CONTROL THEOREMS FOR ORDINARY 2-ADIC FAMILIES OF MODULAR FORMS

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ABSTRACT. We prove a control theorem for Hida's ordinary Hecke algebra for the prime $p = 2$, thereby establishing a uniqueness result for ordinary 2-adic families of cusp forms. As a consequence we show that the possibly finitely many exceptions that arise, in showing that the local Galois representations attached to the arithmetic members of a non-CM ordinary 2-adic family of cuspidal eigenforms are all non-split, contain no CM forms.

1. INTRODUCTION

Hida theory for the prime p is the theory [Hid86a], [Hid86b] that deals with p-ordinary families of elliptic modular forms. While generalizations are now available, for automorphic forms on groups other than GL_2 and base fields other than $\mathbb{Q}$, many authors tend to shy away from the prime $p = 2$.

The goal of this paper is to check that some of the basic results in the literature stated for odd primes p (and originally for $p \geq 5$ in [Hid86a], [Hid86b]) remain valid for the prime $p = 2$. In particular, we shall check that Hida's control theorem (Theorem 8.1) for the ordinary Λ-adic Hecke algebra holds in this setting. Such control theorems are of fundamental importance since

(i) they are connected to the fact that the dimension of the space of ordinary cusp forms $S_k^0(\Gamma_1(Np^r), \mathbb{Z}_p)$ of tame level N coprime to p, and r fixed, is bounded independent of the weight k, even though the dimension of the ambient space of cusp forms grows large with k, and

(ii) they can be used to prove existence and uniqueness results for p-ordinary cusp forms in Hida families.

Both these applications are well known when p is odd. As a result of our work here, we similarly deduce that a 2-stabilized ordinary cuspidal newform of weight at least 2 lives in a unique primitive 2-ordinary cuspidal family, up to Galois conjugacy (cf. Section 9). As a consequence, we are able to separate primitive 2-adic CM families from primitive non-CM families.

The proof of the control theorem in the case $p = 2$ given here uses a melange of techniques from several of Hida's papers. However, since some key facts needed from the theory of mod 2 modular forms do not still seem to be known, we have had to replace these with other ingredients; see in particular the proof of Theorem 5.3.

The reason we decided to embark on this project was to understand to what extent a recent application of Hida theory for odd primes p, namely to understanding the local splitting behaviour of p-ordinary modular Galois representations, continues to hold for the prime $p = 2$. As explained in [Gha04], a direct geometric approach to this problem using the motive attached to the underlying form only seems possible when the weight is 2. Following instead the Hida theoretic approach in [GV04] for odd primes p, and assuming that a modularity result of Buzzard [Buz03] for Artin-like representations holds in sufficient generality for the prime $p = 2$ (see [All12] for some progress on this front), we show that almost all arithmetic members of a primitive non-CM 2-ordinary family of cusp forms have locally non-split Galois representations. The uniqueness result mentioned above implies that none of these possibly finitely many exceptions are CM forms. Thus, as for odd primes p, if an exception occurs in a 2-ordinary non-CM family, it would give a genuine counterexample to the natural guess of Greenberg that p-ordinary modular Galois representations tend to be locally split only if the underlying form has CM.

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2. Preliminaries

We recall some background and notation.

Let Φ be a torsion-free congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then Φ acts freely on the upper half complex plane $\mathbb{H}$ by linear fractional transformations. Let Y be the complex open manifold associated to Φ, i.e., $Y = \Phi\backslash\mathbb{H}$. Let $C(\Phi)$ denote a finite set of representatives for the Φ-equivalence classes of cusps. Let X be the smooth compactification of Y obtained by adjoining the cusps in $C(\Phi)$.

For any $\mathbb{Z}[\Phi]$-module M, equipped with the discrete topology, let $F(M) = \Phi\backslash(\mathbb{H} \times M)$, where $\alpha \in \Phi$ acts on $\mathbb{H} \times M$ by $\alpha(z, m) = (\alpha z, \alpha m)$ for $(z, m) \in \mathbb{H} \times M$. We denote by the same symbol $F(M)$ the sheaf of continuous sections of the natural covering map $F(M) \to Y$. The sheaf cohomology group is denoted by $\mathrm{H}^i(Y, F(M))$ and compactly supported cohomology is denoted by $\mathrm{H}^i_c(Y, F(M))$. The parabolic sheaf cohomology group $\mathrm{H}^1_{\mathrm{p}}(Y, F(M))$ is the image of $\mathrm{H}^1_c(Y, F(M))$ in $\mathrm{H}^1(Y, F(M))$.

We relate the sheaf cohomology group with group cohomology. For $n \geq 0$, let $L_n(\mathbb{Z}) = \mathbb{Z}^{n+1}$ be the n-th symmetric power representation of $\mathrm{SL}_2(\mathbb{Z})$. For an abelian group A, let $L_n(A) := L_n(\mathbb{Z}) \otimes_{\mathbb{Z}} A$. It is naturally an $\mathrm{SL}_2(\mathbb{Z})$-module through its action on the left factor.

It is well-known that
$$\mathrm{H}^i(Y, F(L_n(A))) \simeq \mathrm{H}^i(\Phi, L_n(A)),$$
where $\mathrm{H}^i(\Phi, L_n(A))$ denotes the i-th group cohomology of the $\mathbb{Z}[\Phi]$-module $L_n(A)$. This isomorphism is compatible with the action of the Hecke operators. Moreover, we have
$$\mathrm{H}^1_{\mathrm{p}}(Y, F(L_n(A))) \simeq \mathrm{H}^1_{\mathrm{p}}(\Phi, L_n(A)),$$
where the right hand side is the first parabolic group cohomology group and this isomorphism is again equivariant for the action of the Hecke operators.

When we study the ordinary parts of cohomology groups of $\Gamma_1(Np^r)$ with (p, N) =1, for different r's, we will also need to consider the ordinary parts of cohomology groups of Φ^s_r, for $r \geq s \geq 0$, where
$$\Phi^s_r := \Gamma_1(Np^s) \cap \Gamma_0(p^r) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{Np^r},\ a \equiv 1 \pmod{Np^s} \right\}.$$
From now on $p = 2$, $q = 4$ and $(2, N) = 1$. Let $\Gamma_0 = \Gamma_1 = \mathbb{Z}_p^\times$ and for $r \geq 2$, let Γ_r denote the subgroup $1 + p^r\mathbb{Z}_p$ of Γ, where $\Gamma = \Gamma_2 = 1 + q\mathbb{Z}_p$. There is a short exact sequence of groups
$$0 \to \Gamma_1(N2^r) \to \Phi^s_r \to \Gamma_s/\Gamma_r \to 0,$$
induced by $\Phi^s_r \ni \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \bar{d} \in \Gamma_s/\Gamma_r$. For convenience write Φ_r for Φ^0_r, for $r \geq 0$. In Hida theory, for primes $p \geq 3$, the congruence subgroup Φ_1 plays an important role, but when $p = 2$, the role of this group is played by the congruence subgroup $\Phi_2 = \Gamma_0(4) \cap \Gamma_1(N)$. Note that this last group is torsion-free if $N > 1$.

Let K be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ be the integral closure of $\mathbb{Z}_p$ in K. Let Z denote the group $\varprojlim_r (\mathbb{Z}/Np^r\mathbb{Z})^\times = \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$. We may consider $\Gamma = \Gamma_2 = 1 + 4\mathbb{Z}_p$ as a subgroup of Z; let u denote a generator of Γ_2, often taken to be $1 + q$. By definition, there is a tautological character $\iota : \Gamma \hookrightarrow \Lambda_K = \mathcal{O}_K[[\Gamma]]$, which takes u to itself in Λ_K. For each character $\chi : \Gamma \to \mathcal{O}_K^\times$, the element $P_\chi = \iota(u) - \chi(u)$ is a prime element, and $\Lambda_K/P_\chi\Lambda_K \simeq \mathcal{O}_K$, so that $\iota(u)$ corresponds to $\chi(u)$. If $\chi(u) = \epsilon(u)u^k$, where ϵ is a finite order character of Γ, we write $P_{k,\epsilon}$ for P_χ and simply P_k if ϵ is trivial. We may identify Λ_K with $\mathcal{O}_K[[X]]$ sending u to $1 + X$ and in this case, $P_{k,\epsilon}$ is nothing but $(1 + X) - \epsilon(u)u^k$. When $K = \mathbb{Q}_p$, we denote $\Lambda_{\mathbb{Q}_p}$ by Λ.

Finally, let ω denote the mod 4 cyclotomic character, that is, ω is the mod 4 character defined by $\omega(x) = \pm 1$, for $x \equiv \pm 1 \pmod 4$.

3. Main theorems

Recall that $p = 2$, and N is odd. We only use the congruence subgroup $\Gamma_1(Np^r)$ with $r \geq 2$, which is a torsion-free group. We denote the corresponding complex Riemann surface by Y_r and its compactification by X_r. Let $\mathrm{H}^i(Y_r, M)$ and $\mathrm{H}^i(X_r, M)$ denote the corresponding sheaf cohomology groups for each constant sheaf M of $\mathbb{Z}$-modules. It is well-known that
$$S_2(\Gamma_1(Np^r)) \simeq \mathrm{H}^1(X_r, \mathbb{R}),$$

where the right hand side of the isomorphism, the sheaf cohomology with $\mathbb{R}$ coefficients, can be identified with de Rham cohomology. The above isomorphism is invariant under the Hecke action. The Hecke algebra $\mathfrak{h}_2(\Gamma_1(Np^r),\mathbb{Z})$ acts on $\mathrm{H}^1(X_r,\mathbb{Z})$ and therefore $\mathfrak{h}_2(\Gamma_1(Np^r),\mathbb{Z}_p)$ acts on $\mathrm{H}^1(X_r,\mathbb{Z}_p)$, $\mathrm{H}^1(X_r,\mathbb{Q}_p)$, $\mathrm{H}^1(X_r,\mathbb{T}_p)$, where $\mathrm{H}^1(X_r,M)=\mathrm{H}^1(X_r,\mathbb{Z})\otimes_{\mathbb{Z}}M$ and $\mathbb{T}_p:=\mathbb{Q}_p/\mathbb{Z}_p$.

For every positive integer $r\geq 2$, we simply write

$$\mathcal{V}_r=\mathrm{H}^1(X_r,\mathbb{T}_p),\quad \mathcal{W}_r=\mathrm{H}^1(Y_r,\mathbb{T}_p).$$

Since $\mathrm{H}^1(X_r,\mathbb{T}_p)\simeq \mathrm{H}^1(X_r,\mathbb{Q}_p)/\mathrm{H}^1(X_r,\mathbb{Z}_p)$, we see that $\mathcal{V}_r$ and $\mathcal{W}_r$ are p-divisible modules of finite $\mathbb{Z}_p$-corank. Therefore $\mathrm{End}(\mathcal{V}_r)$ and $\mathrm{End}(\mathcal{W}_r)$ are free of finite rank. Hence one can define Hida's idempotent operator e_r attached to the Hecke operator T_p in $\mathrm{End}(\mathcal{V}_r)$ and $\mathrm{End}(\mathcal{W}_r)$. Define the ordinary parts of $\mathcal{V}_r$ to be $\mathcal{V}_r^0=e_r\mathcal{V}_r$ and similarly for $\mathcal{W}_r$. $\mathcal{V}_r^0$ is a module for $\mathfrak{h}_2^0(\Gamma_1(Np^r),\mathbb{Z}_p)$, the ordinary part of $\mathfrak{h}_2(\Gamma_1(Np^r),\mathbb{Z}_p)$. By abuse of notation, we will use the same notation e for the various e_r's.

There is also an action of $\Gamma_0(Np^r)/\Gamma_1(Np^r)$ on $\mathcal{V}_r^0$ and $\mathcal{W}_r^0$. Let $\mathcal{V}$ denote the direct limit of $\mathcal{V}_r$ and define similarly $\mathcal{W}$, $\mathcal{V}^0$ and $\mathcal{W}^0$. Since $(\mathbb{Z}/Np^r\mathbb{Z})^\times$ acts on $\mathcal{V}_r^0$ and $\mathcal{W}_r^0$, hence Z acts on $\mathcal{V}^0$ and $\mathcal{W}^0$. In particular $\mathcal{V}^0$ and $\mathcal{W}^0$ become continuous modules over the Iwasawa algebra $\Lambda=\mathbb{Z}_p[[\Gamma]]$ if we equip them with the discrete topology. Let $V^0=\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{V}^0,\mathbb{T}_p)$, respectively W^0, be the Pontryagin dual module of $\mathcal{V}^0$, respectively $\mathcal{W}^0$. Then V^0 and W^0 are compact Λ-modules. We can now state one of the main theorems of this article.

Theorem 3.1. *Let $p=2$. We have:*

1. *For each positive integer $r\geq 2$, the restriction morphism of cohomology groups induces an isomorphism of $\mathcal{V}_r^0$ onto $(\mathcal{V}^0)^{\Gamma_r}$. The same result also holds for $\mathcal{W}^0$.*
2. *Let $N>1$. The modules V^0 and W^0 are free modules of finite rank over Λ.*

The first part of Theorem 3.1 gives control of the ordinary parts of the cohomology modules associated with the decreasing sequence of congruence subgroups $\Gamma_1(Np^r)$, for $r\geq 2$, and we refer to such a result as a control theorem (for cohomology).

4. Control theorem for cohomology

In this section, we prove part (1) of Theorem 3.1.

When studying the action of the Hecke operators on cohomology groups or on parabolic cohomology groups, often one needs to decompose certain double coset spaces into a disjoint union of left cosets with a clever choice of coset representatives. Such decompositions can be found in [Hid86b, Lem. 4.3]. We recall with proof only part (ii) of that lemma, since the hypotheses of the original statement are mildly misstated. We refer the reader to [Hid86b] for the other parts, especially since the lemma is only used implicitly below, in the proof of Proposition 4.5.

Lemma 4.1. *Let $r,\,m\geq 1$, $r\geq s$. For every integer $u\in\mathbb{Z}$, let $\alpha_u\in \mathrm{M}_2(\mathbb{Z})$ be such that*

$$\alpha_u\equiv\left(\begin{smallmatrix}1 & u\\ 0 & p^m\end{smallmatrix}\right)\pmod{Np^{\max(m,r)}}\ \text{and}\ \det(\alpha_u)=p^m.$$

Then we have a disjoint decomposition

$$\Phi_r^s\left(\begin{smallmatrix}1 & 0\\ 0 & p^m\end{smallmatrix}\right)\Phi_r^s=\bigcup_{u \bmod p^m}\Phi_r^s\alpha_u.$$

Proof. Suppose that $m\geq r$. The proof in the other case is similar. The group Γ' in (3.3.2) of [Shi71, p. 67], is just Φ_r^s, for $r\geq s$, if we take $t=1$, $\mathfrak{h}$ to be the kernel of $(\mathbb{Z}/Np^r)^\times\to(\mathbb{Z}/Np^s)^\times$ and N to be Np^r. Now by [Shi71, Prop. 3.33], we have that $\Phi_r^s\left(\begin{smallmatrix}1 & 0\\ 0 & p^m\end{smallmatrix}\right)\Phi_r^s=\{\beta\in\Delta'|\det(\beta)=p^m\}$, where Δ' is as in [Shi71, p. 68]. The elements α_u belongs to the right hand side of the above equality. Observe that $u\equiv u'\pmod{p^m}$ if and only if $\alpha_u\equiv\alpha_{u'}\pmod{p^m}$. By the same proposition, the number of left cosets of Φ_r^s is p^m. Thus, the α_u, for $u\pmod{p^m}$, are candidates for the coset representatives. □

Before we start the proof of the control theorem, let us state another lemma.

Lemma 4.2. *Let $\{M_r\}_{r\geq 2}$ be an inductive system of compatible modules over $\mathbb{Z}_p[\Gamma/\Gamma_r]$, respectively. Assume that for all $r\geq t\geq 2$, $M_r^{\Gamma_t}=M_t$. Then $(\varinjlim_r M_r)^{\Gamma_t}=M_t$.*

Proof. Clearly, $\varinjlim_r M_r$ is a module over $\mathbb{Z}_p[[\Gamma]]$. For every integer $q \geq 0$, one knows $\mathrm{H}^q(\Gamma_t, \varinjlim_r M_r) = \varinjlim_r \mathrm{H}^q(\Gamma_t, M_r)$. In particular, when $q = 0$, we have that $(\varinjlim_r M_r)^{\Gamma_t} = \varinjlim_r (M_r)^{\Gamma_t} = M_t$, where the last equality follows from the assumption. □

Now, we start the proof of the control theorem for cohomology.

Lemma 4.3. *If $\Phi_r^s/\Gamma_1(Np^r)$ acts on $\mathbb{T}_p$ trivially, then $\mathrm{H}^2(\Phi_r^s/\Gamma_1(Np^r), \mathbb{T}_p) = 0$.*

Proof. Since $\Phi_r^s/\Gamma_1(Np^r)$ is a finite cyclic group, $\mathrm{H}^2(\Phi_r^s/\Gamma_1(Np^r), \mathbb{T}_p) = \mathbb{T}_p/\mathcal{N}\mathbb{T}_p$, where $\mathcal{N}$ denote the norm map from $\mathbb{T}_p$ to itself. Since $\Phi_r^s/\Gamma_1(Np^r)$ acts on $\mathbb{T}_p$ trivially, $\mathcal{N}$ is multiplication by the index of $\Gamma_1(Np^r)$ in Φ_r^s, hence is surjective. □

Lemma 4.4. *For each $r \geq s \geq 2$, $e\mathrm{H}^1(\Gamma_1(Np^s), \mathbb{T}_p) \simeq e\mathrm{H}^1(\Phi_r^s, \mathbb{T}_p)$, where e is the idempotent operator attached to T_p on the respective groups.*

Proof. Since $\Phi_r^s \subseteq \Gamma_1(Np^s)$, there is a restriction map $\mathrm{H}^1(\Gamma_1(Np^s), \mathbb{T}_p) \to \mathrm{H}^1(\Phi_r^s, \mathbb{T}_p)$. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma_1(Np^s), \mathbb{T}_p) & \xrightarrow{\text{res}} & \mathrm{H}^1(\Phi_r^s, \mathbb{T}_p) \\ \big\downarrow T_p^{r-s} & \swarrow {\scriptstyle [\Phi_r^s \left(\begin{smallmatrix} 1 & 0 \\ 0 & p^{r-s} \end{smallmatrix}\right) \Phi_s^s]} & \big\downarrow T_p^{r-s} \\ \mathrm{H}^1(\Gamma_1(Np^s), \mathbb{T}_p) & \xrightarrow{\text{res}} & \mathrm{H}^1(\Phi_r^s, \mathbb{T}_p). \end{array}$$

By applying the idempotent operator, we get that the vertical morphisms are isomorphisms and hence the diagonal map is an isomorphism. □

For each $r \geq s \geq 2$, we have the inflation-restriction sequence

$$0 \to \mathrm{H}^1(\Phi_r^s/\Gamma_1(Np^r), \mathbb{T}_p) \xrightarrow{\iota} \mathrm{H}^1(\Phi_r^s, \mathbb{T}_p) \to \mathrm{H}^1(\Gamma_1(Np^r), \mathbb{T}_p)^{\Gamma_s} \to \mathrm{H}^2(\Phi_r^s/\Gamma_1(Np^r), \mathbb{T}_p) = 0$$

where the last term vanishes by Lemma 4.3. The image of the group $\mathrm{H}^1(\Phi_r^s/\Gamma_1(Np^r), \mathbb{T}_p)$ inside $\mathrm{H}^1(\Phi_r^s, \mathbb{T}_p)$ is annihilated by the idempotent e attached to T_p by [Hid86b, Lem. 6.1]. Therefore,

$$e\mathrm{H}^1(\Phi_r^s, \mathbb{T}_p) \simeq e\mathrm{H}^1(\Gamma_1(Np^r), \mathbb{T}_p)^{\Gamma_s} = (\mathcal{W}_r^0)^{\Gamma_s}.$$

By Lemma 4.4, we have that $\mathcal{W}_s^0 = e\mathrm{H}^1(\Gamma_1(Np^s), \mathbb{T}_p) \simeq e\mathrm{H}^1(\Phi_r^s, \mathbb{T}_p)$. By combining these isomorphisms, we get

$$\mathcal{W}_s^0 \simeq (\mathcal{W}_r^0)^{\Gamma_s}.$$

By Lemma 4.2, for any $r \geq 2$, we have that $(\mathcal{W}^0)^{\Gamma_r} \simeq \mathcal{W}_r^0$. This finishes the proof of the control theorem for $\mathcal{W}^0$. Note that so far the proof works for all primes $p \geq 2$.

Now, we shall prove the control theorem for $\mathcal{V}^0$, concentrating on what changes need to be made in Hida's original proof when $p = 2$. For a torsion-free congruence subgroup Φ, let $P(\Phi)$ denote a set of generators of Φ_s, the stabilizer of Φ at s, for $s \in C(\Phi)$. Then, for $r \geq 2$, $\mathcal{W}_r$ is given by

$$\left\{ \varphi \in \mathrm{Hom}(\Gamma_1(Np^r), \mathbb{T}_p) \mid \sum_{\pi \in P(\Gamma_1(Np^r))} \varphi(\pi) = 0 \right\},$$

and $\mathcal{V}_r$ is the submodule of $\mathcal{W}_r$ given by

$$\mathcal{V}_r = \left\{ \varphi \in \mathrm{Hom}(\Gamma_1(Np^r), \mathbb{T}_p) \mid \varphi(\pi) = 0, \text{ for } \pi \in P(\Gamma_1(Np^r)) \right\},$$

since the group $\Gamma_1(Np^r)$ acts trivially on $\mathbb{T}_p$. For a detailed proof of the above equalities, see [Hid86b, p. 583].

By Lemma 4.2, it is enough to prove that $(\mathcal{V}_r^0)^{\Gamma_s} = \mathcal{V}_s^0$ for $r \geq s \geq 2$. It is clear that there is a map $\mathcal{V}_s \to \mathcal{V}_r$. By taking the ordinary parts of Γ_s-invariants, we have an inclusion $\mathcal{V}_s^0 \hookrightarrow (\mathcal{V}_r^0)^{\Gamma_s}$. Therefore it is enough to prove the surjectivity of this last map. Since $(\mathcal{W}_r^0)^{\Gamma_s} = \mathcal{W}_s^0$, given a homomorphism $\varphi : \Gamma_1(Np^r) \to \mathbb{T}_p$ invariant under Γ_s and satisfying $\varphi|_e = \varphi$, there exists a homomorphism $\psi : \Gamma_1(Np^s) \to \mathbb{T}_p$ with $\psi|_e = \psi$ such that $\psi = \varphi$ on $\Gamma_1(Np^r)$. Thus, we need to show that $\psi(\pi) = 0$, for all $\pi \in P(\Gamma_1(Np^s))$, i.e., $\psi \in e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^s), \mathbb{T}_p)$, assuming the same holds for φ with r instead of s.

Let $[\psi]$ denote the equivalence class of ψ in the module

$$\mathcal{G}(\Gamma_1(Np^s), \mathbb{T}_p) := \mathrm{H}^1(\Gamma_1(Np^s), \mathbb{T}_p)/\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^s), \mathbb{T}_p).$$

We need to show that $[\psi] = 0$. We know that $[\psi]|_e = [\psi]$. If $[\psi]|_{1-e} = [\psi]$, then $[\psi] = 0$, since e is an idempotent. Hence, it is enough to show that

$$[\psi]|_{1-e} = [\psi]$$

holds. By following the strategy in [Hid86b], this reduces to proving [Hid86b, Thm. 5.8], which characterizes the elements of $(1-e)\mathcal{G}(\Gamma_1(Np^s), \mathbb{T}_p)$ as elements of the set

$$V(\mathbb{T}_p) := \{\psi \in \mathrm{Hom}(\Gamma_1(Np^s)^{\infty}_{\mathrm{ab}}, \mathbb{T}_p) \mid \psi(\pi) = 0, \text{ for all } \pi \in P(\Gamma_1(Np^s)) \text{ corresponding to the unramified cusps}\},$$

where the module $\Gamma_1(Np^s)^{\infty}_{\mathrm{ab}}$ is the free submodule of $\Gamma_1(Np^s)_{\mathrm{ab}}$ generated by the elements of $P(\Gamma_1(Np^s))$. Under the above equality, if $\psi \in V(\mathbb{T}_p)$, then $[\psi] = (1-e)[\psi']$ and hence $[\psi]|_{1-e} = [\psi]$ holds. Thus it suffices to show that $\psi \in V(\mathbb{T}_p)$. Since every unramified cusp of X_s over X_0 is under an unramified cusp of X_r over X_0, the elements of $P(\Gamma_1(Np^s))$ corresponding to unramified cusps in X_s can be taken to be among the elements of $P(\Gamma_1(Np^r))$ corresponding to unramified cusps in X_r. Then $\psi(\pi) = \varphi(\pi) = 0$, for all $\pi \in P(\Gamma_1(Np^s))$, which corresponds to unramified cusps of $\Gamma_1(Np^s)$, as desired.

The proof of [Hid86b, Thm. 5.8] depends, firstly, on various relations between the dimensions of the space of Eisenstein series for $\Gamma_1(Np^r)$ with coefficients in A and the boundary cohomology $\mathrm{G}^1(\Gamma_1(Np^r), L_n(A)) = \oplus_{t \in C(\Gamma_1(Np^r))}\mathrm{H}^1(\Gamma_1(Np^r)_t, L_n(A))$, for any subalgebra A of $\mathbb{C}$ or $\mathbb{C}_p$, and secondly, on the validity of [Hid86b, Prop. 5.7]. The results on the dimensions of the space of Eisenstein series and the space $\mathrm{G}^1(\Gamma_1(Np^r), L_n(A))$ also holds for the prime $p = 2$. But, in the proof of [Hid86b, Prop. 5.7], one crucially uses the fact that $p \neq 2$. Thus to finish the proof of the control theorem for $\mathcal{V}^0$ when $p = 2$, it suffices to check that the proposition holds. Before we do that, let us introduce the notion of regular and irregular cusps.

The stabilizer Φ_s at a cusp $s \in C(\Phi)$ of a torsion-free congruence subgroup Φ is an infinite cyclic group. We fix an element $\alpha = \alpha_s$ in $\mathrm{SL}_2(\mathbb{Z})$ for each $s \in C(\Phi)$ such that $\alpha(\infty) = s$. We can choose a generator $\pi = \pi_s$ of Φ_s so that $\alpha^{-1}\pi\alpha = \pm\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ with $u > 0$.

Definition 1. *When $\alpha^{-1}\pi\alpha = -\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$, we say that the cusp s is irregular, and otherwise, we say that s is regular.*

This definition makes sense since $-1 \notin \Phi$, by assumption. We also remark that some authors define irregularity by the condition $\alpha^{-1}\pi\alpha = \left(\begin{smallmatrix} -1 & u \\ 0 & -1 \end{smallmatrix}\right)$, with $u > 0$. This is easily seen to be equivalent to the above by taking the inverse of the generator π.

Coming back to the proof, the difference between [Hid86b, Prop. 5.7] and the analogous result for $p = 2$ (Proposition 4.5 below) is that in the former case, i.e., when $\Phi = \Gamma_1(Np^r)$ for $p \geq 5$ and $r \geq 1$, the group Φ is torsion-free with regular cusps, whereas in the latter case, i.e., when $\Phi = \Gamma_1(Np^r)$, for $p = 2$ and $r \geq 2$, the group Φ is torsion-free, but its cusps are not necessarily regular. For example, when $N = 1$ and $r = 2$, the group $\Gamma_1(4)$ has both regular and irregular cusps. These irregular cusps create problems in the proof given in [Hid86b, Prop. 5.7] when $p = 2$.

Proposition 4.5. *Let $\Phi = \Gamma_1(Np^r)$ with $r \geq 2$. Let A be either $\mathbb{Z}_p$, $\mathbb{Z}_p/p^i\mathbb{Z}_p$ or any field of characteristic 0. Let s be any* ***unramified*** *cusp of Φ and $\rho_s : \mathrm{G}^1(\Phi, A) := \oplus_{t \in C(\Phi)}\mathrm{H}^1(\Phi_t, A) \to \mathrm{H}^1(\Phi_s, A)$ be the natural projection map. Then for any $c \in \mathrm{G}^1(\Phi, A)$, we have that $\rho_s(c|e) = \rho_s(c)$, where e is the idempotent operator attached to T_p.*

Proof. For any positive integer $M \geq 5$, all the cusps of $\Gamma_1(M)$ are regular. Hence, when $p = 2$, all cusps of $\Gamma_1(Np^r)$, for $N \geq 3$ or $r \geq 3$ are regular (so in particular are the unramified ones). So, it is enough to consider the case when $\Phi = \Gamma_1(4)$. By [Hid86b, Lem. 5.1], $\Gamma_1(4)$ has a unique unramified cusp, namely ∞. We see that this cusp is also regular since otherwise we would have

$$\pi_s = \pi_\infty = -\left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$$

for some $u > 0$, which is not an element of $\Gamma_1(4)$. Hence, when $N = 1$ and $r = 2$, the unique unramified cusp is also regular. Thus Hida's original proof of [Hid86b, Prop. 5.7] for regular cusps applies when $p = 2$ as well. ∎

Remark 1. Subsequently we will only need to consider $N \geq 3$, for applications to p-adic families. Also, in the odd prime case, the statement of [Hid86b, Prop. 5.7] also treats the groups $\Phi = \Phi_r^s$ for $r > s \geq 0$. Since we do not need this part of the proposition when $p = 2$, we ignore it. In fact, if Φ has elements of finite order (this happens for small values of N and s when $p = 2$), it is not clear that the proposition holds for such Φ.

This finishes the proof of the control theorem for cohomology.

5. Freeness

In this section, we prove that the modules V^0 and W^0 are free of finite rank over $\mathbb{Z}_p[[X]]$, completing the proof of Theorem 3.1. We restate this formally as:

Theorem 5.1. *Let $p = 2$, $N > 1$ be odd, and $\Lambda = \mathbb{Z}_p[[X]]$. The modules V^0 and W^0 are free modules of finite rank over Λ.*

The freeness of W^0 follows from [Kum, Thm. 5.3], by Poincare duality. The claim for V^0 is more subtle and requires more machinery and results. We start by recalling without proof a lemma [Hid86b, Lem. 6.3] which is useful in proving the freeness of V^0.

Lemma 5.2. *A compact continuous Λ-module M is free of finite rank r over Λ if and only if there is a subset I of positive integers and infinitely many elements $\{P_n\}_{n\in I}$ in Λ such that $\mathcal{M}[P_n] \simeq \mathbb{T}_p^r$, for all $n \in I$, where $\mathcal{M}$ is the Pontryagin dual of M and $\mathcal{M}[P_n] = \{m \in \mathcal{M} | P_n.m = 0\}$.*

We know that the group $\mathbb{Z}_p^\times$ (recall $p = 2$) acts on $\mathcal{V}^0$. In particular, $\mu_2 = (\mathbb{Z}/q\mathbb{Z})^\times$ acts on $\mathcal{V}^0$ (recall $q = 4$). Write

$$\mathcal{V}^0 = \mathcal{V}^0(0) \oplus \mathcal{V}^0(1),$$

where $\mathcal{V}^0(a) = \{v \in \mathcal{V}^0 |\ v|\zeta = \zeta^a v,\ \text{for } \zeta \in \mu_2\}$. Since the action of Γ commutes with the action of μ_2, $\mathcal{V}^0(a)$ is also a Λ-module, for $a = 0$, 1. Let $V^0(a)$ denote the Pontryagin dual of $\mathcal{V}^0(a)$. We shall show $V^0(a)$ is a free module of rank $2r(a)$ over Λ, where $r(a)$ is the rank of the Hecke algebra $\mathfrak{h}_2^0(\Phi_2, \omega^a, \mathbb{Z}_p)$.

By part (1) of Theorem 3.1, we have that $V^0(a)/\mathfrak{a}_2 V^0(a)$, where $\mathfrak{a}_2$ is the augmentation ideal of $\mathbb{Z}_p[[\Gamma]]$, is a free module of rank $2r(a)$ over $\mathbb{Z}_p$. By Nakayama's lemma, we see that $V^0(a)$ is a finitely generated Λ-module with minimal number of generators $2r(a)$. Hence there is a surjection from $\Lambda^{2r(a)} \twoheadrightarrow V^0(a)$. Hence, by duality, we have

$$\mathcal{V}^0(a)[P_n] \hookrightarrow \mathbb{T}_p^{2r(a)},$$

where P_n is the prime ideal of Λ defined in section 2. Now define

$$\mathrm{H}^1(\Gamma_1(Np^r), n; \mathbb{Z}_p/p^r\mathbb{Z}_p) := \{v \in \mathrm{H}^1(\Gamma_1(Np^r), \mathbb{Z}_p/p^r\mathbb{Z}_p) \mid v|z = z^n v \text{ for } z \in \mathbb{Z}_p^\times\},$$
$$\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), n; \mathbb{Z}_p/p^r\mathbb{Z}_p) := \mathrm{H}^1(\Gamma_1(Np^r), n; \mathbb{Z}_p/p^r\mathbb{Z}_p) \cap \mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), \mathbb{Z}_p/p^r\mathbb{Z}_p).$$

Suppose that the following inclusions and isomorphisms are true for $r \geq 2$:

$$\begin{aligned} e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \otimes \mathbb{Z}_p/p^r\mathbb{Z}_p &\underset{(1)}{\hookrightarrow} e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p/p^r\mathbb{Z}_p)) \\ &\underset{(2)}{\simeq} e\mathrm{H}^1_{\mathrm{p}}(\Phi_r, L_n(\mathbb{Z}_p/p^r\mathbb{Z}_p)) \underset{(3)}{\simeq} e\mathrm{H}^1_{\mathrm{p}}(\Phi_r, \mathbb{Z}_p/p^r\mathbb{Z}_p(n)) \\ &\underset{(4)}{\hookrightarrow} e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), n; \mathbb{Z}_p/p^r\mathbb{Z}_p) \underset{(5)}{\hookrightarrow} \mathcal{V}^0(a)[P_n], \end{aligned} \tag{5.1}$$

where the last inclusion holds only if $n \equiv a \pmod 2$. Then we have

$$e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \otimes \mathbb{Z}_p/p^r\mathbb{Z}_p \hookrightarrow \mathcal{V}^0(a)[P_n] \hookrightarrow \mathbb{T}_p^{2r(a)}.$$

Taking direct limits with respect to r, we have that

$$e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \otimes \mathbb{T}_p \hookrightarrow \mathcal{V}^0(a)[P_n] \hookrightarrow \mathbb{T}_p^{2r(a)}. \tag{5.2}$$

In the next section, we prove that the module $e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p))$ is $\mathbb{Z}_p$-free (see Lemma 6.2). More precisely, we prove in Theorem 6.1 that:

Theorem 5.3. *The $\mathbb{Z}_p$-rank of the module $e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p))$ is $2r(a)$, for $n \equiv a \pmod 2$.*

Proof. For $p \geq 5$, the theorem is proved in [Hid86a, Thm. 3.1 and Cor. 3.2]. In his proof of [Hid86a, Thm. 3.1], Hida uses results from the theory of Katz modular forms, the theory of mod p modular forms and the fact that $p \geq 5$. For the prime $p = 2$, we need different arguments to prove the theorem and we postpone the proof to the next section. □

We complete the proof of Theorem 5.1, assuming Theorem 5.3.

Proof. By Theorem 5.3, we have that $\mathbb{T}_p^{2r(a)} \simeq e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \otimes \mathbb{T}_p$. Hence, $\mathcal{V}^0(a)[P_n] \simeq \mathbb{T}_p^{2r(a)}$, for all $n \equiv a \pmod 2$. The theorem now follows from Lemma 5.2. □

Now, we shall show that the inclusions and isomorphisms in (5.1) hold. This is the content of the next few lemmas and propositions. The following proposition proves the inclusion (1) in (5.1).

Proposition 5.4. *For all $r \geq 1$ and $n \geq 0$, we have*

$$e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \otimes \mathbb{Z}_p/p^r\mathbb{Z}_p \hookrightarrow e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p/p^r\mathbb{Z}_p)).$$

Proof. For any $\mathbb{Z}_p$-module A, the short exact sequence of modules

$$0 \to e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \to e\mathrm{H}^1(\Phi_2, L_n(\mathbb{Z}_p)) \to e\mathrm{H}^1(\Phi_2, L_n(\mathbb{Z}_p))/e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \to 0$$

induces the long exact sequence

$$\mathrm{Tor}(e\mathrm{H}^1(\Phi_2, L_n(\mathbb{Z}_p))/e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)), A) \to e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \otimes A \to e\mathrm{H}^1(\Phi_2, L_n(\mathbb{Z}_p)) \otimes A.$$

If $e\mathrm{H}^1(\Phi_2, L_n(\mathbb{Z}_p))/e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p))$ is $\mathbb{Z}_p$-free, then the first term in the above exact sequence is zero, and in particular this is so when $A = \mathbb{Z}_p/p^r\mathbb{Z}_p$. So the second map above is injective. Now, for any congruence subgroup Φ,

$$e\mathrm{H}^1(\Phi, L_n(\mathbb{Z}_p)) \otimes A \xrightarrow{\sim} e\mathrm{H}^1(\Phi, L_n(A)),$$

and under this identification, this second map above preserves parabolic classes, proving the theorem. The $\mathbb{Z}_p$-freeness of the module $e\mathrm{H}^1(\Phi_2, L_n(\mathbb{Z}_p))/e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p))$ follows from [Hid88a, Prop. 2.3]. Although this proposition was proved there for $\Gamma_1(Np^r)$, for $p \geq 5$, the same proof works for Φ_2 for $p = 2$ and $N \geq 3$ with $(2, N) = 1$. □

The isomorphisms (2) and (3) in (5.1) follows from [Hid86b, Cor. 4.5], noting that the argument given there works for $p = 2$ and for Φ_2, instead of p odd and the Φ_1 there. The following lemma proves the inclusion (4).

Lemma 5.5. *For $r \geq 2$, we have an inclusion*

$$e\mathrm{H}^1_{\mathrm{p}}(\Phi_r, \mathbb{Z}_p/p^r\mathbb{Z}_p(n)) \hookrightarrow e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), n; \mathbb{Z}_p/p^r\mathbb{Z}_p).$$

Proof. We have the following inflation-restriction sequence for the groups $\Gamma_1(Np^r) \subseteq \Phi_r$:

$$\begin{aligned} 0 \to \mathrm{H}^1(\Phi_r/\Gamma_1(Np^r), \mathbb{Z}_p/p^r\mathbb{Z}_p(n)) &\to \mathrm{H}^1(\Phi_r, \mathbb{Z}_p/p^r\mathbb{Z}_p(n)) \\ &\to \mathrm{H}^1(\Gamma_1(Np^r), \mathbb{Z}_p/p^r\mathbb{Z}_p(n))^{\Phi_r/\Gamma_1(Np^r)}. \end{aligned}$$

Since

$$\mathrm{H}^1(\Gamma_1(Np^r), \mathbb{Z}_p/p^r\mathbb{Z}_p(n))^{\Phi_r/\Gamma_1(Np^r)} \hookrightarrow \mathrm{H}^1(\Gamma_1(Np^r), n; \mathbb{Z}_p/p^r\mathbb{Z}_p),$$

we have the following exact sequence

$$0 \to \mathrm{H}^1(\Phi_r/\Gamma_1(Np^r), \mathbb{Z}_p/p^r\mathbb{Z}_p(n)) \to \mathrm{H}^1(\Phi_r, \mathbb{Z}_p/p^r\mathbb{Z}_p(n)) \to \mathrm{H}^1(\Gamma_1(Np^r), n; \mathbb{Z}_p/p^r\mathbb{Z}_p).$$

By [Hid86b, Lem. 6.1], we have

$$e\mathrm{H}^1(\Phi_r, \mathbb{Z}_p/p^r\mathbb{Z}_p(n)) \hookrightarrow e\mathrm{H}^1(\Gamma_1(Np^r), n; \mathbb{Z}_p/p^r\mathbb{Z}_p).$$

Hence, we have the required claim. □

The following lemma proves inclusion (5) in (5.1).

Lemma 5.6. $e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), n; \mathbb{Z}_p/p^r\mathbb{Z}_p) \hookrightarrow \mathcal{V}^0(a)[P_n]$, *if $n \equiv a \pmod 2$.*

Proof. Observe that, $e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), n; \mathbb{Z}_p/p^r\mathbb{Z}_p)$ is the subspace of $e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), \mathbb{Z}_p/p^r\mathbb{Z}_p)$ on which $\mathbb{Z}_p^\times$ act by $v|z = z^n v$, where v is a cohomology class. The group $\mathcal{V}^0(a)[P_n]$ is also the subspace of $\mathcal{V}^0$ such that $\mathbb{Z}_p^\times$ acts by $v|z = z^n v$, where $v \in \mathcal{V}^0$. This is true because μ_2 acts by $\zeta_2^a = \zeta_2^n$ and $\gamma \in \Gamma$ acts by $v|\gamma = \gamma^n v$.

By [Hid86b, p. 584, (5.4)], we have that $e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), \mathbb{Z}_p/p^t\mathbb{Z}_p) \simeq e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), \mathbb{Z}_p) \otimes \mathbb{Z}_p/p^t\mathbb{Z}_p$. Since tensor product commutes with direct limits, we have that

$$\varinjlim_t e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), \mathbb{Z}_p/p^t\mathbb{Z}_p) \simeq e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{T}_p \simeq e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), \mathbb{T}_p).$$

By part (1) of Theorem 3.1 (i.e., $(\mathcal{V}^0)^{\Gamma_r} \simeq \mathcal{V}_r^0$, for every $r \geq 2$), we have that

$$e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), \mathbb{Z}_p/p^r\mathbb{Z}_p) \hookrightarrow \mathcal{V}^0.$$

Now the lemma follows, since this map respects the action of $\mathbb{Z}_p^\times$. □

6. Constant rank

In this section, we prove that the ranks of certain cuspidal ordinary 2-adic Hecke algebras of different weights are all equal to the rank of a weight 2 cuspidal ordinary Hecke algebra. As in the previous section, $p = 2$ and $N > 1$ is odd.

For $a = 0$ or 1, recall that $r(a)$ is the rank of the Hecke algebra $\mathfrak{h}_2^0(\Phi_2, \omega^a, \mathbb{Z}_p)$, where ω denotes the mod 4 cyclotomic character. For simplicity, we write $A(\omega^n)$ for the sheaf with twisted action $L_0(\omega^n, A)$, for any $\mathbb{Z}_p$-module A.

Theorem 6.1. *For each positive integer $n \equiv a \pmod 2$,*

$$\mathrm{rank}_{\mathbb{Z}_p}\ \mathfrak{h}^0_{n+2}(\Phi_2, \mathbb{Z}_p) = r(a).$$

Before proving this theorem, we need to gather some results, which we do now.

Lemma 6.2. *For $r > s \geq 0$, the module $e\mathrm{H}^1(\Phi_r^s, L_n(\mathbb{Z}_p))$ is $\mathbb{Z}_p$-free, for $n \geq 0$.*

Proof. The short exact sequence

$$0 \to L_n(\mathbb{Z}_p) \to L_n(\mathbb{Q}_p) \to L_n(\mathbb{T}_p) \to 0$$

induces a long exact sequence of cohomology groups for the group Φ_r^s

$$\mathrm{H}^0(\Phi_r^s, L_n(\mathbb{Q}_p)) \xrightarrow{\alpha} \mathrm{H}^0(\Phi_r^s, L_n(\mathbb{T}_p)) \xrightarrow{\beta} \mathrm{H}^1(\Phi_r^s, L_n(\mathbb{Z}_p)) \xrightarrow{\gamma} \mathrm{H}^1(\Phi_r^s, L_n(\mathbb{Q}_p)).$$

If $n = 0$, then the map α is surjective and hence the map β is zero. Therefore the map γ is injective and $\mathrm{H}^1(\Phi_r^s, L_n(\mathbb{Z}_p))$ is $\mathbb{Z}_p$-free. Assume that $n > 0$. If we can show that $e\mathrm{H}^0(\Phi_r^s, L_n(\mathbb{T}_p)) = 0$, then the lemma follows. The operator T_p acts on $L_n(\mathbb{T}_p)$ by $x|T_p = \sum_{i=0}^{p-1} \left(\begin{smallmatrix} 1 & -i \\ 0 & p \end{smallmatrix}\right)^\iota x$, where $A^\iota = \mathrm{Adj}(A)$. We see that T_p acts on any p-torsion element of $\mathrm{H}^0(\Phi_r^s, L_n(\mathbb{T}_p))$ by the matrix $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$ and hence T_p^2 acts trivially on such elements, hence the idempotent e annihilates $\mathrm{H}^0(\Phi_r^s, L_n(\mathbb{T}_p))$. □

Corollary 6.3. *For any integer $n \geq 0$, the module $e\mathrm{H}^1(\Phi_2, \mathbb{Z}_p(\omega^n))$ is $\mathbb{Z}_p$-free.*

Proof. If n is even, then $\omega^n = 1$, hence this follows from the lemma and when n is odd, the proof is similar to the proof of the lemma. □

Lemma 6.4. $e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \otimes \mathbb{Z}_p/q\mathbb{Z}_p \simeq\ e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p/q\mathbb{Z}_p))$.

Proof. By Proposition 5.4 with $r = 2$, the map

$$0 \to e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \otimes \mathbb{Z}_p/q\mathbb{Z}_p \to e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p/q\mathbb{Z}_p)).$$

is injective. For the surjectivity, we work with sheaf cohomology instead of group cohomology. Let Y be the complex open manifold associated with Φ_2. Observe that we have the following commutative diagram:

$$\begin{array}{ccccc}
e\mathrm{H}^1_c(Y, F(L_n(\mathbb{Z}_p)))/q & \twoheadrightarrow & e\mathrm{H}^1_{\mathrm{p}}(Y, F(L_n(\mathbb{Z}_p)))/q & \hookrightarrow & e\mathrm{H}^1(Y, F(L_n(\mathbb{Z}_p)))/q \\
\downarrow \wr & & \downarrow & & \downarrow \wr \\
e\mathrm{H}^1_c(Y, F(L_n(\mathbb{Z}_p/q\mathbb{Z}_p))) & \twoheadrightarrow & e\mathrm{H}^1_{\mathrm{p}}(Y, F(L_n(\mathbb{Z}_p/q\mathbb{Z}_p))) & \hookrightarrow & e\mathrm{H}^1(Y, F(L_n(\mathbb{Z}_p/q\mathbb{Z}_p))).
\end{array}$$

By [Hid88a, Cor. 2.2], the first vertical map is an isomorphism. As a result, we get that the middle vertical map is surjective and the lemma follows. □

Lemma 6.5. *For any $n \geq 0$, $e\mathrm{H}^1(Y, F(\mathbb{Z}_p(\omega^n)))/q \simeq e\mathrm{H}^1(Y, F(\mathbb{Z}_p/q\mathbb{Z}_p(\omega^n)))$.*

Proof. The short exact sequence

$$0 \to \mathbb{Z}_p(\omega^n) \xrightarrow{q} \mathbb{Z}_p(\omega^n) \to \mathbb{Z}_p/q\mathbb{Z}_p(\omega^n) \to 0$$

induces another short exact sequence

$$0 \to \mathrm{H}^1(Y, F(\mathbb{Z}_p(\omega^n))) \otimes \mathbb{Z}_p/q\mathbb{Z}_p \to \mathrm{H}^1(Y, F(\mathbb{Z}_p/q\mathbb{Z}_p(\omega^n))) \to \mathrm{H}^2(Y, F(\mathbb{Z}_p(\omega^n)))[q] \to 0,$$

where $\mathrm{H}^2(Y, F(\mathbb{Z}_p(\omega^n)))[q] = \{x \in \mathrm{H}^2(Y, F(\mathbb{Z}_p(\omega^n))) \mid q.x = 0\}$. This last group vanishes, since the cohomological dimension of Φ_r^s is 1. □

Proposition 6.6. *The module $e(\mathrm{H}^1(\Phi_2, \mathbb{Z}_p(\omega))/\mathrm{H}^1_{\mathrm{p}}(\Phi_2, \mathbb{Z}_p(\omega)))$ is $\mathbb{Z}_p$-free.*

Proof. For $i = 0$, 1, define

$$\mathrm{G}^i(\Phi_2, M) = \underset{s \in C(\Phi_2)}{\oplus} \mathrm{H}^i((\Phi_2)_s, M),$$

for any Φ_2-module M. For each $s \in C(\Phi_2)$ and $x \in \mathrm{G}^i(\Phi_2, M)$, we write x_s for the component of x in $\mathrm{H}^i((\Phi_2)_s, M)$. The module $\mathrm{G}^i(\Phi_2, M)$ has a natural action of the Hecke operators and we have an exact sequence of abelian groups for which the maps are compatible with the action of the Hecke operators:

$$0 \to \mathrm{H}^1_{\mathrm{p}}(\Phi_2, M) \to \mathrm{H}^1(\Phi_2, M) \to \mathrm{G}^1(\Phi_2, M).$$

From the exact sequence above we see that if the module $e\mathrm{G}^1(\Phi_2, \mathbb{Z}_p(\omega))$ is $\mathbb{Z}_p$-free, then the proposition follows. Consider the long exact sequence of cohomology groups

$$\mathrm{G}^0(\Phi_2, \mathbb{Q}_p(\omega)) \xrightarrow{\beta} \mathrm{G}^0(\Phi_2, \mathbb{T}_p(\omega)) \to \mathrm{G}^1(\Phi_2, \mathbb{Z}_p(\omega)) \to \mathrm{G}^1(\Phi_2, \mathbb{Q}_p(\omega)),$$

induced by the short exact sequence $0 \to \mathbb{Z}_p(\omega) \to \mathbb{Q}_p(\omega) \to \mathbb{T}_p(\omega) \to 0$.

Since the image of β is p-divisible, it is sufficient to know that for all $x \in \mathrm{G}^0(\Phi_2, \mathbb{T}_p(\omega))[p]$, $x|T_p$ belongs to $\beta(\mathrm{G}^0(\Phi_2, \mathbb{Q}_p(\omega)))$. Then a small computation shows that

$$e(\mathrm{G}^0(\Phi_2, \mathbb{T}_p(\omega))/\beta(\mathrm{G}^0(\Phi_2, \mathbb{Q}_p(\omega)))) = 0,$$

and hence $e\mathrm{G}^1(\Phi_2, L_n(\mathbb{Z}_p(\omega)))$ is $\mathbb{Z}_p$-free. We now prove, for all $x \in \mathrm{G}^0(\Phi_2, \mathbb{T}_p(\omega))[p]$, the element $x|T_p$ belongs to $\beta(\mathrm{G}^0(\Phi_2, \mathbb{Q}_p(\omega)))$.

Since $(2, N) = 1$ and $N \geq 3$, all the cusps of Φ_2 are regular, because irregularity for Φ_2 implies the irregularity for $\Gamma_1(N)$, but there are no irregular cusps for $\Gamma_1(N)$. Let $s \in C(\Phi_2)$ be a cusp of Φ_2. Let $\alpha_s = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\alpha_s(\infty) = s$. If π_s denotes a generator for $(\Phi_2)_s$, we may write

$$\pi_s = \alpha_s \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) \alpha_s^{-1} = \left(\begin{smallmatrix} 1-cau & a^2u \\ -c^2u & 1+cau \end{smallmatrix}\right) \in \Phi_2, \text{ with } u \neq 0. \tag{6.1}$$

The structure of $\mathrm{G}^0(\Phi_2, M(\omega))$ depends on the action π_s on M. In order to study this, let us divide the cusps into two types. If $p \mid u$, then we refer to this cusp as being of type 1, otherwise of type 2. We assume that Φ_2 acts trivially on M, because we are only interested in the cases when $M = \mathbb{Z}_p$, $\mathbb{Q}_p$ or $\mathbb{T}_p$. Let x be an element of $\mathrm{G}^0(\Phi_2, \mathbb{T}_p(\omega))[p]$.

If s is a cusp of type 1, then we see that $\mathrm{H}^0((\Phi_2)_s, M(\omega)) = M$ by (6.1) and moreover the map β_s is surjective, where $\beta_s : \mathrm{H}^0((\Phi_2)_s, \mathbb{Q}_p(\omega)) \to \mathrm{H}^0((\Phi_2)_s, \mathbb{T}_p(\omega))$. Hence $(x|T_p)_s \in \beta(\mathrm{H}^0((\Phi_2)_s, \mathbb{Q}_p(\omega)) = \mathbb{Q}_p)$.

Suppose s is a cusp of type 2. If π_s acts trivially on M, then $\mathrm{H}^0((\Phi_2)_s, M(\omega)) = M$ and if π_s does not act trivially on M, then $\mathrm{H}^0((\Phi_2)_s, M(\omega)) = M[2]$. In the former case, again $(x|T_p)_s \in \beta(\mathbb{Q}_p)$. In the latter case,

$$(x|T_p)_s = \sum_{i=0}^{p-1} \left(\gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \pi_s^i\right)^{\iota} \cdot x_t,$$

where $\gamma \in \Phi_2$ such that $t = \gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)(s) \in C(\Phi_2)$. Since x is 2-torsion, we see that $(x|T_p)_s = \sum_{i=0}^{p-1}(\pm 1) \cdot x_t = \sum_{i=0}^{p-1} x_t = 2x_t = 0 \in \beta(\mathrm{H}^0((\Phi_2)_s, \mathbb{Q}_p(\omega))) = 0$. Hence we have that for any $x \in \mathrm{G}^0(\Phi_2, \mathbb{T}_p(\omega))[p]$, the element $x|T_p$ belongs to $\beta(\mathrm{G}^0(\Phi_2, \mathbb{Q}_p(\omega)))$. □

Remark 2. In the above proof, we have used the fact that $p = 2$.

Corollary 6.7. $e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, \mathbb{Z}_p(\omega^n))/q \hookrightarrow e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, \mathbb{Z}_p/q\mathbb{Z}_p(\omega^n))$.

Proof. When n is even, this follows from Proposition 5.4 with $r = 2$. When n is odd, the injectivity of the first vertical map follows from the following diagram

$$\begin{array}{ccc} e\mathrm{H}^1_{\mathrm{p}}(Y, F(\mathbb{Z}_p(\omega)))/q & \overset{\alpha}{\hookrightarrow} & e\mathrm{H}^1(Y, F(\mathbb{Z}_p(\omega)))/q \\ \big\downarrow & & \big\downarrow \wr \\ e\mathrm{H}^1_{\mathrm{p}}(Y, F(\mathbb{Z}_p/q\mathbb{Z}_p(\omega))) & \hookrightarrow & e\mathrm{H}^1(Y, F(\mathbb{Z}_p/q\mathbb{Z}_p(\omega))), \end{array}$$

since α is injective by Proposition 6.6, and the second vertical map is an isomorphism by Lemma 6.5. □

Now we shall give a proof Theorem 6.1.

Proof. It is enough to prove that the $\mathbb{Z}_p$-rank of $e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p))$ is the same as the $\mathbb{Z}_p$-rank of $e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, \mathbb{Z}_p(\omega^a))$ (the modules are $\mathbb{Z}_p$-free by Lemma 6.2 and by its corollary). By (5.2), we see that the rank of $e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p))$ is less than or equal to the rank of $e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, \mathbb{Z}_p(\omega^a))$.

Again by Lemma 6.2 and by its corollary, it is enough to show the $\mathbb{Z}_p/q\mathbb{Z}_p$-rank of the module $e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \otimes \mathbb{Z}_p/q\mathbb{Z}_p$ is greater than or equal to that of $e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, \mathbb{Z}_p(\omega^a)) \otimes \mathbb{Z}_p/q\mathbb{Z}_p$. We have the following

$$\begin{aligned} e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p)) \otimes \mathbb{Z}_p/q\mathbb{Z}_p &\underset{(1)}{\simeq} e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, L_n(\mathbb{Z}_p/q\mathbb{Z}_p)) \\ &\underset{(2)}{\simeq} e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, \mathbb{Z}_p/q\mathbb{Z}_p(\omega^a)) \underset{(3)}{\hookleftarrow} e\mathrm{H}^1_{\mathrm{p}}(\Phi_2, \mathbb{Z}_p(\omega^a)) \otimes \mathbb{Z}_p/q\mathbb{Z}_p, \end{aligned} \tag{6.2}$$

where the isomorphisms (1), (2) and the inclusion (3) follow from Lemma 6.4, the isomorphism (3) in (5.1) with $r = 2$, and Corollary 6.7, respectively. Hence the theorem is proved. □

7. Λ-adic Hecke algebras

Recall that our aim is to prove a control theorem for Hida's ordinary Hecke algebra, which we now introduce.

Each element f in $S_k(\Gamma_1(Np^r))$, for $r \geq 0$, has the Fourier expansion $f(z) = \sum_n a_n(f)q^n$, for complex constants $a_n(f)$. By means of this, we may embed $S_k(\Gamma_1(Np^r))$ into the power series $\mathbb{C}[[q]]$. One may then give a rational structure on $S_k(\Gamma_1(Np^r))$ by defining the A-rational subspace $S_k(\Gamma_1(Np^r), A)$ for each subalgebra A of $\mathbb{C}$ by $S_k(\Gamma_1(Np^r), A) = S_k(\Gamma_1(Np^r)) \cap A[[q]]$.

For any $r \geq s \geq 1$, we have a commutative diagram for all n:

$$\begin{array}{ccc} S_k(\Gamma_1(Np^s), A) & \longrightarrow & S_k(\Gamma_1(Np^r), A) \\ \big\downarrow T_n & & \big\downarrow T_n \\ S_k(\Gamma_1(Np^s), A) & \longrightarrow & S_k(\Gamma_1(Np^r), A), \end{array}$$

where the horizontal arrows are the natural inclusion. Then the restriction of each Hecke operator in $\mathfrak{h}_k(\Gamma_1(Np^r), A)$ to the subspace $S_k(\Gamma_1(Np^s), A)$ is again contained in the algebra $\mathfrak{h}_k(\Gamma_1(Np^s), A)$. Thus, we have surjective A-algebra homomorphism, $\mathfrak{h}_k(\Gamma_1(Np^r), A) \to \mathfrak{h}_k(\Gamma_1(Np^s), A)$ and since $T_p \mapsto T_p$, we have that $\mathfrak{h}^0_k(\Gamma_1(Np^r), A) \to \mathfrak{h}^0_k(\Gamma_1(Np^s), A)$ for each $r \geq s \geq 1$, where the ordinary part is defined by using Hida's idempotent attached to T_p.

Now, take limits and set:

$$\mathfrak{h}_k(\Gamma_1(Np^\infty), A) := \varprojlim_r \mathfrak{h}_k(\Gamma_1(Np^r), A), \quad \mathfrak{h}^0_k(\Gamma_1(Np^\infty), A) := \varprojlim_r \mathfrak{h}^0_k(\Gamma_1(Np^r), A),$$

$$S_k(Np^\infty, A) := \cup_{r=1}^\infty S_k(\Gamma_1(Np^r), A).$$

In Lemma 7.2 below we show there is a surjection $\mathfrak{h}_{k_1}(\Gamma_1(Np^\infty), A) \twoheadrightarrow \mathfrak{h}_{k_2}(\Gamma_1(Np^\infty), A)$, for weights $k_1 \geq k_2 \geq 2$, and hence on the ordinary parts. Before we state it, we need to define a pairing between certain Hecke algebras and certain spaces of modular forms. Recall that K is a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ is the integral closure of $\mathbb{Z}_p$ in K. Put

$$S_k(Np^r, K/\mathcal{O}_K) = S_k(\Gamma_1(Np^r), K)/S_k(\Gamma_1(Np^r), \mathcal{O}_K).$$

By definition, one can embed this space via q-expansion into the module of formal series $K/\mathcal{O}_K[[q]]$. We take the injective limit:

$$S_k(Np^\infty, K/\mathcal{O}_K) = \varinjlim_r S_k(Np^r, K/\mathcal{O}_K) \to K/\mathcal{O}_K[[q]].$$

Then $S_k(Np^\infty, K/\mathcal{O}_K) \simeq S_k(Np^\infty, K)/S_k(Np^\infty, \mathcal{O}_K)$. The algebra $\mathfrak{h}_k(\Gamma_1(Np^\infty), \mathcal{O}_K)$ acts on $S_k(Np^\infty, K/\mathcal{O}_K)$. Define the pairing

$$(,) : \mathfrak{h}_k(\Gamma_1(Np^\infty), \mathcal{O}_K) \times S_k(Np^\infty, K/\mathcal{O}_K) \to K/\mathcal{O}_K,$$

by $(h, f) = a(1, f|h)$. Then $(h, f|g) = (hg, f)$, for all $h,\ g \in \mathfrak{h}_k(\Gamma_1(Np^\infty), \mathcal{O}_K)$. Equip the space $S_k(Np^\infty, K/\mathcal{O}_K)$ with the discrete topology. We have (cf. [Hid86b, Lem. 7.1]):

Lemma 7.1. *The pairing above shows that* $\mathfrak{h}_k(\Gamma_1(Np^r), \mathbb{Z}_p)$ *and* $S_k(\Gamma_1(Np^r), \mathbb{T}_p)$ *(respectively,* $\mathfrak{h}^0_k(\Gamma_1(Np^r), \mathbb{Z}_p)$ *and* $S^0_k(\Gamma_1(Np^r), \mathbb{T}_p)$*), for* $r = 1, 2, \ldots, \infty$*, are Pontryagin duals.*

Lemma 7.2. *For* $k_1 \geq k_2 \geq 2$*, there exists a surjection*

$$\mathfrak{h}_{k_1}(\Gamma_1(Np^\infty), \mathcal{O}_K) \twoheadrightarrow \mathfrak{h}_{k_2}(\Gamma_1(Np^\infty), \mathcal{O}_K).$$

Proof. The proof is similar to the proof of [Hid86b, Lem. 7.2]. For $p = 2$, we need to work with a different Eisenstein series than the one given in that lemma. For $r \geq 2$, define a formal q-expansion for each $t \in (\mathbb{Z}/p^r\mathbb{Z})^\times$ by

$$G(r, t) = -t_0 p^{-r} + \frac{1}{2} + \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n \\ d\equiv t \pmod{p^r}}} \mathrm{sgn}(d) \right) q^n,$$

where t_0 is an integer satisfying $0 \leq t_0 < p^r$ and $t_0 \equiv t \bmod p^r$. Then, as shown by Hecke, $G(r, t)$ gives the q-expansion of an element of $\mathcal{M}_1(\Gamma_1(Np^r), \mathbb{Q})$ and satisfies

$$G(r, t)|_1 = G(r, at) \text{ for } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(Np^r).$$

Put $E(r, t) = -p^r G(r, t)$. For odd primes p, the congruence $E(r, t) \equiv t \pmod{p^r}$ holds. For the even prime $p = 2$, the congruence that holds is $E(r, t) \equiv t \pmod{p^{r-1}}$. Multiplication by the Eisenstein series $E(r, 1)$ gives an injective morphism

$$\iota_r : S_{k-1}(Np^\infty, \mathbb{T}_p)[p^{r-1}] \to S_k(Np^\infty, \mathbb{T}_p)[p^{r-1}].$$

Using the injective limit of the maps ι_r and Lemma 7.1, we can finish the proof of the lemma along the lines of the proof of [Hid86b, Lem. 7.2]. □

It is known, by [Hid88b, Thm. 3.2], that the map in the above lemma is an isomorphism. This theorem is stated adelically, but includes the case of $p = 2$. Thus, the Hecke algebra $\mathfrak{h}^0_k(\Gamma_1(Np^\infty), \mathcal{O}_K)$ is independent of the weight, for all $k \geq 2$. Denote this Hecke algebra by $\mathfrak{h}^0(N, \mathcal{O}_K)$.

8. Control theorem for ordinary Hecke algebras

In this section, we prove a control theorem for Hida's ordinary Hecke algebras for the prime $p = 2$. Recall K is a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_K$ is integral closure of $\mathbb{Z}_p$ in K. Let ϵ be a character of Γ/Γ_r with values in $\mathcal{O}_K$, with $r \geq 2$. In this section, we write Λ for Λ_K and $Q(\Lambda)$ for the field of fractions of Λ_K.

We know that $\mathfrak{h}^0(N, \mathcal{O}_K)$ acts on the finite free Λ-module $\mathcal{V}^0$. Hence the Λ-module $\mathfrak{h}^0(N, \mathcal{O}_K)$ is finitely generated and torsion-free, since the action is faithful on $\mathcal{V}^0$. By abuse of notation, let $P_{k,\epsilon}$ also denote the prime ideal generated by the prime element $P_{k,\epsilon} = \iota(u) - \epsilon(u)u^k$. By the independence of weight of $\mathfrak{h}^0(N, \mathcal{O}_K)$, there is a surjective homomorphisms of $\mathcal{O}_K$-algebras, respectively, of Λ-algebras:

$$\rho : \mathfrak{h}^0(N, \mathcal{O}_K) \twoheadrightarrow \mathfrak{h}^0_k(\Phi^2_r, \epsilon, \mathcal{O}_K) \text{ and } \Lambda_{P_{k,\epsilon}} \twoheadrightarrow \Lambda_{P_{k,\epsilon}}/P_{k,\epsilon}\Lambda_{P_{k,\epsilon}} = K, \tag{8.1}$$

inducing the map

$$\tilde{\rho}_{k,\epsilon} : \mathfrak{h}^0(N, \mathcal{O}_K) \otimes_\Lambda \Lambda_{P_{k,\epsilon}} \twoheadrightarrow \mathfrak{h}^0_k(\Phi^2_r, \epsilon, \mathcal{O}_K) \otimes_{\mathcal{O}_K} K,$$

which in turn factors via $P_{k,\epsilon}$, to give the map:

$$\rho_{k,\epsilon} : \mathfrak{h}^0(N, \mathcal{O}_K) \otimes_\Lambda \Lambda_{P_{k,\epsilon}}/P_{k,\epsilon} \twoheadrightarrow \mathfrak{h}^0_k(\Phi^2_r, \epsilon, \mathcal{O}_K) \otimes_{\mathcal{O}_K} K \simeq \mathfrak{h}^0_k(\Phi^2_r, \epsilon, K),$$

where $\Lambda_{P_{k,\epsilon}}/P_{k,\epsilon}\Lambda_{P_{k,\epsilon}}$ is identified with K with $\iota(u)$ corresponding to $u^k\epsilon(u)$.

Theorem 8.1. *The natural map*

$$\rho_{k,\epsilon} : \mathfrak{h}^0(N, \mathcal{O}_K) \otimes_\Lambda \Lambda_{P_{k,\epsilon}}/P_{k,\epsilon} \twoheadrightarrow \mathfrak{h}^0_k(\Phi^2_r, \epsilon, K)$$

is an isomorphism.

Proof. Since the module $\mathfrak{h}^0(N, \mathcal{O}_K)$ is finitely generated and torsion-free over Λ, so is $\mathfrak{h}^0(N, \mathcal{O}_K)_{P_{k,\epsilon}}$ over $\Lambda_{P_{k,\epsilon}}$. Since any finitely generated torsion-free module over a discrete valuation ring is free, the module $\mathfrak{h}^0(N, \mathcal{O}_K)_{P_{k,\epsilon}}$ is free and hence it makes sense to speak of its rank. Let $S(k, \epsilon)$ (respectively, $R(k, \epsilon)$) denote the rank of $\mathfrak{h}^0(N, \mathcal{O}_K)_{P_{k,\epsilon}}$ (respectively, $\mathfrak{h}^0_k(\Phi^2_r, \epsilon, K)$). A priori the number $S(k, \epsilon)$ depends on k and ϵ. Since

$$\mathfrak{h}^0(N, \mathcal{O}_K)_{P_{k,\epsilon}} \otimes_{\Lambda_{P_{k,\epsilon}}} Q(\Lambda) \simeq \mathfrak{h}^0(N, \mathcal{O}_K) \otimes_\Lambda Q(\Lambda),$$

we see that $S(k, \epsilon)$ is independent of k and ϵ and we denote this common value by R.

We first prove the theorem for weights $k > 2$ by assuming that it holds for $k = 2$. The Eisenstein series $E(2, 1)$ above has the property that $E(2, 1) \equiv 1 \pmod 2$. Multiplication by $E(2, 1)^{k-2}$ induces an injection

$$S^0_2(\Gamma_1(Np^r), \mathbb{T}_p)[p] \to S^0_k(\Gamma_1(Np^r), \mathbb{T}_p)[p].$$

By duality, we have a surjection

$$\mathfrak{h}^0_k(\Gamma_1(Np^r), \mathbb{Z}_p) \otimes \mathbb{Z}_p/p\mathbb{Z}_p \twoheadrightarrow \mathfrak{h}^0_2(\Gamma_1(Np^r), \mathbb{Z}_p) \otimes \mathbb{Z}_p/p\mathbb{Z}_p.$$

Then

$$\begin{aligned} R[\Gamma : \Gamma_r] \geq \sum_\epsilon R(k, \epsilon) &= \mathrm{rank}_{\mathcal{O}_K}(\mathfrak{h}^0_k(\Gamma_1(Np^r), \mathcal{O}_K)) \\ &\geq \mathrm{rank}_{\mathcal{O}_K}(\mathfrak{h}^0_2(\Gamma_1(Np^r), \mathcal{O}_K)) = R[\Gamma : \Gamma_r], \end{aligned}$$

where the last equality follows by assumption. This can happen only if $R(k, \epsilon) = R$ for all k, ϵ, showing $\rho_{k,\epsilon}$ is an isomorphism.

Now, we shall prove the result for $k = 2$. By Theorem 3.1, we have that the $\mathbb{Z}_p$-rank of $e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Np^r), \mathbb{Z}_p)$ is equal to $2[\Gamma : \Gamma_r]\ \mathrm{rank}_{\mathbb{Z}_p}\mathfrak{h}^0_2(\Gamma_1(Nq), \mathbb{Z}_p)$. Hence,

$$\mathrm{rank}_{\mathbb{Z}_p}\mathfrak{h}^0_2(\Gamma_1(Np^r), \mathbb{Z}_p) = [\Gamma : \Gamma_r]\ \mathrm{rank}_{\mathbb{Z}_p}\mathfrak{h}^0_2(\Gamma_1(Nq), \mathbb{Z}_p).$$

Since $\mathfrak{h}^0_2(\Gamma_1(Np^r), K) = \oplus_\epsilon \mathfrak{h}^0_2(\Phi^2_r, \epsilon, K)$, the left hand side of the equality above is also $\sum_\epsilon R(2, \epsilon)$. If $\mathrm{rank}_{\mathbb{Z}_p}\mathfrak{h}^0_2(\Gamma_1(Nq), \mathbb{Z}_p) = R$, then $[\Gamma : \Gamma_r]R = \sum_\epsilon R(2, \epsilon)$. Since $R \geq R(2, \epsilon)$, we get $R = R(2, \epsilon)$, for each ϵ, as desired. Thus, we need to show that $R = \mathrm{rank}_{\mathbb{Z}_p}\mathfrak{h}^0_2(\Gamma_1(Nq), \mathbb{Z}_p)$. This is proved in Theorem 8.3 below. □

The following lemma is well-known; for the proof refer to [Hid86b, Lem. 6.4].

Lemma 8.2. *For any subfield K of $\mathbb{C}$ or $\mathbb{C}_p$, $\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(M), L_n(K))$ is free of rank 2 over the Hecke algebra $\mathfrak{h}_{n+2}(\Gamma_1(M), K)$ for each positive integer M.*

Set $\epsilon := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The matrix ϵ normalizes $\Gamma_1(Np^r)$ for $r \geq 1$. Let M be a module over the Hecke algebra $\mathfrak{h}^0_2(\Gamma_1(Np^r), \mathbb{Z})$. Let $M^\pm$ denote the subspaces of M defined by $\{m \pm [\epsilon]m \mid m \in M\}$. Since ϵ normalizes $\Gamma_1(Np^r)$, the action of $[\epsilon] = [\Gamma_1(Np^r)\epsilon\Gamma_1(Np^r)]$ commutes with that of the Hecke algebra $\mathfrak{h}^0_2(\Gamma_1(Np^r), \mathbb{Z})$ on M. Therefore, the modules $M^\pm$ are stable under the action of $\mathfrak{h}^0_2(\Gamma_1(Np^r), \mathbb{Z})$. For simplicity, we write $\mathfrak{h}^0_2(N, \mathbb{Z}_p)$ for the weight-2 Λ-adic Hecke algebra $\mathfrak{h}^0_2(\Gamma_1(Np^\infty), \mathbb{Z}_p)$.

Theorem 8.3. *The surjective map*

$$\rho_{2,\mathrm{triv}} : \mathfrak{h}^0_2(N, \mathbb{Z}_p) \otimes_\Lambda \Lambda_{P_2}/P_2 \twoheadrightarrow \mathfrak{h}^0_2(\Gamma_1(Nq), \mathbb{Q}_p)$$

is an isomorphism.

Proof. By Theorem 3.1, we have $(\mathcal{V}^0)^{\Gamma_r} = \mathcal{V}^0_r$, for $r \geq 2$, and in particular $(\mathcal{V}^0)^{\Gamma_2} = \mathcal{V}^0_2$, i.e., $\mathcal{V}^0[P_2] = e\mathrm{H}^1(X_2, \mathbb{T}_p) = e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Nq), \mathbb{T}_p)$, where the last equality follows from [Hid86b, p. 583 (5.3)]. Again by the same theorem, we have

$$V^0/P_2V^0 \simeq \mathrm{Hom}_{\mathbb{Z}_p}(e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Nq), \mathbb{Z}_p), \mathbb{Z}_p). \tag{8.2}$$

Since $\mathcal{V}^0$ is direct limit over $\mathcal{V}^0_r$, we see that $[\epsilon]$ acts on $\mathcal{V}^0$ and the action commutes with that of the Hecke algebra $\mathfrak{h}^0_2(N, \mathbb{Z}_p)$. There is a map $\mathcal{V}^{0+} \oplus \mathcal{V}^{0-} \to \mathcal{V}^0$, which is an isomorphism if p is odd. Since $p = 2$, we tensor this with Λ_{P_2} so that we have an isomorphism $(\mathcal{V}^{0+})_{P_2} \oplus (\mathcal{V}^{0-})_{P_2} \simeq \mathcal{V}^0_{P_2}$.

Let $V^{0\pm}$ denote the Pontryagin dual of $\mathcal{V}^{0\pm}$. Then $(V^{0+})_{P_2} \oplus (V^{0-})_{P_2} \simeq V^0_{P_2}$. We can think of $\mathfrak{h}^0_2(N, \mathbb{Z}_p)_{P_2}$ as a subalgebra of the endomorphism algebra of $(V^{0+})_{P_2}$ and hence we shall restrict ourselves to the module $(V^{0+})_{P_2}$. We now prove that

$$(V^{0+})_{P_2}/P_2(V^{0+})_{P_2} = \mathfrak{h}^0_2(\Gamma_1(Nq), \mathbb{Q}_p).$$

We remark that since $p = 2$ and we work with $(V^{0+})_{P_2}$, the above isomorphism is with $\mathbb{Q}_p$-coefficients, otherwise we would have worked with V^{0+} and the above isomorphism would have been with $\mathbb{Z}_p$-coefficients. Since the functor $\mathrm{Hom}_{\mathbb{Z}_p}(-, \mathbb{T}_p)$ commutes with the $\pm$-action after tensoring with Λ_{P_2}, we see that $V^{0\pm} \otimes_\Lambda \Lambda_{P_2} \simeq (V^0_{P_2})^\pm$ holds. Hence $(V^{0\pm})_{P_2}/P_2(V^{0\pm})_{P_2} \simeq (V^0_{P_2})^\pm/P_2(V^0_{P_2})^\pm \simeq (V^0_{P_2}/P_2V^0_{P_2})^\pm$, where the last isomorphism is an easy check. We have that

$$\begin{aligned}(V^{0+})_{P_2}/P_2(V^{0+})_{P_2} &= (V^0_{P_2}/P_2V^0_{P_2})^+ \\ &\underset{(8.2)}{=} (\mathrm{Hom}_{\mathbb{Z}_p}(e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Nq), \mathbb{Z}_p), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^+ \\ &= \mathrm{Hom}_{\mathbb{Q}_p}(e\mathrm{H}^1_{\mathrm{p}}(\Gamma_1(Nq), \mathbb{Q}_p)^+, \mathbb{Q}_p) = \mathfrak{h}^0_2(\Gamma_1(Nq), \mathbb{Q}_p),\end{aligned} \tag{8.3}$$

where the last equality follows from Lemma 8.2 and the third equality follows from the fact that for any $\mathbb{Q}_p$-module M, $\mathrm{Hom}_{\mathbb{Q}_p}(M, \mathbb{Q}_p)^\pm \simeq \mathrm{Hom}_{\mathbb{Q}_p}(M^\pm, \mathbb{Q}_p)$.

Let v denote the vector in $(V^{0+})_{P_2}$ corresponding to 1 in $\mathfrak{h}^0_2(\Gamma_1(Nq), \mathbb{Q}_p)$ in (8.3). Therefore, we have a map $\mathfrak{h}^0_2(N, \mathbb{Z}_p)_{P_2} \to (V^{0+})_{P_2}$ defined by mapping $h \to hv$. This map is a surjective map by Nakayama's lemma and by (8.3). The map is injective since the Hecke action is faithful on $(V^{0+})_{P_2}$. Therefore, we have $\mathfrak{h}^0_2(N, \mathbb{Z}_p)_{P_2} \simeq (V^{0+})_{P_2}$. Tensoring this isomorphism with Λ_{P_2}/P_2 and using (8.3), we obtain the theorem. □

9. Uniqueness

In this section, we prove a uniqueness result for Hida families. Let f be a p-stabilized newform. Let P_f denote the unique height one prime ideal, induced by f, via the isomorphism in Theorem 8.1. Suppose $Q = P_f$ lies over the prime ideal $P_{k,\epsilon}$, where the integer k and the character ϵ depend on f.

First we show that, for the prime $P = P_{k,\epsilon}$ of Λ, the localized Hecke algebra $\mathfrak{h}^0(N, \mathcal{O}_K)_Q$ is étale over $\Lambda_{P_{k,\epsilon}}$. We deduce the uniqueness result as a consequence. For simplicity, let us denote $\mathfrak{h}^0(N, \mathcal{O}_K)$ by $\mathfrak{h}^0(N)$.

Proposition 9.1. *The localized Hecke algebra $\mathfrak{h}^0(N)_Q$ is étale over Λ_P and $Q\mathfrak{h}^0(N)_Q = P\mathfrak{h}^0(N)_Q$, i.e., $\mathfrak{h}^0(N)_Q$ is a regular local ring.*

Proof. We apply [Nek06, Lem. 12.7.6], with $A = \Lambda$, $B = \mathfrak{h}^0(N)$ and $J = 0$ (and also by switching the roles of P and Q). The first condition of that lemma, namely the Hecke algebra $\mathfrak{h}^0(N)$ is finitely generated and torsion-free over Λ follows, as mentioned earlier, from Theorem 3.1. By Theorem 8.1, the short exact sequence in the second part of that lemma reduces to

$$0 \to \mathcal{P} \to \mathfrak{h}^0_k(\Phi^2_r, \epsilon, K) \xrightarrow{\alpha} \mathbb{Q}_p(a_n(f))_{n=1}^\infty \to 0,$$

where the last map is given by $T_n \to a_n(f)$ and $\mathcal{P}$ denote the kernel of α. By analyzing the proof of that lemma, we see that if $\mathcal{P}_\mathcal{P} = 0$, where $\mathcal{P}_\mathcal{P}$ denote the localisation, then the proposition follows. From the theory of newforms, one knows that $\mathfrak{h}^0_k(\Phi^2_r, \epsilon, K)_\mathcal{P} \xrightarrow{\sim} \mathbb{Q}_p(a_n(f))_{n=1}^\infty$, hence $\mathcal{P}_\mathcal{P} = 0$. □

Now we recall the definition of a 2-adic Λ-adic form. Let $p = 2$. Let L denote the integral closure of Λ in a finite extension of $Q(\Lambda)$. Let ζ denote a p^{r-2}-th root of unity in $\bar{\mathbb{Q}}_p$, the algebraic closure of $\mathbb{Q}_p$, with $r \geq 2$, and let $k \geq 1$ be a positive integer. The assignment $X \to \zeta(1+q)^k - 1$ yields a $\mathbb{Z}_p$-algebra homomorphism $\varphi_{k,\zeta} : \Lambda \to \bar{\mathbb{Q}}_p$. We shall say that a height one prime $P \in \mathrm{Spec}(L)(\bar{\mathbb{Q}}_p)$ has weight k if the corresponding Λ-algebra homomorphism $P : L \to \bar{\mathbb{Q}}_p$ extends $\varphi_{k,\zeta}$ on Λ for some $k \geq 1$ and for some ζ. In addition we say that P is arithmetic if P has weight $k \geq 2$.

Recall N is an integer prime to p. We need some notation for certain Dirichlet characters. Let:

- ψ be a Dirichlet character of level Nq,
- ω be the mod 4 cyclotomic character,
- ϵ be the character χ_ζ mod 2^r for each root of unity ζ of order 2^{r-2} with $r \geq 2$ defined by first decomposing

$$(\mathbb{Z}_p/2^r\mathbb{Z}_p)^\times = (\mathbb{Z}_p/q\mathbb{Z}_p)^\times \times \mathbb{Z}/2^{r-2},$$

where the second factor is generated by $1+q$, and then by setting

$$\chi_\zeta = 1 \text{ on } (\mathbb{Z}_p/q\mathbb{Z}_p)^\times \quad \text{and} \quad \chi_\zeta(1+q) = \zeta.$$

Definition 2. *Let $\mathcal{F} = \sum_{n=1}^{\infty} a(n,\mathcal{F})q^n \in L[[q]]$ be a formal q-expansion with coefficients $a(n,\mathcal{F}) \in I$. We say $\mathcal{F}$ is a Λ-adic form of tame level N and character ψ if for each arithmetic point $P \in Spec(L)(\bar{\mathbb{Q}}_p)$ lying over $\varphi_{k,\zeta}$, with $k \geq 2$ and ζ of order $2^{r-2}, r \geq 2$, the specialization*

$$P(\mathcal{F}) \in \bar{\mathbb{Q}}_p[[q]]$$

of $\mathcal{F}$ at P is the q-expansion of a classical cusp form $f \in S_k(N2^r, \chi)$, where $\chi = \psi\omega^{-k}\chi_\zeta$.

The notion of primitive, ordinary, p-distinguished for Λ-adic forms can be defined similar to the classical case. For definitions, refer to [GV04, §3].

We remark that there is no 2-adic Hida theory when the tame level is 1, showing that our assumption that $N > 1$ in several previous sections loses no generality. Indeed, we have that:

Proposition 9.2. *There are no ordinary Λ-adic eigenforms of tame level 1.*

Proof. If such a Λ-adic eigenform were to exist, then for every integer $k \geq 2$ and $r \geq 2$, its' specialization at $P_{k,\zeta}$, where ζ is a 2^{r-2}-th root of unity, would be an element of $S_k(p^r, \omega^{a-k}\chi_\zeta)$, for some $a \in \mathbb{N}$. For parity reasons, $(-1)^{a-k} = (-1)^k$, hence a is even. But, if $r = 2$ and k is even, then are no 2-ordinary, 2-stabilized Hecke eigenforms in $S_k(4, \mathrm{triv})$. For newforms this follows from [Miy89, Thm. 4.6.17] and for oldforms from *loc. cit.* and the fact that $X_0(2)$ has genus 0, and from Hatada [Hat79]. Since there are no ordinary specializations in even weight, there are no 2-ordinary Λ-adic eigenforms of tame level 1. □

Corollary 9.3. *For odd integers k, the space $eS_k^{2\text{-new}}(4,\omega)$ is zero.*

Proof. This follows immediately from the proposition noting that every 2-ordinary eigenform in the above space must live in a 2-ordinary Hida family of tame level 1. □

Remark 3. It can be checked independently that the dimensions of $eS_k^{2\text{-new}}(4,\omega)$ for $k = 3$, 5, 7, 9, 11, 13, 15, 17, are indeed all zero, whereas the dimensions of $S_k^{2\text{-new}}(4,\omega)$ for $k = 3$, 5, 7, 9, 11, 13, 15, 17 are 0, 1, 2, ..., 7, respectively.

We now turn to the uniqueness result for 2-adic families.

Theorem 9.4. *Any p-ordinary elliptic p-stabilized newform is an arithmetic specialization of a **unique** Hida family, up to Galois conjugacy.*

Proof. By (8.1), we know that any p-ordinary eigenform lives in a Hida family. We want to show that such a family is unique, up to Galois conjugacy (this last caveat is necessary since if a form lies in F by specialization under $P : L \to \bar{\mathbb{Q}}_p$, then it also lies in the conjugate family F^σ, by specializing under $\sigma^{-1} \circ P : L^\sigma \to \bar{\mathbb{Q}}_p$. Note that F and F^σ correspond to the same minimal prime ideal of $\mathfrak{h}^0(N)$).

Assume the contrary. Let λ_1 and λ_2 denote the algebra homomorphisms from $\mathfrak{h}^0(N)$ to L and L' respectively, where L, L' are finite integral extensions of Λ. Let P_1 and P_2 denote the minimal prime ideals of $\mathfrak{h}^0(N)$ which are the respective kernels of these homomorphisms. Since λ_1 and λ_2 have one arithmetic specialization in common, there are two algebra homomorphisms $P : L \to \bar{\mathbb{Q}}_p$ and $P' : L' \to \bar{\mathbb{Q}}_p$ such that $P \circ \lambda_1 = P' \circ \lambda_2 = \lambda_{P,P'}$, say. Then the kernel of $\lambda_{P,P'}$ is a height one prime of $\mathfrak{h}^0(N)$, denote by Q, containing both P_1 and P_2 and lying over $P = P_{k,\zeta}$ for some $k \geq 2$, ζ.

By Proposition 9.1, $\mathfrak{h}^0(N)_Q$ is a regular local ring. But, a regular a local ring is a domain, hence the prime ideals P_1 and P_2 have to be equal. □

As an application of the last result we now show that the notion of CM-ness is pure with respect to families.

Proposition 9.5. *Let $\mathcal{F}$ be a primitive 2-adic Hida family. Then either all arithmetic specializations are CM forms or no arithmetic specialization is a CM form.*

Proof. The proof is the same as for odd prime p, once one has the uniqueness result for $p = 2$. Indeed a CM family is defined to be one which is obtained as the theta series of a Λ-adic Hecke character of an imaginary quadratic field. Clearly all its arithmetic specializations are CM forms. Now start with an arbitrary CM form. Assume it lives in a non-CM family (one which is not a theta series). Then explicit interpolation allows us to also construct a CM family passing through this CM form. Clearly the non-CM family and the CM family are not Galois conjugate, which is a contradiction by Theorem 9.4. □

In view of this result from now on we may and do speak of CM and non-CM 2-adic Hida families.

10. Applications to Galois representations

In [GV04], the splitting of the local Galois representations associated to ordinary eigenforms was studied for odd primes p. We carry out the same analysis for the case of $p = 2$, assuming that the relevant result of Buzzard continues to hold for $p = 2$ in the residually dihedral setting. That is, under this assumption, we prove that in a non-CM 2-adic Hida family, all arithmetic specializations have non-split local Galois representation, except for a possible finite set of exceptions. By Proposition 9.5, we are able to exclude CM forms from this finite exceptional set, but we do not yet know if this set is empty.

Recall p denotes the prime 2 and $q = 4$. We recall some preliminaries on ordinary eigenforms and their associated Galois representations. Let $f = \sum_{n=1}^{\infty} a_n(f)q^n$ be a primitive elliptic modular Hecke eigenform of weight $k \geq 2$ and nebentypus $\chi : (\mathbb{Z}/Np^r)^\times \to \mathbb{C}^\times$, for some $r \geq 0$. (The two usages of q, that in the q-expansion and the natural number 4, should be clear from the context!) Let K_f denote the number field generated by the Fourier coefficients of the cusp form f. Fix an embedding i_p of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$. Let $\wp$ be the prime of $\bar{\mathbb{Q}}$ determined by this embedding. Let $\wp$ also denote the induced prime of K_f, and let $K_{f,\wp}$ be the completion of K_f at $\wp$. Let G_p denote the absolute Galois group of $\mathbb{Q}_p$ and also the decomposition group at $\wp$. There is a Galois representation

$$\rho_f = \rho_{f,\wp} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\wp}),$$

associated to f (and $\wp$) which has the property that for all primes $\ell \nmid Np$,

$$\mathrm{trace}(\rho_f(\mathrm{Frob}_\ell)) = a_\ell(f) \quad \text{and} \quad \det(\rho_f(\mathrm{Frob}_\ell)) = \chi(\ell)\ell^{k-1}.$$

Recall f is ordinary at $\wp$ (or $\wp$-ordinary), if $a_p(f)$ is $\wp$-adic unit. If f is ordinary at $\wp$, then the result of Wiles [Wil88] shows that the restriction of ρ_f to the decomposition group G_p is upper-triangular, i.e.,

$$\rho_f|_{G_p} \sim \begin{pmatrix} \delta & \psi \\ 0 & \epsilon \end{pmatrix},$$

where δ, $\epsilon : G_p \to K_{f,\wp}^\times$ are characters with ϵ unramified and $\psi : G_p \to K_{f,\wp}$ is a continuous function. We say that the ordinary representation $\rho_f|_{G_p}$ splits, if the representation space of ρ_f can be written as direct sum of two G_p-invariant lines.

10.1. Buzzard's result.

We shall assume that a slight strengthening of a result of Buzzard holds. Let $\mathcal{O}$ denote the ring of integers in a finite extension L of $\mathbb{Q}_p$. Let $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation. Let λ denote the maximal ideal of $\mathcal{O}$ and let $\bar{\rho}$ denote the mod λ reduction of ρ. The following result is proved in [Buz03], and we refer to that paper for a detailed explanation of all the hypotheses.

Theorem 10.1 (Buzzard)**.** *Assume that*

(1) *ρ is ramified at finitely many primes and $\bar{\rho}$ is modular,*
(2) *$\bar{\rho}$ is absolutely irreducible when restricted to* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(i))$,
(3) *$\rho|_{G_p}$ is the direct sum of two 1-dimensional characters α and $\beta : G_p \to \mathcal{O}^\times$, such that $\alpha(I_p)$ and $\beta(I_p)$ are finite, and (α/β) mod λ is non-trivial,*
(4) *$\bar{\rho}(c) \neq 1$,*
(5) *$\bar{\rho}(c)$ is both α-modular and β-modular, in the sense that there are eigenforms f_α with T_p-eigenvalue $\bar{\alpha}(\mathrm{Frob}_p)$ and f_β with T_p-eigenvalue $\bar{\beta}(\mathrm{Frob}_p)$ giving rise to $\bar{\rho}$,*
(6) ***The projective image of $\bar{\rho}$ is not dihedral.***

Then ρ is modular, in the sense that there exists an embedding $i : L \hookrightarrow \mathbb{C}$ and a classical weight 1 cuspidal eigenform f such that the composite $i \circ \rho$ is isomorphic to the representation associated to f by Deligne and Serre.

Let us comment on the assumption (6). There is no restriction (6) on the projective image of $\bar{\rho}$ for odd primes p in [Buz03]. For $p = 2$, this assumption was made due to the unavailability of '$R^{\mathrm{red}} = T$ theorems' in the residually dihedral setting. In his recent thesis, Allen [All12] has proved such a theorem, deducing the modularity of nearly ordinary 2-adic residually dihedral Galois representations. He works under some assumptions, the most crucial for us being that he assumes that the prime 2 does not split in the quadratic extension of $\mathbb{Q}$ corresponding to the (dihedral) residual representation. However, for the application of Buzzard's theorem we have in mind below, the prime 2 does split in this extension. It appears that extending Allen's result to the split case might not be possible without a new idea. From now on we therefore **assume** that Theorem 10.1 holds without condition (6).

Remark 4. In [All12], the splitting assumption on the prime 2 is made in order to ensure that the dihedral locus is small in the Hecke algebra. This guarantees the existence of certain 'nice' primes that are needed in order to use a connectivity result of Raynaud in the course of the proof.

10.2. **Λ-adic Galois representations.** We state a few facts about Λ-adic Galois representations. Let $\mathcal{F} \in I[[q]]$ be a primitive Λ-adic form of level N and with character ψ. Let $K_{\mathcal{F}}$ denote the quotient field of I. Then there exists a Galois representation attached to $\mathcal{F}$, constructed by Hida, and Wiles in the case of $p = 2$,

$$\rho_{\mathcal{F}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{\mathcal{F}}),$$

such that for each arithmetic point P of I, $P(\rho_{\mathcal{F}})$, the specialization of $\rho_{\mathcal{F}}$ at P, is isomorphic to the representation ρ_f attached to $f = P(\mathcal{F})$ by Deligne. Note that if ℓ is a prime number such that $\ell \nmid Np$, then

$$\mathrm{trace}(\rho_{\mathcal{F}}(\mathrm{Frob}_\ell)) = a(\ell, \mathcal{F}) \in I, \ \det(\rho_{\mathcal{F}}(\mathrm{Frob}_\ell)) = \psi(\ell)\kappa(\mathrm{Frob}_\ell)\ell^{-1},$$

where $\kappa : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \Lambda^\times$ is the 'Λ-adic cyclotomic character'.

The restriction of $\rho_{\mathcal{F}}$ to G_p also turns out to be 'upper-triangular'. More precisely, the representation $\rho_{\mathcal{F}}|_{G_p}$ has the following shape

$$\rho_{\mathcal{F}}|_{G_p} \sim \begin{pmatrix} \delta_{\mathcal{F}} & u_{\mathcal{F}} \\ 0 & \epsilon_{\mathcal{F}} \end{pmatrix},$$

where $\delta_{\mathcal{F}}, \epsilon_{\mathcal{F}} : G_p \to K_{\mathcal{F}}^\times$ are characters with $\epsilon_{\mathcal{F}}$ unramified, and $u_{\mathcal{F}} : G_p \to K_{\mathcal{F}}$ is a continuous map. Let

$$c_{\mathcal{F}} = \epsilon_{\mathcal{F}}^{-1}.u_{\mathcal{F}} \in \mathrm{Z}^1(G_p, K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}^{-1}))$$

be the associated cocycle. Then the representation

$$\rho_{\mathcal{F}}|_{G_p} \text{ splits if and only if } [c_{\mathcal{F}}] = 0 \text{ in } \mathrm{H}^1(G_p, K_{\mathcal{F}}(\delta_{\mathcal{F}}\epsilon_{\mathcal{F}}^{-1})).$$

We shall shortly show that for a primitive 2-adic family $\mathcal{F}$ whose residual representation satisfies some technical conditions (cf. conditions (1), (2), (3) below), the corresponding representation $\rho_{\mathcal{F}}$ splits at p if and if $\mathcal{F}$ is a CM family. As a consequence, standard descent arguments allow us to conclude the following partial result towards Greenberg's question on the local splitting of ordinary 2-adic modular Galois representations.

Theorem 10.2. *Let $\mathcal{F}$ be a primitive non-CM 2-ordinary Hida family of eigenforms with the property that*

1. *$\bar{\rho}_{\mathcal{F}}$ is p-distinguished,*
2. *$\bar{\rho}_{\mathcal{F}}$ is absolutely irreducible, when restricted to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}(i))$,*
3. *$\bar{\rho}_{\mathcal{F}}(c) \neq 1$ and $\bar{\rho}_{\mathcal{F}}(c)$ is both α-modular and β-modular.*

Then, for all but except possibly finitely many arithmetic members $f \in \mathcal{F}$, the representation $\rho_f|_{G_p}$ is non-split. Moreover the possible exceptions are necessarily non-CM forms.

For the definitions of primitive and p-distinguished, refer to [GV04, §2]. We remark again that the last statement in the theorem is a consequence of uniqueness for 2-adic families, proved in the last section.

10.3. Local splitting for Λ-adic eigenforms.

Proposition 10.3. *Let $\mathcal{F}$ be a primitive 2-adic Λ-adic eigenform of fixed tame level N satisfying conditions (1)-(3) above. Then $\rho_{\mathcal{F}}|_{G_p}$ splits if and only if $\mathcal{F}$ is of CM type.*

Proof. The proof is very similar to that for odd primes given in [GV04, Prop. 14]. One shows that the following statements are equivalent.

(1) $\rho_{\mathcal{F}}|_{G_p}$ splits.
(2) $\mathcal{F}$ has infinitely many weight one classical specializations.
(3) $\mathcal{F}$ has infinitely many weight one classical CM specializations.
(4) $\mathcal{F}$ is of CM type.

For the readers convenience, we prove the implications (1) $\implies$ (2), to show how the strengthened version of Buzzard's result is used. For the remaining implications, we refer the reader to [GV04, Prop. 14], although a shorter proof of the implication (3) $\implies$ (4) can be found in [DG12].

(1) $\implies$ (2): Recall that we have the following characters:

$$\psi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \bar{\mathbb{Q}}_p^\times \quad \text{the character of } \mathcal{F} \text{ of conductor } Nq,$$
$$\kappa : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \Lambda^\times, \quad \text{the } \Lambda\text{-adic cyclotomic character},$$
$$\nu : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times \quad \text{the 2-adic cyclotomic character.}$$

We know that $\det(\rho_{\mathcal{F}})= \psi\kappa\nu^{-1}$. The specialization of $\det(\rho_{\mathcal{F}})$ at $\varphi_{k,\zeta}$ is $\chi\nu^{k-1}$, where $\chi = \psi\omega^{-k}\chi_\zeta$. By assumption $\rho_{\mathcal{F}}|_{G_p}$ splits, i.e.,

$$\rho_{\mathcal{F}}|_{I_p} \sim \begin{pmatrix} \psi\kappa\nu^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Let P be a weight one point of L extending $\varphi_{1,\zeta} : \Lambda \to \bar{\mathbb{Q}}_p$. It follows that $P(\rho_{\mathcal{F}}) = \rho_{P(\mathcal{F})}$ has the following shape on I_p:

$$\rho_{P(\mathcal{F})}|_{I_p} \sim \begin{pmatrix} \psi\omega^{-1}\chi_\zeta & 0 \\ 0 & 1 \end{pmatrix},$$

noting that the characters on the diagonal have finite order. Now by Theorem 10.1, we have that

$$\rho_{P(\mathcal{F})} \sim \rho_f,$$

where f is a primitive weight 1 form of level $N2^r$, with character $\psi\omega^{-1}\chi_\zeta$ where ζ is exactly of order 2^{r-2}, $r \geq 2$. As we vary the point P, and therefore $r \geq 2$, we obtain infinitely many classical weight 1 specializations of $\mathcal{F}$ as required.

We remark that elementary arguments (cf. [GV04, (2) $\implies$ (3) of Prop. 14]), allow us to conclude that infinitely many of these must be of CM type, and in particular the residual representation must necessarily be of dihedral type. Moreover, by ordinariness, the prime 2 will split in the corresponding imaginary quadratic field. This explains why it is crucial to assume that Buzzard's result holds in this case as well.

□

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