimes D \to X_{r'} \to V_2^G \to 0$. Tensoring this with V_2^G, using right exactness of the tensor functor, and using the identity (2.4) twice, we obtain:

$$V_{p-2}^G \otimes D^2 \oplus V_p^G \otimes D \longrightarrow X_{r'} \otimes V_2^G \longrightarrow V_1^G \otimes D \oplus V_3^G \to 0.$$

Since the dimension of X_{r-1} in the Glover range is $2p+2$ by Proposition 11, the inclusion (4.3) shows that we must have:

$$0 \to V_{p-2}^G \otimes D^2 \oplus V_p^G \otimes D \longrightarrow X_{r-1} \longrightarrow V_1^G \otimes D \oplus V_3^G \to 0.$$

In particular, the possible JH factors in soc X_{r-1} do not match those of soc $\bar{V}$, at least if $p \geq 2m+1$. Indeed, inspection of the formulas above shows the weight of maximal dimension in the socle of $\bar{V}$ has dimension $p-4 < p-2$ (and one also easily checks that neither $V_1^G \otimes D$ nor V_3^G can occur in soc $\bar{V}$). In particular, $X_{r-1} \cap \bar{V} = 0$, and we obtain (1).

For part (2) the formulas above show that soc $V_r^{**} \cap$ soc $\bar{\bar{V}} = V_{p-2}^G \otimes D^2$, if $p \geq 2m+1$. Indeed, this JH factor is the only one in (the socle of) V_r^{**} which is not in $\bar{V}$. This forces (2) to hold, since if the intersection in (2) was bigger than just $V_{p-2}^G \otimes D^2$, then V_r/V_r^{**} would have fewer than 4 JH factors (note here that the V_3^G in V_r^{**} is not the V_3^G in $\bar{\bar{V}}$, since otherwise $X_r \subset V_r^{**}$, and $X_r = X_r^{**}$, a contradiction).

Part (3) follows from (1) and (2), noting that $V_{p-2}^G \otimes D^2$ is in X_{r-1}^{**}, from what we have just shown. For (4), we note that $V_r/V_r^{**} = \bar{V}/(V_r^{**} \cap \bar{V}) \oplus \bar{\bar{V}}/(V_r^{**} \cap \bar{\bar{V}}) = 0 \oplus \bar{\bar{V}}/(V_{p-2}^G \otimes D^2)$ has $0 \oplus 4 = 4$ Jordan factors, all of which are in the two PIMs above. Taking a further quotient by X_{r-1} kills 3 of these (by (3)). Thus only the top JH factor in the PIM $P[p-2,2]$ survives in the quotient in (4). □

Remark 16. The primes in the range $m+3 \leq p < 2m+1$ can be treated in a similar fashion by closer inspection of the formulas above. We sketch the argument. One checks that (1) continues to hold as above. (2) is clearly false, since $\bar{\bar{V}}^{**} = V_{(2m-p)p}^G \otimes D^{2-m}$ is also in the intersection. However, one checks none of the JH factors of X_{r-1} meet soc $(\bar{\bar{V}}^{**})$. This implies that $X_{r-1} \subset V_{3p}^G$ and that $\bar{\bar{V}}^{**} \cap V_{3p}^G = 0$. It follows that $V_{3p}^G \cap V_r^{**} = V_{p-2}^G \otimes D^2$, and this implies (3) and (4).

The case $r = 2p$ is a bit different, and very interesting, so we treat it separately now. It requires, for the first time so far in this paper, the use of the Hecke operator T_p, so leads nicely into the computations of the next section. The first remark to make is that part (1) of the previous theorem does not seem to be true. Similarly, parts (2) and (3) are also false since clearly $V_r^{**} = 0$. Even though part (4) seems to be true for $r = 2p$, it is *irrelevant* for our purposes, since when $r < 2p+1$ (the induction of) X_{r-1} may not lie in the kernel of the map (1.1). However X_r always lies in the kernel (if $v(a_p) > 0$), and what we need instead is the following result.

Lemma 17. *Suppose $r = 2p \equiv 2 \mod p-1$. Then, for $p \geq 5$,*

$$V_r/X_r \quad \simeq \quad V_p^G \otimes D \,\oplus\, V_{p-2}^G \otimes D^2$$

has 2 JH factors.

Proof. Indeed, $r^G = 2p+1$. Taking "$k=2$" and "$r=1$" in the local notation of Theorem 5, we obtain $V_r = \bar{V} \oplus \bar{\bar{V}}$, with

$$\text{soc}\,\bar{V} = V_3^G, \quad \bar{V}/(\text{soc}\,\bar{V}) = V_{p-2}^G \otimes D^2 \quad \text{and} \quad \bar{\bar{V}} = V_p^G \otimes D,$$

for $p \gneq 3 = 2+1$. One checks directly that $X_r = V_3^G$ (or see part (i) of Proposition 19 below). The lemma follows. □

Numerically, X_{r-1} seems to have dimension $p+3$, so that one should have $X_{r-1} = V_3^G \oplus V_p^G \otimes D$, making part (4) of the theorem true, but we do not investigate this further as we do not need it.

The weight $r = 2p$ is the first weight we have encountered in this paper for which the quotient module through which the map (1.1) factors (in this case V_r/X_r) has more than 1 JH factor, namely the two JH factors in the lemma, which we label J_1 and J_2. We now use the Hecke operator $T = T_p$ to 'eliminate' the second JH factor $J_2 = V_{p-2}^G \otimes D^2$. Consider the function

$$f_2 = \sum_{\lambda \in \mathbb{F}_p} [g^0_{2,p[\lambda]}, \frac{1}{p}(X^{p-1}Y^{p+1} - Y^{2p})] \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} \mathrm{Sym}^{2p} \bar{\mathbb{Q}}_p^2.$$

Recall $T = T^+ + T^-$. Applying the formula for T^- in Section 2.2 to f_2, one has

$$\begin{aligned} T^- f_2 &= [g^0_{1,0}, \sum_{j=0}^{r} \left(-\frac{1}{p}\binom{r}{j} \sum_{\lambda \in \mathbb{F}_p} [\lambda]^{r-j} \right) X^{r-j}Y^j + pv^-] \\ &= [g^0_{1,0}, -X^{2p-2}Y^2 + 2X^{p-1}Y^{p+1} - Y^{2p}] \mod p \\ &= [g^0_{1,0}, X^{2p-2}Y^2 - 2\theta X^{p-2}Y - Y^{2p}] \mod p, \end{aligned}$$

where both the main term (given by the sum) and v^- are integral vectors. The second equality uses the fact that the power sum $\sum_{\lambda \in \mathbb{F}_p} [\lambda]^i = 0$, unless $i \equiv 0 \mod p-1$, in which case it is $p-1$ (or p, if $i = 0$), as well as the easily checked congruences: $-\frac{p-1}{p}\binom{2p}{2} \equiv -1 \mod p$, $-\frac{p-1}{p}\binom{2p}{p+1} \equiv 2 \mod p$, and, $-\frac{p}{p}\binom{2p}{2p} \equiv -1 \mod p$. We now project the last vector $w = X^{2p-2}Y^2 - 2\theta X^{p-2}Y - Y^{2p}$ to the space $V_r/X_r = V_r/(X_r + V_r^{**})$. Clearly $Y^{2p} \in X_r$ dies. Project w further to J_2, which is also clearly the bottom right entry in the square in (2.1), since $V_r^* = V_{p-1} \otimes D = J_1$. By the commutativity of the lower right 2×2 subsquare in (2.1), the term in w involving θ also dies since it is in V_r^*. Finally we claim that $X^{2p-2}Y^2 \mapsto X^{p-3} \in J_2$. This is easily checked noting that the map $V_r \to J_2$ is given by $V_r \to V_r/V_r^* \to V_{p+1}/V_{p+1}^* \to J_2$, where the middle map is the periodicity isomorphism between the various quotients V_r/V_r^* and is induced by the (inverse of) Glover's ψ map occurring in the proof of [G78, (4.2)], and the last map is Breuil's projection map in [B03b, Lem. 5.3(ii)] (for Breuil's $k = p+3$). Using the explicit description of these maps, we see $X^{2p-2}Y^2 \mapsto X^{p-1}Y^2 \mapsto X^{p-3} \neq 0$, as the reader may easily check. In any case, the vector w maps to a non-zero vector in J_2.

Similarly, one easily checks that $T^+ f_2 \in \sum_\mu [g^0_{3,\mu}, \bar{\mathbb{Z}}_p \cdot X^{r-1}Y + pv^+]$, for various $\mu \in I_3$ (which are not important here), where again, v^+ is an integral vector. Going mod p, and projecting to J_2, and noting that $X^{r-1}Y \mapsto X^pY \mapsto 0$ under the Glover and Breuil maps above, we see that $T^+ f_2 = 0$ in J_2.

Now $a_p f_2$ is integral and dies mod p, if $v(a_p) > 1$, so putting things together, $(T - a_p)f_2$ maps to $[g^0_{1,0}, X^{p-3}]$ when projected to J_2, with $0 \neq X^{p-3} \in J_2$. This shows that (the induction of) J_2 dies in $\bar{\Theta}_{k,a_p}$, and that the map (1.1) factors through (the induction of) the surviving Jordan-Hölder factor, namely $J_1 := V_p^G \otimes D$. Applying [BG09, Prop. 3.3], we see that on inertia $\bar{V}_{k,a_p}$ is $\mathrm{ind}(\omega_2^{2p+1}) \sim \mathrm{ind}(\omega_2^{p+2})$. This proves the $r = 2p$ case of part (8) of Theorem 2.

Remark 18. We have actually proved that the $r = 2p$ case in Theorem 2 holds whenever $v(a_p) > 1$. This is consistent with Theorem 1, part (5), which is valid for $v(a_p) > 2 = \lfloor \frac{2p}{p-1} \rfloor$, when $p \geq 5$.

5. The case $r \equiv a \mod p-1$, for $3 \leq a \leq p-1$

The cases $r \equiv 1, 2 \mod p-1$ are the only cases (in the Glover range) for which the 'final quotient' $Q = V_r/(X_{r-1} + V_r^{**})$ is irreducible, i.e., has exactly 1 JH factor. As we show now, for r in the congruence classes $r \equiv a \mod p-1$ with $3 \leq a \leq p-1$, Q has 2 JH factors. We must then use explicit computations with the Hecke operator T_p (as was done for $r = 2p$) to eliminate exactly one of these JH factors. We do this when $2p \leq r \leq ap$, but the cases $ap < r \leq p^2 - p - 2$ (for $3 \leq a \leq p-3$) are still open. All this is perhaps best summarized by the following picture:

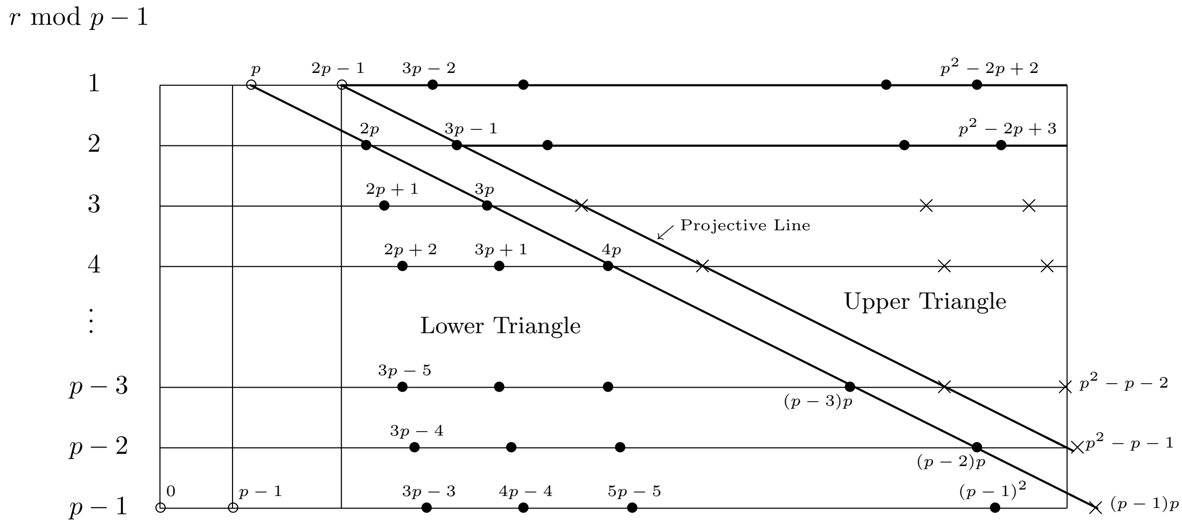


Figure 1: Weights in the Glover Range.

The large (boundary) rectangle depicts all weights in the Glover range, that is, between (the columns) $r = 0$ and $r = p^2 - p - 2$, broken down by congruence classes (rows) mod $p-1$. The weights in the first small vertical rectangle $0 \leq r \leq p-1$ were treated by Fontaine and Edixhoven, whereas the weights in the second vertical rectangle $p \leq r \leq 2p-1$ were treated by Breuil (these endpoints are marked with a circle $\circ$). When the slope is between 1 and 2, the new weights we are able to treat completely are those in the *lower triangle* as well as those on the *diagonal* (by which we mean the line $r = ap$, for $2 \leq a \leq p-2$). These weights are marked with bullets $\bullet$. The first two rows of the *upper triangle* were also treated completely in the two previous sections, so are again marked with bullets. For the remaining weights in the upper triangle we are able to show the 'final quotient' Q has 2 JH factors but so far are unable to use T_p to eliminate one of them (so we mark these weights with a cross $\times$, noting that the last two weights $r = p^2 - p - 1$ and $r = p^2 - p$ are not even in the

Glover range). Nonetheless, we prove a general lemma which shows that exactly one of the two JH factors in Q must occur, giving us a (weaker) result here as well.

5.1. **The Lower Triangle.** We start by treating the lower triangle of weights in the Glover range. The following proposition is the natural place to start.

Proposition 19. *Let $a = 1, 2, \ldots, p-1$. Let $p \leq r \leq ap$ with $r \equiv a \mod p-1$. Then,*

(1) $\dim X_r = a+1$*, and* $X_r \simeq V_{a+1}^G$
(2) *if $p < r < ap$ and $a \geq 2$, then* $\dim X_{r-1} = 2a$ *and* $X_{r-1} \simeq V_{a+1}^G \oplus V_{a-1}^G \otimes D$.

Proof. To prove (1), write $r = (a-n)p + n$, for $a = 1, \ldots, p-1$, and $n = 0, 1, \ldots a-1$. Let $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$. Then

$$g \cdot X^r = (\alpha X^p + \gamma Y^p)^{a-n}(\alpha X + \gamma Y)^n = \sum_{j=0}^{a} \alpha^{a-j}\gamma^j f_j(X, Y),$$

for some polynomials $f_j(X, Y)$, $j = 0, \ldots a$, independent of g. This shows that $\dim X_r \leq a+1$. On the other hand, since $r \geq p$, one has $X_r/X_r^* = V_{a+1}^G$, by (4.5) of [G78], proving (1).

For (2), let $r' = r - 1 \equiv a - 1 \mod p-1$. Then $p \leq r' \leq (a-1)p$ (NB: $r < ap$ and $r \equiv a \mod p-1 \implies r \leq (a-1)p+1$), and $1 \leq a-1 \leq p-2$. By (1), we have $\dim X_{r'} = a$ and $X_{r'} = V_a^G$. But $X_{r-1} \subset \phi(X_{r'} \otimes V_2^G) = \phi(V_a^G \otimes V_2^G) = \phi(V_{a-1}^G \otimes D \oplus V_{a+1}^G)$, by (2.4), where ϕ is Glover's multiplication map. Thus $\dim X_{r-1} \leq 2a$ and X_{r-1} is a homomorphic image of $V_{a-1}^G \otimes D \oplus V_{a+1}^G$. By Proposition 4, $V_{a+1}^G = X_r \subsetneq X_{r-1}$, whence (2). □

Theorem 20. *Let $p \geq 5$ and let $3 \leq a \leq p-1$. Suppose $2p+1 \leq r < ap$ and $r \equiv a \mod p-1$. Write $V_r = V_{r+1}^G = \bar{V} \oplus \bar{\bar{V}}$. Then,*

(1) $X_{r-1} \subset \bar{V}$ *(if $a \neq p-1$)*
(2) $V_r^{**} \supset \bar{\bar{V}}$ *(if $a \neq p-1$)*
(3) $X_{r-1}^{**} = 0$*, so that $X_{r-1} + V_r^{**}$ is a direct sum*
(4) $V_r/(X_{r-1} + V_r^{**}) \simeq V_{p-a+2}^G \otimes D^{a-1} \oplus V_{p-a}^G \otimes D^a$ *has exactly 2 JH factors.*

Proof. Write $r = (a-n)p + n$ with $n = 1, \ldots, a-2$. Then $(a-n)p + n + 1 = kp + r$, in Glover's notation, with "$k = a - n$" and "$r = n + 1$". Note $2 \leq k \leq p-2$ and $1 \leq r \leq p-2$, and

$$k + r = \begin{cases} (i) & a+1 < p \quad \text{if } 3 \leq a \leq p-2 \\ (ii) & p \geq p \quad \text{if } a = p-1. \end{cases}$$

We apply Glover's structure theorem in these two cases. Writing $V_r = \bar{V} \oplus \bar{\bar{V}}$ as in Theorem 5, we have:

Case (i): $3 \leq a \leq p-2$.

$$\begin{aligned}
\operatorname{soc} \bar{V} &= \begin{cases} \bigoplus_{i=0}^{a-n} V_{a+1-2i}^{G} \otimes D^{i} & \text{if } a \leq 2n \\ \bigoplus_{i=0}^{n} V_{a+1-2i}^{G} \otimes D^{i} & \text{if } a \geq 2n+1, \end{cases} \\
\bar{V}/(\operatorname{soc} \bar{V}) &= \begin{cases} \bigoplus_{i=0}^{a-n-1} V_{p+a-2n-2-2i}^{G} \otimes D^{n+1+i} & \text{if } a \leq 2n \\ \bigoplus_{i=0}^{n} V_{p-a+2n-2i}^{G} \otimes D^{a-n+i} & \text{if } a \geq 2n+1, \end{cases}
\end{aligned}$$

and

$$\bar{\bar{V}} = \begin{cases} 0 & \text{if } a \leq 2n \\ V_{(a-2n-1)p}^{G} \otimes D^{n+1} & \text{if } a \geq 2n+1. \end{cases}$$

Now $V_r^{**} = V_{(a-n-2)p+n-1}^{G} \otimes D^{2}$, so in Glover's notation we have "$k = a-n-2$", "$r = n-1$" and $k + r \leq p-4 < p$. By Theorem 5, $V_r^{**} = \bar{V}^{**} \oplus \bar{\bar{V}}^{**}$, with

$$\begin{aligned}
\operatorname{soc} \bar{V}^{**} &= \begin{cases} \bigoplus_{i=0}^{a-n-2} V_{a-3-2i}^{G} \otimes D^{2+i} & \text{if } a \leq 2n \\ \bigoplus_{i=0}^{n-2} V_{a-3-2i}^{G} \otimes D^{2+i} & \text{if } a \geq 2n+1, \end{cases} \\
\bar{V}^{**}/(\operatorname{soc} \bar{V}^{**}) &= \begin{cases} \bigoplus_{i=0}^{a-n-3} V_{p+a-2n-2-2i}^{G} \otimes D^{n+1+i} & \text{if } a \leq 2n \\ \bigoplus_{i=0}^{n-2} V_{p-a+2n-2i}^{G} \otimes D^{a-n+i} & \text{if } \ a \geq 2n+1, \end{cases}
\end{aligned}$$

and

$$\bar{\bar{V}}^{**} = \begin{cases} 0 & \text{if } a \leq 2n \\ V_{(a-2n-1)p}^{G} \otimes D^{n+1} & \text{if } a \geq 2n+1. \end{cases}$$

Thus V_r/V_r^{**} has four JH factors, all coming from $\bar{V}$, with $V_{a+1}^{G} \oplus V_{a-1}^{G} \otimes D$ as a submodule and $V_{p-a+2}^{G} \otimes D^{a-1} \oplus V_{p-a}^{G} \otimes D^{a}$ as a quotient module. By parts (1) and (2) of Proposition 19, we obtain part (1) of the theorem. Parts (2), (3), (4) can now also be easily checked.

Case (ii): $a = p-1$.

$$\begin{aligned}
\operatorname{soc} \bar{V} &= \begin{cases} \bigoplus_{i=1}^{p-n-1} V_{p-2i}^{G} \otimes D^{i} & \text{if } p \leq 2n+1 \\ \bigoplus_{i=1}^{n} V_{p-2i}^{G} \otimes D^{i} & \text{if } p \geq 2n+2, \end{cases} \\
\bar{V}/(\operatorname{soc} \bar{V}) &= \begin{cases} \bigoplus_{i=0}^{p-n-2} V_{2p-2n-3-2i}^{G} \otimes D^{n+1+i} & \text{if } p \leq 2n+1 \\ \bigoplus_{i=0}^{n} V_{2n+1-2i}^{G} \otimes D^{-n+i} & \text{if } p \geq 2n+2, \end{cases}
\end{aligned}$$

and

$$\bar{\bar{V}} = \begin{cases} V_{p}^{G} & \text{if } p \leq 2n+1 \\ V_{p}^{G} \oplus V_{(p-2n-2)p}^{G} \otimes D^{n+1} & \text{if } p \geq 2n+2. \end{cases}$$

Similarly $V_r^{**} = V^G_{(p-3-n)p+n-1} = \bar{V}^{**} \oplus \bar{\bar{V}}^{**}$, with "$k = p-3-n$", "$r = n-1$" and $k+r = p-4 < p$. So

$$\begin{aligned}
\operatorname{soc} \bar{V}^{**} &= \begin{cases} \bigoplus_{i=0}^{p-n-3} V^G_{p-4-2i} \otimes D^{2+i} & \text{if } p \leq 2n+1 \\ \bigoplus_{i=0}^{n-2} V^G_{p-4-2i} \otimes D^{2+i} & \text{if } p \geq 2n+2, \end{cases} \\
\bar{V}^{**}/(\operatorname{soc} \bar{V}^{**}) &= \begin{cases} \bigoplus_{i=0}^{p-n-4} V^G_{2p-2n-3-2i} \otimes D^{n+1+i} & \text{if } p \leq 2n+1 \\ \bigoplus_{i=0}^{n-2} V^G_{2n+1-2i} \otimes D^{-n+i} & \text{if } \ p \geq 2n+2, \end{cases}
\end{aligned}$$

and

$$\bar{\bar{V}}^{**} = \begin{cases} 0 & \text{if } p \leq 2n+1 \\ V^G_{(p-2n-2)p} \otimes D^{n+1} & \text{if } p \geq 2n+2. \end{cases}$$

Thus, for all $p \geq 5$, one sees that V_r/V_r^{**} has 4 JH factors, with $V_p^G \oplus V_{p-2}^G \otimes D$ as a submodule, and $V_3 \otimes D^{-1} \oplus V_1$ as a quotient. By Proposition 19, the former is just X_{r-1}, and then parts (3) and (4) can be easily checked to hold. □

5.2. **The Diagonal.** We now turn to the diagonal $r = ap$ between the lower and upper triangles of weights. The following gives the behaviour of X_{r-1} on this line (cf. Proposition 19).

Proposition 21. *Let $3 \leq a \leq p-2$, and let $r = ap$. Then $\dim X_{r-1} = a + p + 2$, and X_{r-1} fits in the exact sequence:*

$$0 \to V^G_{a+1} \oplus V^G_{p-a+2} \otimes D^{a-1} \to X_{r-1} \to V^G_{a-1} \otimes D \to 0.$$

Proof. When $r = p$, $\dim X_{r-1} = p+1$, and when $r = 2p$, $\dim X_{r-1}$ is $p+3$ (numerically), so the result is not true for $a = 1, 2$. When $r = 3p$, the proposition is true, and we leave this case as an exercise for the reader. So assume $4 \leq a \leq p-2$. As usual, let $r' = r-1 = ap-1 = (a-1)p+p-1 \equiv a-1 \mod p-1$, with $3 \leq a-1 \leq p-3$. Note that r' is in the upper triangle! By Proposition 24 (to be proved shortly) applied with $a-1$ in place of a, there is an exact sequence

$$0 \to V^G_{p-(a-1)} \otimes D^{a-1} \to X_{r'} \to V^G_{(a-1)+1} \to 0.$$

Tensoring this with V_2^G and applying (2.4) twice, we obtain

$$V^G_{p-a} \otimes D^a \oplus V^G_{p-a+2} \otimes D^{a-1} \to X_{r'} \otimes V_2^G \to V^G_{a-1} \otimes D \oplus V^G_{a+1} \to 0. \tag{5.1}$$

Since $X_{r-1} \subset \phi(X_{r'} \otimes V_2^G)$, we obtain that the JH factors of X_{r-1} are a subset of the 4 JH factors occurring in (5.1). We now claim that the 'first' JH factor $V^G_{p-a} \otimes D^a$ in (5.1) does not occur as a sub of X_{r-1}. Indeed, write $V_r = \bar{V} \oplus \bar{\bar{V}}$ as in Glover's structure theorem. By the proof of Theorem 20, Case (i) (which is clearly also valid for $n = 0$ in the notation there) we see that if $p \geq a+2$, then

$$\begin{aligned}
\operatorname{soc} \bar{V} &= V^G_{a+1} \\
\bar{V}/(\operatorname{soc} \bar{V}) &= V^G_{p-a} \otimes D^a \\
\bar{\bar{V}} &= V^G_{(a-1)p} \otimes D = \oplus_{m=0}^{\lfloor (a-2)/2 \rfloor} P[p-a+2+2m, a-1-m].
\end{aligned} \tag{5.2}$$

Since the socles of the PIMs above have dimension at least $p-a+2 > p-a$, the claim follows. Thus X_{r-1} has at most 3 JH factors. The 'fourth' JH factor V_{a+1}^G in (5.1) always occurs in X_{r-1} (it is actually just X_r, by part (1) of Proposition 19, and so also occurs as a sub of X_{r-1}). We now claim that the 'second' JH factor $V_{p-a+2}^G \otimes D^{a-1}$ in (5.1) must also occur as a sub of X_{r-1}. If not, $X_{r-1} \simeq V_{a-1}^G \otimes D \oplus V_{a+1}^G$, since $X_{r-1} \supsetneq X_r = V_{a+1}^G$, by Proposition 4. But then $V_{a-1}^G \otimes D$ lies in the socle of X_{r-1}, hence in the socle of V_r. But an inspection of the formulas above shows that it lies neither in soc $(\bar{V})$ nor in the socle of any of the PIMs (indeed, one would need to simultaneously have $p-a+2+2m = a-1 \iff m = a - \frac{p+3}{2}$ and $a-m-1 \equiv 1 \mod p-1$, which is impossible). We also claim that the third JH factor $V_{a-1} \otimes D$ in (5.1) always occurs, since if not we'd have $X_{r-1} = V_{p-a+2}^G \otimes D^{a-1} \oplus V_{a+1}^G \subset V_r^{**} + X_r$, again contradicting Proposition 4. Indeed, the factor $V_{p-a+2}^G \otimes D^{a-1}$ is equal to V_r^{**} when $a = 3$ $(r = 3p)$ and is in the socle of V_r^{**} by (5.3) below, when $a \gneq 3$. This proves the proposition. $\square$

Remark 22. We did not treat $a = p-1$ above, because $r = (p-1)p$ is slightly outside the Glover range $r \leq p^2 - p - 2$, and also because we needed to use $a - 1 \leq p - 3$ in the proof.

Theorem 23. *Let $p \geq 5$, $3 \leq a \leq p-2$ and $r = ap$. Write $V_r = V_{r+1}^G = \bar{V} \oplus \bar{\bar{V}}$. Then,*

(1) $X_{r-1} \not\subset \bar{V}$ *and* $X_{r-1} \not\subset \bar{\bar{V}}$
(2) $V_r^{**} \subset \bar{\bar{V}}$
(3) $X_{r-1}^{**} = V_{p-a+2}^G \otimes D^{a-1}$, *so* $X_{r-1} + V_r^{**}$ *is not a direct sum.*
(4) $V_r/(X_{r-1} + V_r^{**}) \simeq V_{p-a+2}^G \otimes D^{a-1} \oplus V_{p-a}^G \otimes D^a$ *has exactly* 2 *JH factors.*

Proof. The JH factors of the modules involved in the decomposition $V_r = \bar{V} \oplus \bar{\bar{V}}$ have already been worked out in (5.2). Let us therefore turn our attention to $V_r^{**} = V_{(a-3)p+p-1}^G \otimes D^2$. In Glover's notation, we have "$k = a-3$", "$r = p-1$", and $k + r > p$, at least if $a \gneq 3$. We leave the case $a = 3$ $(r = 3p)$ as an exercise for the reader. So assume that $4 \leq a \leq p-2$. In this case, we have

$$
\begin{aligned}
\operatorname{soc} \bar{V}^{**} &= V_{p-a+2}^G \otimes D^{a-1} \\
\bar{V}^{**}/(\operatorname{soc} \bar{V}^{**}) &= V_{a-3}^G \otimes D^2 \qquad (5.3) \\
\bar{\bar{V}}^{**} &= V_{(a-3)p}^G \otimes D^2.
\end{aligned}
$$

Now $V_{(a-1)p}^G \otimes D = V_{(a-3)p}^G \otimes D^2 \oplus P[p-a+2, a-1]$ where the PIM has socle filtration:

$$V_{p-a+2}^G \otimes D^{a-1} \quad \text{---} \quad V_{a-1}^G \otimes D \oplus V_{a-3}^G \otimes D^2 \quad \text{---} \quad V_{p-a+2}^G \otimes D^{a-1}.$$

Part (1) of the theorem now follows from the structure of X_{r-1} given in Proposition 21. Part (2) follows from the structure of the PIM just given, and the fact that soc $(\bar{V}^{**})$ does not meet soc $(\bar{V})$. Part (3) is also clear since $V_{p-a+2}^G \otimes D^{a-1}$ is in the socle of both X_{r-1} and V_r^{**} (one also checks that V_{a+1}^G is not in the socle of the latter, and that $V_{a-1}^G \otimes D$ is not in X_{r-1}^{**} by Proposition 4). Comparing (5.2) with (5.3) gives us the 4 JH factors V_r/V_r^{**}. Part (4) then follows immediately by inspection of the various JH factors involved. $\square$

5.3. **The Upper Triangle.** The following result is the natural complement to Proposition 19.

Proposition 24. *Let $3 \leq a \leq p-3$ and suppose $ap < r \leq p^2 - p - 2$ is in the Glover range with $r \equiv a \mod p-1$. Then*

(1) $\dim X_r = p+1$, *and there is an exact sequence:*
$$0 \to V_{p-a}^G \otimes D^a \to X_r \to V_{a+1}^G \to 0.$$

(2) $\dim X_{r-1} = 2p+2$, *and X_{r-1} fits in the exact sequence:*
$$0 \to V_{p-a}^G \otimes D^a \oplus V_{p-a+2}^G \otimes D^{a-1} \to X_{r-1} \to V_{a-1}^G \otimes D \oplus V_{a+1}^G \to 0.$$

Proof. The proof is by induction on a. The base case is when $a = 2$, for which (1) and (2) were treated completely in Section 4. For the inductive step assume part (1) of the proposition holds for some $2 \leq a \leq p-4$. Setting $r' = r - 1 \equiv a - 1 \mod p-1$, for $3 \leq a \leq p-3$, we have
$$0 \to V_{p-a+1}^G \otimes D^{a-1} \to X_{r'} \to V_a^G \to 0,$$
by assumption. We use the usual Glover tensor multiplication map $\phi = \phi_{r',1}$ to deduce the structure of X_{r-1} and X_r from that of $X_{r'}$, just as we did going from $a = 1$ to $a = 2$ in Section 4. Tensoring the exact sequence above with V_2^G and using (2.4) twice, we have
$$V_{p-a}^G \otimes D^a \oplus V_{p-a+2}^G \otimes D^{a-1} \to X_{r'} \otimes V_2^G \to V_{a-1}^G \otimes D \oplus V_{a+1}^G \to 0.$$
Since $X_{r-1} \subset \phi(X_{r'} \otimes V_2^G)$ has dimension $2p+2$ by Proposition 13, it fits in the exact sequence
$$0 \to V_{p-a}^G \otimes D^a \oplus V_{p-a+2}^G \otimes D^{a-1} \to X_{r-1} \to V_{a-1}^G \otimes D \oplus V_{a+1}^G \to 0,$$
proving part (2). The formulas below show that none of the 4 JH factors in the sequence above is in $\bar{V}$, so $X_{r-1} \subset \bar{\bar{V}}$. But $X_r \subset X_{r-1}$ and X_r always has $X_r/X_r^* = V_{a+1}^G$ as a quotient, by [G78, (4.5)]. Now the intersection $\operatorname{soc} X_r \cap \operatorname{soc} \bar{\bar{V}}$ is non-zero, and by inspection of the formulas, does not contain V_{a+1}^G. So $X_r^* \neq 0$ and thus $X_r^* = V_{p-a}^G \otimes D^a$, for dimension reasons. Thus X_r has the structure claimed in part (1), completing the proof of the inductive step. □

Theorem 25. *Suppose $p \geq 7$. Say that $3 \leq a \leq p-3$ and $ap < r \leq p^2 - p - 2$ with $r \equiv a \mod p-1$. Write $V_r = V_{r+1}^G = \bar{V} \oplus \bar{\bar{V}}$. Then,*

(1) $X_{r-1} \subsetneq \bar{\bar{V}}$
(2) *The JH factors of $V_r^{**} \cap \bar{\bar{V}}$ are $V_{p-a+2}^G \otimes D^{a-1}$, $V_{p-a}^G \otimes D^a$, $V_{a-1}^G \otimes D$, $V_{a-3}^G \otimes D^2$, other than those in the submodule $V_{(a-3)p}^G \otimes D^2$, for $p \gg 0$*
(3) $X_{r-1}^{**} = V_{p-a+2}^G \otimes D^{a-1} \oplus V_{p-a}^G \otimes D^a$, *so that $X_{r-1} + V_r^{**}$ is not a direct sum*
(4) $V_r/(X_{r-1} + V_r^{**}) \simeq V_{p-a+2}^G \otimes D^{a-1} \oplus V_{p-a}^G \otimes D^a$ *has exactly 2 JH factors.*

Proof. Write $r = (m+a)p - m$ for $2 \leq m \leq p-a-1$. Then $V_r = V_{(m+a)p-m+1}^G$. Write $(m+a)p - m+1 = (m+a-1)\cdot p + (p-m+1)$ so that "$k = m+a-1$" and "$r = p-m+1$" in the local notation of the structure theorem. Again $0 \leq k \leq p-2 \iff p \geq m+a+1$ and $0 \leq r \leq p-1 \iff m \geq 2$. Note also that $k+r = p+a \geq p$. Applying Theorem 5, we obtain $V_r = \bar{V} \oplus \bar{\bar{V}}$, with

$$\begin{aligned}
\operatorname{soc} \bar{V} &= \begin{cases} \bigoplus_{i=1}^{m-1} V_{p-a-2i}^G \otimes D^{a+i} & \text{if } p \geq 2m+a-1 \\ \bigoplus_{i=1}^{p-m-a} V_{p-a-2i}^G \otimes D^{a+i} & \text{if } m+a+1 \leq p < 2m+a-1, \end{cases} \\
\bar{V}/(\operatorname{soc} \bar{V}) &= \begin{cases} \bigoplus_{i=0}^{m-2} V_{2m+a-3-2i}^G \otimes D^{2-m+i} & \text{if } p \geq 2m+a-1 \\ \bigoplus_{i=0}^{p-m-a} V_{2p-2m-a+1-2i}^G \otimes D^{m+a-1+i} & \text{if } m+a+1 \leq p < 2m+a-1, \end{cases}
\end{aligned}$$

and

$$\bar{\bar{V}} = \begin{cases} V_{(a+1)p}^G & \text{if } p \geq 2m+a-1 \\ V_{(a+1)p}^G \oplus V_{(2m-p+a-2)p}^G \otimes D^{2-m} & \text{if } m+a+1 \leq p < 2m+a-1, \end{cases}$$

where

$$V_{(a+1)p}^G = V_{(a-3)p}^G \otimes D^2 \oplus P[p-a+2, a-1] \oplus P[p-a, a], \tag{5.4}$$

and the socle filtrations of the PIMs are, respectively,

$$V_{p-a+2}^G \otimes D^{a-1} \;\text{——}\; V_{a-1}^G \otimes D \oplus V_{a-3}^G \otimes D^2 \;\text{——}\; V_{p-a+2}^G \otimes D^{a-1}, \tag{5.5}$$

$$V_{p-a}^G \otimes D^a \;\text{——}\; V_{a+1}^G \oplus V_{a-1}^G \otimes D \;\text{——}\; V_{p-a}^G \otimes D^a. \tag{5.6}$$

Turning our attention to $V_r^{**} = V_{(m+a-2)p-m-1}^G \otimes D^2$, write $(m+a-2)p-m-1 = (m+a-3) \cdot p + p - m - 1$ so that this time "$k = m+a-3$", "$r = p-m-1$" and $k+r = p+a-4$ in the notation of Theorem 5. Note that $k+r < p$ if $a = 3$, and $k+r \geq p$ otherwise. Applying that result, we obtain $V_r^{**} = \bar{V}^{**} \oplus \bar{\bar{V}}^{**}$, with

$$\begin{aligned}
\operatorname{soc} \bar{V}^{**} &= \begin{cases} \bigoplus_{i=0}^{m} V_{p-1-2i}^G \otimes D^{2+i} & \text{if } a=3,\ p \geq 2m+2 \\ \bigoplus_{i=1}^{m+1} V_{p-a+4-2i}^G \otimes D^{a-2+i} & \text{if } a \geq 4,\ p \geq 2m+a-1 \\ \bigoplus_{i=0}^{p-m-2} V_{p-1-2i}^G \otimes D^{2+i} & \text{if } a=3,\ m+4 \leq p < 2m+2 \\ \bigoplus_{i=1}^{p-m-a+2} V_{p-a+4-2i}^G \otimes D^{a-2+i} & \text{if } a \geq 4,\ m+a+1 \leq p < 2m+a-1, \end{cases} \\
\bar{V}^{**}/(\operatorname{soc} \bar{V}^{**}) &= \begin{cases} \bigoplus_{i=0}^{m-1} V_{2m-2i}^G \otimes D^{2-m+i} & \text{if } a=3,\ p \geq 2m+2 \\ \bigoplus_{i=0}^{m} V_{2m+a-3-2i}^G \otimes D^{2-m+i} & \text{if } a \geq 4,\ p \geq 2m+a-1 \\ \bigoplus_{i=0}^{p-m-2} V_{2p-2m-4-2i}^G \otimes D^{2+m+i} & \text{if } a=3,\ m+4 \leq p < 2m+2 \\ \bigoplus_{i=0}^{p-m-a+2} V_{2p-2m-a-1-2i}^G \otimes D^{m+a-1+i} & \text{if } a \geq 4, m+a+1 \leq p < 2m+a-1, \end{cases}
\end{aligned}$$

and, for $a \geq 3$,

$$\bar{\bar{V}}^{**} = \begin{cases} V_{(a-3)p}^G \otimes D^2 & \text{if } p \geq 2m+a-1 \\ V_{(a-3)p}^G \otimes D^2 \oplus V_{(2m-p+a-2)p}^G \otimes D^{2-m} & \text{if } m+a+1 \leq p < 2m+a-1. \end{cases}$$

We now prove (1). By part (2) of Proposition 24, we have an exact sequence

$$0 \to V_{p-a}^G \otimes D^a \oplus V_{p-a+2}^G \otimes D^{a-1} \to X_{r-1} \to V_{a-1}^G \otimes D \oplus V_{a+1}^G \to 0.$$

The possible JH factors above, and in particular those in $\operatorname{soc} X_{r-1}$, do not overlap with those in $\operatorname{soc} \bar{V}$, for $a \geq 3$. Indeed, inspection of the formulas above shows the weight of maximal dimension in the socle of $\bar{V}$ has dimension $p-a-2 < p-a$. Similarly, one easily checks that neither $V_{a-1}^G \otimes D$ nor V_{a+1}^G can occur in $\operatorname{soc} \bar{V}$, since $4 \leq a+i \leq p-2$, since $i \leq m-1$ or $p-m-a$, hence $a+i \not\equiv 0, 1 \mod p-1$, and the power of D doesn't match. In particular, $X_{r-1} \cap \bar{V} = 0$, and we obtain (1).

For part (2), the formulas above show $V_{(a-3)p}^G \otimes D^2$ lies in the intersection in (2), at least if $p \geq 2m+a-1$. They also show that the first two JH factors mentioned in (2) are also part of the intersection. Indeed, these 2 factors are the only ones in the socle of (the non-projective part of) V_r^{**} which are not in the socle of $\bar{V}$ (and their direct sum is in $\bar{\bar{V}}$, as the sum of the socles of the two PIMs above). Moreover, $\bar{V}^{**}/\mathrm{soc}$ also has 1, if $a=3$, or 2, if $a \geq 4$, JH factors which an easy check shows are not in $\bar{V}$, and so these must also be in $\bar{\bar{V}}$, and hence in the intersection. These factors are $V_2^G \otimes D$, when $a=3$, and $V_{a-1}^G \otimes D$, $V_{a-3}^G \otimes D^2$, when $a \geq 4$. Now, other than these $2+2=4$ ($2+1=3$, when $a=3$) JH factors, the rest of the non-projective part of V_r^{**} is equal to the non-projective part of V_r (note here that the V_{a+1}^G in $\bar{V}^{**}/\mathrm{soc}$ is not the $V_{a+1}^G = X_r/X_r^*$ in $\bar{\bar{V}}$, since if it were, noting $X_r^* \subset V_r^{**}$, we'd have $X_r \subset V_r^{**}$, implying $X_r = X_r^* = X_r^{**}$, a contradiction). Similarly, the projective part of V_r^{**} is equal to the projective part of V_r, except for the two PIMs, which have 7 (line plus diamond) or 8 (diamond plus diamond) Jordan-Hölder factors depending on whether $a=3$ or $a \geq 4$. Thus, the quotient V_r/V_r^{**} has $7-3=4$ JH factors, when $a=3$, or $8-4=4$ JH factors, when $a \geq 4$, as it should. This forces (2) to hold, since if the intersection in (2) had any more JH factors, then V_r/V_r^{**} would have fewer than 4 JH factors, a contradiction.

For part (3), we note $X_{r-1} \subset \bar{\bar{V}}$ by (1). An easy check shows that none of the 4 JH factors in X_{r-1} is in the socle of $V_{(a-3)p}^G \otimes D^2$, so that we have X_{r-1} is contained in the direct sum of the two PIMs. By (2), we see that the factor V_{a+1}^G in X_{r-1} is not in X_{r-1}^{**}. If the factor $V_{a-1}^G \otimes D$ in X_{r-1} was the same as the one in V_r^{**}, then we would have $X_{r-1} \subset X_r + V_r^{**}$, contradicting what we proved in Proposition 4. Thus this JH factor in X_{r-1} is also not in X_{r-1}^{**}. The remaining two JH factors in X_{r-1}, namely $V_{p-a+2}^G \otimes D^{a-1}$ and $V_{p-a}^G \otimes D^a$, are clearly in (the socle of) $X_{r-1}^{**} = X_{r-1} \cap V_r^{**}$, and we obtain (3).

For (4), we have $V_r/V_r^{**} = \bar{V}/(V_r^{**} \cap \bar{V}) \oplus \bar{\bar{V}}/(V_r^{**} \cap \bar{\bar{V}})$ has $0 \oplus 4 = 4$ Jordan-Hölder factors, all of which are in the two PIMs above. Taking a further quotient by X_{r-1} kills exactly 2 more of these, by (3). Thus, exactly the top JH factors in the two PIMs $P[p-a+2, a-1]$ and $P[p-a, a]$ survive in the quotient in (4), proving the theorem, for $m \geq 2$.

The attentive reader will notice that for notational reasons (we took $m \neq 1$ above) the proof did not treat the the super-diagonal just above the diagonal $r = ap$, namely the line given by $r = (a+1)p-1$, for $3 \leq a \leq p-3$. Note $V_{r+1}^G = V_{(a+1)p}^G$ is a projective Γ-module on this line, since $(a+1)p$ is a multiple of p, and so we refer to the super-diagonal as the projective line. We treat this case briefly now. Part (1) is immediate since $\bar{V} = 0$, by projectivity. To analyze the veracity of parts (2), (3) and (4), we need to write down the structure of V_r^{**}. Note $(a+1)p-(2p-2) = (a-2)p+(p-2)$, and we take "$k = a-2$" and "$r = p-2$" in Theorem 5. Since $k+r = a+p-4 \geq p \iff a \geq 4$, one again has to consider the case $a = 3$ separately. We obtain:

$$\begin{aligned}
\operatorname{soc} \bar{V}^{**} &= \begin{cases} V_{p-a+2}^{G} \otimes D^{a-1} \oplus V_{p-a}^{G} \otimes D^{a} & \text{if } 3 \leq a \leq p-3, \end{cases} \\
\bar{V}^{**}/(\operatorname{soc} \bar{V}^{**}) &= \begin{cases} V_2^{G} \otimes D & \text{if } a = 3 \\ V_{a-1}^{G} \otimes D \oplus V_{a-3}^{G} \otimes D^2 & \text{if } 4 \leq a \leq p-3, \end{cases} \\
\bar{\bar{V}}^{**} &= \begin{cases} 0 & \text{if } a = 3 \\ V_{(a-3)p}^{G} \otimes D^2 & \text{if } 4 \leq a \leq p-3. \end{cases}
\end{aligned}$$

Clearly, then, part (2) also holds. Parts (3) and (4) also follow from the structure of V_r^{**}, Proposition 24, and the formulas (5.4), (5.5), (5.6) above. One also has to use Proposition 4 again. We leave all of this as an exercise, since the arguments are a subset of those given just above. $\square$

Remark 26. For the smaller primes $m+a+1 \leq p < 2m+a-1$, (2) cannot possibly be true because of the extra projective module $V_{(2m-p+a-2)p}^{G} \otimes D^{2-m}$ in the intersection (for $m \geq 2$), but one checks that (1), (3) and (4) follow in a similar manner as above.

5.4. **Eliminating 1 JH factor using the Hecke operator T_p.** Recall that by the theorems above we have the 'final quotient' is the sum of two JH factors:

$$V_r/(X_{r-1} + V_r^{**}) \simeq J_1 \oplus J_2, \quad \text{with} \quad J_1 = V_{p-a+2}^{G} \otimes D^{a-1} \ \text{ and } \ J_2 = V_{p-a}^{G} \otimes D^{a}.$$

We now show how to eliminate J_2 in the lower triangle and on the diagonal. Recall $r = k-2$.

Theorem 27. *Let $3 \leq a \leq p-1$ and let $r = (a-n)p + n$, with $0 \leq n \leq a-2$ (and $1 \leq n \leq a-2$, if $a = p-1$). The natural surjection $\operatorname{ind}_{\mathrm{KZ}}^{\mathrm{G}} V_r \twoheadrightarrow \bar{\Theta}_{k,a_p}$ in* (1.1) *factors as follows:*

$$\operatorname{ind}_{\mathrm{KZ}}^{\mathrm{G}} V_r \twoheadrightarrow \operatorname{ind}_{\mathrm{KZ}}^{\mathrm{G}} V_r/(X_{r-1} + V_r^{**}) \twoheadrightarrow \operatorname{ind}_{\mathrm{KZ}}^{\mathrm{G}} J_1 \twoheadrightarrow \bar{\Theta}_{k,a_p}.$$

Proof. We consider the function $f_2 \in \operatorname{ind}_{\mathrm{KZ}}^{\mathrm{G}} \operatorname{Sym}^r \bar{\mathbb{Q}}_p^2$ defined by

$$f_2 = \sum_{\lambda \in \mathbb{F}_p} [g_{2,p[\lambda]}^0, \frac{1}{p}(X^{p-1}Y^{r-p+1} - Y^r)].$$

Assume momentarily that $a \neq p-1$.

Lemma 28. *Let $T = T_p$ be the Hecke operator. Then $Tf_2 = T^+ f_2 + T^- f_2$ with*

1. $T^+ f_2 \in \operatorname{ind}_{\mathrm{KZ}}^{\mathrm{G}} \langle XY^{r-1} \rangle_{/\bar{\mathbb{Z}}_p} + p \cdot \operatorname{ind}_{\mathrm{KZ}}^{\mathrm{G}} \operatorname{Sym}^r(\bar{\mathbb{Z}}_p^2)$.
2. $T^- f_2 = [g_{1,0}^0, w]$, *for a vector $w \in \operatorname{Sym}^r(\bar{\mathbb{Z}}_p^2)$ whose image in V_r/V_r^* is $\frac{a-n}{a} \cdot X^{r-a}Y^a - Y^r$.*

Proof. Part (1) is a standard computation. For part (2), note that the formula for the Hecke operator T^- gives that $w = \sum_{j=0}^{r} d_j X^{r-j} Y^j$ with

$$d_j \quad = \quad -\frac{1}{p}\binom{r}{j}\sum_{\lambda\in\mathbb{F}_p}[\lambda]^{r-j} = \begin{cases} -\binom{r}{j}\frac{p-1}{p} & \text{if } j \equiv a \mod p-1 \text{ and } a \le j < r \\ -1 & \text{if } j = r \equiv a \mod p-1 \\ 0 & \text{if } j \not\equiv a \mod p-1. \end{cases}$$

Since $\theta = X^pY - XY^p$, for $p-1 < j < r$, we have $X^{r-j}Y^j = -\theta\cdot X^{r-j-1}Y^{j-p} + X^{r-j+(p-1)}Y^{j-(p-1)}$. Thus mod p and mod θ, we may substitute all but the last non-zero monomial with the first non-zero monomial, and write $w = c\cdot X^{r-a}Y^a - Y^r$ where $c = \frac{1}{p}\sum_{l=0}^{a-n-1}\binom{r}{a+l(p-1)} \equiv \frac{a-n}{a} \mod p$, where the last congruence is proved in 6.4 in the Appendix. □

Let us return to the proof of the theorem. It is very similar to the argument already given for $r = 2p$, so we will be brief. One checks that the image of the vector $X^{r-a}Y^a$ in J_2 is the 'top monomial' X^{p-a-1}. Indeed, the map $V_r \to J_2$ factors as $V_r \to V_r/V_r^* \to V_{a+p-1}/V_{a+p-1}^* \to V_{p-a-1}\otimes D^a = J_2$, where the middle map is Glover's periodicity isomorphism (cf. [G78, (4.2)]), and the last map is the quotient map in [B03b, Lem. 5.3(ii)], applied with $k = a+p+1$ (which, incidentally, satisfies $p+4 \le k \le 2p$ since $3 \le a \le p-1$). Since Y^r $(\in X_r)$ and X_{r-1} both map to 0 in the final quotient Q, it follows from parts (1) and (2) of the lemma that $(T-a_p)f_2 \in \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}}\mathrm{Sym}^r(\bar{\mathbb{Z}}_p^2)$ and that the projection of this function to $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} J_2$ is $[g_{1,0}^0, \frac{a-n}{a}\cdot X^{p-a-1}]$. Since $\frac{a-n}{a} \not\equiv 0 \mod p$, we see J_2 dies, and the natural surjection $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} V_r \twoheadrightarrow \bar{\Theta}_{k,a_p}$ factors via the map $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} J_1 \twoheadrightarrow \bar{\Theta}_{k,a_p}$, as desired.

When $a = p-1$, part (2) of the lemma is not quite true, since $d_0 = -\frac{p-1}{p}$ is not integral and so w is non-integral. We therefore modify f_2 slightly. Set $f = f_0 + f_2$, with $f_0 = [\mathrm{Id}, \frac{1}{p}(X^{r-p+1}Y^{p-1} - X^r)]$. One checks that T^-f_0 is integral and vanishes mod p, and that $T^+f_0 + T^-f_2 = [g_{1,0}^0, w']$ with w' integral. Thus, $(T-a_p)f$ is integral and the proof proceeds as before with w replaced by w'. □

Parts (9) and (10) of Theorem 2 now follow immediately from Theorem 27 and [BG09, Prop. 3.3] applied to the JH factor $J_1 = V_{p-a+2}^G \otimes D^{a-1}$. We note here that when $a = 3$, one is in the exceptional case of that proposition, hence there are two possibilities for the reduction on inertia.

Based on what happens for $a = 1$ and $a = 2$, one might also conjecture that one can eliminate J_1 in the upper triangle. We prove the weaker result that the map (1.1) factors through exactly one of the two Jordan-Hölder factors in Q, namely J_1 or J_2. We need the following general lemma:

Lemma 29. *Say $Q = V_r/(X_{r-1} + V_r^{**}) = J_1 \oplus J_2$ is a direct sum of two weights $J_i = V_{r_i}\otimes D^{n_i}$, with $0 \le r_i \le p-1$, $0 \le n_i \le p-2$, for $i = 1, 2$. Assume that $r_1 \not\equiv r_2 \mod p-1$ or $n_1 \ne n_2$. Set $r_i^G = r_i + 1$. If one of the following two conditions fails:*

(1) $r_1^G + r_2^G \equiv 0 \mod p-1$, *or,*
(2) $n_2 - n_1 \equiv r_1^G \mod p-1$,

then the natural map $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} V_r \twoheadrightarrow \mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} Q \twoheadrightarrow \bar{\Theta}_{k,a_p}$ factors through one of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} J_1$ or $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}} J_2$.

Proof. Before we start, we remark that condition (2) is symmetric: it is equivalent to $n_1 - n_2 \equiv r_2^G$ mod $p-1$, under (1). Now, let F_i be the image of $\mathrm{ind}_{\mathrm{KZ}}^{\mathrm{G}}\, J_i$ in $\bar{\Theta}_{k,a_p}$, for $i = 1, 2$. Thus, the F_i have finite length. If F_1 vanishes, then we are done. So assume that $F_1 \neq 0$. If $\bar{\Theta}_{k,a_p}$ is irreducible, then $F_1 = \bar{\Theta}_{k,a_p}$, and so again we are done. So we may assume that $F_2 \neq 0$ and that $\bar{\Theta}_{k,a_p}$ is reducible, in which case:

$$\bar{\Theta}^{\mathrm{ss}}_{k,a_p} = \pi(r, \lambda, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], \lambda^{-1}, \eta\omega^{r+1})^{\mathrm{ss}},$$

for some $\lambda \neq 0$, and some character η, since $\bar{\Theta}_{k,a_p}$ is in the image of the mod p Local Langlands Correspondence. By [BG09, Lem. 3.2], which one checks also holds with twists, every JH factor of F_i is also a JH factor of $\pi(r_i, \lambda_i, \omega^{n_i})$, for some λ_i, $i = 1, 2$, where λ_i may depend on the JH factor of F_i under consideration (and r_i, n_i are determined by $J_i = V_{r_i} \otimes D^{n_i}$). Now suppose $\pi_1 := \pi(r, \lambda, \eta)$ has a JH factor in common with F_1 and (possibly another) JH factor in common with F_2. By [BG09, Lem. 3.1], one has $r_1 \equiv r \equiv r_2 \mod p-1$ and $n_1 - n_2 \equiv 0 \mod p-1$, contradicting our hypothesis. A symmetric argument applies to $\pi_2 := \pi([p-3-r], \lambda^{-1}, \eta\omega^{r+1})$. We conclude that π_1 has a JH factor in common with F_1 and π_2 has a JH factor in common with F_2 (or vice versa). The lemmas just quoted now give:

$$\begin{array}{cccc} r_1 \equiv r \mod p-1 & \text{and} & r_2 \equiv p-3-r \mod p-1 \\ \dfrac{\omega^{n_1}}{\eta} \text{ is unramified} & \text{and} & \dfrac{\omega^{n_2}}{\eta\omega^{r+1}} \text{ is unramified} \end{array}$$

(or vice versa). Adding the terms in the first row gives (1), and dividing the terms in the second row gives (2). $\square$

Parts (9)' and (10)' of Theorem 2 now follow immediately, applying the lemma with $J_1 = V^G_{p-a+2} \otimes D^{a-1}$ and $J_2 = V^G_{p-a} \otimes D^a$, noting that $3 \leq a \leq p-3$, and applying [BG09, Prop. 3.3] as usual (to both J_1 and J_2 separately).

6. Appendix

6.1. Let $r \geq 2p+2$. Write $r = (m+1)p - a$, with $3 \leq m+1 \leq p$ and $1 \leq a \leq p$.

6.1.1. If $mp \leq r \leq (m+1)p - (m+1)$, then

$$\binom{r}{bp-(b-1)} = \begin{cases} 0 & \text{for } 2 \leq b \leq m \\ m & \text{for } b = 1 \\ r & \text{for } b = 0. \end{cases}$$

Proof. By Lucas' identity, given integers $0 \leq r, s < p^2$, with $r = r_1 p + r_0$, $s = s_1 p + s_0$, with $0 \leq r_i, s_i, \leq p-1$,

$$\binom{r}{s} \equiv \binom{r_1}{s_1}\binom{r_0}{s_0} \equiv 0 \mod p,$$

iff some $s_i > r_i$. Hence, the binomial coefficient in question, for $b \geq 2$, is

$$\binom{m}{b-1}\binom{p-a}{p-(b-1)} \mod p,$$

and the second factor vanishes since $b-1 < a$ (noting $b \leq m \leq a-1$). The case $b = 1$ is similar; we obtain instead

$$\binom{m}{1}\binom{p-a}{0} = m \mod p,$$

as desired. The case $b = 0$ is obvious. □

6.1.2. If $r = (m+1)p - a$ with $3 \leq m+1 \leq p-1$ and $1 \leq a \leq m-2$, then

$$\binom{r}{bp-(b-1)} = \begin{cases} (-1)^{a+b-3}\binom{m}{b-1}\binom{b-2}{a-1} & \text{for } b = 2, 3, \ldots, m+1 \\ m & \text{for } b = 1 \\ r & \text{for } b = 0. \end{cases}$$

Proof. The cases $b = 0$, 1 were just done above, so we assume $b \geq 2$. By Lucas' identity it suffices to prove that

$$\binom{p-a}{p-(b-1)} \equiv (-1)^{a+b-3}\binom{b-2}{a-1} \mod p.$$

Both sides vanish when $b-1 < a$, and both sides are 1 when $b-1 = a$, so assume that $b-1 > a$. Replacing $b-1$ by b we must show

$$\binom{p-a}{p-b} \equiv (-1)^{a+b}\binom{b-1}{a-1} \mod p.$$

for $2 \leq a+1 \leq b \leq m \leq p-2$. This is proved by induction on b. First assume that $a = 1$. The base case $b = 2$ is then obvious. For general b, note

$$\binom{p-1}{p-b} \equiv (-1)\cdot\binom{p-1}{p-(b-1)} \equiv (-1)\cdot(-1)^{1+b-1}\binom{b-2}{0} = (-1)^{b+1} \mod p,$$

as desired, where the second equivalence is by the inductive hypothesis. Now assume $a \geq 2$, so that $a-1 \neq 0$. Again, we induct on b. The base case is then $b = a+1$, which is again obvious. For general b, we have:

$$\binom{p-a}{p-b} \equiv \frac{b-1}{a-1}\cdot\binom{p-(a-1)}{p-(b-1)} \equiv \frac{b-1}{a-1}\cdot(-1)^{a-1+b-1}\binom{b-2}{a-2} = (-1)^{a+b}\binom{b-1}{a-1} \mod p,$$

where again the second equivalence follows from the inductive hypothesis. □

6.1.3. With notation as in the previous subsection ($1 \le a \le m-2$), we have:

$$(m-1)\binom{m}{m}\binom{m-1}{a-1} - (m-2)\binom{m}{m-1}\binom{m-2}{a-1} + (m-3)\binom{m}{m-2}\binom{m-3}{a-1} - \cdots + (-1)^m\binom{m}{2}\binom{1}{a-1}$$

$$= (-1)^{a+m-1}.$$

Proof. The proof is by induction on m. The base case is $m = 3$, and hence $a = 1$. We first prove the result for $a = 1$ (and any $m \geq 3$). Expanding $(1-t)^m$, and its derivative with respect to t, we obtain the two well-known identities

$$\sum_{i=2}^{m}(-1)^i\binom{m}{i} = m-1 \qquad \text{and} \qquad \textstyle\sum_{i=2}^{m}(-1)^i i\binom{m}{i} = m,$$

by substituting $t = 1$. But the LHS of 6.1.3 when $a = 1$ is

$$(-1)^m\sum_{i=2}^{m}(-1)^i(i-1)\binom{m}{i} = (-1)^m(m-(m-1)) = (-1)^m,$$

as desired. So assume now that $2 \le a \le m-2$. Then the LHS of 6.1.3 is

$$(-1)\cdot\sum_{j=1}^{m-1}(-1)^{m-j}j\binom{m}{j+1}\binom{j}{a-1} =$$
$$\sum_{j=1}^{m-2}(-1)^{m-1-j}j\binom{m-1}{j+1}\binom{j}{a-1} - \sum_{j=1}^{m-1}(-1)^{m-j}j\binom{m-1}{j}\binom{j}{a-1},$$

noting that $\binom{m}{j+1} = \binom{m-1}{j+1} + \binom{m-1}{j}$, and that $\binom{m-1}{m} = 0$ in the first sum on the right just above. Letting $j = k+1$, the second sum on the right further splits as

$$-\sum_{k=0}^{m-2}(-1)^{m-1-k}k\binom{m-1}{k+1}\binom{k+1}{a-1} - \sum_{j=1}^{m-1}(-1)^{m-j}1\binom{m-1}{j}\binom{j}{a-1} =$$
$$-\sum_{k=1}^{m-2}(-1)^{m-1-k}k\binom{m-1}{k+1}\binom{k}{a-1} - \sum_{k=1}^{m-2}(-1)^{m-1-k}k\binom{m-1}{k+1}\binom{k}{a-2} - \sum_{j=1}^{m-1}(-1)^{m-j}\binom{m-1}{j}\binom{j}{a-1},$$

where again we've used the identity $\binom{k+1}{a-1} = \binom{k}{a-1} + \binom{k}{a-2}$. Putting this together, we obtain that the LHS of 6.1.3 is the sum/difference of 4 sums. However, the first two of these cancel. Noting that $a-1 \le (m-1)-2$, the third sum is just $(-1)^{(a-1)+(m-1)-1}$, by the inductive hypothesis. Calling the fourth sum $S(m-1,a)$ we obtain that the LHS of 6.1.3 is $(-1)^{a+m} - S(m-1,a)$. We claim that $S(m-1,a) = 0$, thereby completing the proof. This is proved by induction on a. The base case is $a = 2$, and we have $S(m-1,2) =$

$$\sum_{j=1}^{m-1}(-1)^{m-j}j\binom{m-1}{j} = (-1)^m(m-1)\sum_{j=1}^{m-1}(-1)^j\binom{m-2}{j-1} = (-1)^{m+1}(m-1)\sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k} = 0,$$

by the first of the two identities at the beginning of the proof (for $m-2$). For general $a \geq 3$, note

$$\begin{aligned} S(m-1,a) &= \sum_{j=1}^{m-1} (-1)^{m-j} \binom{m-1}{j} \binom{j}{a-1} \\ &= \sum_{j=1}^{m-1} (-1)^{m-j} \left(\frac{m-1}{a-1}\right) \binom{m-2}{j-1} \binom{j-1}{a-2} \\ &= \frac{m-1}{a-1} \cdot \sum_{k=1}^{m-2} (-1)^{m-1-k} \binom{m-2}{k} \binom{k}{a-2} \\ &= \frac{m-1}{a-1} \cdot S(m-2,a-1) = 0, \end{aligned}$$

by the inductive hypothesis, as claimed. □

6.2. Let $2p-1 \leq r \leq p^2-p+1$ and $r \equiv 1 \mod p-1$. Then

$$\binom{r-1}{b} = 0 \qquad \text{for } b \equiv 0 \mod p-1, \quad b \neq 0, r-1.$$

Proof. Write $r-1 = mp + (p-m-1)$, for $1 \leq m \leq p-1$, and $b = b_1 p + b_0$, with $0 \leq b_i \leq p-1$. By Lucas' identity, the binomial coefficient is

$$\binom{m}{b_1} \binom{p-m-1}{b_0} \mod p,$$

But if $b \equiv 0 \mod p-1$, $b \neq 0, r-1$, then $b_0 > p-m-1$, so the second coefficient vanishes. □

6.3. Let $r = (m+a)p - m$ for $1 \leq m \leq p-a$. We have

$$\binom{r-1}{b} = 0,$$

for b in the successive ranges $p-m \leq b \leq p-1$; $2p-m \leq b \leq 2p-1$; $\ldots$; $(m+a-1)p-m \leq b \leq (m+a-1)p-1$, and is non-zero for the other b in the range $0 \leq b \leq r-1$.

Proof. This is again a direct application of Lucas' identity. Write $r-1 = (m+a-1)p + p - m - 1$, and $b = b_1 p + b_0$, so that the binomial coefficient is

$$\binom{m+a-1}{b_1} \binom{p-m-1}{b_0} \mod p.$$

Then, for b as above, $b_0 \geq p-m > p-m-1$, so the second binomial coefficient vanishes. On the other hand, if b is not in the above ranges, then $b_0 \leq p-m-1$, and the second binomial coefficient does not vanishes, and neither does the first since $m+a-1 \leq p-1$. □

6.4. Let $3 \leq a \leq p-1$ and let $r = (a-n)p+n$, with $0 \leq n \leq a-2$ (and $n \neq 0$, if $a = p-1$). Then

$$\frac{1}{p} \sum_{l=0}^{a-n-1} \binom{r}{a+l(p-1)} \equiv \frac{a-n}{a} \mod p$$

and in particular is non-zero mod p.

Proof. We sketch the proof. Using the identity $\binom{m}{k} = \frac{m}{k}\binom{m-1}{k-1}$ several times (until one can cancel the p in the denominator) and then Lucas' identity to show that the LHS is the same as the sum:

$$\frac{1}{\binom{a}{n}} \sum_{l=0}^{a-n-1} \binom{a}{l} \binom{p-1}{a-n-1-l} \mod p.$$

But $\binom{p-1}{t} \equiv (-1)^t \mod p$, so this reduces further to:

$$\frac{(-1)^{a+n+1}}{\binom{a}{n}} \sum_{l=0}^{a-n-1} (-1)^l \binom{a}{l} \mod p.$$

We consider the case $n = 0$, leaving the $n \geq 1$ cases as an exercise (use, e.g., proof by induction). But this is immediate: plugging $x = 1$ into $(1-x)^a = \sum_{l=0}^{a-1}(-1)^l\binom{a}{l}x^l + (-x)^a$, we see that the last expression above is identically 1. $\square$

7. Summary

We summarize some of our results in tabular form, for the weights in the upper triangle $ap < r \leq p^2 - p - 2$, for $1 \leq a \leq p-3$ (and $r \neq 2p-1$, when $a = 1$). We remark that the increasing order of complexity of the computations in this paper is largely because of the increasing complexity in the shapes of the 'last two' PIMs in the projective part of V_r as a grows.

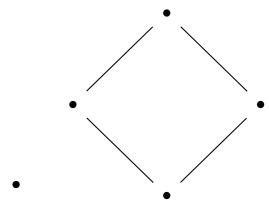
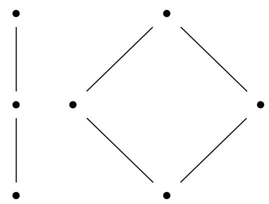
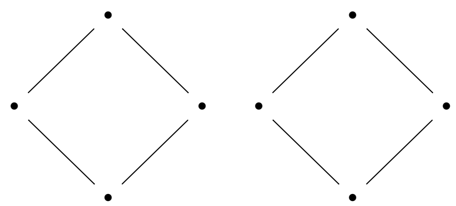
r mod $p-1$	$\dim X_r$ $\dim X_{r-1}$ X_{r-1}/X_r	mixed or full	Last two PIMs in $\bar{V}_r$	$Q = V_r/(X_{r-1} + V_r^{**})$	Final # of JH factors
1	$p+1$ $2p$ $V_{p-2} \otimes D$	mixed		V_1	1
2	$p+1$ $2p+2$ $D \oplus V_{p-3} \otimes D^2$	full		$V_{p-3} \otimes D^2$	1
3	$p+1$ $2p+2$ extension of $V_1 \otimes D$ by $V_{p-2} \otimes D^2$	full		$V_{p-2} \otimes D^2$ $\oplus$ $V_{p-4} \otimes D^3$	2
$4 \leq a \leq p-3$	$p+1$ $2p+2$ extension of $V_{a-2} \otimes D$ by $V_{p-a+1} \otimes D^{a-1}$	full		$V_{p-a+1} \otimes D^{a-1}$ $\oplus$ $V_{p-a-1} \otimes D^a$	2

Table 1: JH factors in the final quotient Q in the upper triangle.

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