The space of degenerate Whittaker models for $GL(4)$ over p-adic fields

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1 Introduction

Let $G = GL_{2n}(k)$ where k is a non-Archimedean local field. Let P be the (n, n) parabolic in G with Levi subgroup $GL_n(k) \times GL_n(k)$ and unipotent radical $N = M_n(k)$. Let ψ_0 be a non-trivial additive character $\psi_0 : k \rightarrow \mathbf{C}^*$. Let $\psi(X) = \psi_0(\mathrm{tr}X)$ be the additive character on $N = M_n(k)$. Let V be an irreducible admissible representation of G. Let $V_{N,\psi}$ be the largest quotient of V on which N operates via ψ:

$$V_{N,\psi} = \frac{V}{\{n \cdot v - \psi(n)v | n \in N, v \in V\}}.$$

Since $\mathrm{tr}(gXg^{-1}) = \mathrm{tr}(X)$, it follows that $V_{N,\psi}$ is a representation space for $H = \Delta GL_n(k) \hookrightarrow GL_n(k) \times GL_n(k)$. The space $V_{N,\psi}$ will be referred to as the space of degenerate Whittaker models, or sometimes also as the twisted Jacquet functor of the representation V. These considerations also work for the group $G = GL_2(D)$ where D is any central simple algebra with center k. Again to any irreducible admissible representation V of $GL_2(D)$, one can define $V_{N,\psi}$ which will now be a representation space for $H = D^*$.

The aim of this work is to understand the structure of $V_{N,\psi}$ as a representation space for $GL_n(k)$. In an earlier work, cf. [P4], we had done this in the case of finite fields. The case of p-adic field seems much more difficult, and it appears that $V_{N,\psi}$ has interesting structure only for $n = 4$ where the following multiplicity 1 theorem due to Rallis holds.

Theorem 1 *(Rallis) Let V be an irreducible admissible representation of $G = GL_4(k)$ (resp. $GL_2(D)$, D a quaternion division algebra) and W an irreducible admissible representation of $H = GL_2(k)$ (resp. D^*). Then*

$$\dim \mathrm{Hom}_H[V_{N,\psi}, W] \leq 1.$$

In this paper we make a conjecture about the structure of $V_{N,\psi}$, when the group is $GL_4(k)$, or $GL_2(D)$ where D is a quaternion division algebra. Before we state our conjecture which tells exactly which representations W of $GL_2(k)$ or D^* appear in $V_{N,\psi}$, we recall that by Langlands correspondence for $G = GL_4(k)$ or $GL_2(D)$, for any irreducible admissible representation V of G there is a natural 4 dimensional representation of the Weil-Deligne group W_k of k which will be denoted by σ_V.

Here is the main conjecture. The statement of the conjecture involves epsilon factors attached to representations of the Weil-Deligne group of k for which we refer to the article [Ta] of Tate. We are able to prove this conjecture only for those representations which are irreducibly induced from a proper parabolic subgroup in which case the conjecture reduces to author's earlier work on the trilinear forms for representations of GL_2, cf. [P1]. We will also reformulate the conjecture for many other representations so as to not involve epsilon factors directly.

Conjecture 1 *Let V be an irreducible admissible generic representation of $G = GL_4(k)$ (resp. of $GL_2(D)$ whose Jacquet-Langlands lift to GL_4 is generic) and W an irreducible admissible generic representation of $H = GL_2(k)$ (resp. D^*). Assume that the central characters of V and W are the same. Then $\mathrm{Hom}_{GL_2(k)}[V_{N,\psi}, W] \neq 0$ if and only if $\epsilon[(\Lambda^2\sigma_V) \otimes \sigma_W^*] = (\det \sigma_W)(-1)$, and $\mathrm{Hom}_{D^*}[V_{N,\psi}, W] \neq 0$ if and only if $\epsilon[(\Lambda^2\sigma_V) \otimes \sigma_W^*] = -(\det \sigma_W)(-1)$.*

Remark : The above conjecture is essentially conjecture 6.9 of [G-P] for the particular case when the orthogonal group is of 6 variables which is closely related to $GL(4)$. The motivation for the present work comes from some work which the author has done with A. Raghuram in [P-R] which is an attempt to develop Kirillov theory for $GL_2(D)$ in which the space of degenerate Whittaker models plays a prominent role. The global analogue of the space of degenerate Whittaker models that we consider will consist in looking at the following period integral:

$$\int_{GL_2(k)\backslash GL_2(\mathbf{A}) \times M_2(k)\backslash M_2(\mathbf{A})} F\begin{pmatrix} g & gX \\ 0 & g \end{pmatrix} G(g)\psi(X)dgdX$$

where F is a cusp form belonging to an automorphic representation π_1 on GL_4 over a global field k with P as the $(2,2)$ maximal parabolic with $P(\mathbf{A})$ as its adelic points; the function G belongs to a cuspidal automorphic representation π_2 of GL_2. The analogue of our main conjecture will relate the non-vanishing of this integral to the non-vanishing at the central critical value of $L(\Lambda^2\pi_1 \otimes \pi_2^*, \frac{1}{2})$. We refer to the paper [JS] of Jacquet and Shalika for some related work.

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2 Calculation of degenerate Whittaker models for Principal Series

Let π_1 and π_2 be irreducible representations of $GL_2(k)$. Denote by $Ps(\pi_1, \pi_2)$ the principal series representation of $GL_4(k)$ induced from the $(2,2)$ parabolic with Levi subgroup $GL_2(k) \times GL_2(k)$. In this section we calculate the twisted Jacquet functor of $Ps(\pi_1, \pi_2)$.

Theorem 2 *The twisted Jacquet functor $Ps(\pi_1, \pi_2)_{N,\psi}$ of $Ps(\pi_1, \pi_2)$ where π_1 and π_2 are irreducible representations of $GL_2(k)$ neither of which is 1 dimensional, and with central characters ω_1 and ω_2 sits in the following exact sequence*

$$0 \to \pi_1 \otimes \pi_2 \to Ps(\pi_1, \pi_2)_{N,\psi} \to Ps(\omega_1, \omega_2) \to 0.$$

Here $Ps(\omega_1, \omega_2)$ is the principal series representation of $GL_2(k)$ induced from the character (ω_1, ω_2) of $k^* \times k^*$.

Proof : Let P denote the $(2,2)$ parabolic stabilising the 2 dimensional subspace $\{e_1, e_2\}$ of the 4 dimensional space $\{e_1, e_2, e_3, e_4\}$. The set $GL_4(k)/P$ can be identified to the set of 2 dimensional subspaces of $\{e_1, e_2, e_3, e_4\}$; two elements of $GL_4(k)/P$ are in the same orbit of P if and only if the corresponding subspaces intersect $\{e_1, e_2\}$ in the same dimensional subspaces of $\{e_1, e_2\}$. It follows that there are three orbits of P on $GL_4(k)/P$ corresponding to the dimension of intersection 0,1,2.

Denote by ω the automorphism which takes e_1 to e_3, e_2 to e_4, e_3 to e_1, and e_4 to e_2. Also, denote by ω_{23} the automorphism which takes e_1 to e_1, e_2 to e_3, e_3 to e_2, and e_4 to e_4. It follows that we have the decomposition

$$GL_4(k) = P \coprod P\omega_{23}P \coprod P\omega P.$$

By Mackey theory, the restriction of $Ps(\pi_1, \pi_2)$ to P has

$$A = \pi_1 \otimes \pi_2, B = \mathrm{Ind}^P_{P\cap\omega_{23}P\omega_{23}}(\pi_1 \otimes \pi_2), C = \mathrm{Ind}^P_{P\cap wPw}(\pi_1 \otimes \pi_2),$$

as Jordan-Holder factors. Since $A = \pi_1 \otimes \pi_2$ is a representation of P on which N operates trivially, this summand does not contribute to twisted Jacquet functor. Since $P \cap \omega P\omega = GL_2(k) \times GL_2(k)$, it is easy to see that

$$C = \mathrm{Ind}^P_{GL_2(k)\times GL_2(k)}(\pi_1 \otimes \pi_2) \cong \pi_1 \otimes \pi_2 \otimes \mathbf{C}[M_2(k)]$$

as a representation space for $N = M_2(k)$. From this isomorphism it is easy to see that the twisted Jacquet functor of $C = \mathrm{Ind}^P_{GL_2(k)\times GL_2(k)}(\pi_1 \otimes \pi_2)$ is $\pi_1 \otimes \pi_2$ as a representation space for $GL_2(k)$. Finally we calculate the twisted Jacquet functor of $B = \mathrm{Ind}^P_{P\cap\omega_{23}P\omega_{23}}(\pi_1 \otimes \pi_2)$. For this, we first need to calculate $P \cap \omega_{23}P\omega_{23}$. For this purpose, we note that since P is the stabiliser of $\{e_1, e_2\}$, $\omega_{23}P\omega_{23}$ is the stabiliser of the 2 dimensional subspace $\{e_1, e_3\}$. Therefore $P \cap \omega_{23}P\omega_{23}$ is the stabiliser of the pair of planes $\{e_1, e_2\}$ and $\{e_1, e_3\}$. It follows that $P \cap \omega_{23}P\omega_{23}$ is exactly the set of matrices of the form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & 0 & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix}.$$

It is easy to see that

$$\omega_{23} \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & 0 & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \omega_{23} = \begin{pmatrix} x_{11} & x_{13} & x_{12} & x_{14} \\ 0 & x_{33} & 0 & x_{34} \\ 0 & 0 & x_{22} & x_{24} \\ 0 & 0 & 0 & x_{44} \end{pmatrix}.$$

We note that in the induced representation $\mathrm{Ind}^P_{P\cap\omega_{23}P\omega_{23}}(\pi_1 \otimes \pi_2)$, $\pi_1 \otimes \pi_2$ is considered as a representation space of $P \cap \omega_{23}P\omega_{23}$ via the inclusion of

$$P \cap \omega_{23}P\omega_{23} \hookrightarrow P$$

by $x \to \omega_{23} x \omega_{23}$.

Observe that since π_1 is not 1 dimensional, the representation π_1 has a Whittaker model, and hence there is exactly a 1 dimensional space of linear forms, generated by ℓ_1, on which the upper-triangular unipotent matrices operate via the character ψ. Similarly we find a linear form ℓ_2 on π_2. Therefore recalling the expression for $\omega_{23} p \omega_{23}$ given earlier, the set of matrices of the form

$$\begin{pmatrix} x_{11} & x_{13} & x_{12} & x_{14} \\ 0 & x_{22} & 0 & x_{34} \\ 0 & 0 & x_{11} & x_{24} \\ 0 & 0 & 0 & x_{22} \end{pmatrix}$$

operate on the linear form $\ell_1 \otimes \ell_2$ on $\pi_1 \otimes \pi_2$ by

$$\omega_1(x_{11})\omega_2(x_{22})\psi(x_{12})\psi(x_{34}),$$

from which it is easy to see by Frobenius reciprocity that the twisted Jacquet functor of $\mathrm{Ind}_{P \cap \omega_{23} P \omega_{23}}^{P}(\pi_1 \otimes \pi_2)$ is $Ps(\omega_1, \omega_2)$, completing the proof of the theorem.

3 Principal series representations

In this section we prove conjecture 1 for those representations V of $GL_4(k)$ which are induced from a representation, say $\pi_1 \otimes \pi_2$ of the Levi subgroup $GL_2(k) \times GL_2(k)$ of the $(2, 2)$ parabolic. If the Langlands parameters of the representations π_1 and π_2 of $GL_2(k)$ are σ_1 and σ_2, then the Langlands parameter σ_V of V is $\sigma_1 \oplus \sigma_2$. Therefore,

$$\begin{aligned} \Lambda^2 \sigma_V &\cong \Lambda^2(\sigma_1 \oplus \sigma_2) \\ &\cong \Lambda^2\sigma_1 \oplus \Lambda^2\sigma_2 \oplus \sigma_1 \otimes \sigma_2. \end{aligned}$$

Therefore for a representation W of $GL_2(k)$ with the same central character as V,

$$\epsilon(\Lambda^2\sigma_V \otimes \sigma_W^*) = \epsilon([\Lambda^2\sigma_1] \otimes \sigma_W^*) \cdot \epsilon([\Lambda^2\sigma_2] \otimes \sigma_W^*) \cdot \epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_W^*).$$

Since the central characters of V and W are the same, we have

$$\Lambda^2\sigma_1 \cdot \Lambda^2\sigma_2 = \Lambda^2\sigma_W.$$

It follows that

$$[(\Lambda^2\sigma_1) \otimes \sigma_W^*]^* \cong (\Lambda^2\sigma_2) \otimes \sigma_W^*.$$

Therefore,

$$\begin{aligned}\epsilon([\Lambda^2\sigma_1] \otimes \sigma_W^*) \cdot \epsilon([\Lambda^2\sigma_2] \otimes \sigma_W^*) &= \det[(\Lambda^2\sigma_1) \cdot \sigma_W^*](-1) \\ &= \det(\sigma_W)(-1).\end{aligned}$$

It follows that

$$\begin{aligned}\epsilon(\Lambda^2\sigma_V \otimes \sigma_W^*) &= \epsilon([\Lambda^2\sigma_1] \otimes \sigma_W^*) \cdot \epsilon([\Lambda^2\sigma_2] \otimes \sigma_W^*)\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_W^*) \\ &= \det(\sigma_W)(-1)\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_W^*).\end{aligned}$$

Therefore,

$$\epsilon(\Lambda^2\sigma_V \otimes \sigma_W^*) = \det(\sigma_W)(-1)$$

if and only if

$$\epsilon(\sigma_1 \otimes \sigma_2 \otimes \sigma_W^*) = 1.$$

From theorem 1.4 of [P1] this is exactly the condition for the appearance of the representation W of $GL_2(k)$ as a quotient of $\pi_1 \otimes \pi_2$. We therefore need to check that the representations of $GL_2(k)$ which appear as a quotient of $Ps(\pi_1, \pi_2)_{N,\psi}$ are exactly those which arise as a quotient of $\pi_1 \otimes \pi_2$.

By theorem 2, the twisted Jacquet functor of $Ps(\pi_1, \pi_2)$ sits in the following exact sequence,

$$0 \to \pi_1 \otimes \pi_2 \to Ps(\pi_1, \pi_2)_{N,\psi} \to Ps(\omega_1, \omega_2) \to 0.$$

From this we get the following long exact sequence,

$$\begin{aligned}0 &\to \mathrm{Hom}_{GL_2(k)}[Ps(\omega_1, \omega_2), W] \to \mathrm{Hom}_{GL_2(k)}[Ps(\pi_1, \pi_2)_{N,\psi}, W] \\ &\to \mathrm{Hom}_{GL_2(k)}[\pi_1 \otimes \pi_2, W] \to \mathrm{Ext}^1_{GL_2(k)}[Ps(\omega_1, \omega_2), W] \to \cdots\end{aligned}$$

From the corollary 5.9 of [P1], it follows that for an irreducible representation W of $GL_2(k)$,

$$\mathrm{Ext}^1_{GL_2(k)}[Ps(\omega_1, \omega_2), W] \neq 0, \quad \text{if and only if} \quad \mathrm{Hom}_{GL_2(k)}[Ps(\omega_1, \omega_2), W] \neq 0.$$

Therefore if we can check that non-triviality of $\mathrm{Hom}_{GL_2(k)}[Ps(\omega_1, \omega_2), W]$ implies the non-triviality of $\mathrm{Hom}_{GL_2(k)}[\pi_1 \otimes \pi_2, W]$, we will have proved that

$\mathrm{Hom}_{GL_2(k)}[Ps(\pi_1,\pi_2)_{N,\psi}, W]$ is nonzero if and only if W is a quotient of $\pi_1 \otimes \pi_2$ which implies our conjecture in this case. It follows from [P1] that an irreducible principal series (of right central character) is a quotient of $\pi_1 \otimes \pi_2$ for any choice of π_1 and π_2. We therefore need only to take care of when $Ps(\omega_1, \omega_2)$ has the Steinberg representation as a quotient. Again it follows from [P1] that the Steinberg representation of $GL_2(k)$ appears as a quotient of $\pi_1 \otimes \pi_2$ unless $\pi_1 = \alpha||^{1/4}St$ and $\pi_2 = \alpha^{-1}||^{-1/4}St$ where α is a quadratic character of k^*, and St denotes the Steinberg representation of GL_2. (We are using here the fact that $Ps(\omega_1, \omega_2)$ has the Steinberg representation as a quotient.) It can be seen that for these choices of π_1 and π_2, the principal series representation $Ps(\alpha||^{1/4}St, \alpha^{-1}||^{-1/4}St)$ is actually reducible which we are omitting from our considerations here.

This completes the proof of Conjecture 1 for those principal series representations of $GL_4(k)$ which are irreducibly induced from the $(2,2)$ parabolic.

Remark : It is easy to see that if π is a principal series representation of $GL_4(k)$ induced from the $(3,1)$ parabolic, then every irreducible generic representation W of $GL_2(k)$ with the same central character as π appears as a quotient in $\pi_{N,\psi}$, and moreover, $\epsilon[(\Lambda^2\sigma_V) \otimes \sigma_W^*] = (\det \sigma_W)(-1)$. Hence conjecture 1 is true for this case of parabolic induction too, and is a case where ϵ factors play no role. We omit the details of the argument which a re via standard application of the Mackey orbit theory.

4 Supercuspidal representations

In this section we will make an equivalent formulation of conjecture 1 for those representations V of $GL_4(k)$ which are obtained by automorphic induction of a representation, say Π, of $GL_2(K)$ where K is a quadratic extension of k. When the residue characteristic of k is odd, it is known that all the supercuspidal representations of $GL_4(k)$ are obtained in this manner. This equivalent form will describe the space of degenerate Whittaker models without any explicit mention of epsilon factors.

If the Langlands parameter of a representation Π of $GL_2(K)$ is the 2 dimensional representation σ of the Weil-Deligne group W_K of K, then the Langlands parameter of the representation V of $GL_4(k)$ which is obtained by automorphic induction from Π, is

$$\sigma_V = \mathrm{Ind}_{W_K}^{W_k} \sigma.$$

In this case

$$\Lambda^2 \sigma_V = \mathrm{Ind}_{W_K}^{W_k} (\Lambda^2 \sigma) \oplus M_K^k \sigma,$$

where $M_K^k \sigma$ is a 4 dimensional representation of W_k obtained from the index 2 subgroup W_K by the process of multiplicative or tensor induction described in [P2].

If ω_W is the central character of the representation W, and $\omega_{K/k}$ denotes the quadratic character of k^* associated to the quadratic extension K, then by a theorem due to Saito [S] and Tunnell [Tu],

$$\epsilon(\mathrm{Ind}_K^k [\Lambda^2 \sigma] \otimes \sigma_W^*) = \omega_W(-1) \cdot \omega_{K/k}(-1)$$

if and only if the character $\chi = \Lambda^2 \sigma$ of K^* appears in W.

By theorem D of [P2],

$$\epsilon(M_K^k \sigma \otimes \sigma_W^*) = \omega_{K/k}(-1)$$

if and only if the representation W of $GL_2(k)$ appears as a quotient in the representation Π of $GL_2(K)$ when restricted to $GL_2(k)$. Combining these two theorems, we can interpret the condition

$$\epsilon(\Lambda^2 \sigma_V \otimes \sigma_W^*) = \omega_W(-1),$$

and hence conjecture 1 as follows.

Consequence 1 of conjecture 1. *Let V be supercuspidal representation of $GL_4(k)$ which is obtained by automorphic induction of a representation Π of $GL_2(K)$ where K is a quadratic extension of k. Then a representation W of $GL_2(k)$ with the same central character as that of V appears in $V_{N,\psi}$ if and only if either,*

(a) The character ω_Π of K^* appears in the representation W of $GL_2(k)$ and the representation W of $GL_2(k)$ appears as a quotient when the representation Π of $GL_2(K)$ is restricted to $GL_2(k)$.

Or,

(b) The character ω_Π of K^* does not appear in the representation W of $GL_2(k)$ and the representation W of $GL_2(k)$ also does not appear as a quotient when the representation Π of $GL_2(K)$ is restricted to $GL_2(k)$.

5 Generalised Steinberg representations

Suppose π is a cuspidal representation of $GL_2(k)$. Then it is known that the principal series representation of $GL_4(k)$ induced from the representation $\pi|\cdot|^{1/2} \times \pi|\cdot|^{-1/2}$ of the $(2,2)$ parabolic of $GL_4(k)$ with Levi subgroup $GL_2(k) \times GL_2(k)$ has length 2 with a unique irreducible quotient which is a discrete series representation of $GL_4(k)$, called generalised Steinberg and denoted by $St(\pi)$. We will denote the unique subrepresentation of this principal series by $Sp(\pi)$. This theorem due to Bernstein-Zelevinsky has also been proved in the context of $GL_2(D)$ by Tadic in [T] with exactly analogous statement.

The Langlands parameter of the representation $St(\pi)$ is $\sigma \otimes s_2$ where σ is the Langlands parameter of the representation π of $GL_2(k)$, and for any $n \geq 1$, s_n denotes the unique irreducible representation of $SL_2(\mathbf{C})$ of dimension $= n$.

We interpret what conjecture 1 says about the space of degenerate Whittaker functionals in this case which we will divide into two separate cases. We will often use the following relation about epsilon factors

$$\epsilon(\tau \otimes s_n, \psi_0) = \epsilon(\tau, \psi_0)^n \cdot \det(-F, \tau^I)^{n-1} \qquad (*)$$

where τ is a representation of the Weil group, F is a Frobenius element of the Weil group and τ^I denotes the subspace of τ on which the inertia group acts trivially.

5.1 The case when W is not a twist of Steinberg

For vector spaces V_1 and V_2, there is a natural isomorphism of complex vector spaces,

$$\Lambda^2(V_1 \otimes V_2) \cong \Lambda^2(V_1) \otimes \mathrm{Sym}^2(V_2) \oplus \mathrm{Sym}^2(V_1) \otimes \Lambda^2(V_2).$$

It follows that

$$\begin{aligned} \Lambda^2(\sigma \otimes s_2) &= \Lambda^2\sigma \otimes \mathrm{Sym}^2 s_2 \oplus \mathrm{Sym}^2\sigma \otimes \Lambda^2 s_2 \\ &= \det\sigma \cdot s_3 \oplus \mathrm{Sym}^2\sigma. \end{aligned}$$

Using this and taking into account $(*)$, we have

$$\begin{aligned} \epsilon(\Lambda^2(\sigma \otimes s_2) \otimes \sigma_W^*) &= \epsilon(\det\sigma \cdot s_3 \otimes \sigma_W^*)\epsilon(\mathrm{Sym}^2\sigma \otimes \sigma_W^*) \\ &= \epsilon(\det\sigma \cdot \sigma_W^*)^3 \det(-F, [\det\sigma \cdot \sigma_W^*]^I)^2 \cdot \epsilon(\mathrm{Sym}^2\sigma \otimes \sigma_W^*). \end{aligned}$$

Since the central character of $St(\pi)$ is $(\det\sigma)^2$, from the condition on the central characters,

$$(\det\sigma)^2 = \det\sigma_W.$$

It follows that

$$[\det\sigma\cdot\sigma_W^*]^* \cong \det\sigma\cdot\sigma_W^*.$$

Therefore $\epsilon(\det\sigma\cdot\sigma_W^*)^2 = \omega_W(-1)$. Moreover note that since we are assuming that σ_W is an irreducible representation of the Weil group of dimension 2, there are no invariants under the inertia group in $\det\sigma\cdot\sigma_W^*$. Therefore,

$$\begin{aligned}
\epsilon(\Lambda^2(\sigma\otimes s_2)\otimes\sigma_W^*) &= \epsilon(\det\sigma\cdot\sigma_W^*)^3\det(-F,[\det\sigma\cdot\sigma_W^*]^I)^2\cdot\epsilon(\mathrm{Sym}^2\sigma\otimes\sigma_W^*)\\
&= \omega_W(-1)\epsilon(\det\sigma\cdot\sigma_W^*)\cdot\epsilon(\mathrm{Sym}^2\sigma\otimes\sigma_W^*)\\
&= \omega_W(-1)\epsilon(\sigma\otimes\sigma\otimes\sigma_W^*)
\end{aligned}$$

It follows that $\epsilon(\Lambda^2(\sigma\otimes s_2)\otimes\sigma_W^*) = \omega_W(-1)$ if and only if

$$\epsilon(\sigma\otimes\sigma\otimes\sigma_W^*) = 1.$$

Therefore by theorem 1.4 of [P1], conjecture 1 reduces to the following statement.

Consequence 2 of conjecture 1 *Suppose that π is an irreducible admissible cuspidal representation of $GL_2(k)$. Then a representation W of $GL_2(k)$ which is not a twist of the Steinberg appears as a quotient in the degenerate Whittaker model of the generalised Steinberg representation $St(\pi)$ of $GL_4(k)$ if and only if it appears as a quotient in $\pi\otimes\pi$.*

5.2 The case when W is a twist of the Steinberg

In this subsection we analyse what conjecture 1 implies for W, a twist of the Steinberg representation St on $GL_2(k)$ by a character χ of k^*. In this case the Langlands parameter σ_W is given by $\sigma_W = \chi\cdot s_2$ with the determinant condition $(\det\sigma)^2 = \chi^2$. Let $\chi = \omega\det\sigma$ with ω a quadratic character of k^*. We have,

$$\begin{aligned}
\Lambda^2(\sigma\otimes s_2)\otimes\sigma_W^* &= [\det\sigma\cdot s_3\oplus\mathrm{Sym}^2\sigma]\otimes s_2\chi^{-1}\\
&= \omega[s_4\oplus s_2]\oplus\chi^{-1}\mathrm{Sym}^2\sigma\otimes s_2.
\end{aligned}$$

Therefore,

$$\begin{aligned}\epsilon(\Lambda^2(\sigma \otimes s_2) \otimes \sigma_W^*) &= \epsilon(\omega)^6 \det(-F, \omega^I)^4 \epsilon(\chi^{-1}\mathrm{Sym}^2\sigma)^2 \det(-F, [\chi^{-1}\mathrm{Sym}^2\sigma]^I) \\ &= \omega(-1) \cdot \chi(-1) \cdot \det\sigma(-1) \cdot \det(-F, [\chi^{-1}\mathrm{Sym}^2\sigma]^I) \\ &= \det(-F, [\chi^{-1}\mathrm{Sym}^2\sigma]^I).\end{aligned}$$

Here we have used the relation $\epsilon(\omega)^2 = \omega(-1)$, and $\det(-F, \omega^I)^2 = 1$, both arising because ω is a quadratic character.

Since W_k/I is a cyclic group, the subspace on which the inertia group acts trivially can be decomposed as a sum of W_k-invariant lines. It is easy to see that if σ is an irreducible but non-dihedral representation, then $\mathrm{Sym}^2\sigma$ is an irreducible representation of W_k, and therefore has no I-invariants. If on the other hand, σ is a dihedral representation obtained by inducing a character, say μ on K^*, for K a quadratic extension of k, then $\mathrm{Sym}^2\sigma$ has a unique W_k invariant line on which W_k acts by the restriction of μ to k^*. Therefore $\chi^{-1}\mathrm{Sym}^2\sigma$ has an I-invariant vector if and only if $\mu\chi^{-1}$ is trivial on the inertia subgroup, and $-F$ acts by -1 on the corresponding line if and only if μ restricted to k^* is χ. We therefore obtain that,

$$\epsilon(\Lambda^2(\sigma \otimes s_2) \otimes \sigma_W^*) = -1$$

if and only if σ is a dihedral representation obtained by inducing a character, say μ on K^*, for K a quadratic extension of k, with $\mu = \chi$ on k^*. Conjecture 1 therefore reduces to the following in this case.

Consequence 3 of conjecture 1 *For a character χ of k^*, the twist of the Steinberg representation $\chi \otimes St$ appears in the space of degenerate Whittaker models of the generalised Steinberg representation $St(\pi)$ on $GL_4(k)$ for a cuspidal representation π on $GL_2(k)$ with $\omega_\pi^2 = \chi^2$ if and only if either the representation π does not come from a quadratic extension, or if the representation π comes from a quadratic extension K of k obtained by inducing a character, say μ on K^*, for K a quadratic extension of k, then $\mu \neq \chi$ on k^* (but $\mu^2 = \chi^2$ on k^* by the condition on central characters).*

It is easier to state what conjecture 1 reduces to for $GL_2(D)$.

Consequence 4 of conjecture 1 *Let π be an irreducible representation of D^* of dimension > 1 and central character ω_π. Then the space of degenerate Whittaker models of $Sp(\pi)$ is the 1 dimensional representation of D^* obtained from the character ω_π by composing with the reduced norm mapping.*

Proof : Let $Ps(\pi|\cdot|^{1/2}, \pi|\cdot|^{-1/2})$ be the principal series representation of $GL_2(D)$ obtained by inducing the representation $\pi|\cdot|^{1/2} \times \pi|\cdot|^{-1/2}$ of the minimal parabolic of $GL_2(D)$ with Levi subgroup $D^* \times D^*$. By work of Tadic [T], it is known that if $\dim(\pi) > 1$, $Ps(\pi|\cdot|^{1/2}, \pi|\cdot|^{-1/2})$ has length 2 with a unique irreducible quotient which is a discrete series representation of $GL_2(D)$, called generalised Steinberg and denoted by $St(\pi)$. We will denote the unique subrepresentation of this principal series by $Sp(\pi)$. We have therefore an exact sequence of representations

$$0 \rightarrow Sp(\pi) \rightarrow Ps(\pi|\cdot|^{1/2}, \pi|\cdot|^{-1/2}) \rightarrow St(\pi) \rightarrow 0.$$

Since the twisted Jacquet functor is an exact functor, and since the twisted Jacquet functor of $Ps(\pi|\cdot|^{1/2}, \pi|\cdot|^{-1/2})$ is $\pi \otimes \pi$, we have the exact sequence of D^* representations

$$0 \rightarrow Sp(\pi)_{N,\psi} \rightarrow \pi \otimes \pi \rightarrow St(\pi)_{N,\psi} \rightarrow 0.$$

Therefore the twisted Jacquet functor of $Sp(\pi)$ consists of those irreducible representations of D^* which appear in $\pi \otimes \pi$ but not in $St(\pi)_{N,\psi}$. By our calculation of epsilon factors, all the irreducible representations of D^* of dimension > 1 appearing in $\pi \otimes \pi$ also appear in $St(\pi)_{N,\psi}$ (as by theorem 1.4 of [P1], the condition for appearance in the two representations is the same). This proves that no representations of D^* of dimension > 1 appears in $Sp(\pi)_{N,\psi}$. Since $\pi \cong \omega_\pi \cdot \pi^*$, it follows that $\pi \otimes \pi$ always contains the character ω_π of k^*, and is the only character of k^* it contains unless π comes from a character of a quadratic field extension K in which case it also contains $\omega_\pi \cdot \omega_{K/k}$. Therefore from consequence 3 of conjecture 1, this corollary follows. (One needs to know that if π is a dihedral representation of D^* obtained by inducing a character, say μ on K^*, for K a quadratic extension of k, then the central character of such a representation is $\mu|_{k^*} \cdot \omega_{K/k}$).

Consequence 5 of conjecture 1 *The generalised Steinberg representation $St(\pi)$ of $GL(2, D)$ has a Shalika model if and only if the representation π is self-dual with non-trivial central character.*

Remark : The above corollary was conjectured in [P3].

6 Relation to triple product epsilon factor

There is an intertwining operator between the principal series representations $Ps(\pi_1, \pi_2)$ and $Ps(\pi_2, \pi_1)$ where π_i are either representations of $GL_2(k)$ or of D^* for D a quaternion division algebra over k. The intertwining operator is defined in terms of an integral over the unipotent radical of the opposite parabolic to the $(2, 2)$ parabolic, and is in particular over a non-compact space, and depends on a certain complex parameter s. The integral converges in a certain region of values for s, and is defined for all representations π_1, π_2 by analytic continuation.

The action of the intertwining operator from the principal series $Ps(\pi_1, \pi_2)$ to $Ps(\pi_2, \pi_1)$ seems closely related to the triple product epsilon factor. We make this suggestion more precise. We will assume in this section that D is either a quaternion division algebra or a 2×2 matrix algebra over a p-adic field k. Let π_1 and π_2 be two irreducible representations of D^* neither of which is 1 dimensional if D^* is isomorphic to $GL_2(k)$.

The intertwining operator induces an action on the twisted Jacquet functor which as we have seen before for $Ps(\pi_1, \pi_2)$ is essentially $\pi_1 \otimes \pi_2$. Therefore the intertwining operator induces a D^*-equivariant mapping from $\pi_1 \otimes \pi_2$ to $\pi_2 \otimes \pi_1$. Composing this with the mapping from $\pi_1 \otimes \pi_2$ to $\pi_2 \otimes \pi_1$ given by $v_1 \otimes v_2 \to v_2 \otimes v_1$, we now have an intertwining operator, call it I, from $\pi_1 \otimes \pi_2$ to itself. If π_3 is an irreducible representation of D^*, then by the multiplicity 1 theorem of [P1], the space of D^*-invariant maps from $\pi_1 \otimes \pi_2$ to π_3 is at most 1 dimensional. The intertwining operator I acts on this 1 dimensional vector space, and therefore the action of I on this 1 dimensional space is by multiplication by a complex number $I(\pi_1, \pi_2, \pi_3)$.

Conjecture 2 $I(\pi_1, \pi_2, \pi_3) = c\epsilon(\pi_1 \otimes \pi_2 \otimes \pi_3)$ *where c is a constant independent of π_1, π_2, π_3. (We remark that the intertwining operator from the principal series $Ps(\pi_1, \pi_2)$ to the principal series $Ps(\pi_2, \pi_1)$ itself depends on the choice of Haar measure on N^-, and therefore the constant c depends on the Haar measure on N^-.)*

Remark : The conjecture above is analogous to the works of Shahidi in which he relates the action of intertwining operators on the Whittaker functional to local constants. We refer to the paper [Sh] of Shahidi for one such case.

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