Let \( G \) be a reductive algebraic group over a local field \( k \), and \( K \) a quadratic extension of \( k \). The aim of this section is to formulate a conjecture about representations of \( G(K) \) distinguished by \( G(k) \) in terms of Langlands parameters, or what is also called the Langlands-Vogan parametrization. A special case of this conjecture is for the degenerate case of \( K = k + k \), in which case the question amounts to understanding the contragredient of a representation of \( G(k) \) in terms of Langlands-Vogan parameters, which in fact we will take up first; results about \( \text{SL}_2(k) \subset \text{SL}_2(K) \) from [AP03] in terms of Whittaker models (involving nontrivial \( L \)-packets for both groups) was an important guide, besides works of Jacquet on distinction of representations of \( \text{GL}_n(K) \) by \( \text{U}_n \).

Let \( W_k \) be the Weil group of \( k \), and \( W'_k \) the Weil-Deligne group of \( k \). Let \( \hat{L}(C) = \hat{G}(C) \rtimes W_k \), be the \( L \)-group of \( G \) which comes equipped with a map onto \( W_k \). An admissible homomorphism \( \varphi_k : W'_k \rightarrow \hat{L} \) is called a Langlands parameter for \( G \). To an admissible homomorphism \( \varphi_k \), is associated the group of connected components \( C_{\varphi_k} = Z_{\varphi_k} / Z_{0,\varphi_k} \) where \( Z_{\varphi_k} \) is the centralizer of \( \varphi_k \) in \( \hat{G}(C) \), and \( Z_{0,\varphi_k} \) is its connected component of identity.

According to the Langlands-Vogan parametrization, to an irreducible admissible representation \( \pi \) of \( G(k) \), there corresponds a pair \((\varphi_k, \mu)\) consisting of an admissible homomorphism \( \varphi_k : W'_k \rightarrow \hat{L} \) and a representation \( \mu \) of its group of connected components \( C_{\varphi_k} \). The pair \((\varphi_k, \mu)\) determines \( \pi \) uniquely, and one knows which pairs arise in this way. The Langlands-Vogan parametrization depends on fixing a base point, consisting of a pair \((\psi, N)\) where \( \psi \) is a nondegenerate character on \( N \) which is a maximal unipotent subgroup in a quasi-split innerform of \( G(k) \).

Before we come to the description of the Langlands parameter of \( \pi^\vee \) in terms of that of \( \pi \), we recall that for a simple algebraic group \( G \) over \( C \), one has either:

1. \( G(C) \) has no outer automorphism, or \( G(C) = D_{2n} \) for some \( n \); in these cases, all irreducible finite dimensional representations of \( G(C) \) are self-dual; or,
2. \( G(C) \) has an outer automorphism of order 2 which has the property that it takes all irreducible finite dimensional representations of \( G(C) \) to their contragredient.

The above results about simple algebraic groups (over \( C \)) can be extended to all semisimple algebraic groups, and then further to all reductive algebraic groups, and give in this generality of reductive algebraic groups an automorphism \( \iota \) of \( G(C) \) of order 1 or 2, unique up to inner automorphisms, which takes any finite dimensional representation of \( G(C) \) to its dual. (For example, for a torus \( T \), \( \iota \) is just the inversion, \( \iota : z \rightarrow z^{-1}, z \in T \).)
One can also define such an automorphism $\iota$ for a quasi-split group $G(k)$, for $k$ any field, which is well-defined as an element of $\text{Aut}(G(k)/G(k)$ with $G(k) = G(k)/\mathbb{Z}(k)$ sitting inside the automorphism group of $G$ by inner-automorphisms. The element $\iota$ is defined in the usual way using a based root datum for $G(k)$. We will call this automorphism $\iota$, the duality automorphism of $G(k)$. For $G = \hat{G}$, this automorphism of $G(\mathbb{C}) = \hat{G}(\mathbb{C})$ extends to an automorphism of $\hat{L}G(\mathbb{C}) = \hat{G}(\mathbb{C}) \rtimes W_k$, which will again be denoted by $\iota$.

We next recall that in [GGP], section 9, denoting $G^{\text{ad}}$ the adjoint group of $G$ (assumed without loss of generality at this point to be quasi-split since we want to construct something for the $L$-group), there is constructed a homomorphism from $G^{\text{ad}}(k)/G(k) \to \hat{C}_{\varphi_k}$, denoted $g \mapsto \eta_g$. Let $g_0$ be the unique conjugacy class in $G^{\text{ad}}(k)$ representing an element in $T^{\text{ad}}(k)$ (with $T^{\text{ad}}$ a maximally split, maximal torus in $G^{\text{ad}}(k)$) which acts by $-1$ on all simple root spaces. Denote the corresponding $\eta_{g_0}$ by $\eta_{\iota^{-1}}$, a character on $C_{\varphi_k}$, which will be the trivial character for example if $g_0$ can be lifted to $G(k)$.

Conjecture 1. For an irreducible admissible representation $\pi$ of $G(k)$ with Langlands-Vogan parameter $(\varphi_k, \mu)$, the Langlands-Vogan parameter of $\pi^\vee$ is $(\iota \circ \varphi_k, (\iota \circ \mu)^\vee \otimes \eta_{\iota^{-1}})$, where $\iota$ is the duality automorphism of $L^G(\mathbb{C})$.

For $G$ quasi-split over $k$, and $\iota$ now an automorphism of $G$ defined over $k$ (well-defined up to inner-automorphisms by elements of $G(k)$), if $C_{\varphi_k}$ is an elementary abelian 2-group (as is usually the case), the dual representation $\pi^\vee$ is obtained by using the automorphism $\iota$ of $G(k)$, and then conjugating by the element $g_0$ in $G^{\text{ad}}(k)$.

Remark: (a) Note that the conjecture allows for the possibility, for groups such as $G_2(k)$, $F_4(k)$, or $E_8(k)$ to have non-selfdual representations arising out of component groups (which can be $\mathbb{Z}/3, \mathbb{Z}/4,$ and $\mathbb{Z}/5$ in these respective cases). And indeed $G_2(k)$, $F_4(k)$, and $E_8(k)$ are known to have non self-dual representations.

(b) A particular case of the conjecture is that the map $\pi \to \pi^\vee$ takes $L$-packet of representations to $L$-packet of representations which if it stabilizes an $L$-packet, acts either as identity on it, or without fixed points.

(c) The conjecture above generalizes the [MVW] description of contragredient of representations of classical groups, such as for $\text{Sp}(W), U(V)$, to all groups, and suggests an approach via analysis of conjugacy classes in $G(k)$.

We now return to the context of $G(k) \subset G(K)$. The $L$-group of $G(K)$ is $(\hat{G}(\mathbb{C}) \times \hat{G}(\mathbb{C})) \rtimes W_k$, leaving the precise description of the semi-direct product to the reader’s imagination. (If $G$ is split over $k$, this is of course standard.) The group $\hat{G}(\mathbb{C})$ comes equipped with an automorphism $\iota$ of order 2, allowing us to define a subgroup of $(\hat{G}(\mathbb{C}) \times \hat{G}(\mathbb{C})) \rtimes W_k$, to be

$$\Delta'(G)(\mathbb{C}) \rtimes W_k = \{ (g, \iota(g)) | g \in \hat{G}(\mathbb{C}) \} \rtimes W_k.$$  

Clearly, $\Delta'(G)(\mathbb{C}) \rtimes W_k$ is the $L$-group of either $G$ if $\iota$ is trivial, or is the $L$-group of the unique quasi-split but not split outer form of $G$ which splits over $K$; in either case, we have defined a companion form of $G$, to be denoted by $G^{\text{out}}$; for example, for $G = \text{GL}_n$, $G^{\text{out}} = U_n$. 


A character of $G(k)$ of order 2: In an earlier paper of the 2nd author [P01], there is the construction of a character $\chi_K : G(k) \to \mathbb{Z}/2$ associated to any quadratic extension $K$ of $k$ which plays an important role in questions about distinction; the character $\chi_K$ is functorial under maps of reductive algebraic groups with abelian kernel and cokernel. We will review the construction of $\chi_K$ here. Let $G^{\text{ad}}$ denote the adjoint group of $G$, i.e., $G$ modulo center, and $G^{\text{sc}}$ the simply connected cover of $G^{\text{ad}}$. Let $Z_{sc}$ be the center of $G^{\text{sc}}$. Then we have an exact sequence of groups,

$$1 \to Z_{sc}(k) \to G^{sc}(k) \to G^{\text{ad}}(k) \to H^1(\text{Gal}(\bar{k}/k), Z_{sc}) \to \cdots$$

The character $\chi_K$ factors through a character of $G^{\text{ad}}(k)$ via the natural map, $G(k) \to G^{\text{ad}}(k)$, so we need to construct one for $G^{\text{ad}}(k)$, which arises from the previous exact sequence from a character of $H^1(\text{Gal}(\bar{k}/k), Z_{sc})$, which by Tate-Nakayama duality amounts to constructing an element of $H^1(\text{Gal}(\bar{k}/k), \hat{Z}_{sc})$, where $\hat{Z}_{sc}$ is the Cartier dual of $Z_{sc}$.

Let $\hat{G}'$ be the connected component of the $L$-group of $G'=G^{\text{ad}}$. It is clear that one can choose a regular unipotent in $\hat{G}'$ such that the corresponding Jacobson-Morozov embedding of $\text{SL}_2(\mathbb{C})$ into $\hat{G}'$ is invariant under (pinned) outer automorphisms of $\hat{G}'$. The center of $\text{SL}_2(\mathbb{C})$ under this embedding goes to the center of $\hat{G}'$ which is nothing but $\hat{Z}_{sc}$, inducing a map $H^1(\text{Gal}(\bar{k}/k), Z_{sc}) \to H^1(\text{Gal}(\bar{k}/k), \hat{Z}_{sc})$. Since $H^1(\text{Gal}(\bar{k}/k), Z/2)$ parametrizes quadratic etale extensions of $k$ of degree 2, we have finally constructed the character $\chi_K : G(k) \to Z/2$ associated to any quadratic extension $K$ of $k$.

Example: (a) For $G = \text{GL}_n$, $\chi_K = \omega_{K/k} \circ \text{det}$ for $n$ even and trivial for $n$ odd.

(b) For $G = \text{U}_n$, defined using a hermitian form over $K$, $\chi_K$ is trivial for all $n$.

Conjecture 2. An irreducible admissible representation $\pi$ of $G(K)$ with Langlands-Vogan parameter $(\varphi_k, \mu)$ is distinguished by the character $\chi_K$ of $G(k)$ if and only if

1. The parameter $\varphi_k : W_k \to (\hat{G}(\mathbb{C}) \times \hat{G}(\mathbb{C})) \rtimes W_k$, factors through $\Delta^i(G)(\mathbb{C}) \rtimes W_k$.
2. $\mu \circ \varphi_k = \mu \rtimes \eta_{-1}$.
3. For representations $\pi$ inside an $L$-packet of representations of $G(K)$ with character of the component group $\mu$ satisfying (2) above, the dimension of $\text{Hom}_{G(k)}[\pi, \mathbb{C}]$ is independent of $\mu$, and is equal to the number of in-equivalent ways of lifting the parameter $\varphi_k : W_k \to (\hat{G}(\mathbb{C}) \times \hat{G}(\mathbb{C})) \rtimes W_k$, to a parameter $\varphi'_k : W_k \to \Delta^i(G)(\mathbb{C}) \rtimes W_k$:

$$\Delta^i(G)(\mathbb{C}) \rtimes W_k,$$

modulo the equivalence relation on these parameters $\varphi'_k : W_k \to \Delta^i(G)(\mathbb{C}) \rtimes W_k$, induced by twisting representations (or parameters) by characters of $Z_Q(k)$, where $Z_Q$ is the maximal abelian quotient of $G^{\text{out}}$ as an algebraic
group over $k$, whose base change to $\mathbb{Z}_Q(K)$ are trivial; in fact, one should consider the degree of the corresponding map from the set of homomorphisms $W'_k \to \Delta'(G)(\mathbb{C}) \times W_k$, up to equivalence to its image in the set of homomorphisms up to equivalence $W'_k \to (\hat{G}(\mathbb{C}) \times \hat{G}(\mathbb{C})) \times W_k$, taking the twisting by central characters into account.

Remark: (a) The condition (1) in the conjecture fixes those $L$-packets of $G(K)$ for which there is a possible representation with $G(k)$-invariant form (by the quadratic character $\chi_K$). Condition (2) then works inside the $L$-packet to fix those that have $G(k)$-invariant form. Condition (3) gives the multiplicity, which is then either 0 or a number dependent only on the $L$-packet. Abelian twists have been used since one would like multiplicity one for $T(K)$, for a torus $T$, see below for some details; its necessity is also reminded by a conjecture of Jacquet about multiplicity formula for the pair $(GL_n(K), U_n(k))$ that we come to later.

(b) For $SL_2(k) \subset SL_2(K)$, we know by [AP03] that a representation $\pi$ of $SL_2(K)$ is distinguished by $SL_2(k)$ if and only if $\pi$ has a Whittaker model with respect to a character of $K/k$. This condition about Whittaker model can be written as $\psi^\sigma(x) = \psi(-x)$, which is the condition (2) above.

(c) The mysterious role of the character $\chi_K$ in the conjecture is related eventually to the symmetry of the bilinear form $B : \pi \times \pi^\sigma \to \mathbb{C}$; indeed the character $\chi_K$ is constructed via an element of the center of $\hat{G}$ which determines symmetry property of selfdual representations of $\hat{G}$.

(d) It seems curious that in this paper, we had many occasions to consider the question about fibers of functorial lifts.

Evidence for the multiplicity formula:

(1) For a torus $T$ over $k$, it is easy to see that $T^{\text{out}}$ is the torus which sits in the exact sequence,

$$1 \to T \to R_{K/k}(T) \to T^{\text{out}} \to 1.$$ 

Thus, at the level of $k$-rational points, we have,

$$1 \to T(k) \to T(K) \to T^{\text{out}}(k) \to H^1(\text{Gal}(\bar{k}/k), T) \to H^1(\text{Gal}(\bar{k}/K), T) \to \cdots$$

It follows that characters of $T(K)$ which are trivial on $T(k)$ arise from restriction of characters of $T^{\text{out}}(k)$, and that if $T$ splits over $K$, there are $H^1(\text{Gal}(\bar{k}/k), T)$ many characters of $T^{\text{out}}(k)$ giving rise to the same character of $T(K)$.

(2) The pair $(GL_n(K), GL_n(k))$. In this case, $G^{\text{out}}$, is the unitary group $U_n$ defined by $K/k$. Our conjecture above says that representations of $GL_n(K)$ distinguished by $GL_n(k)$ are precisely those which arise as base change of a representation of $U_n(k)$. It has been known, cf. [GGP], that the base change map taking the Langlands parameter of a representation of $U_n(k)$ to one of $GL_n(K)$ is an injective map; since multiplicity one for the pair $(GL_n(K), GL_n(k))$ is well-known (an elementary result based on the method
of Gelfand pairs, cf. [Fli91]), our multiplicity formula matches well with known results in this case.

(3) The pair \((\text{GL}_n(K), U_n(k))\). It is known that if a representation \(\pi\) of \(\text{GL}_n(K)\) is distinguished by \(U_n(k)\), then it must arise as a base change of a representation of \(\text{GL}_n(k)\). It has been conjectured by Jacquet in [Jac01] that if \(n\) is odd, then \(\dim \text{Hom}_{U_n(k)}[\pi, \mathbb{C}]\) is equal to half the number of representations of \(\text{GL}_n(k)\) which base change to the representation \(\pi\) of \(\text{GL}_n(K)\). Our conjectures fit well with this, and suggest that for \(n\) even, \(\dim \text{Hom}_{U_n(k)}[\pi, \mathbb{C}]\) is equal to the number of the equivalence classes of representations of \(\text{GL}_n(k)\) under the equivalence \(V \sim V \otimes \omega_{K/k}\) which base change to \(\pi\) (note that \(V\) and \(V \otimes \omega_{K/k}\) have the same base change to \(\text{GL}_n(K)\)).

(4) In the case of \((\text{SL}_2(K), \text{SL}_2(k))\), the multiplicity of the space of \(\text{SL}_2(k)\)-invariant linear forms on a representation of \(\text{SL}_2(K)\) was studied in [AP03] in detail, and it was found that \(\dim \text{Hom}_{\text{SL}_2(k)}[\pi, \mathbb{C}]\), as \(\pi\) runs over an \(L\)-packet of representations of \(\text{SL}_2(K)\), is either \(d_\pi\) or 0 for an integer \(d_\pi\) which depends only on the \(L\)-packet of \(\pi\), and in the notation used earlier in the paper, is given by

\[
d_\pi = \frac{X_\pi}{Z_\pi/Y_\pi}.
\]

It suffices then to note the following lemma.

**Lemma 1.** The basechange map \(\pi \to \Pi = \text{BC}(\pi)\) from irreducible admissible representations of \(\text{GL}_2(k)\) to representations of \(\text{GL}_2(K)\), for \(K/k\) quadratic, descends to a map — call it \(TBC\) (for twisted basechange) — from irreducible admissible representations of \(\text{GL}_2(k)\), considered up to twists by characters, to irreducible admissible representations of \(\text{GL}_2(K)\) considered up to twists by characters. Then the fibers of \(TBC\), which is nothing but the fibers of the base change map for \(\text{SL}_2\), given in terms of the \(L\)-group as liftings:

\[
PGL_2(\mathbb{C}) \xrightarrow{\text{W}_k} (PGL_2(\mathbb{C}) \times PGL_2(\mathbb{C})) \rtimes \text{Gal}(K/k),
\]

has order given by,

\[
\frac{X_\Pi}{Z_\Pi/Y_\Pi}.
\]

**Proof.** : Consider the group,

\[
A_\Pi = \left\{ \chi : K^\times \to \mathbb{C}^\times \mid (\Pi \otimes \chi)^\sigma \cong \Pi \otimes \chi \right\} / \left\{ \chi : K^\times \to \mathbb{C}^\times \mid \chi = \chi^\sigma \right\}.
\]

For \(\chi \in A_\Pi\), let \(\pi_{\chi^{-1}\chi^\sigma}\) denote the class of pair of representations \(\{\pi', \pi' \otimes \omega_{E/F}\}\) of \(\text{GL}_2(k)\) such that \(\Pi \otimes \chi = \text{BC}(\pi')\). Note that \(Z_\Pi/Y_\Pi\) acts freely on \(\{\pi_{\chi^{-1}\chi^\sigma} \mid \chi \in A_\Pi\}\). The fibers of \(TBC\) is given by orbits under the above action:

\[
\left\{ \pi_{\chi^{-1}\chi^\sigma} \mid \chi \in A_\Pi \right\} / Z_\Pi/Y_\Pi.
\]
We note that $A_{II}$ is in bijection with $Y_{II}$ under the map $\chi \mapsto \chi^{-1} \chi^\sigma$, and that if $\Pi$ is in the discrete series, then
(a) The set on the numerator above is in bijection with $A_{II}$, and
(b) $Y_{II}$ is in bijection with $X_{II}$.

This proves the lemma if $\Pi$ is a discrete series representation. For principal series representations, both (1) and (2) are wrong in general, thus the lemma is more subtle; we verify it in a case-by-case check below. □

Examples : We illustrate the multiplicity formula with examples of principal series representations of $\text{SL}_2(K)$ taken from [AP03]. In what follows, we introduce the notation $\pi_1 \sim \pi_2$ for two representations of $\text{GL}_2(K)$ (or $\text{GL}_2(k)$) which are twists of each other by a character.

Let $V$ be an irreducible admissible representation of $\text{SL}_2(K)$ that occurs in the restriction of a principal series representation $\pi = \text{Ps}(\chi_1, \chi_2)$ of $\text{GL}_2(K)$. Suppose that $V$ is distinguished with respect to $\text{SL}_2(k)$ (and therefore $\chi_1 \chi_2^{-1} |_{k^\times} = 1$ or $\chi_1 \chi_2^{-1} = (\chi_1 \chi_2^{-1})^\sigma$). Then we have,

(a) $\dim_{C} \text{Hom}_{\text{SL}_2(k)}(V, 1) = 1$, if $\chi_1 \chi_2^{-1} |_{k^\times} = 1$, and $\chi_1 \neq \chi_2^2$. In this case, since $\chi_1/\chi_2$ is trivial on $k^\times$, there is a character $\chi$ of $K^\times$ such that $\chi_1/\chi_2 = \chi/\chi^\sigma$, and so

$$\pi = \text{Ps}(\chi_1, \chi_2) \sim \text{Ps}(\chi_1/\chi_2, 1) = \text{Ps}(\chi/\chi^\sigma, 1) \sim \text{Ps}(\chi, \chi^\sigma),$$

comes as the base change of a unique representation of $\text{GL}_2(k)$ (up to twists) corresponding to the character $\chi$ of $K^\times$.

(b) $\dim_{C} \text{Hom}_{\text{SL}_2(k)}(V, 1) = 1$, if $\chi_1 \chi_2^{-1} |_{k^\times} = \omega_{K/k}^2 \chi_1^2 = \chi_2^2$, $\chi_1 \neq \chi_2^2$. In this case, $\chi_1/\chi_2$ is a Galois invariant quadratic character of $K^\times$, so $\chi_1/\chi_2^2 = \mu \circ N$ for a character $\mu$ of $k^\times$ with $\mu^2 = \omega_{K/k}$. Hence the representation $\pi$ (up to twists) is the base change of a unique principal series representation of $\text{GL}_2(k)$ (up to twists).

(c) $\dim_{C} \text{Hom}_{\text{SL}_2(k)}(V, 1) = 2$, if either $\chi_1 \chi_2^{-1} = (\chi_1 \chi_2^{-1})^\sigma = \mu \circ N$ and $\chi_1^2 \neq \chi_2^2$, or $\chi_1 = \chi_2$. In this case,

$$\pi = \text{Ps}(\chi_1, \chi_2) \sim \text{Ps}(\chi_1/\chi_2, 1) = \text{Ps}(\mu \circ N, 1),$$

hence, $\pi = \text{Ps}(\chi_1, \chi_2)$ arises as base change of two principal series representations of $\text{GL}_2(k)$ which are $\text{Ps}(\mu, 1)$, and $\text{Ps}(\mu, \omega_{K/k})$.

(d) $\dim_{C} \text{Hom}_{\text{SL}_2(k)}(V, 1) = 3$, if $\chi_1 \chi_2^{-1} |_{k^\times} = 1$, $\chi_1^2 = \chi_2^2$, $\chi_1 \neq \chi_2$. In this case, the representation $\pi$ (up to twists) is the base change of a unique discrete series representation of $\text{GL}_2(k)$, and two principal series representation of $\text{GL}_2(k)$ (up to twists).

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REFERENCES


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