Annotated list of publications of Dipendra Prasad

A first draft written in March 2015

(1) Trilinear forms for representations of GL(2) and local epsilon factors, *Compositio Math* 75, 1-46 (1990).

This was my thesis work done at Harvard, roughly from January, 1988 to April, 1989. I have often been asked what was my motivation for this work. It was not for any great goals! I had come across a paper of Joe Repka (Tensor products of unitary representations of $\mathrm{SL}_2(\mathbb{R})$; Amer. J. Math. 100 (1978), no. 4, 747–774) which has in particular tensor product of holomorphic and anti-holomorphic representations of $\mathrm{SL}_2(\mathbb{R})$. I noticed that if instead of $\mathrm{SL}_2(\mathbb{R})$, one takes the tensor product of representations of $\mathrm{GL}_2(\mathbb{R})$, the results looked much prettier, specially if one also combined these results with the tensor product of (finite dimensional) representations of $\mathbb{H}^\times$. This suggested a clear possibility of doing a similar analysis in the p-adic case, which is what my thesis was all about. Initially, I only considered discrete summands in the tensor product of supercuspidal representations, and only later realized that the results for (direct summands in the) tensor products of supercuspidal representations could be considered for *all* irreducible admissible infinite dimensional representations of $\mathrm{GL}_2(F)$, F a p-adic field, if I considered invariant trilinear forms $\ell : \pi_1 \times \pi_2 \times \pi_3 \longrightarrow \mathbb{C}$.

Through Gross's works, I was acquainted with epsilon factors, and Tunnell's theorem, and was calculating the triple product epsilon factors in the Archimedean case, and its relation with Clebsch-Gordan theorem. Around this time, Michael Harris too suggested the possibility of the triple product epsilon factors in these questions.

This work was written when I was just beginning to learn the representation theory of p-adic groups. So the paper is very brute force calculation using the compact induction picture of supercuspidals due to Phil Kutzko. But explicit calculations in this paper may have their own advantages as it calculates exactly when a trilinear form exists, and then calculates quite independently the triple product epsilon factors.

As an example, it is proved that just like the Clebsch-Gordon theorem for $\mathrm{SU}_2(\mathbb{R})$, for a representation π_χ of $D^\times$, D the unique quaternion division algebra over a local field F, which arises from a character χ of $E^\times$ for E a quadratic extension of F, and π any irreducible representation of $D^\times$ of conductor less than that of π_χ

$$\pi \otimes \pi_\chi = \sum_\eta \pi_{\chi\eta},$$

where η's are the characters of $E^\times$ appearing in π.

I was pleased with an example which arose in this paper: of finding possible irreducible sub-representations of $\pi_1 \otimes \pi_2$ which are not supercuspidal. I construct an embedding of the Steinberg in $\pi \otimes \pi$ where π is an irreducible principal series of $\mathrm{PGL}_2(F)$. In my later work, I prove that this is the only non-supercuspidal sub-representation (up to character twists).

I attended my first conference as a student at Harvard in Michigan, Ann Arbor in the summer of 1988 (organized by Clozel and Milne), where I met Dinakar Ramakrishnan for the first time. He introduced me to H. Jacquet who took quite some interest in what I was doing, and proposed the global analogue of what I was doing. This came to be called Jacquet's conjecture, which was proved soon afterwards by M. Harris and S. Kudla. Since it was not clear at all then that I was doing anything interesting, as no one else had looked at similar questions of restricting an infinite dimensional irreducible, admissible representation of either real or p-adic group to a non-compact subgroup (Borel whom I met at an afternoon tea at IAS in 1988 said exactly that), it was certainly an encouragement to see that the work had at least some takers.

Jacquet also suggested that I should also consider 'twisted' cases of my trilinear forms work, i.e., invariant forms on representations of $\mathrm{GL}_2(K)$ for K any cubic algebra over k.

(2) (With B.H. Gross) Test Vectors for linear forms, *Maths Annalen* 291, 343-355 (1991).

This work was partly done while I was still a student at Harvard. Later I visited Gross in Paris for 2-3 days in the summer of 1990 from Basel taking an overnight train with Jurg Kramer whom I was visiting in Basel.

To me, this work on finding 'test vectors' dealing with representations with rather small ramification seemed only a first step towards a more 'all inclusive' theory, and therefore seemed unsatisfactory as a publication. However, since in applications one is often dealing with square free conductors, this is good enough for many purposes—something which my inexprience did not allow me to appreciate then; indeed this work has been invoked for many applications where one wants to calculate period integrals explicitly.

'Test vectors' in more ramified cases have still not been constructed, but see the paper, of Dimitrov and Nyssen (Test vectors for trilinear forms when at least one representation is not supercuspidal. Manuscripta Math. 133 (2010), no. 3–4, 479–504).

Before one can find a good test vector in all cases, one of the questions which is really begging to be answered is to construct test vectors for the triple product in the finite field case, i.e., for $\mathrm{GL}_2(\mathbb{F}_q)$. This is of course a question for an undergraduate student!

(3) Invariant linear forms for representations of GL(2) over a local field, *American J. of Maths* 114, 1317-1363 (1992).

This paper was the result of adapting my thesis work to general cubic algebras as suggested by Jacquet in 1988 in Ann Arbor. It was written during the year 1989-90 after returning to Tata Institute in Mumbai. I had the luxury of no duties at TIFR, and no other projects to think about!

In this paper, I used an idea which I like to call 'cancellation of singularities' according to which the difference of the characters of two infinite dimensional irreducible admissible representations of $\mathrm{GL}_2(F)$ of the same central character is smooth at all points of $\mathrm{GL}_2(F)$. This allowed me to calculate multiplicities in branching laws by the usual Schur orthogonality relations (plugging in matrix coefficients, and eventually using Selberg principle: orbital integrals of matrix coefficients of supercuspidal representations are characters on elliptic elements, which are then related on the division algebra side by the Jacquet-Langlands character identity), reducing end result to one for principal series which is known by Mackey theory. However, I have not been able to use this idea for groups of rank greater than 1.

Except for the above conceptual part, this paper continued a very explicit approach to the problem of decomposing a representation restricted to a subgroup either by Mackey theory of orbits if the representation is a principal series, or by explicit compact induction if the representation is supercuspidal.

In dealing with the Steinberg representation, I was reduced to an equivalent question: a discrete series representation of $\mathrm{GL}_2(E)$ for E/F quadratic, is distinguished by $\mathrm{GL}_2(F)$ if and only if it is distinguished by $D^\times$. This too was proved by brute force, and to date (2015), I do not know of an elegant and simple argument—except that during my stay in San Diego in 2008, Wee Teck pointed out that it is a simple consequence of theta correspondence. This theorem was also independently proved by Jeff Hakim in his thesis at Columbia with Jacquet. It is a local analogue of a theorem of Jacquet and Lai (Compositio Math. 54 (1985), no. 2, 243–310), but their global theorem was formulated so as to avoid having to deal with this local problem, which was later addressed by Flicker and Hakim (Amer. J. Math. 116 (1994), no. 3, 683–736).

This paper, as well as my Compositio paper based on my thesis, had to understand a bit of the extensions of representations offered by the Mackey theory, and I was struggling to understand if the following extension of G-modules is always non-split (I believed and still believe this is the case):

$$0 \to \mathbb{S}(U) \to \mathbb{S}(X) \to \mathbb{S}(Z) \to 0,$$

where $X = X(F)$ is an irreducible algebraic variety over a non-Archimedean local field F together with an action of a reductive algebraic group $G = G(F)$, Z is a closed subvariety of X, $U = X - Z$, and it is assumed that both U and Z are homogeneous spaces for the group G (over the algebraic closure of F); $\mathbb{S}(X)$ denoted the Schwartz space of locally constant compactly supported functions on $X(F)$. More specifically, I wanted to understand if the following sequence is never exact whenever $H_0(G(F), \mathbb{S}(U)) \neq 0$:

$$0 \to H_0(G(F), \mathbb{S}(U)) \to H_0(G(F), \mathbb{S}(X)) \to H_0(G(F), \mathbb{S}(Z)) \to 0.$$

For the epsilon factor in the twisted case, I discovered 'multiplicative' or 'tensor' induction which from a representation W of dimension m of a subgroup H of a group G of index d produces a representation of G of dimension m^d, in a way very analogous to the usual induction. Tate whom I told this was very happy, and even wrote a page of calculation on its character. Later I found out that it was (naturally) known, as for instance in Curtis-Riener.

(4) (With B.H. Gross) On the decomposition of a representation of $\mathrm{SO}(n)$ when restricted to $\mathrm{SO}(n-1)$, *Canadian J. of Maths* 44, 974–1002 (1992).

This paper was written during the year 1991-92, and as has been said on several occasions, it was inspired by a paper of Harris and Kudla (Arithmetic automorphic forms for the nonholomorphic discrete series of GSp(2). Duke Math. J. 66 (1992), no. 1, 59–121) where among many other things, they calculate the branching of discrete series representations of $\mathrm{Sp}_4(\mathbb{R})$ restricted to $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Realizing that $\mathrm{Sp}_4(\mathbb{R})$ is essentially $\mathrm{SO}(3,2)$, this branching amounted to restriction from $\mathrm{SO}(3,2)$ to $\mathrm{SO}(2,2)$; combining this with similar, but much easier branchings known from $\mathrm{SO}(1,4)$ to $\mathrm{SO}(0,4)$ and $\mathrm{SO}(5,0)$ to $\mathrm{SO}(4,0)$, it became clear that we were going up the ladder of $\mathrm{SO}(2) \subset \mathrm{SO}(3)$ treated by Saito and Tunnell locally, and Waldspurger globally, then $\mathrm{SO}(3) \subset \mathrm{SO}(4)$ treated in my thesis locally, and Harris-Kudla globally, and now $\mathrm{SO}(4) \subset \mathrm{SO}(5)$ treated by Harris-Kudla in the Archimedean case.

Among the key geometric properties which go in making the branching laws from $G = \mathrm{SO}_{n+1}(F)$ to $H = \mathrm{SO}_n(F)$ have such fantastic properties is the fact that, assuming G and H to be quasi-split, $H(F)$ has a unique open orbit on the (F-points of the) flag variety of $G \times H$, with trivial stabilizer. For example, for $G = \mathrm{SO}(V)$ and $H = \mathrm{SO}(W)$ defined by a codimension one non-degenerate subspace W of an odd dimensional quadratic space V with basis $\{e_1, \cdots, e_n, e_0, f_1, \cdots, f_n\}$ (with standard quadratic form which has e_i, f_i isotropic for $i = 1, \cdots, n$, and $\langle e_i, f_j \rangle = \delta_{i,j}$) if we take the stabilizer of maximal isotropic flags,

$$\mathcal{F}_1 = \{\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \cdots, e_n \rangle\},$$

and,

$$\mathcal{F}_2 = \{\langle f_1 \rangle \subset \langle f_1, f_2 \rangle \subset \cdots \subset \langle f_1, \cdots, f_n \rangle\},$$

then the intersection of the stabilizers of $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathrm{SO}(V)$ is the maximal torus T which acts on e_i by λ_i, and on f_i by λ_i^{-1}, and trivially on e_0. Let W then be the codimension one subspace of V which is the orthogonal complement of the vector $e_0 + e_1 + \cdots + e_n$. The flag $\mathcal{F}_1$ lies in W, therefore the intersection of the Borel subgroup in $\mathrm{SO}(W)$ defined by $\mathcal{F}_1$ with the Borel subgroup in $\mathrm{SO}(V)$ defined by $\mathcal{F}_2$ is that part of the torus T which belongs to the subgroup $\mathrm{SO}(W)$, i.e., which acts by 1 one on $e_0 + e_1 + \cdots + e_n$.

Since the torus T operates on e_0 by one, and by scaling on e_i, it follows that $T \cap \mathrm{SO}(W) = 1$.

The other crucial geometric property being that there is a unique $H(F)$ closed orbit on the (F-points of the) flag variety of $G \times H$, except when both G and H are split groups and H is defined by an even dimensional quadratic space W, in which case there are two closed orbits which are permuted by $\mathrm{O}(W)$). (Recall that if we define a Borel subgroup B' in $H' = \mathrm{O}_n$ to be the normalizer of a Borel subgroup B in $H = \mathrm{SO}_n$, then $H'/B' = H/B$, and hence on the flag variety of $G \times H$, the group $\mathrm{O}_n(F)$ operates.)

In this paper, it was important to fix for the pair of groups $(\mathrm{SO}(V), \mathrm{SO}(W))$, with W a codimension one nondegenerate subspace of V, a canonical choice of a generic character on the maximal unipotent subgroups of $\mathrm{SO}(V)$ and $\mathrm{SO}(W)$ (assuming that both the groups are quasi-split). It is easy to see that if the space V or W has odd dimension, then any two choices of a generic character on $\mathrm{SO}(V)$ or $\mathrm{SO}(W)$, as the case may be, are conjugate. It suffices then to construct such a canonical choice of a generic character on $\mathrm{SO}(V)$ when V has even dimension and $\mathrm{SO}(V)$ is quasi-split, assuming that V comes equipped with a line on which the quadratic form is non-trivial. There is a construction in this paper, as well as in [–]. I offer another one here.

Suppose that $\dim(V) = 2n$, and e_0 is a non-isotropic vector in V with W as its orthogonal complement. Since $\mathrm{SO}(V)$ is quasi-split, W must have an isotropic subspace $\langle e_1, \cdots, e_{n-1} \rangle$ of dimension $n-1$. A Borel subgroup B in $\mathrm{SO}(V)$ can be taken to be the stabilizer of the flag (note that this is not necessarily a maximal isotropic flag, which is better since quasi-split groups do not have one!):

$$\mathcal{F} = \{\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \cdots, e_{n-1} \rangle\}.$$

Let θ be the reflection in $\mathrm{O}(V)$ around $\langle e_0 \rangle$. Clearly θ preserves B, and acts trivially on $\mathrm{SO}(W)$. It can be seen that generic characters on the unipotent radical U of B which are left invariant under θ are conjugate to each other under the action of the part B which lies in $\mathrm{SO}(W)$, or more simply by the maximal torus T_W in W which operates by scaling on a standard set of isotropic vectors $e_i, f_j, 1 \leq i, j \leq n-1$ inside $\mathrm{SO}(W)$.

It may be recalled that when the paper was written in 1991-92, the Local Langlands conjecture was not proven even for $\mathrm{GL}_n(F)$, and here we were talking of parametrization of L-packets on $\mathrm{SO}_n(F)$ in terms of characters of component groups, so the paper was proposing conjectures which depended on conjectures which seemed very much in the distant future. It is very striking then that developments in automorphic forms in the next two decades proved the Local Langlands correspondence in the precise way needed here, and the local conjectures themselves got proven by Waldspurger and Moeglin-Waldspurger for non-Archimedean fields. We refer the readers to the Asterisque volumes 346 and 347 containing some of the papers of Waldspurger and Moeglin-Waldspurger proving these conjectures.

(5) On the decomposition of a representation of GL(3) restricted to GL(2), *Duke Math J.* 69, 167–177 (1993).

This paper was written in response to some correspondence with Flicker. I came up also with a general conjecture of when a representation of $\mathrm{GL}_n(F)$ is distinguished by $\mathrm{GL}_{n-1}(F)$, which was proved by Venketasubramanian (On representations of GL(n) distinguished by GL($n-1$) over a p-adic field. Israel J. Math. 194 (2013), no. 1, 1–44).

As some of my earlier papers, I had to deal with certain questions on Extensions of representations. In particular, I had a Lemma (Lemma 7 on page 175) according to which extension of the Steinberg by the Steinberg is trivial for PGL(2) (it was actually wrongly asserted for GL(2)). The proof setters of my article changed this assertion in English to '....the following Lemma by Steinberg about extension of the same name...' instead of 'extension of the Steinberg by the Steinberg...'!

(6) Bezout's theorem for simple abelian varieties, *Expositiones Math.* 11, 465-467 (1993).

In this short note I give a proof of the analogue of Bezout's theorem for abelian varieties: any two subvarieties of complementary dimensions in a simple abelian variety intersect.

The question was casually posed by Madhav Nori, and the (very short) paper was actually concieved in 1984, but written in 1991 only after I returned from Harvard. From referees to this paper, I learnt that the result in this paper was already known, for example due to Barth (Fortsetzung, meromorpher Funktionen in Tori und komplex-projektiven Rumen. (Italian) Invent. Math. 5 1968 4262), but the proof presented here was different anyway. I had considerable difficulty getting this note of 2 pages published.

It appears to me that the result is still not such a well-known result, and in particular, it is still not known in positive characteristic. This paper of mine has never been cited, so perhaps no one has noticed it.

(7) Weil representation, Howe duality, and the theta correspondence, *AMS and CRM proceeding and lecture notes*, 105-126 (1993).

These are the notes of a lecture course given at McGill University in the Fall of 1991 at the invitation of Ram Murty who was a Professor at McGill then. I was learning the subject as I was lecturing, and was fortunate in having Steve Gelbart in my audience who was very encouraging as well as helpful. Thank you Steve!

I was already thinking about the functorial nature of the Howe correspondence in my course at McGill, where I tried to propose an answer for dual pairs of type II, i.e., $(\mathrm{GL}_n, \mathrm{GL}_m)$, which unfortunately was marred by some typographical issues. The exact nature of the Howe correspondence for this pair $(\mathrm{GL}_n, \mathrm{GL}_m)$ was established by Alberto Minguez in (Correspondance de Howe explicite: paires duales de type II. Ann. Sci. c. Norm. Supr. (4) 41 (2008), no. 5, 717–741).

(8) On the local Howe duality correspondence, *IMRN*, No. 11, 279-287 (1993).

This paper was written in Toronto in 1993 where I was an NSERC post-doctoral fellow with Kumar Murty. I realized that although Howe duality correspondence in general was very complicated, and not functorial in obvious ways, when the groups involved had a similar size, the correspondence behaved much better, in particular it is functorial, and in fact this paper is meant to suggest that within an L-packet too it behaved beautifully!

There was some work of Jeff Adams on the L-parameters involved, and there is also a letter of Langlands to Roger Howe on this topic (see his website). What I did was to propose how the Howe correspondence works inside an L-packet for groups of similar size using Vogan's point of view of 'pure inner forms', and character of component groups, and made some explicit consequences independent of the parametrization. This was done in this paper for the pairs $(\mathrm{SO}_{2n}, \mathrm{Sp}_{2n})$, and $(\mathrm{SO}_{2n+2}, \mathrm{Sp}_{2n})$.

For example, it was suggested in this paper that the theta lift from $\mathrm{Sp}(2n-2)$ to $\mathrm{O}(2n)$ gives a one-to-two map on Vogan packets of representations unless the rep'n on $\mathrm{Sp}(2n-2)$ came as theta lift from $\mathrm{O}(2n-2)$ in which case it gives a one-to-one map, with a similar statement for the correspondence from $\mathrm{O}(2n)$ to $\mathrm{Sp}(2n)$.

This is now proven by Gan and Ichino (Theorem C.5 of Gan-Ichino's paper in Invent. Math. 195 (2014)).

The full local conjecture in the paper is now proven by Hiraku Atobe (The local theta correspondence and the local Gan-Gross-Prasad conjecture for the symplectic-metaplectic case; arXiv: 1502.03528).

In this paper, I also suggest that the global non-vanishing in the almost equal rank case studied here is controlled only by local non-vanishing for tempered representations. It is not clear to me if there is an explicit theorem in the literature to this effect.

(9) (With B.H. Gross) On irreducible representations of $\mathrm{SO}(2n+1) \times \mathrm{SO}(2m)$, *Canadian Journal of Mathematics*, vol 46(5), 930-950 (1994).

The notion of Bessel models appeared first in the work of M. E. Novodvorsky [C. R. Acad. Sci. Paris Sér. A-B 280 (1975), A1421–A1422] to construct the L-function for $\mathrm{GL}(1) \times \mathrm{O}(V)$ (with $\dim V$ odd), where the automorphic representation on $\mathrm{O}(V)$ was assumed generic.

See also the voluminous work (Ginzburg, D., Piatetski-Shapiro, I.; Rallis, S. L functions for the orthogonal group. Mem. Amer. Math. Soc. 128 (1997), no. 611) but which curiously does not use the word 'Bessel model'!

It was only in the later paper with Wee Teck Gan and Gross that we realized that the questions on Bessel models being studied in this paper are actually special cases of the branching problem from SO_{n+1} to SO_n (for appropriate principal series representations induced from maximal parabolics which we recall for $\mathrm{SO}_n(F)$ have Levi subgroups of the form $\mathrm{GL}_d(F) \times \mathrm{SO}_{n-2d}(F)$; one takes principal series representations which on the $\mathrm{GL}_d(F)$ part of the Levi subgroup is a supercuspidal representation).

There is an suggestion (Note 9.3) in this paper on the possible Langlands parameter of the Jordan-Holder factors of a principal series representation, which seems still not addressed in the literature. It generalizes a recipe of Rodier when the inducing character is regular.

This paper was a gift from Gross to me!

(10) On an extension of a theorem of Tunnell, *Compositio Math.*, 94, 19-28(1994).

This paper was written in 1992-93, and extended Tunnell's theorem for those representations of $\mathrm{GL}_2(F)$ which arise from a character of a quadratic extension E/F. In this case, the representation of $\mathrm{GL}_2(F)$ when restricted to the subgroup $\mathrm{GL}_2(F)_+$ of index 2 (containing $E^\times$) with determinants belonging to norms from $E^\times$ decomposes into 2 components $\pi = \pi^+ + \pi^-$ when restricted to $\mathrm{GL}_2(F)_+$, and the epsilon factor $\epsilon(\pi \otimes \mathrm{Ind}\,_E^F \chi)$ too factorizes. The extension of Tunnell's theorem in this paper dealt with the question of which characters of $E^\times$ appear in π^+ and π^- in terms of these 'partial' epsilon factors.

The paper was inspired by a question of M. Harris.

(11) (With D. Ramakrishnan) Lifting orthogonal representations to spin groups and local root numbers, *Proc. of the Indian Academy of Science,* vol 105, 259-267 (1995).

I visited Dinakar in the summer of 1993 at Cal Tech where this paper was written. I do not remember what brought us to this paper, but anyhow we analysed in this paper when a finite dimensional irreducible representation of $D^\times/F^\times$ (where F is a non-Archimedean local field, and D is the unique quaternion division algebra) with values in $\mathrm{O}_n(\mathbb{C})$ can be lifted to the Pin group $\mathrm{P}in_n(\mathbb{C})$.

Although the results of the paper did not turn out to be so elegant — so have not been pursued further by us or others, except Franco-Fernandez and Snaith (Stiefel-Whitney classes and symplectic local root numbers. J. Algebra Appl. 5 (2006), no. 4, 403–416), we did formulate a general question of when a selfdual representation of $D^\times$, now D is a general division algebra over a local field F, is orthogonal. Based on considerations of Poincare duality on the middle dimensional cohomology of a certain rigid analytic space, Dinakar Ramakrishnan and I conjectured that a representation of division algebra is orthogonal if and only if the associated representation of the Galois group is symplectic. This was proved by us later, and appeared in American J. of Math in 2012.

This paper also analyzed when an irreducible selfdual orthogonal representation of $D^\times$ (so of $D^\times/F^\times$) lands inside SO instead of just O (so has trivial determinant). Later I proposed to Jayasree and Rajat Tandon from Central University of Hyderabad in the mid 90's, to calculate the determinant of an irreducible finite dimensional representation of $D^\times$, for general division algebras. Since characters of $D^\times$ arise from characters of $F^\times$ via the reduced norm, we thus have to each irreducible Galois representation of dimension n of $\mathrm{Gal}(\bar{F}/F)$ with determinant ω, a character μ of $F^\times$ with $\mu^n = \omega^d$ where d is the dimension of the representation of $D^\times$ (by the condition on central characters of the Local Langlands correspondence). This gives, for example, for every irreducible Galois representation of dimension n of trivial determinant, a canonically associated character of $F^\times$ of order (a divisor of) n. The question is what is this character of $F^\times$ from the Galois theoretic point of view?

The work of Jayasree and Tandon calculates this character explicitly using Howe's construction of all irreducible representations of $D^\times$, but they could not interpret the result so obtained in any nice way. (Jayasree, S.; Tandon, R. Determinant of representations of division algebras of prime degree over local fields. Number theory (Tiruchirapalli, 1996), 109–133, Contemp. Math., 210, Amer. Math. Soc., Providence, RI, 1998.)

(12) Some applications of seesaw duality to branching laws, *Maths Annalen*, vol. 304, 1-20 (1996).

This paper was also written in the year 1993 before I returned to India at the end of 1993 from Toronto. I was now using theta correspondence for the purpose I originally learnt: to do branching laws!

In this paper, branching laws were being considered from the group

$$\mathrm{G}[\mathrm{SL}_2(F)\times\mathrm{SL}_2(F)] = \{(g_1,g_2)\in \mathrm{GL}_2(F)\times\mathrm{GL}_2(F) \mid \det g_1 = \det g_2\}$$

to the diagonally embedded $\mathrm{GL}_2(F)$, as well as from $\mathrm{GSp}_4(F)$ to the subgroup $\mathrm{G}[\mathrm{SL}_2(F)\times\mathrm{SL}_2(F)]$, and also to the subgroup

$$\mathrm{G}[\mathrm{SL}_2(E)] = \{g\in\mathrm{GL}_2(E) \mid \det g\in F^\times\}$$

for E a quadratic extension of F.

Noting that,
(a) $\mathrm{GSpin}(2) = E^\times$,
(b) $\mathrm{GSpin}(3) = \mathrm{GL}_2(F)$,
(c) $\mathrm{GSpin}(4) = \mathrm{G}[\mathrm{SL}_2(E)]$,
(d) $\mathrm{GSpin}(5) = \mathrm{GSp}_4(F)$,
(all these Spin groups are quasi-split, and the even Spin groups are associated to a quadratic algebra E/F) we find that the branching laws in this paper are particular cases of branching laws from $\mathrm{GSpin}(n+1) = \mathbb{G}_m \times_{\mu_2} \mathrm{Spin}(n+1)$

to GSpin(n) = $\mathbb{G}_m \times_{\mu_2}$ Spin(n) for $n = 3, 4$, and the case $n = 2$ corresponds to the case studied by Saito and Tunnell.

Observe that the L-group of GSpin($2n + 1$) is GSp($2n, \mathbb{C}$), whereas the L-group of GSpin($2n$) is GSO($2n, \mathbb{C}$). Now there is a natural mapping,

$$\mathrm{GSp}(2n, \mathbb{C}) \times \mathrm{GSO}(2m, \mathbb{C}) \longrightarrow \mathrm{GSp}(4mn, \mathbb{C}),$$

which lands inside Sp($4mn, \mathbb{C}$) if the product of the similitude characters is 1, which is the case in the study of branching laws from GSpin($n + 1$) to GSpin(n) since the central characters must be the same.

It therefore looks like a reasonable problem to study branching laws from GSpin($n+1$) to GSpin(n) in general generalizing the work done here, where multiplicity one should hold good, and where symplectic epsilon factors should play a role as in the case for SO($n + 1$) to SO(n).

I had discussions with Gross in 2007 at San Diego on this, but we never pursued this matter.

In another direction, for a representation π of $\mathrm{GL}_2(F)$, given the decomposition, $\pi \otimes \pi = \mathrm{Sym}^2(\pi) \oplus \Lambda^2(\pi)$, we refine the branching law for $\pi \otimes \pi$ to consider which representations of $\mathrm{GL}_2(F)$ appear as a quotient of $\mathrm{Sym}^2(\pi)$, and which appear as a quotient of $\Lambda^2(\pi)$. It was noted in my Duke paper (2007) that this amounts to refining branching laws from SO($n+1$) to SO(n) to one from O($n + 1$) to O(n).

It may be easily proved that multiplicity one from SO($n + 1$) to SO(n) implies multiplicity one from O($n + 1$) to O(n). What is not obvious is that multiplicity one from O($n+1$) to O(n) (which is what was first proven historically) also implies multiplicity one from SO($n + 1$) to SO(n) (as was proved by Waldspurger in the Asterisque volume 346). On the other hand, branching laws from O($n + 1$) to O(n) are finer than branching laws from SO($n+1$) to SO(n) (incorporating for example the information on which representations of $\mathrm{GL}_2(F)$ appear as a quotient of $\mathrm{Sym}^2(\pi)$, and which appear as a quotient of $\Lambda^2(\pi)$).

To extend branching laws from SO($n + 1$) to SO(n) as proved by Waldspurger and Moeglin-Waldspurger to branching laws from O($n + 1$) to O(n), one will need to extend the Local Langlands correspondence from SO(n) to O(n) (this was already proposed for example in my IMRN (1993) paper), and then extend the work of Waldspurger and Moeglin-Waldspurger.

At the very end of the paper, I formulate a conjecture about multiplicity one (even in the strong sense, i.e., for L-packets) for Fourier-Jacobi models. This was proved by Binyong Sun (Multiplicity one theorems for Fourier-Jacobi models. Amer. J. Math. 134 (2012), no. 6, 1655–1678)

(13) Ribet's Theorem: Shimura-Taniyama-Weil implies Fermat, Proceedings of the seminar on Fermat's Last Theorem at Fields Institute, edited by V. Kumar Murty, *CMS Conference Proceedings*, vol. 17, 155–177 (1995).

This was an expository paper that I wrote on Ribet's level lowering and level raising theorems, and appeared in a volume of proceedings of a Fields

Institute activity which I participated in 1993 while in Toronto. This was my first attempt at learning this kind of mathematics in which I was greatly helped by discussions with C. Khare at Harish-Chandra Research Institute (then Mehta Research Institute) in 1995.

(14) (With C. Khare) Extending local representations to global representations, *Kyoto J. of Mathematics*, vol 36, 471-480 (1996).

There is a well known theorem of Deligne about estimates on the Fourier coefficients of modular forms. In this paper with C. Khare, we study whether the converse is true, i.e. if given finitely many algebraic integers satisfying Deligne bounds, there exists an eigenform of Hecke operators with these algebraic integers as Fourier coefficients. One simple case of this problem (for modular forms of weight 2 with integral Fourier coefficients) is solved by an application of Wiles's theorem about the Shimura-Taniyama conjecture.

(15) On the self-dual representations of finite groups of Lie type, *Journal of Algebra*, vol 210, 298–310 (1998).

I have taken some interest in understanding which irreducible selfdual representations of a group G carry an invariant form which is symmetric (when the representation could be said to be orthogonal), and when it carries an invariant form which is skew-symmetric (when the representation could be said to be symplectic). For example, it is known that all irreducible representations of the symmetric group S_n, and of the orthogonal groups $\mathrm{O}_n(\mathbb{F}_q)$ are orthogonal, and all irreducible selfdual representations of $\mathrm{GL}_n(\mathbb{F}_q)$ are orthogonal.

For a compact connected Lie group it is a theorem due to Malcev that an irreducible, self-dual representation carries an invariant symmetic or skew-symmetric bilinear form depending on the action of a certain element in the center of the group. In this paper, I have generalised this result to certain representations of finite groups of Lie type. These results are, however, proved only for generic representations and a condition on the group: the group contains an element which operates by -1 on all simple roots.

The situation seems to be fairly well understood for all classical groups, i.e., $\mathrm{GL}_n(\mathbb{F}_q), \mathrm{O}_n(\mathbb{F}_q), \mathrm{SO}_n(\mathbb{F}_q), \mathrm{U}_n(\mathbb{F}_q)$, and $\mathrm{Sp}_n(\mathbb{F}_q)$, however, when this paper was written, there were no general results for $G(\mathbb{F}_q)$ for G a reductive algebraic group over a finite field $\mathbb{F}_q$, and even $\mathrm{SL}_n(\mathbb{F}_q)$ was understood in all cases. In fact, one of the observations of this paper was that there is an irreducible self-dual representation of $\mathrm{SL}_6(\mathbb{F}_q)$ for $q \equiv 3 \mod 4$ which has trivial central character, but is of symplectic kind. The paper essentially had one general Lemma which allowed conclusions on irreducible selfdual representations which are generic.

Alex Turull went much further in this question for $\mathrm{SL}_n(\mathbb{F}_q)$ in (The Schur indices of the irreducible characters of the special linear groups. J. Algebra 235 (2001), no. 1, 275–314).

(16) A brief survey on the Theta correspondence, Proceedings of the Trichy conference edited by K. Murty and M. Waldschmidt, *Contemporary Maths*, AMS, vol. 210, 171-193 (1997).

After my lectures at Montreal in 1991, I got another opportunity to lecture and think about theta correspondence in a conference at Trichy in January 1996 which was inaugurated by a young IAS officer.

In these notes based on my lecture in Trichy, I have included a result on the K-type of the Weil representation for $K = \mathrm{Sp}_{2n}(\mathcal{O}_F)$ where F is a non-Archimedean local field of odd residue characteristic in which case the two fold metaplectic covering splits over $K = \mathrm{Sp}_{2n}(\mathcal{O}_F)$.

There is also the character formula for the Weil representation for $\mathrm{SL}_2(F)$ obtained by combining a result of Rogawski (and Moen in odd residue characteristic) which tells about which characters of E^1 where E is a quadratic field extension of F appear in the Weil representation, and a result of my own which allowed one to make a closed formula for the sum of these one dimensional characters of E^1. The result is a transfer factor for $\mathrm{SL}_2(F)$ which I have not seen in any greater generality.

(17) Some remarks on representations of a division algebra and of Galois groups of local fields, *Journal of Number Theory*, vol 74, 73-97 (1999).

This paper was motivated by a conjecture of Dinakar Ramakrishnan and myself which tells the parity of an irreducible selfdual representation of $D^\times$ (D a division algebra over a non-Archimedean local field F of index n) in terms of the parity of the corresponding Galois representation of dimension n (which is known to be selfdual). In particular, the paper makes the elementary observation that unless n is even or the residue characteristic of F is even, there are no irreducible selfdual representations of $D^\times$ of dimension > 1; same on the Galois side.

In the paper, we also count number of exceptional two dimensional representations of 2-adic local fields, reproving a theorem of Weil: there are $\frac{4}{3}(q^{2\mathrm{val}(2)} - 1)$ many of them up to twisting.

The paper also formulates another kind of generalization of Saito-Tunnell theorem: for D a division algebra of index $2n$ over a local field F, and E a quadratic extension of F sitting inside D, let D_E be the centralizer of E inside D which is a central division algebra of index n. The question formulated was: what are the characters of $D_E^\times$ which appear inside an irreducible finite dimensional representation of $D^\times$? Note that characters of $D_E^\times$ are obtained by characters of $E^\times$ through the reduced norm mapping.

(18) Distinguished representations for quadratic extensions, *Compositio Math.*, vol. 119(3), 343-354 (1999).

This paper studies the question of when a representation of $G(K)$ has a $G(k)$-invariant vector for K a quadratic extension of k for k either a finite or a p-adic field. For finite fields, the basic theorem is proved in the generality of connected groups without reductivity hypothesis. Since the result was not stated in its ideal form, I do it here.

Before I do this, recall that for a connected algebraic group G over a finite field $\mathbb{F}_q$, by Lang's theorem, every element x of $G(\mathbb{F}_q)$ can be written as $x = y^{-1}y^{[q]}$ for $y \in G(\bar{\mathbb{F}}_q)$, and $y \to y^{[q]}$ the Frobenius map on $G(\bar{\mathbb{F}}_q)$; the choice of y in expressing $x = y^{-1}y^{[q]}$ is unique up to left translation by $G(\mathbb{F}_q)$. For $x = y^{-1}y^{[q]}$, define the Shintani transform $Sh(x) = y^{[q]}y^{-1}$. The Shintani transform $x \to Sh(x)$ defines a well-defined map on the set of $G(\mathbb{F}_q)$ conjugacy classes in $G(\mathbb{F}_q)$ to the set of $G(\mathbb{F}_q)$ conjugacy classes in $G(\mathbb{F}_q)$. Since any semi-simple element of $G(\mathbb{F}_q)$ belongs to a torus, the Shintani transform is the identity map on semi-simple elements.

Theorem. *Let G be a connected algebraic group over a finite field $\mathbb{F}_q$, and π an irreducible $\mathbb{C}$-representation of $G(\mathbb{F}_{q^2})$. Assume that the character of π takes the same value at x as at $Sh(x)$ for $x \in G(\mathbb{F}_{q^2})$. Then the representation π has a $G(\mathbb{F}_q)$ fixed vector if and only if $\pi^\sigma \cong \pi^\vee$. If $\pi^\sigma \cong \pi^\vee$, then π has a one dimensional space of fixed vectors under $G(\mathbb{F}_q)$, and the representation $\pi \otimes \pi^\sigma$ which is canonically a representation of $G(\mathbb{F}_{q^2}) \rtimes \mathbb{Z}/2$ has a $G(\mathbb{F}_q) \rtimes \mathbb{Z}/2$ fixed vector.*

Corollary. *Let G be a connected reductive group over a finite field $\mathbb{F}_q$, and π an irreducible Deligne-Lusztig representation of $G(\mathbb{F}_{q^2})$ induced from a torus. Then the representation π has a $G(\mathbb{F}_q)$ fixed vector if and only if $\pi^\sigma \cong \pi^\vee$. If $\pi^\sigma \cong \pi^\vee$, then π has a one dimensional space of fixed vectors under $G(\mathbb{F}_q)$, and the representation $\pi \otimes \pi^\sigma$ which is canonically a representation of $G(\mathbb{F}_{q^2}) \rtimes \mathbb{Z}/2$ has a $G(\mathbb{F}_q) \rtimes \mathbb{Z}/2$ fixed vector.*

Proof. The proof of the corollary follows from a theorem of Digne-Michel that for that the character of a Deligne-Lusztig representation of $G(\mathbb{F}_{q^2})$ takes the same value at x as at $Sh(x)$ for $x \in G(\mathbb{F}_{q^2})$. □

Lusztig with whom I discussed the results of this paper during his visit to Allahabad in 1998-99, followed up the theme of this paper in his paper ($G(\mathbb{F}_q)$-invariants in irreducible $G(\mathbb{F}_{q^2})$-modules. Representation Theory 4 (2000), 446–465). He even offered to have me as a joint author, but I really had played no direct role, so how could I!

(19) (with D. Ramakrishnan) On the global root numbers of $\mathrm{GL}(n) \times \mathrm{GL}(m)$, *Proceedings of Symposia in Pure Maths of the AMS*, vol. 66, 311-330, (1999).

(20) On the self-dual representations of p-adic groups, *IMRN* vol. 8, 443-452 (1999).

(21) (With Kumar Murty) Tate cycles on a product of two Hilbert Modular Surfaces, *Journal of Number Theory*, vol. 80, 25-43 (2000).

In this paper with Kumar Murty, we parametrise Tate cycles on products of two Hilbert modular surfaces in terms of Hilbert modular forms, including the precise information about the field of rationality.

(22) Theta correspondence for Unitary groups, *Pacific Journal of Math*, vol. 194, No. 2, 427-438(2000).

This paper was inspired by a paper of Harris, Kudla, and Sweet (Theta dichotomy for unitary groups. J. Amer. Math. Soc. 9 (1996), no. 4, 941–1004.)
The proposed conjecture in this paper on precise nature of theta correspondence is now proven by Gan and Ichino (The Gross-Prasad conjecture and local theta correspondence).

(23) (with A. Raghuram) Kirillov theory of $\mathrm{GL}_2(D)$ where D is a division algebra over a non-Archimedean local field, *Duke Math. J.*, vol. 104, No. 1, 19-44(2000).

In this work done with A. Raghuram, we develop Kirillov theory for irreducible admissible representations of $\mathrm{GL}_2(D)$ where D is a division algebra over a non-Archimedean local field. This work is in close analogy with the work of Jacquet-Langlands done in the case when D is a field, and realises any irreducible admissible representation of $\mathrm{GL}_2(D)$ on a space of functions of D^* with values in what may be called the space of degenerate Whittaker models which is the largest quotient of the representation on which the unipotent radical of the minimal parabolic which is isomorphic to D acts via a non-trivial character of D.

(24) The space of degenerate Whittaker Models for GL_4 over p-adic fields, Proceedings of the TIFR conference on *Automorphic Forms*, AMS (2001).

In this paper in the conference proceedings of a conference at the Tata Institute on Automorphic forms, I elaborate on a conjecture with B. Gross which gives a very precise structure for the space of degenerate Whittaker models on $\mathrm{GL}_2(D)$ when D is a quaternion division algebra. There is also a proposal in this paper to interpret triple product epsilon factors (for GL(2)) in terms of intertwining operators.

(25) Comparison of germ expansion on inner forms of GL_n, *Manuscripta Math.*, vol. 102 , 263-268(2000).

(26) The space of degenerate Whittaker models for general linear groups over finite field, *IMRN*, No. 11, 579-595(2000).

This paper studies the space of degenerate Whittaker models for finite fields obtaining a rather pretty result about the space of degenerate Whittaker model for a cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{F})$ with respect to the (n, n) parabolic with unipotent radical $M_n(\mathbb{F})$.

(27) (With C. Khare) On the Steinitz module and capitulation of ideals, *Nagoya Math. Journal,* vol. 160 (2000), 1-15.

(28) (with C.S.Yogananda) Bounding the torsion in CM elliptic curves, *Comptes Rendus Mathematiques*, Mathematical Reports of the Academy of Sciences of Canada, vol. 23 (2001), 1-5.

L. Merel had proved an important theorem stating that the order of torsion on elliptic curves over a number field are bounded independent of the elliptic curve and the field, and depends only on the degree of the field over $\mathbb{Q}$. However, there are still no good bounds. In an attempt to see what might be the best bound, in this note with Yogananda, we estimate the bounds on torsion on CM elliptic curves.

This paper was actually concieved in 1984, but not written till 1997 when Yogananda came to visit HRI. We had considerable difficulty getting the paper published since the referees to various journals thought that since much more general results for arbitrary CM abelian varieties were known by Silverberg, it had little value. It turns out that the paper did have some interest for some people, see for instance:

Clark, Pete L.; Cook, Brian; Stankewicz, James: Torsion points on elliptic curves with complex multiplication (with an appendix by Alex Rice). Int. J. Number Theory 9 (2013), no. 2, 447–479.

(29) On a conjecture of Jacquet about distinguished representations of GL_n, *Duke Math. J.*, vol. 109 (2001) 67-78.

Let K/k be a quadratic extension of non-Archimedean local fields of characteristic zero. Suppose that π is an irreducible, square-integrable representation of $G = \mathrm{GL}_n(K)$ whose central character is trivial on $k^\times$. Let $g \to g^\sigma$ be the automorphism of G obtained by the nontrivial Galois automorphism of K/k. Then $\pi^\sigma(g) = \pi(g^\sigma)$ defines a representation of G on the space of π. Define π to be *distinguished* if $\mathrm{Hom}_H(\pi, 1) \neq 0$, where $H = \mathrm{GL}_n(k)$.

The conjecture of Jacquet referred to in the title of the paper predicts that

(a) If n is odd then $\pi^\sigma \cong \pi^\vee$ if and only if it is distinguished.

(b) If n is even then $\pi^\sigma \cong \pi^\vee$ if and only if either π or $\pi \otimes \chi_{K/k}$ is distinguished, where $\chi_{K/k}$ is the character of $k^\times$ attached to the quadratic extension K/k by class field theory.

There is a similar statement for distinction by the Unitary subgroup of $\mathrm{GL}_n(K)$.

I have used the methods of my paper in Compositio (1999) to prove this conjecture of Jacquet about distinguished representations of GL_n and U_n in the case when K is an unramified quadratic extension of a non-Archimedean local field k.

A little later, Anthony Kable gave a global and most satisfying proof for Jacquet's conjecture (Asai L-functions and Jacquet's conjecture. Amer. J. Math. 126 (2004), no. 4, 789–820).

This paper also formulates a question of when the Steinberg representation of $G(K)$ has a $G(k)$-invariant linear form for K a quadratic extension of k for k a p-adic field. The Steinberg representation was singled out partly because its character is stable, and therefore the theorem in my Compositio paper for finite fields, is most likely to hold for such representations; but even here there is a twist to the story! 'Dual' to the canonical element in the center of a reductive group of order one or two which decides if a finite dimensional selfdual algebraic representation of the group is orthogonal or symplectic, I define for any group reductive group G over a local field k, a character $\chi_G : G(k) \to \mathbb{Z}/2$. The conjecture formulated in the paper is that the Steinberg representation of $G(K)$ has exactly one dimensional space of χ_G-invariant linear form for the subgroup $G(k)$, and that the space of χ-invariant linear form for $\chi : G(k) \to \mathbb{C}^\times$ is zero if $\chi \neq \chi_G$.

This conjecture on the Steinberg has now been proven in several cases by Broussous and Courtes, see for instance (Distinction of the Steinberg representation, IMRN 2014, no. 11, 3140–3157), and several more forthcoming papers of these authors.

I realized later that there was no reason for me to restrict myself to quasi-split groups that I did for this conjecture on Steinberg. The way the character was defined, it makes sense for all reductive groups, and so does the Steinberg. Let me recall the definition of the character χ_G to make another remark.

Look at the exact sequence of algebraic groups (with G^{sc} the simply connected cover of the derived group of G, Z its center, and G^{ad} the adjoint group)

$$1 \to Z \to G^{sc} \to G^{ad} \to 1,$$

and the associated exact sequence of Galois cohomology (taking into account vanishing of H^1 of simply connected groups due to Kneser and Bruhat-Tits)

$$1 \to Z(k) \to G^{sc}(k) \to G^{ad}(k) \to H^1(k, Z) \to 1.$$

The character $\chi_G : G(k) \to \mathbb{Z}/2$ was defined by using a chacter on $H^1(k, Z)$, but the center Z of G^{sc} remains the same under inner twisting, so the definition of the character χ_G makes sense for all reductive groups. Note that assuming G^{sc} to be isotropic, it is its own derived group by the Kneser-Tits conjecture (a theorem of Platonov in this case), so any character of G^{ad} in fact comes from a character of $H^1(k, Z)$ if G^{sc} is isotropic, but it is amusing to note that this conclusion is true even if G^{sc} is not isotropic, in which case it is known that G^{sc} is $\mathrm{SL}_1(D)$, and $G^{ad} = D^\times/k^\times$, and that any character of $D^\times$ is trivial on $\mathrm{SL}_1(D)$ (Matsushima's theorem)!

It seems quite reasonable to expect that the distinction of representations of $G(K)$ by $G(k)$ for K/k quadratic, and G an inner form of GL_n is independent of the inner form when we consider representations on $G(K)$ to be related by the Jacquet-Langlands correspondence. For this it is important to observe that the sign which relates the character of $G(K)$ and of $G'(K)$ for G and G' inner forms of GL_n is the same. We do this checking here.

Suppose $G = \mathrm{GL}_n(D)$ and $G' = \mathrm{GL}_{mn}(k)$, where D is a central division algebra of index m. Therefore, $G(K) = \mathrm{GL}_n(D \otimes K)$, and $G'(K) = \mathrm{GL}_{mn}(K)$.

If π and π' are representations of $G(K) = \mathrm{GL}_n(D \otimes K)$ and $G'(K) = \mathrm{GL}_{mn}(K)$, with characters Θ_π and $\Theta_{\pi'}$, then it is known that on matching conjugacy classes, if m is odd, so $D \otimes K$ remains a division algebra,

$$(-1)^n \Theta_\pi(x) = (-1)^{mn} \Theta_{\pi'}(x').$$

Note that if m is odd, then $(-1)^n = (-1)^{mn}$.

On the other hand, if m is even, then $D \otimes K = M_2(D')$ where D' is a central division algebra of index $m/2$ over K. In this case, $G(K) = \mathrm{GL}_n(D \otimes K) = \mathrm{GL}_{2n}(D')$ and $G'(K) = \mathrm{GL}_{mn}(K)$, and so on matching conjugacy classes, we have:

$$(-1)^{2n} \Theta_\pi(x) = (-1)^{mn} \Theta_{\pi'}(x').$$

Since m is even, we have $(-1)^{2n} = (-1)^{mn}$; thus the sign relating the character of π and π' is the same whether m is even or odd.

In fact the following general lemma holds (which will be relevant to the suggestion on Steinberg for general reductive groups).

Lemma. *Let G, G' be reductive algebraic groups over a local field k which are innerforms of each other, and let K be a quadratic extension of k. Then the rank of the groups G and G' over K are the same modulo 2.*

Proof. We have carried out the proof already for inner forms of GL_n above. For Unitary groups U_n defined in terms of Hermitian forms, if n is odd, then any two such unitary groups become quasi-split (or split) at any quadratic extension of k. For n even, the groups are quasi-split to begin with. For other classical groups defined in terms of Hermitian forms over division algebra, similar analysis gives a proof. In general, it is Corollary 4 on page 295 of Kottwitz's paper (Sign changes in harmonic analysis on reductive groups. Trans. Amer. Math. Soc. 278 (1983), no. 1, 289–297). □

(30) Locally algebraic representations of p-adic groups, appendix to the paper by P.Schneider and J.Teitelbaum, $U(\mathfrak{g})$-finite locally analytic representations, Electronic Journal, *Representation Theory* 5(2001) 111-128.

This short note was inspired by a lecture I heard at IHP, Paris in 2000 of Peter Schneider in which he said that they is no reasonable category of representations of p-adic groups which contains both Algebraic representations, and smooth representations. Perhaps I suggested the notion of locally algebraic right at the end of the lecture. Although it seems not well-known that the concept was first formulated in this appendix, it has become a rather important and basic concept which I had certainly not anticipated.

There is a suggestion in this paper of an analogue of the Harish-Chandra sub-quotient theorem for p-adic representations of p-adic groups.

(31) (with Nilabh Sanat) On the restriction of cuspidal representations to unipotent elements, *Math. Proceedings of Cambridge Phil. Society* 132, No. 1, (2002) 35-56.

This paper with my student Nilabh Sanat decomposes an irreducible Deligne-Lustztig representation of a reductive group over a finite fields associated to the Coxeter torus when restricted to its maximal unipotent subgroup as an alternating sum of certain explicit unipotent representations. It was inspired by the observation that for $\mathrm{GL}_n(\mathbb{F}_q)$, for n small, the dimension of irreducible cuspidal representation is an alternating sum of dimensions of certain unipotent representations, for instance:
$(q-1) = q-1,$
$(q-1)(q^2-1) = q^3 - (q^2+q) + 1,$
$(q-1)(q^2-1)(q^3-1) = q^6 - (q^5+q^4+q^3) + (q^3+q^2+q) - 1,$
and each term in the right hand side represents the dimension of an irreducible unipotent representation of $\mathrm{GL}_n(\mathbb{F}_q)$. This equality between characters of a cuspidal representation, and that of an alternating sum of characters of irreducible unipotent representations (of n of them for $\mathrm{GL}_n(\mathbb{F}_q)$), persisted at all unipotent elements too. The paper proves this and deals with a similar phenomenon for all reductive groups over finite fields.

An important step was to consider the decomposition in irreducible components of unipotent representations:

$$R[\chi] = \frac{1}{|W|} \sum_{w \in W} \chi(w) R^G_{T_w}(1),$$

where $R^G_{T_w}(1)$ is the (unipotent) Deligne-Lusztig representation associated to the trivial character of the torus T_w which is associated to the Weyl group element w; this was relevant to our work for those characters χ which took nonzero value on the Coxeter conjugacy class (which is a very small set of characters of W).

This work with Nilabh was the only piece of work where I delved a bit deeper into Lusztig's works. A little later, Nilabh quit mathematics to seek other careers in the USA. I have continued to take interest in representation theory of finite groups of Lie type, even running workshops etc., but have not had occasion to go into details of it as here.

(32) (with CS Rajan) On an Archimedean analogue of Tate's conjecture, *J. of Number Theory*, vol. 99 (2003) 180-184.

In this paper written with CS Rajan is a re-look at Sunada's theorem about isospectral Riemannian manifolds where we deduce it as a consequence of a simple lemma in group theory. In this paper we also conjecture, and verify in several cases, that the Jacobians of two Riemann surfaces with the same spectrum for Laplacian are isogenous (after an extension of the base field), and propose this as an Archimedean analogue of Tate's conjecture.

(33) On an analogue of a conjecture of Mazur: A question in Diophantine approximation, *Contributions to automorphic forms, geometry, and number theory,* 699–709, Johns Hopkins Univ. Press, Baltimore, MD, 2004

The conjecture of Mazur in the title of this paper is about the topological closure (in the Euclidean topology) of $A(\mathbb{Q})$ inside $A(\mathbb{R})$ for an abelian variety defined over $\mathbb{Q}$; the conjecture of Mazur is that the connected component of identity of this topological closure is the set of real points of an abelian subvariety of A defined over $\mathbb{R}$. There is no progress on this conjecture to date.

In this paper, I consider an anisotropic torus T defined over $\mathbb{Q}$, so $T(\mathbb{R})/T(\mathbb{Z})$ is a compact torus (isomorphic to $(\mathbb{S}^1)^n$ but non-algebraic!). I prove that — under the Schanuel hypothesis— the connected component of identity of the topological closure of any finitely generated subgroup of $T(\mathbb{R})/T(\mathbb{Z})$ by elements of $T(\mathbb{Q})$ is the set of real points of a sub-torus of T defined over $\mathbb{R}$.

(34) (with C. Khare) Reduction of abstract homomorphisms of lattices mod p, and rigidity; *Journal of Number Theory*, vol. 105, no.2 (2004), 322-332.

In this paper with C. Khare, we prove that an abstract homomorphism between the Mordell-Weil group of abelian varieties over a number field which respects reduction mod p, in fact arises from homomorphism of abelian varieties.

(35) (with Anandavardhanan) Distinguished representations for $SL(2)$; *Mathematics Research Letters*, vol. 10, no. 5-6 (2004) 867-878.

Anand arrived as a post-doc at TIFR in the beginning of 2003 just as I was starting my second innings at TIFR, and it has been a pleasure collaborating with him.

A theme which was well known by this time is that for E a quadratic extension of a local or global field F, a representation of $\mathrm{GL}_n(E)$ (or automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$) with trivial central character restricted to $F^\times$ ($\mathbb{A}_F^\times$ globally), is distinguished by $\mathrm{GL}_n(F)$ (has nonzero period integral of $\mathrm{GL}_n(\mathbb{A}_F)/\mathbb{A}_F^\times$ if and only if its Asai L-function has pole at $s = 0$ (resp. $s = 1$).

This paper analyses similar question for $\mathrm{SL}_2(E)$, the driving question being: how does the multiplicity $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}[\pi, \mathbb{C}]$ vary as we vary π inside an L-packet of $\mathrm{SL}_2(E)$. Multiplicities do vary, and in somewhat complicated way, and there was a particular case when this multiplicity is 3! This work will be an inspiration for my later work in [] where the number 3 gets explained as well.

(36) (with O. Juriaans and IBS Passi) Hyperbolic Unit Groups; *Proceedings of the AMS*, vol. 133 (2005) 415-423.

Prof. Passi was my senior colleague at HRI. In my last year at HRI (2002), he proposed on my birthday that we should try to write a paper classifying finite groups for which the units in the integral group ring form a Hyperbolic group. I knew something about units in integral groups rings since they give a good class of Arithmetic groups, and I knew that inside a Hyperbolic group any two commuting elements were contained in a cyclic group. This means that there are very few options for elements of finite order in the group, for example it could not contain any element of odd order greater than 4, and no element of order 8 or more (Dirichlet unit theorem which I knew being a Number theorist!). I believe this is all the input we used in the paper about Hyperbolic groups, rest being group theory of the kind very familar to Prof Passi, and Juriaans who became my collaborator without having met him, and they wrote the whole paper. Here is the review by Peter A. Linnell as it appears in Math Reviews.

"Let G be a group and let $\Gamma = U_1(\mathbb{Z}G)$ denote the group of normalized units of the group ring $\mathbb{Z}G$; that is, those units which map to 1 under the augmentation map induced by $g \to 1 : \mathbb{Z}G \to \mathbb{Z}$. Of course G itself is a subgroup of Γ. This paper investigates for which groups G the unit group Γ is hyperbolic (that is, word hyperbolic in the sense of Gromov); it also investigates the possible subgroups of Γ in this case. The authors obtain surprisingly complete results for this problem. Here is a summary of their investigations.

For any group H, we shall let $T(H)$ denote its elements of finite order. Recall that a nonabelian group H is called Hamiltonian if and only if all its subgroups are normal. If H is a finite Hamiltonian group, then $H \cong Q_8 \times A \times E$, where Q_8 denotes the quaternion group of order 8, A is an abelian group of odd order, and E is an elementary abelian 2-group. Then Theorem 3 determines which finite groups can appear as subgroups of hyperbolic unit groups. It states: If Γ is hyperbolic, then a torsion subgroup of Γ must be finite and isomorphic to one of the following groups: C_5, C_8, C_{12}, an abelian group of exponent dividing 4 or 6, a Hamiltonian 2-group, $S_3, D_4, Q_{12}, C_4 \rtimes C_4$. Conversely if G is isomorphic to one of the aforementioned groups, then Γ is hyperbolic.

The next result goes a step further by determining the polycyclic-by-finite groups which embed into a hyperbolic unit group. Let H be a polycyclic-by-finite group and assume that H is infinite (the case H finite is covered by Theorem 3). Then Theorem 4 states:

Suppose Γ is hyperbolic. Then H embeds in Γ if and only if

(a) $T(H)$ is a subgroup of H;
(b) $H \cong T(H) \rtimes \mathbb{Z}$;
(c) $U_1(\mathbb{Z}T(H)) = T(H)$.

Finally the groups G for which Γ is finitely generated and virtually free are determined. Theorem 5 states:

Γ is finitely generated and virtually free if and only if G is one of the following groups:

(a) $H \rtimes F$, where F is a finitely generated free group and H is a finite abelian group of exponent dividing 4 or 6, or a finite Hamiltonian 2-group.
(b) $C_5, C_8, C_{12}, S_3, D_4, Q_{12}, C_4 \rtimes C_4$.

Furthermore, in the case in which G is infinite, $\Gamma = G$.

A key result in proving these theorems is Lemma 3, which states: Let H be any group, and let $x, y \in H$ such that x has finite order, y has order at least 5, $\langle x \rangle \cap \langle y \rangle = 1$, and $x^y \notin \langle x \rangle$. Then $\mathbb{Z}^2$ embeds in $U_1(\mathbb{Z}H)$.

Since no subgroup of a hyperbolic group is isomorphic to $\mathbb{Z}^2$, this imposes strong restrictions on G when Γ is hyperbolic. Another key result is the following, which is contained in Lemma 5, and is derived with the help of Lemma 3:

Suppose Γ is hyperbolic. If G is not a torsion group, then $T(G)$ is a finite Hamiltonian group and $U_1(\mathbb{Z}T(G)) = T(G)$.

This is a clearly written paper with many interesting results."

(37) (with. Jeff Adler) On certain mutiplicity one theorems; *Israel J. of Mathematics* 153, 221-245 (2006).

In this paper paper with Jeff. Adler, we prove several multiplicity 1 theorems; in particular we show that an irreducible representation of GSp($2n$) when restricted to Sp($2n$) decomposes with mutiplicity 1 for p-adic fields. This was the original motivation for this paper, a question that Sally had asked me, and perhaps Jeff too. We were together in Singapore, and shared an office, and for some reason or the other got started on this. A proof using uniqueness of the Fourier-Jacobi models for GSp was proposed, and in the next two or three years, we wrote this out. In the end, actually this turned out to be rather simple consequence of known lemmas due to Ryan Vinroot who actually had a similar linear algebra result for GO, and proved this for GSp at our request.

(38) (with Anandavardhanan) On period integrals for SL(2); *American J. of Mathematics*, vol. 128, 1429-1453 (2006).

This paper with Anandavardhanan was the global 'twin' of the local paper in MRL analysing period integrals on $\mathrm{SL}_2(\mathbb{A}_F)$ of automorphic forms on $\mathrm{SL}_2(\mathbb{A}_E)$.

In this work we prove factorization of period integrals in a situation where there is no local multiplicity one. What was really nice in this work was that the global factorization happens exactly for those global representations which are Galois theoretically analogous to local representations for which multiplicity one is true! This result is thus analogous to one of Jacquet where he discovers similar global factorization for cusp forms, when locally for cuspidal representations, there is indeed multiplicity one; this is striking since a global cuspidal representations is not cuspidal at most of the places.

This was my first paper where I really was dealing with a global question on automorphic forms—even if of rather simple kind, and it was also a place where I realized how useful and powerful uniqueness of Whittaker model is!

(39) Relating invariant linear form and local epsilon factors via global mehods, with an appendix by H. Saito, *Duke Journal of Math*, vol. 138, No. 2, 233-261 (2007).

There are many parallels between global period integrals, expressed in many situations as special value of L-functions, and local branching laws expressed in terms of epsilon factors. In this paper this theme has been carried out, giving a global proof of the decomposition of tensor product of two representations of GL(2) in terms of epsilon factors studied in my thesis as well as in the American J. paper in 1992. The proof in this paper uses

(a) The Jacquet conjecture, i.e, the triple product global period integral is nonzero if and only if the triple product L-function does not vanish at $1/2$, besides the local conditions. Actually, I needed just one direction of this: if the period integral is nonzero, the central L-value is nonzero.
(b) The Dichotomy principle: the trilinear form is nonzero either on GL_2 or the representations are discrete series, and the trilinear form is nonzero on the division algebra side.

Given this, it is proved in the paper that the trilinear form exists exactly on one of GL_2 or the division algebra, and which of the two possibilities is dictated by the local root number.

The method followed in this paper is rather flexible, and gives an effortless proof of the theorem of Saito-Tunnell from Waldspurger's theorem, so it is a bit surprising that it was not noticed before. It also gives various other refinements that I considered of the Tunnell's theorem, or of the triple product in my Math Annalen paper of 1996.

My earlier papers were rather incomplete on the epsilon factor part not only in even residue characteristic, but even in the odd residue characteristic in the cases dealt with in the AJM (1992) paper. Later Wee Teck Gan found a direct way of proving the local results on epsilon factors in (Trilinear forms and triple product epsilon factors. Int. Math. Res. Not. IMRN 2008, no. 15.) imitating Harris-Kudla's global proof in the local context.

(40) (with R. Schulze-Pillot) Generalised form of a conjecture of Jacquet, and a local consequence; *Crelle Journal*, vol. 616, 219-236 (2008).

In this paper written with Schulze-Pillot, we generalise Jacquet's conjecture to general cubic algebras, and deduce the local analogue, so very much like my Duke paper of 2007 for which the global input of proving Jacquet's conjecture for general cubic algebras was being supplied by this paper.

This paper also proves a very general globalisation theorem of local representations which has been quite useful.

(41) (with Shrawan Kumar and G. Lusztig) Characters of simplylaced nonconnected groups versus characters of nonsimplylaced connected groups; Contemporary Math., vol 478, AMS, pp. 99-101 (2009).

This paper with S. Kumar and G. Lusztig relates representation theory of disconnected groups such as $\mathrm{O}(2n)$ on the 'other' connected component (i.e., at the set of elements of $\mathrm{O}_{2n}(\mathbb{C})$ with $\det = -1$) to $\mathrm{Sp}(2n-2)$, and a similar phenomenon for any simple group with a diagram automorphism, thus for example between $\mathrm{SL}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n+1}(\mathbb{C})$, between $\mathrm{SL}_{2n+1}(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$, or $E_6(\mathbb{C})$ and $F_4(\mathbb{C})$. (Thus the dual group of the 'smaller' group is the fixed points of the corresponding involution on the dual group of the 'bigger' group). We discovered after the work that much of this was essentially known, which was disappointing since I was quite excited about this work, especially as it is a nice case of *endoscopy*. We submitted a longer paper to a journal, but withdrew after learning about it being already known; a much shorter version was submitted to a conference proceedings, but a longer version is still there on my webpage.

(42) (with Ramin Takloo-Bighash) Bessel models for GSp(4); *Crelle Journal* vol. 655, pages 189-243 (2011).

This paper was inspired by a lecture of Ramin in Princeton where he spoke on his work (Spinor L-functions, theta correspondence, and Bessel coefficients. Forum Math. 19 (2007), no. 3, 487–554) which to me amounted to understanding the possible Bessel models for $\mathrm{GSp}_4(\mathbb{R})$, and seemed to prove (using global methods, supplied by Philip Michel in the appendix of the Forum paper) my conjectures with Gross over $\mathbb{R}$. I was convined that if Archimedean prime can be treated, most certainly finite primes can be treated too, and that eventually the proof is local. It took us in fact another year to write this up when I was first at IAS, Princeton in 2006-07, and then at UCSD in 2007-08, where Ramin came to visit me.

The following is the review of the paper in Math Reviews by Christian Zorn.

"In the paper under review, the authors prove some conjectures of B. H. Gross and D. Prasad [Canad. J. Math. 44 (1992), no. 5, 974–1002; MR1186476 (93j:22031)] for the pair $(\mathrm{SO}(2), \mathrm{SO}(5))$ by using methods from the theory of theta correspondences to analyze Bessel models for the group GSp(4). The paper also relies heavily on several "accidental" isomorphisms such as those that relate various forms of PGSp(4) to forms of SO(5). To explain this connection, as well as the authors' results, in a little more depth, let us recall the definition of a Bessel model for GSp(4) in the local field setting.

Let k be a local field with residue characteristic not 2. Further, let $\mathrm{GSp}_4(k)$ denote the split form of GSp(4) over k and let $\mathrm{GSp}_4^D(k)$ be the non-split (rank 1) form of GSp(4) over k (here D is a quaternion division algebra over k). For $\mathrm{GSp}_4(k)$ (resp. $\mathrm{GSp}_4^D(k)$) there exists a maximal parabolic subgroup (the so-called Siegel parabolic subgroup) having Levi factor M isomorphic to $\mathrm{GL}_2(k) \times k^\times$ (resp. isomorphic to $D^\times \times k^\times$) and abelian unipotent radical N isomorphic to $\mathrm{Sym}^2(k)$ (resp. $\{n \in D | n + n^- = 0\}$ where $n \to n^-$ is the standard involution on D). For a choice of nondegenerate additive character $\psi : N \to \mathbb{C}^\times$, let $M_{K^\times} \subset M$ be the subgroup that stabilized ψ (where the action of M on characters of N is induced by the action of M on N). One can show that $M_{K^\times}$ is isomorphic to the group of units $K^\times$ of a quadratic algebra K over k.

For a one-dimensional representation $\chi : K^\times \to \mathbb{C}^\times$, define a representation of the subgroup $R = M_{K^\times} \times N$ by $m(k)n \to \chi(k)\psi(n)$. The paper under review denotes this representation as χ as well since ψ is fixed throughout the paper. Finally, for an admissible representation π of $\mathrm{GSp}_4(k)$ or $\mathrm{GSp}_4^D(k)$, say that π admits a Bessel model for the character χ if $\mathrm{Hom}_R(\pi, \chi) \neq 0$. Recalling some extensions of arguments of Novodvorsky, the authors state that $\dim_{\mathbb{C}}(\mathrm{Hom}_R(\pi, \chi)) \leq 1$. It is worth noting that $K^\times$ is isomorphic to a form of GSO(2) over k and $\mathrm{GSp}_4(k)$ (resp. $\mathrm{GSp}_4^D(k)$) is isomorphic to the split (resp. rank 1) form of GSO(5). Hence we see how this addresses the conjectures of Gross and Prasad by way of using the group GSp(4). We next aim to describe the main theorems of the work under review. These theorems (Theorems 2 and 3) have rather extensive statements, so we summarize them as follows.

Theorem 2 is a local theorem that we describe using the notation introduced above. Given an irreducible admissible generic L-packet $\{\pi\}$ of $\mathrm{GSp}_4(k)$ (resp. $\{\pi'\}$ of $\mathrm{GSp}_4^D(k)$), at most one element of the L-packet has a Bessel model for the character χ of R. Moreover, such a Bessel model actually exists for the L-packet $\{\pi\}$ (resp. $\{\pi'\}$) if and only if $\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = 1$ (resp. $\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = 1$) where σ_π is the Langlands parameter for π.

We now describe some additional material underpinning Theorem 3. Let D be a quaternion algebra over a number field F with adèle ring $\mathbb{A}_F$. Let Π_1 and Π_2 be two automorphic representations of $D^\times(\mathbb{A}_F)$ with the same central character so that $\Pi_1 \boxtimes \Pi_2$ is a representation of $\mathrm{GSO}_4(\mathbb{A}_F)$ (note that $[D^\times \times D^\times]/\Delta\mathrm{G}_m$ is a form of GSO(4)). Let Π be the theta lift of $\Pi_1 \boxtimes \Pi_2$ to $\mathrm{GSp}_4(\mathbb{A}_F)$, E/F a quadratic extension, and χ a Grössencharakter on $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_F^\times$ is the central character of π. If $\Psi : N(\mathbb{A}_F)/N(F) \to \mathbb{C}$ (with N the unipotent radical as above) is a character normalized by $\mathbb{A}_E^\times$, then the pair (χ, ψ) defines a character on $R(\mathbb{A}_F) \cong \mathbb{A}_E^\times N(\mathbb{A}_F)$ which we denote as χ by abuse of notation. Finally, one can define a period integral on Π as

$$\int_{R(F)\mathbb{A}_F^\times R(\mathbb{A}_F)} f(g)\chi^{-1}(g)dg \qquad (1),$$

for $f \in \Pi$.

Theorem 3 has two main parts. First it establishes that necessary and sufficient conditions for (1) not identically vanishing on Π are the non-vanishing of the (twisted) base change L-function $L(s, BCE(\Pi_i) \otimes \chi^{-1})$ at $s = 1/2$ for both $i \in \{1, 2\}$. Second, the main theorem takes automorphic representations Π_1 and Π_2 on $\mathrm{GL}_2(\mathbb{A}_F)$ for which $L(1/2, BCE(\Pi_i) \otimes \chi^{-1}) \neq 0$ for both $i \in \{1, 2\}$ and constructs a quaternion algebra D as well as an automorphic representation of $\mathrm{GSp}_4^D(\mathbb{A}_F)$ for which the period integral (1) is not identically zero.

The authors then use the results and techniques to deduce a theorem (Theorem 4 in the text) regarding the restriction of a representation of $GL_4(k)$ to the subgroup $\mathrm{GL}_2(K)$ for K/k a quadratic algebra. They develop the arguments in great detail and with several examples along the way. The techniques contained in the paper include constructing Bessel models and twisted Jacquet functors of the Weil representation and utilizing seesaw dual pairs. The construction of Bessel models for the Weil representation (Theorem 5 in Section 7) seems especially novel and interesting to the reviewer. Ultimately, the paper under review effectively ties together material from Bessel models, theta correspondences, period integrals, and special values of L-functions to prove a conjecture of Gross and Prasad. Furthermore, it contains a sufficient amount of material to be both clear and comprehensive."

(43) (with D. Ramakrishnan) On the self-dual representations of division algebras over local fields; *American J. of Mathematics*, vol. 134, no. 3, 749-772 (2012).

The project with Dinakar of understanding whether a selfdual representation of $D^\times$ is orthogonal or symplectic where D is a division over a local field, was started in 1993, and got completed only in this paper. In fact we are able to handle more generally, representations of $\mathrm{GL}(n, D)$.

The proof is via global means: embedding the local representation in an automorphic representation, using *multiplicity one* of automorphic representation, to reduce the problem to a representation of a simpler kind which one can handle.

A key global input is provided by the appendix to this paper by Jiang and Soudry which constructs global selfdual representations with specified local selfdual components of a fixed parity.

Theorem. *Let $v_1, ..., v_r$ be r finite places of the number field k. Let $\tau_1, ..., \tau_r$ be r irreducible, self-dual, supercuspidal representations of* $\mathrm{GL}_m(k_{v_1}), ..., \mathrm{GL}_m(k_{v_r})$, *respectively. Assume that $L(\tau_i, \rho, s)$ has a pole at $s = 0$, for all $i \leq r$, where $\rho = \Lambda^2$, sym^2. Then we can globalize $\tau_1 \otimes \cdots \otimes \tau_r$ to an irreducible, self-dual, automorphic, cuspidal representation T of* $\mathrm{GL}_m(\mathbb{A})$, *such that $L(T, \rho, s)$ has a pole at $s = 1$.*

(44) (with W. Gan and B. Gross) Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of Classical groups; *Asterisque*, vol 346, pages 1-109 (2012).

The writing on this paper was started in 2007 when I was visiting Wee Teck in San Diago.

It was already known to Gross and to me in the 90's that there are branching laws from $\mathrm{U}(n+1)$ to $\mathrm{U}(n)$ very much analogous to one from $\mathrm{SO}(n+1)$ to $\mathrm{SO}(n)$, including the fact that there is a symplectic representation of the L-group of $\mathrm{U}(n+1) \times \mathrm{U}(n)$, and our initial proposal was to simply write this out carefully since there were several subtleties involving the fact that the epsilon factors in this case (for conjugate-symplectic representations) depended on a choice of an additive character, wheras in the case of $\mathrm{SO}(n+1)$ to $\mathrm{SO}(n)$ there was no such dependence.

Along the way, we found that the Bessel models for $\mathrm{SO}(n+1)$ were particular cases of branching from $\mathrm{SO}(n+1)$ to $\mathrm{SO}(n)$, and then the Fourier-Jacobi models were brought too, making this into a much wider project than initially anticipated.

Wee Teck's role was essential on all Metaplectic matters!

(45) (with Wee Teck Gan and Benedict H. Gross) Restriction of representations of classical groups: Examples; *Asterisque*, vol 346, pages 111-170 (2012).

In the previous paper, we considered several restriction problems in the representation theory of classical groups over local and global fields. Assuming the Langlands-Vogan parameterization of irreducible representations, we formulated precise conjectures for the solutions of these restriction problems. In the local case, our conjectural answer is given in terms of Langlands parameters and certain natural symplectic root numbers associated to them. In the global case, the conjectural answer is expressed in terms of the central critical value or derivative of a global L-function. In this paper, using methods of base change and the theta correspondence, we test our conjectures for depth zero supercuspidal representations of unitary groups, and for more general representations of groups of low rank.

(46) (with Jeff Adler) Extensions of representations of p-adic groups, *Nagoya J. of Math.* Special volume dedicated to the memory of Prof. Hiroshi Saito, vol. 208, pp. 171-199 (2012).

The classification of irreducible admissible representations of groups over local fields has been a very active and successful branch of mathematics. One next step in the subject would be to understand all possible extensions between irreducible representations. Many results of a general kind are known about extensions between admissible representations of p-adic groups, most notably the notion of the Bernstein center and many other results of Bernstein and Casselman. These results reduce the question to one between components of one parabolically induced representation.

Specific calculations seem not to have attracted attention except for the groups $\mathrm{Ext}^i_G(\mathbb{C},\mathbb{C})$, which is the cohomology $H^i(G,\mathbb{C})$ of G in terms of measurable cochains; besides these, extensions of generalized Steinberg representations are studied.

The key question responsible for this paper was: given a reducible unitary principal series representation of $\mathrm{SL}_2(k)$ with irreducible components π_1 and π_2 what are the Ext groups $\mathrm{Ext}^1(\pi_1,\pi_2)$, $\mathrm{Ext}^1(\pi_1,\pi_1)$, $\mathrm{Ext}^1(\pi_2,\pi_2)$? At the time the paper was concieved, I believe the answer to this question (for SL_2) was not (well-) known.

In this paper, we calculate $\mathrm{Ext}^i_G(\pi_1,\pi_2)$, abbreviated to $\mathrm{Ext}^i(\pi_1,\pi_2)$, between certain irreducible admissible representations π_1,π_2 of $G=\mathsf{G}(k)$ where G is a connected reductive algebraic group over a non-archimedean local field k of characteristic 0.

(47) (with Dinakar Ramakrishnan) On the cuspidality criterion for the Asai transfer to GL(4); an appendix to a paper by M. Krishnamurthy, *Journal of Number Theory*, Volume 132, Issue 6, Pages 1359-1384 (2012).

(48) (with U.K. Anandavardhanan) A local global question in Automorphic forms; *Compositio Math*, vol. 149, 959-995 (2013).

In this paper, we consider the SL(2) analogue of two well-known theorems about period integrals of automorphic forms on GL(2): one due to Harder-Langlands-Rapoport, and the other due to Waldspurger.

The paper was several years in the making, and although "SL(2) analogue of well-known theorems on GL(2)" might give the impression that it was some small book-keeping that was involved, it certainly was a non-trivial work, but quite unsatisfactory too.

We were inspired to consider this work by a question of Vinayak Vatsal about the SL(2) analogue of Waldspurger's theorem, in which he also suggested that since the L-function that appears in Waldspurger's theorem does not make sense for SL(2), there should be no L-function condition for the non-vanishing of toric period integrals for SL(2)! We have shown here that although this would be a consequence of a 'standard conjecture' in analytic number theory, we have not managed to prove an unconditional theorem except in the case of split torus.

This paper makes a contribution to the following question:

Question Let $\Pi = \otimes_v \Pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ such that each of the representations Π_v of $G(F_v)$ is distinguished by $H(F_v)$. Is there an automorphic representation, say Π', in the global L-packet of $G(\mathbb{A}_F)$ determined by Π which is globally distinguished by $H(\mathbb{A}_F)$?

If we take it that the answer to the local and global parts in Question above are in terms of the Langlands parameters associated with $\Pi = \otimes \Pi_v$ to factor through ${}^LG_H \longrightarrow {}^LG$, we are led to questions about local versus global factoring of parameters through this mapping of L-groups.

But there seems no general context, say for representations of an abstract group W, with subgroups W_v which generate W, where one wants to force a representation of W with values in ${}^LG$ to be conjugated to lie inside ${}^LG_H$, under the map ${}^LG_H \longrightarrow {}^LG$, given that the representation of W restricted to W_v can be conjugated to lie inside ${}^LG_H$; this is exactly what we will achieve in the SL(2) analogue of the case dealt with by Harder-Langlands-Rapoport, although, as there is no template for this work (of forcing representations to lie inside a subgroup through local conditions), we have to contend ourselves with a sample theorem in which we restrict either the global representation to be non-CM, or the local representation to be discrete series of a certain kind. In fact, this paper emphasizes the role that a discrete series local component of an automorphic representation might make to a global result: a local condition with a global effect, and we also know that the global result fails without some local conditions.

A key ingredient in our analysis is the multiplicity one theorem for automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ due to Dinakar Ramakrishnan, as well as the exact determination of the fibers of the Asai lift from automorphic representations on $\mathrm{GL}_2(\mathbb{A}_E)$ to automorphic representations on $\mathrm{GL}_4(\mathbb{A}_F)$, completing an earlier work of Krishnamurthy.

The analysis in the paper suggested us the following general question.

Question: Suppose E is a number field, and $\Pi = \otimes \Pi_v$ is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_E)$, and $\Pi' = \otimes \Pi'_v$ is an automorphic representation of $\mathrm{GL}_m(\mathbb{A}_E)$.

Suppose that $\Pi \boxtimes \Pi'$ is automorphic. Then is there an automorphic representation Π'' of $\mathrm{GL}_n(\mathbb{A}_E)$ with $\Pi'' \boxtimes \Pi' = \Pi \boxtimes \Pi'$? What are the various automorphic representations Π'' of $\mathrm{GL}_n(\mathbb{A}_E)$ with this property? (This part of the question generalizes the notion of self-twists of automorphic representations.)

(49) (with Shrawan Kumar) Dimension of zero weight space: an algebro-geometric approach. *J. Algebra* 403 (2014).

The study of zero weight spaces in irreducible representations of compact or Complex Lie groups has been a topic of considerable interest; there are many works which study the zero weight space as a representation space for the Weyl group. In this paper, using algebro-geometric methods, we study the variation on the dimension of the zero weight space as the irreducible representation varies over the set of dominant integral weights for a maximal torus T.

The theorem proved here asserts that the zero weight spaces have dimensions which are piecewise polynomial functions on the polyhedral cone of dominant integral weights which are lattice points in an Euclidean space.

It appears that inspite of considerable importance of the problem on the dimension of the zero weight space, explicit examples are not easy to come by, and so not easily available. Calculating these for SL_3, and Sp_4 was easy enough using known branching laws to smaller groups, which for SL_3 was to SL_2, and for Sp_4 which is a covering of SO_5, to was to the subgroup SO_4; however, even for the rank 2 group G_2, in the absence of a good subgroup where branching laws are known, the computations seem (very) hard, and are due to Michele Vergne who kindly supplied them. The computations on the zero weight space for GL_4 although seems in principle simple enough since branching laws to from GL_4 to GL_3 are easy enough, to actually find the final formula, it was quite a task, which I would not have succeeded had I not been helped by Vinay Wagh.

(50) Half the sum of positive roots, the Coxeter element, and a theorem of Kostant; to appear in *Forum Math.*

The theorem of Kostant in the title of this paper deals with the character of a finite dimensional irreducible representation of a semi-simple algebraic group G over $\mathbb{C}$ at a very special element, the Coxeter element, and asserts that these character values can be only $-1, 0, 1$.

Recall that one usually defines a Coxeter element — or, rather a conjugacy class — in a Weyl group (as a product of simple reflections), in this case in $N(T)/T$, where T is a maximal torus in G, with $N(T)$ its normalizer in G. The first observation is that if we lift the Coxeter conjugacy class in $N(T)/T$ to $N(T)$ arbitrarily, we get a well-defined conjugacy class in G; we will denote this conjugacy class in G by $c(G)$, and call it the Coxeter conjugacy class in G. The conjugacy class $c(G)$ in G has the distinguishing property that it is the unique regular semi-simple conjugacy class in G (i.e., the connected component of its centralizer in G is a torus) such that its image in G/Z is of the smallest order in the adjoint group G/Z.

Interchanging the character and co-character groups of a torus T over a field k introduces a contravariant functor $T \to T^\vee$. Interpreting $\rho : T(\mathbb{C}) \to \mathbb{C}^\times$, half the sum of positive roots for T, a maximal torus in a simply connected semi-simple group G (over $\mathbb{C}$) using this duality, we get a co-character $\rho^\vee : \mathbb{C}^\times \to T^\vee(\mathbb{C})$ for which $\rho^\vee(e^{\frac{2\pi i}{h}})$ (h the Coxeter number) is the Coxeter conjugacy class of the dual group $G^\vee(\mathbb{C})$. This point of view gives a rather transparent proof of this theorem of Kostant on the character values of irreducible finite dimensional representations of $G(\mathbb{C})$ at the Coxeter conjugacy class: the proof amounting to the fact that in $G^\vee_{\mathrm{sc}}(\mathbb{C})$, the simply connected cover of $G^\vee(\mathbb{C})$, there is a unique regular conjugacy class whose image in $G^\vee(\mathbb{C})$ has order h (which is the Coxeter conjugacy class).

(51) A character relationship on GL_n; to appear in *Israel Journal.*

Studying representation theory of disconnected groups in the context of real and p-adic groups has been an important topic of study, finding impressive applications such as to the theory of basechange, and which has been a key instrument in all recent proofs of reciprocity theorems in number theory, and eventually to the Fermat's last theorem!

Representation theory of disconnected algebraic groups can also be studied in a similar vein, see for instance my paper with S. Kumar and G. Lusztig.

In this paper, we prove a simple result on the restriction of representations of connected reductive algebraic group (actually we are able to handle only $\mathrm{GL}_n(\mathbb{C})$ here) to suitable disconnected algebraic subgroups whose connected component of identity has 'large enough' normalizer. Unfortuntely, I have not managed to find a general class of examples where the kind of character relationship we prove here for $\mathrm{GL}_n(\mathbb{C})$ holds.

More, precisely, we consider the character of an irreducible finite dimensional algebraic representation of $\mathrm{GL}_{mn}(\mathbb{C})$ restricted to a particular disconnected component of the normalizer of the Levi subgroup $\mathrm{GL}_m(\mathbb{C})^n$ of $\mathrm{GL}_{mn}(\mathbb{C})$, generalizing a theorem of Kostant on the character values at the Coxeter element.

(52) (with Shiv Prakash Patel) Multiplicity formula for restriction of representations of $\widetilde{\mathrm{GL}_2}(E)$ to $\widetilde{\mathrm{SL}_2}(E)$; submitted.

In this note we prove a certain multiplicity formula regarding the restriction of an irreducible admissible genuine representation of a 2-fold cover $\widetilde{\mathrm{GL}}_2(E)$ of $\mathrm{GL}_2(E)$ to the 2-fold cover $\widetilde{\mathrm{SL}}_2(E)$ of $\mathrm{SL}_2(E)$, and find in particular that this multiplicity may not be one, a result that seems to have been noticed before. The proofs follow the standard path via Waldspurger's analysis of theta correspondence between $\widetilde{\mathrm{SL}}_2(E)$ and $\mathrm{PGL}_2(E)$.

(53) A refined notion of arithmetically equivalent number fields, and curves with isomorphic Jacobians; submitted.

We construct examples of number fields which are not isomorphic but for which their idele class groups are isomorphic. We also construct examples of projective algebraic curves which are not isomorphic but for which their Jacobian varieties are isomorphic. Both are constructed using an example in group theory provided by Leonard Scott of a finite group G and subgroups H1 and H2 which are not conjugate in G but for which the G-module [G/H1] is isomorphic to [G/H2].

(54) A 'relative' local Langlands conjecture.
(55) (with B. Gross and W.T. Gan) Branching laws: The non-tempered case.
(56) Ext-analogues of branching laws; preprint.

We consider the Ext-analogues of branching laws of representations of a group to its subgroups in the context of p-adic groups. Branching laws can be considered either for sub-representations, or for quotient representations, although in practice, and also in the theory of period integrals, it is just one possibility that of quotients that presents itself. The Ext-analogues make sense for both the options, and the two possibilities seem to get related in the higher Ext-groups through a duality analogous to Serre duality for coherent sheaves on Schemes. These considerations have also inspired us to make a general duality conjecture for any reductive p-adic group.