SOME QUESTIONS ABOUT REPRESENTATIONS OF ALGEBRAIC GROUPS

DIPENDRA PRASAD

Disclaimer : *Here are a few questions I compiled at the suggestion of Shripad Garge. It is possible that some of these are still open questions, but I take no responsibility to the possibility that these might be well-known. For most part, these should be do-able questions, even exercises, although I myself have not done that.*

(1) In analogy with complex representations of $\mathrm{O}_n(\mathbb{F}_q)$, is it true that all its irreducible modular representations, say in ch p, orthogonal?

(2) Is the criterion for an irreducible, self-dual, modular algebraic representation of a semi-simple algebraic group in characteristic p being orthogonal or symplectic the same as in characteristic zero where there is an element in the center of G of order 1 or 2 whose action dictates that the representation is orthogonal or symplectic. What about characteristic 2?

(3) Given a representation V of a finite group G, it is easy to get a formula in characteristic zero for the dimension of the space of G-invariant vectors in $\mathrm{Sym}^{\mathrm{k}}(V)$. In particular, it is a polynomial in k. What about characteristic p? How about experimenting with some examples, such as for $\mathrm{Sym}^{\mathrm{k}}(V)$ where $G = S_n$, and V the standard $(n-1)$-dimensional representation of S_n? Is there a better formula if we take not only invariants but all subquotients of $\mathrm{Sym}^{\mathrm{k}}(V)$ on which G acts trivially?

(4) Completely determine which representations, $\Lambda^i(V), \mathrm{Sym}^i(V)$, of $\mathrm{GL}(V)$ are irreducible in characteristic $p > 0$, and in each case, determine completely the Jordan-Holder factors.

(5) Schur-Weyl duality in Characteristic p? If an irreducible representation of S_n in characteristic 0 remains irreducible in characteristic $p > 0$, what can one say about the corresponding representation of $GL(V)$? (The $\Lambda^i(V), \mathrm{Sym}^i(V)$, of the previous question of course correspond to the sign and the trivial representation of S_n.)

(6) It is known that $\Lambda^i(V)$ is an irreducible representation of $\mathrm{O}(V)$, and $\mathrm{Sym}^i(V)$ are irreducible representations of $\mathrm{Sp}(V)$ in characteristic 0. Determine analogous result in positive characteristic, paying attention to characteristic 2.

(7) It is a well-known exercise that if a finite group G operates on a finite set X doubly trasitively, i.e, transitively on X and $X \times X - X$, then the representation of G on $\mathbb{C}[X]$ is the trivial representation plus an irreducible representation. (This proves that the $(n-1)$-dimensional representation of S_n is irreducible.) How far is this true in characteristic p?

(8) To every quadratic form is associated a Clifford algebra which is a semi-simple algebra. The clifford algebra carries an anti-involution, and hence the semi-simple algebra has order atmost 2 in the Brauer group. The question is whether a quadratic form (of trivial discriminant) is split if and only if the Clifford algebra (or its even part) is split? Notice that over a number field, a semisimple algebra of order ≤ 2 can be split by a quadratic extension. So this leads one to ask: is every quadratic space (of trivial discriminant) over a number field split by a quadratic extension? (Actually Shripad Garge observed that this follows from Hasse principle, given that it is obviously true for local fields.)

(9) We know that $\mathrm{SO}(V)$ for V even dim'l can be extended to $\mathrm{O}(V)$, i.e., there is an outer automorphism of order 2. It appears to me that $\mathrm{SL}_1(D)$, with D of index 4, does not have an outer automorphism eventhough it is a form of $\mathrm{SO}(6)$. So the question is: how do you decide that a group G has an outer automorphism, knowing that it has one over algebraic closure? (Recall that $\mathrm{SO}(V)$ always has an outer automorphism no matter what is the quadratic space V; so the group G need not be split...)

(10) The group $\mathrm{GL}(V)$ has well-known irreducible representations in $Sym^k(V)$,$\Lambda^k(V)$ etc. Let D be a central simple algebra of dimension $\dim(V)^2$, i.e., a form of $\mathrm{GL}(V)$. Some of these representations will be defined for $D^\times$. There is a well-known recipe of Tits: it says for example that if $d[D] = 0$ in the Brauer group, then $Sym^k(V)$,$\Lambda^k(V)$ are defined as representations of $D^\times$ for $d|k$. Is it possible to make the 'Galois descent' explicit, i.e., define these representations of $D^\times$ directly with D without invoking V?

(11) Show that the group $\mathbb{B}^\times$ where $\mathbb{B}$ is a central simple algebra of dimension 16, has a representation into $\mathrm{GL}_3(D)$ for D of dimension 4. (This problem corresponds to the isomorphism $\mathrm{PGL}_4 = \mathrm{SO}_6/\pm 1$.)

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