Characters of simplylaced nonconnected groups versus characters of nonsimplylaced connected groups

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Dedicated to Robert Langlands on his 70th birthday

Abstract

Let G be a connected, simply-connected, almost simple algebraic group of simplylaced type over $\mathbf{C}$ and let σ be a diagram automorphism of G. Let $G_\sigma = {}^L({}^\sigma({}^LG))$ be the L-dual of the fixed point subgroup of σ on the L-dual of G. In this paper we relate the character of a finite dimensional irreducible representation of the disconnected algebraic group $G\langle\sigma\rangle$ (generated by G and σ) on the connected component $G\sigma \subset G\langle\sigma\rangle$ to the character of a certain finite dimensional irreducible representation of G_σ.

1 Introduction and statement of the main theorem

Let G be a connected, simply-connected, almost simple algebraic group of simplylaced type over $\mathbf{C}$. We fix a Borel subgroup B and a maximal torus T contained in B. Let σ be a nontrivial diagram automorphism of G (cf. §2). In particular, σ keeps T and B stable and hence acts on the character group X of T. Let $X^+ \subset X$ be the set of dominant weights; then X^+ is stable under the action of σ. Let $G_\sigma = {}^L({}^\sigma({}^LG))$ be the L-dual (=Langlands dual) of the fixed point subgroup of σ on the L-dual of G. (See also §2.) Then G_σ is again a (connected) simply-connected, almost simple algebraic group. (It was introduced in [L2, 6.26] in a different but equivalent way.) We can assume that $T_\sigma := T/\{\sigma(x)x^{-1}; x \in T\}$ is a maximal torus of G_σ. Let $\varpi : T \to T_\sigma$ be the canonical homomorphism.

Generally, if f is a permutation of a set S we write ${}^f S = \{s \in S; f(s) = s\}$. Now the character group of T_σ can be naturally identified with ${}^\sigma X$; under this identification, the set of dominants weights of G_σ becomes ${}^\sigma X^+$.

For $\lambda \in {}^\sigma X^+$, let λ_σ denote the corresponding dominant weight of G_σ.

Let $\langle\sigma\rangle$ be the (finite) subgroup of the automorphism group of G generated by σ and let $G\langle\sigma\rangle$ be the semidirect product of G with $\langle\sigma\rangle$.

Following is our main theorem.

Theorem 1.1 *Let π be an irreducible representation of G with highest weight $\lambda \in {}^\sigma X^+$ and let $\hat{\pi}$ be its unique extension to an irreducible representation of $G\langle\sigma\rangle$ such that σ acts trivially on the highest weight space of π. Let π_σ be the irreducible representation of G_σ with highest weight λ_σ. Then for any $t \in T$ we have*

$$\mathrm{tr}(t\sigma, \hat{\pi}) = \mathrm{tr}(\varpi(t), \pi_\sigma).$$

Note that any semisimple element in the coset $G\sigma$ is G-conjugate to an element $t\sigma$ as above, so the theorem describes the character of $\hat{\pi}$ at any semisimple element of $G\sigma$. Note also that any irreducible finite dimensional representation of $G\langle\sigma\rangle$ is either

(i) obtained from a $\hat{\pi}$ as in the theorem by tensoring with a one dimensional representation, or

(ii) is obtained by inducing an irreducible representation of G whose highest weight is not σ-invariant, in which case its character on $G\sigma$ is identically zero.

We give three different proofs of the theorem. The first proof, given in §4, relies on an extension of the Weyl's character formula for disconnected groups due to Kostant. The second proof, given in §5, relies on the Schur orthogonality relations and what we call 'Basic Lemma' (Lemma 5.1) which is a consequence of the comparison of Weyl integration formula on G_σ with the twisted form of it on G; the strategy of this proof is the same as that of H. Weyl for his character formula where he proves that his character is the character of an irreducible representation (without having to identify it). The third proof, given in §6, relies on the theory of canonical bases. We have included these different proofs since they provide varied insight into our theorem and hopefully they can be used to extend our theorem in different contexts.

Note that the analogue of the theorem with **C** replaced by an algebraically closed field k of characteristic $p > 0$ is false. More precisely, if we take for π the irreducible representation of G (over k) with highest weight $\lambda \in {}^{\sigma}X^{+}$ and for π_σ the irreducible representation of G_σ with highest weight λ_σ, then the character equality in the theorem is false in general.

Let $\mathcal{X}_G(G\sigma)$ be the set of G-conjugacy classes of semisimple elements in $G\sigma$ (a subset of $G\langle\sigma\rangle$). Let $\mathcal{X}_{G_\sigma}(G_\sigma)$ be the set of G_σ-conjugacy classes of semisimple elements in G_σ. We have the following result (see [L2, 6.26]):

(a) *There is a unique bijection* $\Phi : \mathcal{X}_G(G\sigma) \to \mathcal{X}_{G_\sigma}(G_\sigma)$ *such that for any* $t \in {}^{\sigma}T$ *the* G*-conjugacy class of* $t\sigma$ *is carried by* Φ *to the* G_σ*-conjugacy class of* $\varpi(t)$.

A proof is sketched in §3.

Let $\mathcal{F}_G(G\sigma)$ be the set of functions from $\mathcal{X}_G(G\sigma) \to \mathbf{C}$. Similarly, let $\mathcal{F}_{G_\sigma}(G_\sigma)$ be the set of functions from $\mathcal{X}_{G_\sigma}(G_\sigma) \to \mathbf{C}$. Hence, composition with Φ defines an algebra isomorphism $\tilde{\Phi} : \mathcal{F}_{G_\sigma}(G_\sigma) \to \mathcal{F}_G(G\sigma)$.

Theorem 1.1 implies that $\tilde{\Phi}$ carries the (linearly independent) subset of $\mathcal{F}_{G_\sigma}(G_\sigma)$ consisting of characters of the finite dimensional irreducible representations of G_σ bijectively onto the (linearly independent) subset of $\mathcal{F}_G(G_\sigma)$ consisting of characters of the finite dimensional irreducible representations of the form $\hat{\pi}$ (for various irreducible finite dimensional representations π of G with σ-invariant highest weight) restricted to $G\sigma$.

Acknowledgement: This work was inspired by a remark of Katz and Sarnak in [KS] where they observed that the Weyl integration formula for $O(2n+2)$ has 2 components, one for $SO(2n+2)$ and the other which looks exactly the same as that for $Sp(2n)$; this turned out to be the crucial component of one of the proofs (the second proof), generalized in our Basic Lemma 5.1.

Note that the kind of phenomenon contained in this paper is reminiscent of results which occur in the theory of Endoscopy.

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Fund.

2 Review of the group G_σ

Let G be a connected, simply-connected, almost simple algebraic group of simplylaced type over the field $\mathbf{C}$ of complex numbers. Let $\mathfrak{g}$ be the Lie algebra of G. We fix a Borel subgroup B of G and a maximal torus T contained in B. Let X be the group of characters of T; let Y be the group of one parameter suubgroups of T and let $\langle,\rangle : Y \times X \to \mathbf{Z}$ be the standard pairing. Let $R \subset X$ be the set of roots of (G,T), $R_+ \subset R$ the set of positive roots defined by B and $\Pi = \{\alpha_i; i \in I\} \subset R_+$ the set of simple roots. For each $i \in I$ let $\check{\alpha}_i \in Y$ be the simple coroot corresponding to α_i. Note that $(Y, X, \langle,\rangle, \check{\alpha}_i, \alpha_i(i \in I))$ is the root datum of G.

For $\alpha \in R$ let $\mathfrak{g}_\alpha$ be the root subspace of $\mathfrak{g}$ corresponding to α.

For $i \in I$ let $x_i : \mathbf{C} \to G$, $y_i : \mathbf{C} \to G$ be root homomorphisms (corresponding to $\alpha_i, -\alpha_i$) which together with T, B form an "épinglage" of G.

We fix a nontrivial automorphism σ of G such that $\sigma(T) = T$, $\sigma(B) = B$ and such that for some permutation $i \mapsto \tilde{i}$ of I we have $\sigma(x_i(a)) = x_{\tilde{i}}(a)$, $\sigma(y_i(a)) = y_{\tilde{i}}(a)$ for all $a \in \mathbf{C}$. We say that such a σ is a diagram automorphism of G. For $i \in I$ we write $\sigma(i) = \tilde{i}$. Note that σ induces automorphisms of X, Y denoted again by σ, that R, R_+ are stabilized by σ and that $\sigma(\alpha_i) = \alpha_{\sigma(i)}, \sigma(\check{\alpha}_i) = \check{\alpha}_{\sigma(i)}$ for $i \in I$.

We set $Y_\sigma = Y/(\sigma - 1)Y$, ${}^\sigma X = \{\lambda \in X; \sigma(\lambda) = \lambda\}$. Note that $\langle,\rangle : Y \times X \to \mathbf{Z}$ induces a perfect pairing $Y_\sigma \times {}^\sigma X \to \mathbf{Z}$ denoted again by $\langle,\rangle$. Let I_σ be the set of σ-orbits on I. For any $\eta \in I_\sigma$ let $\check{\alpha}_\eta \in Y_\sigma$ be the image of $\check{\alpha}_i$ under $Y \to Y_\sigma$ where i is any element of η. Since $\{\check{\alpha}_i; i \in I\}$ is a $\mathbf{Z}$-basis of Y we see that $\{\check{\alpha}_\eta; \eta \in I_\sigma\}$ is a $\mathbf{Z}$-basis of Y_σ. For any $\eta \in I_\sigma$ let $\alpha_\eta = 2^h \sum_{i \in \eta} \alpha_i \in {}^\sigma X$ where h is the number of unordered pairs (i,j) such that $i,j \in \eta$, $\alpha_i + \alpha_j \in R$. Note that $h = 0$ except when G is of type A_{2n} when $h = 0$ for all η but one and $h = 1$ for one η.

We show that

(a) $(Y_\sigma, {}^\sigma X, \check{\alpha}_\eta, \alpha_\eta(\eta \in I_\sigma))$ *is a root datum.*

In the case where G is not of type A_{2n} this can be seen from the argument in the first paragraph of 6.6. Now assume that G is of type A_{2n}. We may identify $Y = \{(y_0, y_1, \dots, y_{2n}) \in \mathbf{Z}^{2n+1}; y_0 + y_1 + \dots + y_{2n} = 0\}$, $X = \{(x_0, x_1, \dots, x_{2n}) \in \mathbf{Z}^{2n+1}\}/\mathbf{Z}(1, 1, \dots, 1)$; the simple coroots are

$(0,\dots,0,1,-1,0,\dots,0)$ with $1,-1$ on position $j,j+1$ $(0 \le j \le 2n-1)$; the simple roots are the images of the same vectors in X; the pairing $\langle,\rangle$ is induced by $(y_j),(x_j) \mapsto \sum_j y_j x_j$. The action of σ on X and Y is induced by $(x_j) \mapsto (x'_j)$ where $x'_j = -x_{2n-j}$. We may identify $Y_\sigma = \{(y'_0, y'_1, \dots, y'_{n-1}) \in \mathbf{Z}^n\}$ so that the canonical map $Y \to Y_\sigma$ becomes $(y_j) \mapsto (y'_j)$ with $y'_j = y_j - y_{2n-j}$ for $0 \le j \le n-1$. We may identify ${}^\sigma X = \{(x'_0, x'_1, \dots, x'_{n-1}) \in \mathbf{Z}^n\}$ so that the canonical inclusion ${}^\sigma X \hookrightarrow X$ becomes $(x'_j) \mapsto (x_j)$ with $x_j = x'_j$ if $j \le n-1$, $x_j = -x'_{2n-j}$ if $j \ge n+1$, $x_n = 0$. The pairing $\langle,\rangle : Y_\sigma \times {}^\sigma X \to \mathbf{Z}$ becomes $(y'_j),(x'_j) \mapsto \sum_j y'_j x'_j$. The elements $\check\alpha_\eta \in Y_\sigma$ become the elements $(1,-1,0,\dots,0),\dots,(0,\dots,0,1,-1),(0,\dots,0,0,1)$. The elements $\alpha_\eta \in {}^\sigma X$ become the elements $(1,-1,0,\dots,0),\dots,(0,\dots,0,1,-1),(0,\dots,0,0,2)$. Now (a) is clear: we have a root datum of type C_n.

Definition 2.1 *Let G_σ denote the connected semisimple group over $\mathbf{C}$ with root datum (a).*

The definition of G_σ was given in [L2, 6.26]. It may be alternatively described in terms of Langlands duals as in §1. Note that G_σ is simply-connected.

Now $T_\sigma = T/\{\sigma(t)t^{-1}; t \in T\} = \mathbf{C}^* \otimes Y_\sigma$ can be regarded as a maximal torus of G_σ and ${}^\sigma X$ is the group of characters $T_\sigma \to \mathbf{C}^*$ while Y_σ can be regarded as the group of one parameter subgroups of T_σ.

Let W be the Weyl group of G with respect to T. For $i \in I$ let $s_i \in W$ be the simple relection in W corresponding to α_i. Let $l : W \to \mathbf{N}$ be the standard length function.

Clearly, $\sigma : G \to G$ induces an automorphism of W denoted again by σ. Note that $\sigma(s_i) = s_{\sigma(i)}$ for any $i \in I$.

Let ${}^\sigma W = \{w \in W; \sigma(w) = w\}$. For any $\eta \in I_\sigma$ we define $s_\eta \in {}^\sigma W$ to be the longest element in the subgroup of W generated by $\{s_i; i \in \eta\}$. It is well known that ${}^\sigma W$ is a Coxeter group on the generators $\{s_\eta; \eta \in I_\sigma\}$. Let $l_\sigma : {}^\sigma W \to \mathbf{N}$ be the standard length function of the Coxeter group ${}^\sigma W$. It is well known that, if w, w' are elements of ${}^\sigma W$ such that $l_\sigma(ww') = l_\sigma(w) + l_\sigma(w')$ then we have $l(ww') = l(w) + l(w')$.

Note that if $w \in {}^\sigma W$ then the automorphism $w : Y \to Y$ descends to an automorphism of Y_σ denoted again by w. We show:

(b) *the automorphism $w : Y_\sigma \to Y_\sigma$ belongs to the Weyl group of G_σ with respect to T_σ.*

To see this we may assume that $w = s_\eta$ for some $\eta \in I_\sigma$. If η consists of a single element then $w : Y \to Y$ and the reflection in Y_σ defined by

w are given by the same formula and (b) is clear. Assume now that $\eta = \{i,j\}$ where s_i, s_j are distinct and commute. Then $w : Y \to Y$ is given by $y \mapsto y - \langle y, \alpha_i\rangle\check{\alpha}_i - \langle y, \alpha_j\rangle\check{\alpha}_j$ and the reflection in Y_σ defined by w is $y \mapsto y - \langle y, \alpha_i + \alpha_j\rangle\check{\alpha}_i$. It is enough to show that for $y \in Y$ we have

$$\langle y, \alpha_i\rangle\check{\alpha}_i + \langle y, \alpha_j\rangle\check{\alpha}_j = \langle y, \alpha_i + \alpha_j\rangle\check{\alpha}_i \mod (\sigma - 1)Y$$

or that $\langle y, \alpha_j\rangle(\check{\alpha}_j - \check{\alpha}_i) \in (\sigma - 1)Y$. This is clear since $\check{\alpha}_j = \sigma(\check{\alpha}_i)$. The case where $\eta = \{i, j, k\}$ where s_i, s_j, s_k are distinct and commute is entirely similar.

Finally assume that $\eta = \{i, j\}$ where s_i, s_j are distinct and $s_i s_j s_i = s_j s_i s_j$. Then $w : Y \to Y$ is given by

$$y \mapsto y - \langle y, \alpha_i\rangle\check{\alpha}_i - \langle y, \alpha_j\rangle\check{\alpha}_j - \langle y, \alpha_i\rangle\check{\alpha}_j - \langle y, \alpha_j\rangle\check{\alpha}_i$$

and the reflection in Y_σ defined by w is $y \mapsto y - \langle y, 2\alpha_i + 2\alpha_j\rangle\check{\alpha}_i$. It is enough to show that

$$\langle y, \alpha_i\rangle\check{\alpha}_i + \langle y, \alpha_j\rangle\check{\alpha}_j + \langle y, \alpha_i\rangle\check{\alpha}_j + \langle y, \alpha_j\rangle\check{\alpha}_i = \langle y, 2\alpha_i + 2\alpha_j\rangle\check{\alpha}_i \mod (\sigma - 1)Y$$

or that $\langle y, \alpha_i + \alpha_j\rangle(\check{\alpha}_j - \check{\alpha}_i) \in (\sigma - 1)Y$. This is clear since $\check{\alpha}_j = \sigma(\check{\alpha}_i)$. This proves (b).

We see that

(c) *the homomorphism* ${}^\sigma W \to \mathrm{Aut}(Y_\sigma)$ *described above identifies* ${}^\sigma W$ *with the Weyl group of* G_σ.

Let R_σ be the set of roots of G_σ with respect to T_σ. From the results above one can easily obtain an explicit description of R_σ in terms of the action of σ on R.

(d) *If* G *is not of type* A_{2n} *then* R_σ *is in bijection with the set of* σ*-orbits on* R*: this bijection attaches to a* σ*-orbit* ω *on* R *the element* $\sum_{\alpha\in\omega} \alpha \in R_\sigma$. *If* G *is of type* A_{2n} *then* R_σ *is in bijection with the set of* σ*-orbits of size* 2 *on* R*: this bijection attaches to a* σ*-orbit* ω *on* R *of size* 2 *the element* $\sum_{\alpha\in\omega} \alpha \in R_\sigma$ *(if* $\sum_{\alpha\in\omega} \alpha \notin R$*) and* $2\sum_{\alpha\in\omega} \alpha \in R_\sigma$ *(if* $\sum_{\alpha\in\omega} \alpha \in R$*).*

We now describe the type of G_σ for the various possible σ. Let d be the order of σ.

If $d = 2$ and G is of type A_{2n-1} ($n \geq 2$) then G_σ is of type B_n.
If $d = 2$ and G is of type A_{2n} ($n \geq 1$) then G_σ is of type C_n.

If $d = 2$ and G is of type D_n $(n \geq 4)$ then G_σ is of type C_{n-1}.
If $d = 2$ and G is of type E_6 then G_σ is of type F_4.
If $d = 3$ and G is of type D_4 then G_σ is of type G_2.
(See [L2, 6.4].)

3 Semisimple conjugacy classes in $G\sigma$ and in G_σ

We preserve the setup of §1, §2. In this section we sketch a proof of statement §1(a).

Note that ${}^\sigma G$ is a connected semisimple group with maximal torus ${}^\sigma T$. Its Weyl group can be identified with ${}^\sigma W$.

We set $\mathcal{N} = \{g \in G; g({}^\sigma T\sigma)g^{-1} = {}^\sigma T\sigma\}$. Now $\mathcal{N}$ acts by conjugation on ${}^\sigma T\sigma$. The following statement can be found in [Se]. (Actually that reference deals with compact groups instead of reductive groups but this makes no difference for the proof.)

(a) *The inclusion* ${}^\sigma T\sigma \subset G\sigma$ *induces a bijection* $\{\mathcal{N} - \textit{orbits on } {}^\sigma T\sigma\} \to \mathcal{X}_G(G\sigma)$.

We set $N = \{g' \in G_\sigma; g'T_\sigma g'^{-1} = T_\sigma\}$. Now N acts by conjugation on T_σ. The following result is well known.

(b) *The inclusion* $T_\sigma \subset G_\sigma$ *induces a bijection* $\{N - \textit{orbits on } T_\sigma\} \to \mathcal{X}_{G_\sigma}(G_\sigma)$.

Let $\mathcal{N}' = \{\tau \in T; \tau\sigma(\tau)^{-1} \in {}^\sigma T\} = \mathcal{N} \cap T$, $\mathcal{N}'' = \{g \in {}^\sigma G; g({}^\sigma T)g^{-1} = {}^\sigma T\} \subset \mathcal{N}$. By [L2, 6.8]) we have $\mathcal{N} = \mathcal{N}'\mathcal{N}''$. Hence if $t\sigma, t'\sigma$ in ${}^\sigma T\sigma$ are in the same $\mathcal{N}$-orbit then we have $t'\sigma = \tau w(t)\sigma\tau^{-1} = \tau w(t)\sigma(\tau)^{-1}\sigma$ for some $w \in {}^\sigma W$ and some $\tau \in \mathcal{N}'$. Hence $\varpi(t') = \varpi(\tau w(t)\sigma(\tau)^{-1}) = \varpi(w(t))\varpi(\tau\sigma(\tau)^{-1}) = \varpi(w(t))$. But ϖ is compatible with the ${}^\sigma W$ actions on T, T_σ (see §2(b),(c)) hence $\varpi(t') = w(\varpi(t))$. Thus $\varpi(t), \varpi(t')$ are in the same N-orbit (or ${}^\sigma W$-orbit). We see that the map ${}^\sigma T\sigma \to T_\sigma$ given by $t\sigma \mapsto \varpi(t)$ induces a map $\psi : \{\mathcal{N} - \text{orbits on } {}^\sigma T\sigma\} \to \{N - \text{orbits on } T_\sigma\}$.

In view of (a),(b), it is enough to show that ψ is a bijection. Clearly the restriction of ϖ is a surjective map ${}^\sigma T \to T_\sigma$ (in fact an isogeny). It follows that ψ is surjective. Thus, it is enough to show that, if $t\sigma, t'\sigma$ in ${}^\sigma T\sigma$ are such that $\varpi(t) = w(\varpi(t'))$ for some $w \in {}^\sigma W$ then $t\sigma, t'\sigma$ are in the same $\mathcal{N}$-orbit. Replacing t' by $w^{-1}(t')$ we may assume that $\varpi(t) = \varpi(t')$. Then

$t' = t\tau\sigma(\tau)^{-1}$ for some $\tau \in T$. Since $t^{-1}t' \in {}^{\sigma}T$ we have $\tau \in \mathcal{N}'$. Hence $t'\sigma = \tau t\sigma\tau^{-1}$ and $t\sigma, t'\sigma$ are in the same $\mathcal{N}$-orbit. This proves statement §1(a).

We now show:

(c) Φ *carries the* G*-conjugacy class of* $t\sigma$ *(for any* $t \in T$ *not necessarily in* ${}^{\sigma}T$*) to the* G_{σ}*-conjugacy class of* $\varpi(t)$*.*

Note that the homomorphism ${}^{\sigma}T \times T \to T$ given by $(t_1, t_2) \mapsto t_2 t_1 \sigma(t_2)^{-1}$ is surjective. (For a proof see for example [L3, 1.2].) In particular, we have $t = t_2 t_1 \sigma(t_2)^{-1}$ for some $t_1 \in {}^{\sigma}T$ and $t_2 \in T$. Then $t\sigma = t_2 t_1 \sigma t_2^{-1}$. By definition, Φ carries the G-conjugacy class of $t\sigma$ to the G_{σ}-conjugacy class of $\varpi(t_1)$. It is enough to show that $\varpi(t_1) = \varpi(t)$ or that $\varpi(t_2\sigma(t_2)^{-1}) = 1$. This is clear from the definition of ϖ.

4 First proof of the main theorem: using character formula for disconnected groups

We follow the same notation and assumptions as in §§1,2. For any $w \in W$, let $\Phi_w := wR_- \cap R_+$, where $R_- = -R_+$. Let $\mathfrak{n}_w^-$ be the Lie subalgebra of the Lie algebra $\mathfrak{g}$ of G defined by

$$\mathfrak{n}_w^- := \oplus_{\alpha \in \Phi_w} \mathfrak{g}_{-\alpha}.$$

Since for any $w \in {}^{\sigma}W$, σ keeps the set Φ_w stable and hence σ keeps $\mathfrak{n}_w^-$ stable under the adjoint action of $G\langle\sigma\rangle$ on $\mathfrak{g}$. Thus, the element σ acts via a scalar (denoted by d_w) on the top exterior power $\wedge^{\text{top}}(\mathfrak{n}_w^-)$ of $\mathfrak{n}_w^-$.

We shall denote by $\lambda \mapsto e^{\lambda}$ the identity isomorphism from X (with additive notation) to X (with multiplicative notation).

With this notation, we first recall the following extension of the Weyl's character formula for disconnected groups proved by Kostant [Ko, Theorem 7.5]. (Actually, his result is more general but we restrict to the situation we need.)

Theorem 4.1 *For any* $t \in T$*, and any irreducible representation* π *of* G *with highest weight* $\lambda \in {}^{\sigma}X^+$*, we have*

$$\operatorname{tr}(t\sigma, \hat{\pi}) \cdot \Big(\sum_{w \in {}^{\sigma}W} (-1)^{l(w)} d_w e^{w\rho - \rho}(t) \Big) = \sum_{w \in {}^{\sigma}W} (-1)^{l(w)} d_w e^{w(\lambda+\rho)-\rho}(t),$$

where ρ *is half the sum of the positive roots of* G*.*

As a preparation towards the proof of Theorem 1.1, we prove the following.

Lemma 4.2 *For any* $u \in {}^\sigma W$ *we have* $(-1)^{l_\sigma(u)} = d_u(-1)^{l(u)}$.

Proof: Recall from §2 that for any $u, v \in {}^\sigma W$ such that $l_\sigma(uv) = l_\sigma(u) + l_\sigma(v)$ we have $l(uv) = l(u) + l(v)$. Further, since $\Phi_{uv} = \Phi_u \sqcup \mathrm{Ad}(\dot{u})(\Phi_v)$ where $\dot{u}$ is a representative of u in ${}^\sigma G$, we get that

$$d_{uv} = d_u d_v.$$

Thus, it suffices to prove the lemma for the simple reflection $w = s_\eta$ ($\eta \in I_\sigma$) of ${}^\sigma W$. We must be in one of the following four cases.

Case (a). $|\eta| = 1$. In this case $l(w) = 1$ and, moreover, $d_w = 1$ (since σ acts trivially on the simple root spaces $\mathfrak{g}_{\pm\alpha_i}$ such that $\sigma(i) = i$).

Case (b). $|\eta| = 2$ *and* $\eta = \{i, j\}$ *with* $s_i s_j = s_j s_i$. In this case $l(w) = 2$ and moreover, $d_w = -1$. To prove the latter, take a nonzero root vector $x \in \mathfrak{g}_{-\alpha_i}$ and consider the element $a = x \wedge \sigma(x) \in \wedge^2(\mathfrak{n}_w^-)$. Clearly, σ acts by -1 on a.

Case (c). $|\eta| = 2$ *and* $\eta = \{i, j\}$ *with* $s_i s_j s_i = s_j s_i s_j$. In this case G is necessarily of type A_{2n}. Further, $l(w) = 3$ and, moreover, $d_w = 1$. To prove the latter, take a nonzero root vector $x \in \mathfrak{g}_{-\alpha_i}$ and consider the element $a = x \wedge \sigma(x) \wedge [x, \sigma(x)] \in \wedge^3(\mathfrak{n}_w^-)$. Clearly, σ acts by 1 on a.

Case (d). $|\eta| = 3$. In this case $l(w) = 3$ and σ is necessarily of order 3. Further, since σ is a diagram automorphism, it keeps stable the standard integral form of $\mathfrak{g}$. In particular, the action of σ on $\wedge^3(\mathfrak{n}_w^-)$ is trivial (being a third root of unity which is, in addition, an integer). Thus, $d_w = 1$.

The lemma is proved. □

With these preparations, we are now ready to give the first proof of our main theorem.

Proof of Theorem 1.1. Since σ permutes the positive roots, the character e^ρ of T is σ-invariant and hence it descends to a character of T_σ. Let ρ_σ denote half the sum of positive roots of G_σ. Then, we claim that the character e^ρ of T_σ coincides with e^{ρ_σ}. Since ρ is the unique element of X such that it takes value 1 on each simple coroot $\check{\alpha}_i$ of G, it suffices to show that ρ takes value 1 on each simple coroot $\check{\alpha}_\eta$ of G_σ ($\eta \in I_\sigma$). But this follows from the definition of $\check{\alpha}_\eta$. Let T_* be the inverse image under $\varpi : T \to T_\sigma$ of the set of elements in T_σ which are regular with respect to G_σ. Note that T_* is an open dense subset of T. Let $t \in T_*$. We set $\delta(\varpi(t)) = \sum_{w \in {}^\sigma W} (-1)^{l_\sigma(w)} e^{w\rho_\sigma - \rho_\sigma}(\varpi(t))$. Using the discussion above and also Lemma 4.2 and §2(b),(c), we see that Kostant's theorem 4.1 implies

$$\mathrm{tr}(t\sigma, \hat{\pi})\delta(\varpi(t)) = \sum_{w \in {}^\sigma W} (-1)^{l_\sigma(w)} e^{w(\lambda_\sigma + \rho_\sigma) - \rho_\sigma}(\varpi(t)).$$

Now the right hand side can be computed from the standard Weyl character formula for the group G_σ applied to the representation π_σ; we obtain $\mathrm{tr}(t\sigma, \hat{\pi})\delta(\varpi(t)) = \mathrm{tr}(\varpi(t), \pi_\sigma)\delta(\varpi(t))$. Since $\varpi(t)$ is regular we have $\delta(\varpi(t)) \neq 0$ hence $\mathrm{tr}(t\sigma, \hat{\pi}) = \mathrm{tr}(\varpi(t), \pi_\sigma)$. Since here t is an arbitrary element in the open dense subset T_* of T, the previous equality holds by continuity for all $t \in T$. This completes the first proof of the theorem. □

5 Second proof of the main theorem: using Weyl's integration formula

We follow the same notation and assumptions as in §1, §2. Let $\Theta_{\hat{\pi}}$ be the character of $\hat{\pi}$. We begin by noting that $\Theta_{\hat{\pi}}$, being $G\langle\sigma\rangle$-invariant under the conjugation action, its restriction to $T \times \sigma$ can be considered to be a function θ_π on T_σ which is invariant under ${}^\sigma W$. It suffices to prove that θ_π is the character of an irreducible representation of G_σ, because then the resulting irreducible representation of G_σ could be nothing else but the representation π_σ from the theory of highest weights as it contains the same highest weight as that of π_σ.

Note that θ_π is a sum of (one dimensional) characters of T_σ with coefficients in $\mathbf{Z}[\zeta_d]$, where d is the order of σ and ζ_d is a primitive d-th root of unity. Therefore,

$$\theta_\pi = \sum_i n_i \chi_i,$$

a finite sum, where χ_i are irreducible characters of G_σ and $n_i \in \mathbf{Z}[\zeta_d]$.

We use the Schur orthogonality relation to prove that this sum in fact consists of a single term. To apply the Schur orthogonality, take a σ-stable maximal compact subgroup $K \subset G$ and compatible $K_\sigma \subset G_\sigma$ and let H (respectively, H_σ) be the maximal torus in K (respectively, K_σ) defined as $H := T \cap K$ (respectively, $H_\sigma := T_\sigma \cap K_\sigma$).

For the class function f' on G_σ whose restriction to T_σ is θ_π,

$$\int_{K_\sigma} |f'(g)|^2 = \sum_i |n_i|^2.$$

By (subsequent) Lemma 5.2, we know that

$$\int_{K\times\sigma} |\Theta_{\hat{\pi}}(g)|^2 = 1,$$

therefore, by the Basic Lemma, Lemma 5.1 below, $\int_{K_\sigma} |f'(g)|^2 = 1$. This implies that $\sum_i |n_i|^2 = 1$. Since $|n_i| \geq 1$ for any nonzero $n_i \in \mathbf{Z}[\zeta_d]$, all but one of the $n_i's$ is 0. This gives that

$$\Theta_{\hat{\pi}}(g\sigma) = \epsilon\chi(g),$$

for some $\epsilon \in \mathbf{Z}[\zeta_d]$ and an irreducible character χ of K_σ. Looking at the highest weight, we find that in fact $\epsilon = 1$, completing the second proof of our main theorem.

Here is the Basic Lemma.

Lemma 5.1 *Let f be a class function on $K\langle\sigma\rangle$ with the associated class function f' on K_σ. Then, for Haar measures on K and K_σ giving these groups volume 1, we have*

$$\int_{K\times\sigma} f(g) = \int_{K_\sigma} f'(g).$$

Proof: The proof consists in comparing the Weyl integration formula on K_σ with its twisted analogue of K. One knows that there is a certain open subset H^o of ${}^\sigma H$ such that the map

$$\begin{aligned} K/{}^\sigma H \times H^o &\rightarrow K \\ (g,t) &\rightarrow gt\sigma(g^{-1}) \end{aligned}$$

is a covering projection onto its image. Thus, for a function f on K which is invariant under σ-conjugation, i.e., $f(x) = f(yx\sigma(y)^{-1})$ for all $x, y \in K$,

$$\int_K f(g) = \int_{{}^\sigma H} f(t)\delta_\sigma(t)dt,$$

where $\delta_\sigma(t)$ is the absolute value of the determinant of $t \in {}^\sigma H$ acting on the tangent space of $K/{}^\sigma H$ at the identity element by $[1 - \sigma \cdot \mathrm{A}d(t)]$. We do not give a detailed proof of this which is exactly as for the Weyl integration formula (which consists in lifting the top degree invariant differential form on K to $K/{}^\sigma H \times H$ under the map $(g,t) \to gt\sigma(g^{-1})$) except to point out that the derivative of the map $g \to gt\sigma(g)^{-1}t^{-1}$ is $1-\sigma \cdot Ad(t)$. We note that the equality of integrals presumes having a certain normalized Haar measure on ${}^\sigma H$.

Let $\mathcal{O}$ be an orbit for the action of σ on the set of roots R, say of cardinality r. Then there exist roots $e_1, \cdots, e_r$ such that $\sigma e_1 = e_2, \sigma e_2 = e_3, \cdots, \sigma e_r = e_1$. Suppose that $t \in T$ operates by $\chi_i(t)$ on e_i. Then $\sigma \cdot \mathrm{A}d(t)e_1 = \chi_1(t)e_2, \sigma \cdot \mathrm{A}d(t)e_2 = \chi_2(t)e_3, \cdots, \sigma \cdot \mathrm{A}d(t)e_r = \chi_r(t)e_1$. An easy calculation yields the determinant of $[1 - \sigma \cdot \mathrm{A}d(t)]$ restricted to the r-dimensional space spanned by $\{e_1, \cdots, e_r\}$ to be $(1 - \chi_1 \cdots \chi_r)$. Note that the character $\prod_i \chi_i$ of T in fact factors through the canonical map $T \to T_\sigma$, and thus $\chi_{\mathcal{O}} = \prod_{i=1}^r \chi_i$ can be considered to be a character of T_σ. We have,

$$\delta_\sigma(t) = \prod(1 - \prod \chi_i) = \prod(1 - \chi_{\mathcal{O}}),$$

where the inner product in the second term is over an orbit $\mathcal{O}$ of σ on the set of roots, and the outer product (in the second term) is over the set of orbits $\mathcal{O}$ in R.

By §2(d), $\chi_{\mathcal{O}}$ is a root of K_σ unless K is of type A_{2n} and the orbit consists either of 1 element, or of 2 elements $\{\alpha, \sigma\alpha\}$ such that $\alpha + \sigma\alpha$ is also a root. Therefore, except in these cases, we have identified a piece of $\delta_\sigma(t)$ to the corresponding factor $(1-\chi_{\mathcal{O}})$ in the Weyl integration formula for K_σ. In the remaining cases, which occurs only for the group A_{2n} with orbits of size 1 or of 2 consisting of $\{\alpha, \sigma\alpha\}$ with $\alpha + \sigma\alpha$ a root, we still need to identify the corresponding factors in the Weyl integration formula. Suppose the orbit has two elements $\{\alpha, \sigma\alpha\}$ with $\alpha+\sigma\alpha$ a root, then the determinant of $[1-\sigma\cdot\mathrm{A}d(t)]$ restricted to the two dimensional piece is $(1 - \alpha \cdot \sigma\alpha)$. In this case $\alpha + \sigma\alpha$ is a root of K, stabilised by σ, such that on the corresponding root space σ operates by -1. As a result, $[1 - \sigma \cdot \mathrm{A}d(t)]$ on this 1 dimensional piece is $(1+\alpha\sigma\alpha)$. Therefore, the determinant of $[1-\sigma\cdot\mathrm{A}d(t)]$ on the 3 dimensional piece consisting of root spaces $\{\alpha, \sigma\alpha, \alpha+\sigma\alpha\}$ is $(1 - \alpha \cdot \sigma\alpha)(1 + \alpha \cdot \sigma\alpha) = (1-\alpha^2 \cdot \sigma\alpha^2)$, but according to §2(d), $2(\alpha+\sigma\alpha)$ is the root of G_σ in this case, and we are done with the identification of $\delta_\sigma(t)$ with the corresponding factor appearing in the Weyl integration formula for K_σ. Furthermore, observe that *every* root of K fixed by σ is uniquely of the form $\alpha + \sigma\alpha$ for a root α.

To complete the argument, we also observe that the absolute value of the determinant of $t \in {}^{\sigma}H$ acting on the tangent space of $H/{}^{\sigma}H$ at the identity element by $[1 - \sigma \cdot \mathrm{A}d(t)]$ gives rise to a certain constant which can be absorbed in the choice of Haar measures which the conscientious reader will easily identify to be the degree of the isogeny ${}^{\sigma}H \to H_{\sigma}$, although this fact plays no role in the proof, and can in fact be deduced as a result of it.

Finally, note that for a function f on K which is invariant under σ-conjugation, f as well as δ_{σ} restricted to ${}^{\sigma}H$ in fact descend to functions on H_{σ}, where ${}^{\sigma}H \to H_{\sigma}$ is an isogeny. Therefore,

$$\int_K f(g) = \int_{{}^{\sigma}H} f(t)\delta_{\sigma}(t)dt = \int_{H_{\sigma}} f(t)\delta_{\sigma}(t)dt = \int_{K\sigma} f',$$

where choice of Haar measures is so made that at each stage there is equality. □

Lemma 5.2 *Let L be a compact (not necessarily connected) group, and M a closed subgroup. Let V be a finite dimensional irreducible representation of L which remains irreducible restricted to M. Let χ denote the character of V. Then, for a Haar measure on M giving it volume 1,*

$$\int_M |\chi(mg)|^2 dm = 1,$$

for any g in L.

Proof: This is a variant of the Schur orthogonality for which the standard proof, in fact, works. We equip V with a positive definite L-invariant Hermitian form $\langle \, , \rangle$. Let $\{e_i\}$ be an orthonormal basis of V. For any vectors e_i, e_j, define a linear map $\phi_{i,j} : V \to V$, $v \mapsto \int_M \langle v, mg \cdot e_i\rangle (mg \cdot e_j) dm$. Now,

$$\begin{aligned}
\operatorname{tr} \phi_{i,j} &= \sum_k \int_M \langle e_k, mg \cdot e_i\rangle \langle mg \cdot e_j, e_k\rangle dm \\
&= \int_M \langle mg \cdot e_j, mg \cdot e_i\rangle dm \\
&= \int_M \delta_{i,j} dm \\
&= \delta_{i,j}.
\end{aligned}$$

As the operator $\phi_{i,j}$ is M-equivariant and V is irreducible as an M-module, it is a multiple of the identity operator I. Thus,

$$\phi_{i,j} = \frac{1}{\dim V}\delta_{i,j} I.$$

Hence,

$$\begin{aligned}
\int_M |\chi(mg)|^2 dm &= \sum_{k,l} \int_M \langle mg \cdot e_k, e_k \rangle \langle e_l, mg \cdot e_l \rangle dm \\
&= \sum_{k,l} \langle \phi_{l,k}(e_l), e_k \rangle \\
&= \frac{1}{\dim V} \sum_k \langle e_k, e_k \rangle \\
&= 1.
\end{aligned}$$

□

6 Third proof of the main theorem: using the canonical basis

6.1

A version of Theorem 1.1 is that the trace of σ on any σ-stable weight space of π is a natural number, equal to the dimension of a weight space of π_σ. The ideal explanation of such a statement would be to have a basis for any σ-stable weight space of π which is permuted by the action of σ and is such that the set of σ-fixed points on this basis indexes a basis for a certain weight space of π_σ. In fact this approach works and is a simple application of the (geometric) theory of canonical bases. This approach has the advantage that the statement above can also be obtained in the context of Kac-Moody Lie algebras with nondegenerate Cartan datum (unlike the first and second proof). On the other hand, this method cannot handle the case of A_{2n} with σ of order 2 due to a hypothesis of admissibility for σ which was made in [L1].

6.2

Let I be a finite set. Let Y be the free abelian group $\mathbf{Z}[I]$ with I as a basis. Let $Y \times Y \to \mathbf{Z}$, $(y_1, y_2) \mapsto y_1 \cdot y_2$ be a symmetric bilinear form such that $(I, \cdot)$ is a Cartan datum in the sense of [L1, 1.1.1]. Let Y^+ be the submonoid of Y consisting of $\mathbf{N}$-linear combinations of elements of I, where $\mathbf{N}$ is the set of nonnegative integers. Let B be the set ("canonical basis") attached to $(I, \cdot)$ in [L1, 14.4.2]. For any $i \in I$ we have a partition $B = \sqcup_{n \in \mathbf{N}} B^{i,n}$, where $B^{i,n}$ is what in [L1, 14.4.6] is denoted by $\sigma(B_{i;n})$. (Note that σ in *loc. cit.* is not the same as σ in §2.) For $\nu \in Y^+$ let B_ν be the subset of B defined as in [L1, 14.4.2]; we have $B = \sqcup_{\nu \in Y^+} B_\nu$. Let $X := \mathrm{Hom}(Y, \mathbf{Z})$ and let $\langle , \rangle : Y \times X \to \mathbf{Z}$ be the bilinear pairing given by evaluation. Let $y \mapsto y'$ be the homomorphism $Y \to X$ such that $\langle i, y' \rangle = 2i \cdot y / i \cdot i$ for any $i \in I, y \in Y$. Let $X^+ := \{\lambda \in X; \langle i, \lambda \rangle \in \mathbf{N} \quad \forall i \in I\}$. For any $\lambda \in X^+$ let $B(\lambda) = \cap_{i \in I} \cap_{\{n; 0 \le n \le \langle i, \lambda \rangle\}} B^{i,n}$. We have $B(\lambda) = \sqcup_{\lambda_1 \in X} B(\lambda)^{\lambda_1}$, where, for $\lambda \in X^+, \lambda_1 \in X$ we set $B(\lambda)^{\lambda_1} = \sqcup_{\{\nu \in Y^+; \lambda = \lambda_1 + \nu'\}} B(\lambda) \cap B_\nu$. Let $(Y, X, <, >, \cdots)$ be the root datum [L1, 2.2.1] in which $\cdots$ refers to the obvious imbedding $I \subset Y$ and the imbedding $i \mapsto i'$ of I into X.

6.3

In this subsection we assume that $(I, \cdot)$ is of finite type, i.e., the symmetric matrix $(i \cdot j)$ indexed by $I \times I$ is positive definite. To the root datum $(Y, X, <, >, \cdots)$ one can associate canonically a connected, simply-connected, semisimple algebraic group G over $\mathbf{C}$ with maximal torus $T = \mathbf{C}^* \otimes Y$ and given imbeddings $x_i : \mathbf{C} \to G$, $y_i : \mathbf{C} \to G$ for $i \in I$ which form an "épinglage" of type $(Y, X, <, >, \cdots)$. In particular, we can identify X with $\mathrm{Hom}(T, \mathbf{C}^*)$ in such a way that $\{i' \in X; i \in I\}$ are the simple roots of G with respect to T. Let $e_i, f_i (i \in I)$ be the elements of the Lie algebra $\mathfrak{g}$ of G given by the derivatives of x_i, y_i. For $\lambda \in X^+$, let $[B(\lambda)]$ be the $\mathbf{C}$-vector space spanned by $B(\lambda)$; in our case this vector space is finite dimensional. For $\lambda_1 \in X$ let $[B(\lambda)^{\lambda_1}]$ be the subspace of $[B(\lambda)]$ spanned by $B(\lambda)^{\lambda_1}$. We have

$$[B(\lambda)] = \oplus_{\lambda_1 \in X} [B(\lambda)^{\lambda_1}].$$

We can regard $[B(\lambda)]$ as a $\mathfrak{g}$-module as follows. Let $[[B(\lambda)]]$ be the $\mathbf{Q}(v)$-vector space (v an indeterminate) with basis $B(\lambda)$. By [L1, 14.4.11] we may identify $[[B(\lambda)]]$ with the underlying vector space of the $\mathbf{U}$-module Λ_λ, where $\mathbf{U}$ is the $\mathbf{Q}(v)$-algebra with generators $E_i, F_i (i \in I), K_y (y \in Y)$ attached to

$(Y, X, \langle,\rangle, \cdots)$ as in [L1, 3.1.1]) and Λ_λ is as in [L1, 3.5.6]. Let $R := \mathbf{Q}[v, v^{-1}]$. We regard $\mathbf{C}$ as an R-algebra via $v \mapsto 1$. Now, the R-submodule $\mathcal{L}_\lambda$ of $[[B(\lambda)]]$ spanned by $[B(\lambda)]$ is stable under E_i, F_i. Hence, E_i, F_i induce operators (denoted by e_i, f_i) on $\mathbf{C} \otimes_R \mathcal{L}_\lambda = [B(\lambda)]$. These operators define the required $\mathfrak{g}$-module structure on $[B(\lambda)]$. This integrates uniquely to a homomorphism of algebraic groups $G \to GL([B(\lambda)])$ which is an irreducible G-module with highest weight λ. Using [L1, 3.4.5] we see that for $\lambda_1 \in X$ and $t \in T$ the action of t on the subspace $[B(\lambda)^{\lambda_1}]$ is scalar multiplication by $\lambda_1(t)$.

6.4

In the remainder of this paper we fix a graph (I, H, ϕ) in the sense of [L1, 9.1.1]. Thus, I is a finite set (vertices), H is a finite set (edges) and ϕ is a map which to each $h \in H$ associates a two-element subset $[h]$ of I consisting of its vertices. We assume that we are given an admissible automorphism σ of this graph, i.e., a permutation $\sigma : I \to I$ and a permutation $\sigma : H \to H$ such that for any $h \in H$ we have $[\sigma(h)] = \sigma[h]$ and $[h]$ is not contained in any σ-orbit in I (see [L1, 12.1.1]). Let I_σ denote the set of σ-orbits in I. To (I, H, ϕ) we associate a Cartan datum $(I, \cdot)$ by $i_1 \cdot i_2 = -|\{h \in H; [h] = \{i_1, i_2\}\}|$ if $i_1 \neq i_2$ in I and $i \cdot i = 2$ if $i \in I$. As in [L1, 14.1.1], to $\{(I, H, \phi), \sigma\}$ we associate a Cartan datum $(I_\sigma, \cdot)$ by $\eta_1 \cdot \eta_2 = -|\{h \in H; [h] \subset \eta_1 \cup \eta_2\}|$ if η_1, η_2 in I_σ are distinct and $\eta \cdot \eta = 2|\eta|$ if $\eta \in I_\sigma$. From the definitions we have

(a) $\sigma(i_1) \cdot \sigma(i_2) = i_1 \cdot i_2$, for any i_1, i_2 in I;

(b) $2\eta_1 \cdot \eta_2 / \eta_1 \cdot \eta_1 = \sum_{i \in \eta_2} i_1 \cdot i$, for any η_1, η_2 in I_σ,
where i_1 is any element of η_1.

Define $Y, Y^+, B, B^{i,n} \quad (i \in I, n \in \mathbf{N}), B_\nu \quad (\nu \in Y^+), X, \langle,\rangle, Y \to X \quad (y \mapsto y'), X^+, B(\lambda) \quad (\lambda \in X^+), B(\lambda)^{\lambda_1} \quad (\lambda \in X^+, \lambda_1 \in X)$ in terms of $(I, \cdot)$ as in §6.2. Let $\mathbf{Y}, \mathbf{Y}^+, \mathbf{B}, \mathbf{B}^{\eta,n} \quad (\eta \in I_\sigma, n \in \mathbf{N}), \mathbf{B}_\mu \quad (\mu \in \mathbf{Y}^+), \mathbf{X}, (,), \mathbf{Y} \to \mathbf{X} \quad (\mathbf{y} \mapsto \mathbf{y}'), \mathbf{X}^+, \mathbf{B}(\boldsymbol{\lambda}) \quad (\boldsymbol{\lambda} \in \mathbf{X}^+), \mathbf{B}(\boldsymbol{\lambda})^{\boldsymbol{\lambda}_1} \quad (\boldsymbol{\lambda} \in \mathbf{X}^+, \boldsymbol{\lambda}_1 \in \mathbf{X})$ be the analogous objects defined in terms of $(I_\sigma, \cdot)$.

Now $\sigma : I \to I$ extends to a group isomorphism $\sigma : Y \to Y$ and this restricts to a monoid isomorphism $\sigma : Y^+ \to Y^+$. From (a) we see that σ is an automorphism of the Cartan datum $(I, \cdot)$. Since B is canonically attached to $(I, \cdot)$, $\sigma : I \to I$ induces a permutation $\sigma : B \to B$. From the definitions, we have $\sigma(B^{i,n}) = B^{\sigma(i),n}$ for $i \in I, n \in \mathbf{N}$ and $\sigma(B_\nu) = B_{\sigma(\nu)}$ for $\nu \in Y^+$. Define a group isomorphism $\sigma : X \to X$ by the requirement that $\langle\sigma(y),\sigma(x)\rangle=\langle y,x\rangle$ for $y\in Y,x\in X$. Then, $Y\to X, y\mapsto y'$, is compatible with $\sigma:X\to X,\sigma:Y\to Y$. Moreover, $\sigma:X\to X$ restricts to a permutation $\sigma:X^+\to X^+$. For $\lambda\in X^+$ and $\lambda_1\in X$, we have $\sigma(B(\lambda))=B(\sigma(\lambda))$ and $\sigma(B(\lambda)^{\lambda_1})=B(\sigma(\lambda))^{\sigma(\lambda_1)}$.

We see that the fixed point sets ${}^\sigma Y, {}^\sigma Y^+, {}^\sigma B, {}^\sigma X, {}^\sigma X^+$ are defined. Also, if $\lambda\in{}^\sigma X^+$ then $\sigma:B\to B$ restricts to a permutation $\sigma:B(\lambda)\to B(\lambda)$ and thus the fixed point set ${}^\sigma B(\lambda)$ is defined; if, in addition, $\lambda_1\in{}^\sigma X$ then $\sigma:B(\lambda)\to B(\lambda)$ restricts to a permutation $\sigma:B(\lambda)^{\lambda_1}\to B(\lambda)^{\lambda_1}$ and thus the fixed point set ${}^\sigma B(\lambda)^{\lambda_1}$ is defined.

Define a (surjective) homomorphism $\zeta:Y\to\mathbf{Y}$ by $i\mapsto\eta$ where $i\in\eta$. We have $\ker\zeta=(\sigma-1)Y$. Let ${}^t\zeta:\mathbf{X}\to X$ be the homomorphism defined by $\langle y,{}^t\zeta(\mathbf{x})\rangle=(\zeta(y),\mathbf{x})$ for any $y\in Y,\mathbf{x}\in\mathbf{X}$. Then, ${}^t\zeta$ restricts to an isomorphism $\mathbf{X}\xrightarrow{\sim}{}^\sigma X$ and to a bijection $\mathbf{X}^+\xrightarrow{\sim}{}^\sigma X^+$.

Define an injective homomorphism $q:\mathbf{Y}\to Y$ by $\eta\mapsto\sum_{i\in\eta}i$. This defines bijections $\mathbf{Y}\xrightarrow{\sim}{}^\sigma Y$, $\mathbf{Y}^+\xrightarrow{\sim}{}^\sigma Y^+$. We have

(c) ${}^t\zeta(\mathbf{y}')=q(\mathbf{y})'$ for any $\mathbf{y}\in\mathbf{Y}$.

To prove (c), it is enough to show that for any $i_1\in I,\eta\in I_\sigma$ we have $\langle i_1,{}^t\zeta(\eta')\rangle=\langle i_1,\sum_{i\in\eta}i'\rangle$, i.e., $(\eta_1,\eta')=\langle i_1,\sum_{i\in\eta}i'\rangle$, where $\eta_1\in I_\sigma$ is such that $i_1\in\eta_1$. But this follows from (b).

6.5

We preserve the setup in §6.4. Let $\Psi:\mathbf{B}\xrightarrow{\sim}{}^\sigma B$ be the bijection in [L1, 14.4.9]. From the definition we have $\Psi(\mathbf{B}^{\eta,n})={}^\sigma B\cap B^{i,n}$ for any $n\in\mathbf{N},\eta\in I_\sigma,i\in\eta$ and, moreover, $\Psi(\mathbf{B}_\mu)={}^\sigma B_{q(\mu)}$ for any $\mu\in\mathbf{Y}^+$. For $\boldsymbol{\lambda}\in\mathbf{X}^+$, define $\lambda\in{}^\sigma X^+$ by $\lambda={}^t\zeta(\boldsymbol{\lambda})$. We show:

(a) $\Psi(\mathbf{B}(\boldsymbol{\lambda}))={}^\sigma B(\lambda)$.

We have

$$\begin{aligned}\Psi(\mathbf{B}(\boldsymbol{\lambda}))&=\cap_{\eta\in I_\sigma}\cap_{\{n;0\le n\le(\eta,\boldsymbol{\lambda})\}}\ \Psi(\mathbf{B}^{\eta,n})\\&=\cap_{\eta\in I_\sigma}\cap_{\{n;0\le n\le(\eta,\boldsymbol{\lambda})\}}\ \cap_{i\in\eta}({}^\sigma B\cap B^{i,n})\\&=\cap_{i\in I}\big(\cap_{\{n;0\le n\le\langle i,\lambda\rangle\}}({}^\sigma B\cap B^{i,n})\big)\\&={}^\sigma B\cap B(\lambda).\end{aligned}$$

(We have used $(\eta,\boldsymbol{\lambda})=\langle i,\lambda\rangle$ for $\eta\in I_\sigma,i\in\eta$.) This proves (a).

Let $\boldsymbol{\lambda} \in \mathbf{X}^+, \boldsymbol{\lambda}_1 \in \mathbf{X}$; define $\lambda \in {}^\sigma X^+, \lambda_1 \in {}^\sigma X$ by $\lambda = {}^t\zeta(\boldsymbol{\lambda}), \lambda_1 = {}^t\zeta(\boldsymbol{\lambda}_1)$. We show:

(b) $\Psi(\mathbf{B}(\boldsymbol{\lambda})^{\boldsymbol{\lambda}_1}) = {}^\sigma B(\lambda)^{\lambda_1}$.

We have ${}^\sigma B(\lambda)^{\lambda_1} = \cup_{\{\nu \in Y^+; \lambda = \lambda_1 + \nu'\}} {}^\sigma B \cap B(\lambda) \cap B_\nu$. Since ${}^\sigma B \cap B_\nu = \emptyset$ unless $\nu \in {}^\sigma Y^+$, we may replace in the previous union $\nu \in Y^+$ by $\nu \in {}^\sigma Y^+$, i.e., $\nu = q(\mu)$ for some well defined $\mu \in \mathbf{Y}^+$. Thus,

$$ {}^\sigma B(\lambda)^{\lambda_1} = \cup_{\{\mu \in \mathbf{Y}^+; {}^t\zeta(\boldsymbol{\lambda}) = {}^t\zeta(\boldsymbol{\lambda}_1) + q(\mu)'\}} {}^\sigma B \cap B(\lambda) \cap B_{q(\mu)}. $$

Using 6.4(c) we may replace $q(\mu)'$ by ${}^t\zeta(\mu')$. The resulting equation ${}^t\zeta(\boldsymbol{\lambda}) = {}^t\zeta(\boldsymbol{\lambda}_1) + {}^t\zeta(\mu')$ is equivalent to the equation $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1 + \mu'$ (by the injectivity of ${}^t\zeta$). Thus, we see that

$$ {}^\sigma B(\lambda)^{\lambda_1} = \cup_{\{\mu \in \mathbf{Y}^+; \boldsymbol{\lambda} = \boldsymbol{\lambda}_1 + \mu'\}} {}^\sigma B(\lambda) \cap B_{q(\mu)}. $$

On the other hand using (a) we have

$$ \begin{aligned} \Psi(\mathbf{B}(\boldsymbol{\lambda})^{\boldsymbol{\lambda}_1}) &= \cup_{\{\mu \in \mathbf{Y}^+; \boldsymbol{\lambda} = \boldsymbol{\lambda}_1 + \mu'\}} \Psi(\mathbf{B}(\boldsymbol{\lambda})) \cap \Psi(\mathbf{B}_\mu) \\ &= \cup_{\{\mu \in \mathbf{Y}^+; \boldsymbol{\lambda} = \boldsymbol{\lambda}_1 + \mu'\}} {}^\sigma B(\lambda) \cap {}^\sigma B_{q(\mu)}. \end{aligned} $$

This proves (b).

6.6

We preserve the setup of §6.4. Assume in addition that $(I, \cdot)$ is of finite type. We show that $(I_\sigma, \cdot)$ is of finite type. Let $(r_\eta) \in \mathbf{R}^{I_\sigma} - \{0\}$. Define $(\tilde{r}_i) \in \mathbf{R}^I - \{0\}$ by $\tilde{r}_i = r_\eta$ where $\eta \in I_\sigma, i \in \eta$. Using 6.4(b) we have

$$ \sum_{\eta_1, \eta_2 \in I_\sigma} (\eta_1 \cdot \eta_2) r_{\eta_1} r_{\eta_2} = \sum_{i_1, i_2 \in I} (i_1 \cdot i_2) \tilde{r}_{i_1} \tilde{r}_{i_2}. $$

This is > 0 since $(I, \cdot)$ is of finite type. Thus, we see that $(I_\sigma, \cdot)$ is of finite type.

Define G, T in terms of $(I, \cdot)$ as in §6.3. Define (a connected, simply-connected, semisimple algebraic group over $\mathbf{C}$) $\mathbf{G}$ and a maximal torus $\mathbf{T}$ of $\mathbf{G}$ in terms of $(I_\sigma, \cdot)$ in the same way as G, T were defined in terms of $(I, \cdot)$. We have $T = \mathbf{C}^* \otimes Y$, $\mathbf{T} = \mathbf{C}^* \otimes \mathbf{Y}$. Let $\lambda \in {}^\sigma X^+$ and $\boldsymbol{\lambda} \in \mathbf{X}^+$

be such that $\lambda = {}^t\zeta(\boldsymbol{\lambda})$. Recall from §6.3 that the $\mathbf{C}$-vector space $[B(\lambda)]$ is naturally an irreducible representation of G with highest weight λ. Similarly, the $\mathbf{C}$-vector space $[\mathbf{B}(\boldsymbol{\lambda})]$ is naturally an irreducible representation of $\mathbf{G}$ with highest weight $\boldsymbol{\lambda}$. There is a unique automorphism $\sigma : G \to G$ such that $\sigma(x_i(z)) = x_{\sigma(i)}(z)$, $\sigma(y_i(z)) = y_{\sigma(i)}(z)$ for any $i \in I, z \in \mathbf{C}^*$. The restriction of this automorphism to T is the automorphism $1 \otimes \sigma : \mathbf{C}^* \otimes Y \to \mathbf{C}^* \otimes Y$. The permutation $\sigma : B(\lambda) \to B(\lambda)$ extends linearly to a vector space automorphism $\sigma : [B(\lambda)] \to [B(\lambda)]$; for any $g \in G, \xi \in [B(\lambda)]$ we have $\sigma(g\xi) = \sigma(g)\sigma(\xi)$. Hence, for any $s \in T$ the composition $s\sigma$ of the automorphisms of $[B(\lambda)]$ defined by s and σ acts as a monomial matrix with respect to the basis $B(\lambda)$ of $[B(\lambda)]$. In fact, σ acts as a permutation matrix and s acts as a diagonal matrix (it acts on an element in $B(\lambda)^{\lambda_1}$ as multiplication by $\lambda_1(s)$). It follows that

$$\mathrm{tr}(s\sigma; [B(\lambda)]) = \sum_{\lambda_1 \in X} |{}^\sigma B \cap B(\lambda)^{\lambda_1}| \lambda_1(s) = \sum_{\lambda_1 \in {}^\sigma X} |{}^\sigma B(\lambda)^{\lambda_1}| \lambda_1(s).$$

Using here the bijection ${}^t\zeta : \mathbf{X} \xrightarrow{\sim} {}^\sigma X$ and the equality $|{}^\sigma B(\lambda)^{\lambda_1}| = |\mathbf{B}(\boldsymbol{\lambda})^{\boldsymbol{\lambda}_1}|$ (cf. §6.5(b)), we obtain

$$\mathrm{tr}(s\sigma; [B(\lambda)]) = \sum_{\boldsymbol{\lambda}_1 \in \mathbf{X}} |\mathbf{B}(\boldsymbol{\lambda})^{\boldsymbol{\lambda}_1}| ({}^t\zeta\boldsymbol{\lambda}_1)(s). \tag{1}$$

Similarly, any $\mathbf{s} \in \mathbf{T}$ acts as a diagonal matrix with respect to the basis $\mathbf{B}(\boldsymbol{\lambda})$ of $[\mathbf{B}(\boldsymbol{\lambda})]$; more precisely, it acts on an element in $\mathbf{B}(\boldsymbol{\lambda})^{\boldsymbol{\lambda}_1}$ as multiplication by $\boldsymbol{\lambda}_1(\mathbf{s})$. (As earlier, we identify $\mathbf{X} = \mathrm{Hom}(\mathbf{T}, \mathbf{C}^*)$.) It follows that

$$\mathrm{tr}(\mathbf{s}; [\mathbf{B}(\boldsymbol{\lambda})]) = \sum_{\boldsymbol{\lambda}_1 \in \mathbf{X}} |\mathbf{B}(\boldsymbol{\lambda})^{\boldsymbol{\lambda}_1}| \boldsymbol{\lambda}_1(\mathbf{s}). \tag{2}$$

Now, the homomorphism $\zeta : Y \to \mathbf{Y}$ induces (after application of $\mathbf{C}^* \otimes$) a homomorphism of tori $\zeta_o : T \to \mathbf{T}$ satisfying $({}^t\zeta\boldsymbol{\lambda}_1)(s) = \boldsymbol{\lambda}_1(\zeta_o(s))$ for any $\boldsymbol{\lambda}_1 \in \mathbf{X}$, $s \in T$. Comparing (1) and (2), we see that

$$\mathrm{tr}(s\sigma; [B(\lambda)]) = \mathrm{tr}(\zeta_o(s); [\mathbf{B}(\boldsymbol{\lambda})]) \tag{3}$$

for any $s \in T$.

Finally, we observe that (following the notation of §3) we may canonically identify $\mathbf{G} = G_\sigma$ and $\mathbf{T} = T_\sigma$ so that $\zeta_o : T \to \mathbf{T}$ becomes the canonical homomorphism $\varpi : T \to T_\sigma$.

Thus, from (3), Theorem 1.1 follows for any G of type different from A_{2n}. □

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