CHAPTER 23

AN ANALOGUE OF A CONJECTURE OF MAZUR: A QUESTION IN DIOPHANTINE APPROXIMATION ON TORI

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To Professor Shalika, with admiration

Abstract. B. Mazur has considered the question of density in the Euclidean topology of the set of $\mathbb{Q}$-rational points on a variety X defined over $\mathbb{Q}$, in particular for Abelian varieties. In this paper we consider the question of closures of the image of finitely generated subgroups of $T(\mathbb{Q})$ in $\Gamma\backslash T(\mathbb{R})$ where T is a torus defined over $\mathbb{Q}$, Γ an arithmetic subgroup such that $\Gamma\backslash T(\mathbb{R})$ is compact. Assuming Schanuel's conjecture, we prove that the closures correspond to *algebraic* sub-tori of T.

Let V be a smooth algebraic variety over $\mathbb{Q}$. The set $V(\mathbb{R})$ acquires a topological structure from the Euclidean topology of $\mathbb{R}$. It is known that $V(\mathbb{R})$ has finitely many connected components. If $V(\mathbb{Q})$ is Zariski dense in V, it was conjectured by B. Mazur, cf. [M1] and [M2], that the closure of $V(\mathbb{Q})$ in $V(\mathbb{R})$ is a finite union of connected components of $V(\mathbb{R})$. This conjecture was shown to be false in this generality for an elliptic surface by Colliot-Thélène, Skorobogatov, and Swinnerton-Dyer, who have proposed a slightly reformulated conjecture, cf. [CSS]. However, the present evidence seems to suggest that the following special case of Mazur's conjecture is true.

Conjecture 1 (Mazur's conjecture for Abelian varieties). Let A be an abelian variety over $\mathbb{Q}$, and G a subgroup of $A(\mathbb{Q})$. Then the closure of G in the Euclidean topology of $A(\mathbb{R})$ contains $B(\mathbb{R})^0$ as a subgroup of finite index for a certain abelian subvariety B defined over $\mathbb{Q}$.

The following theorem of M. Waldschmidt [W1] is the best result known toward Mazur's conjecture.

Theorem 1 (Waldschmidt). *If A is a simple abelian variety over $\mathbb{Q}$ of dimension d and if rank of $A(\mathbb{Q})$ is $\geq d^2 - d + 1$, then the closure of $A(\mathbb{Q})$ in the Euclidean topology contains $A(\mathbb{R})^0$.*

1. The conjecture about tori. In this section we propose the following analogue of Mazur's conjecture for tori. We begin by recalling certain standard definitions.

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Let S be a torus defined over $\mathbb{Q}$, i.e., let S be a commutative linear algebraic group over $\mathbb{Q}$ which becomes isomorphic to $\mathbb{G}_m^n$ over the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ for a certain integer $n \geq 0$. The torus S is called isotropic over $\mathbb{Q}$ if there exists a homomorphism of algebraic groups $S \to \mathbb{G}_m$ defined over $\mathbb{Q}$. If S is not isotropic over $\mathbb{Q}$, it is called anisotropic.

Given a linear algebraic group over $\mathbb{Q}$ such as S, it makes sense to talk of arithmetic subgroups Γ of $S(\mathbb{Q})$. Any two arithmetic subgroups Γ_1 and Γ_2 are commensurable, i.e., $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2. It is a consequence of Dirichlet unit theorem (or the general theorem due to Borel and Harish-Chandra) that if S is an anisotropic torus over $\mathbb{Q}$, then $\Gamma\backslash S(\mathbb{R})$ is a compact abelian group for Γ any arithmetic subgroup of $S(\mathbb{Q})$. The connected component of identity of $\Gamma\backslash S(\mathbb{R})$ is a torus in the usual sense of the word, i.e., a topological group isomorphic to $(S^1)^n$.

Unlike most other algebraic groups, tori have the property that there is a unique *maximal* arithmetic group. This unique maximal arithmetic subgroup is the subgroup Γ of $T(\mathbb{Q})$ defined as follows:

$$\Gamma = \left\{ \gamma \in T(\mathbb{Q}) \,\middle|\, \begin{array}{l} \chi(\gamma) = \text{ a unit in the ring of integers of } \bar{\mathbb{Q}}^* \\ \text{for all characters } \chi : T \to \mathbb{G}_m \end{array} \right\}.$$

We now make the following analogue of Mazur's conjecture for tori.

Conjecture 2. Let S be an algebraic anisotropic torus defined over $\mathbb{Q}$. Let F be a finitely generated subgroup of $S(\mathbb{Q})$. For Γ any arithmetic subgroup of $S(\mathbb{Q})$, the connected component of identity of the closure of the image of F in $\Gamma\backslash S(\mathbb{R})$ equals connected component of identity of $\Gamma_T\backslash T(\mathbb{R})$ for a certain subtorus T of S defined over $\mathbb{Q}$ with $\Gamma_T = \Gamma \cap T(\mathbb{Q})$. Equivalently, the closure of the image of F is dense in the Euclidean topology in the identity component of $\Gamma\backslash S(\mathbb{R})$ if and only if any subgroup G of $S(\mathbb{Q})$ surjecting on the image of F in $\Gamma\backslash S(\mathbb{R})$ is Zariski dense in S.

Remark 1. To any finitely generated subgroup F of $S(\mathbb{Q})$, there is a natural subtorus T of S over $\mathbb{Q}$ defined as the (connected component of the identity of the) kernel of the group of characters

$$X_F = \{\chi : S \to \mathbb{G}_m | \chi(a) = \text{ a unit in the ring of integers of } \bar{\mathbb{Q}}^* \text{ for all } a \in F\}.$$

By embedding an anisotropic torus in a product of norm 1 tori (as in Lemma 3 below), it is easy to see that the torus T defined here is the same as that which appears in the conjecture above. We are sloppy here, as well as in other places in the paper, about making such assertions only up to connected components.

Remark 2. Let the finitely generated group F be generated by $\{a_1, a_2, \ldots, a_n\}$. Since the product of compact subgroups is compact, it is clear that the closure of the image of F in $\Gamma\backslash S(\mathbb{R})$, Γ an arithmetic subgroup of $S(\mathbb{R})$, is the product of the closures of cyclic groups generated by a_i. Hence it suffices to prove the conjecture for cyclic groups F. By embedding an anisotropic torus in a product of norm 1 tori (as in Lemma 3 below), we are further reduced to proving the conjecture for a cyclic subgroup in a product of norm 1 tori.

Remark 3. The conjecture above can also be formulated as the closure of a finitely generated subgroup in $S(\mathbb{Q})$ containing an arithmetic subgroup in $S(\mathbb{R})$. We note that since an arithmetic group in an anisotropic torus over $\mathbb{Q}$, which is split over $\mathbb{R}$, is Zariski dense, so will this finitely generated subgroup. Thus in this sense, our conjecture has a different flavor than Mazur's conjecture. We note that there is also a conjecture, different from the one formulated here, due to Waldschmidt [W3, Conjecture 3.5 of Chapter 3] about the closure in the Euclidean topology of a finitely generated subgroup of $S(\mathbb{Q})$.

Remark 4. In this paper we will often be using without explicit mention the trivial remark that the connected component of identity of the closure in either Euclidean or Zariski topology of a finitely generated subgroup F (of a topological group or an Algebraic group) or a subgroup G of F of finite index is the same.

Example 1. The simplest case of our conjecture is when T is a torus over $\mathbb{Q}$ such that $T(\mathbb{R})$ itself is compact. In this case we can take $T(\mathbb{Z})$ to be the trivial subgroup of $T(\mathbb{Q})$. We will thus be comparing the closures in Euclidean topology and Zariski topology of a finitely generated subgroup of $T(\mathbb{Q})$. That the two closures are the same follows from the easily proven assertion that any continuous homomorphism from the compact group $T(\mathbb{R})$ to S^1 is the restriction to $T(\mathbb{R})$ of an algebraic character (defined over $\mathbb{C}$) from $T(\mathbb{C})$ to $\mathbb{C}^*$.

Example 2. Let k be a totally real cubic extension of $\mathbb{Q}$. Let $T = k^1$ denote the group of norm 1 elements of k. So $T(\mathbb{Q}) = k^1$, and the group of units of (the ring of integers of) k of norm 1 can be taken to be an arithmetic subgroup of $T(\mathbb{Q})$. We have,

$$\begin{aligned} T(\mathbb{R}) &= [k \otimes \mathbb{R}]^1 \\ &= [\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}]^1 \\ &\cong \mathbb{R}^* \times \mathbb{R}^*. \end{aligned}$$

We note that $(x, y) \to (\log x, \log y)$ gives an isomorphism of the product of two copies of positive reals (under multiplication) with real numbers (under addition). We have thus a homomorphism from $T(\mathbb{R})$ to $\mathbb{R} \times \mathbb{R}$ such that the image of the group of units of norm 1 is a discrete cocompact subgroup with quotient $S^1 \times S^1$.

It is easy to see that the torus T of norm 1 elements of a cubic field has no nontrivial subtorus defined over $\mathbb{Q}$. Our conjecture in this case will therefore say that any element of $T(\mathbb{Q})$ which is not of finite order will generate a dense subgroup of $T(\mathbb{Z})\backslash T(\mathbb{R})$. We make this more concrete.

Observe that if $\Lambda \subset \mathbb{R}^2$ is a lattice in $\mathbb{R}^2$ and v is a vector in $\mathbb{R}^2$, then integral multiples of v is dense in $\Lambda\backslash\mathbb{R}^2$ if and only if there does not exist a nonzero integer q, and a lattice point $\lambda \in \Lambda$ such that $qv + \lambda$ is real multiple of a vector in Λ. Suppose if possible, $qv + \lambda = r\lambda_1$ for an integer q, and a real number r. By looking at the coordinates of the vectors on the two sides of the equality, it is easily seen that it suffices to prove that if $(\log|x_1|, \log|x_2|)$ is a real multiple of $(\log|\epsilon_1|, \log|\epsilon_2|)$, then it is a rational multiple of $(\log|\epsilon_1|, \log|\epsilon_2|)$. Here x_1 and x_2 are the images of an element (corresponding to $qv + r$) in k^1 under two fixed embeddings into the reals, and ϵ_1 and ϵ_2 are the images of a unit element in k^1 under the same embeddings into the reals.

This will be a consequence of the following well-known conjecture, cf. [W2].

Conjecture 3. (4 Exponential Conjecture due to Schneider, Lang, and Ramachandra) Let M be a 2×2 matrix consisting of a logarithm of algebraic numbers. Assume that the rows of the matrix are linearly independent over $\mathbb{Q}$, and also that the columns of the matrix are linearly independent over $\mathbb{Q}$, then the determinant of M is nonzero.

The following theorem is a step toward the proof of the 4 exponential conjecture.

THEOREM 2 (LANG AND RAMACHANDRA). *Let M be a 2×3 matrix consisting of a logarithm of algebraic numbers. Assume that the rows of the matrix are linearly independent over $\mathbb{Q}$, and also that the columns of the matrix are linearly independent over $\mathbb{Q}$. Then the rank of M is 2.*

2. Some lemmas about Tori. In this section we collect together some elementary lemmas about tori. We will be considering closures of finitely generated subgroups in the Euclidean and Zariski topologies.

LEMMA 1. (a) *For a discrete subgroup $\Lambda \subset \mathbb{R}^n$ with $\mathbb{R}^n/\Lambda$ compact, the integral multiples of a point $x \in \mathbb{R}^n$ are dense inside $\mathbb{R}^n/\Lambda$ if and only if no nontrivial continuous homomorphism of $\mathbb{R}^n/\Lambda$ to S^1 takes x to the identity element of S^1.*

(b) *The integral multiples of $x \in \mathbb{R}^n$ are dense inside $\mathbb{R}^n/\Lambda$ if and only if $rx + \lambda$ does not belong to $\Lambda_1 \otimes_{\mathbb{Z}} \mathbb{R}$ for any subgroup Λ_1 of Λ with rank $_{\mathbb{Z}}\Lambda_1 < n$, any nonzero integer r, and any element $\lambda \in \Lambda$.*

(c) *The integral multiples of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ are dense inside $\mathbb{R}^n/\Lambda$ if and only if for $\ell_1 = (\ell_{11}, \ldots, \ell_{1n}), \ldots, \ell_{n-1} = (\ell_{n-1,1}, \ldots, \ell_{n-1,n})$, belonging to*

Λ and generating a rank $(n-1)$ subgroup of Λ, the matrix

$$\begin{pmatrix} rx_1+\lambda_1 & rx_2+\lambda_2 & \cdots & rx_n+\lambda_n \\ \ell_{11} & \ell_{12} & \cdot & \ell_{1n} \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot\cdot & \cdot \\ \ell_{n-1,1} & \ell_{n-1,2} & \cdot\cdot & \ell_{n-1,n} \end{pmatrix}$$

is nonsingular, i.e., the determinant is nonzero, for any nonzero integer r, and any $\lambda=(\lambda_1,\dots,\lambda_n)\in\Lambda$.

Now we have a lemma about density of the abelian group generated by a point on a torus in the Zariski topology. In this lemma for a number field K, we will be looking at the torus $T=R_{K/\mathbb{Q}}\mathbb{G}_m$ defined over $\mathbb{Q}$ to be the algebraic group whose group of rational points over any $\mathbb{Q}$-algebra A is $T(A)=(K\otimes_{\mathbb{Q}}A)^*$; in particular $T(\mathbb{Q})=K^*$.

LEMMA 2. (a) *An element $x\in T(\mathbb{Q})=K^*$ generates K (i.e., K is the smallest field extension of $\mathbb{Q}$ containing x) if and only if all its conjugates (i.e., the image of x under all the distinct embeddings of K in $\mathbb{C}$) $\{x_1,\dots,x_n\}$ are distinct.*

(b) *An element $x\in T(\mathbb{Q})=K^*$ lies in no proper algebraic subgroup defined over $\mathbb{Q}$ if and only if the abelian subgroup generated by $\{x_1,\dots,x_n\}$ is free abelian of rank n.*

Proof. Part (a) is clear. For part (b) note that any algebraic character of $T(\mathbb{Q})=K^*$ (defined over the algebraic closure) is defined by $z\to z_1^{d_1}\cdot z_2^{d_2}\cdots z_n^{d_n}$, where z_i denotes the image of z under the various embeddings of K into $\mathbb{C}$. Since for an element belonging to a proper algebraic subgroup of T, there is a character of T trivial on that element, therefore if x belongs to a proper algebraic subgroup, the subgroup generated by $\{x_1,\dots,x_n\}$ will not be free.

Conversely, if the subgroup generated by $\{x_1,\dots,x_n\}$ is not free abelian, x belongs to the kernel of a nontrivial character χ of T defined over $\overline{\mathbb{Q}}$. Hence $x\in T(\mathbb{Q})$ lies in $S(\overline{\mathbb{Q}})$ for an algebraic subgroup S of T. By Galois conjugation, $x\in S^\sigma(\overline{\mathbb{Q}})$ for all Galois conjugates of S. Hence $x\in\cap(S^\sigma)(\overline{\mathbb{Q}})$. However, $A=\cap(S^\sigma)$ is an algebraic group defined over $\mathbb{Q}$. Hence x belongs to $A(\mathbb{Q})$ for A a proper algebraic subgroup of T. □

LEMMA 3. *For any anisotropic torus T over $\mathbb{Q}$, there are field extensions $K_1,\dots,K_d$ of $\mathbb{Q}$, such that if S denotes the product of the norm 1 tori associated to K_i, then there is an embedding of T into S.*

Proof. As is well known, there is an equivalence of categories between tori over $\mathbb{Q}$ and finitely generated free $\mathbb{Z}$-module with an action of the Galois group Gal $(\bar{\mathbb{Q}}/\mathbb{Q})$ of the algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. The equivalence is given by associating to any torus T, its character group $X^*(T)$. Choose a $\mathbb{Z}$-basis, say $\{e_1, \ldots, e_d\}$ of the character group of T. Suppose that H_i is the subgroup of Gal $(\bar{\mathbb{Q}}/\mathbb{Q})$ which stabilizes the vector e_i. The mapping $g \to g \cdot e_i$ from Gal $(\bar{\mathbb{Q}}/\mathbb{Q})$ to $X^*(T)$ gives a mapping from $\mathbb{Z}[G/H_i]$ to $X^*(T)$. Summing over i, we get a surjective map from $\sum_i \mathbb{Z}[G/H_i]$ to $X^*(T)$. This gives an embedding from T to $\prod_i R_{K_i/\mathbb{Q}}\mathbb{G}_m$ where K_i is the fixed field of H_i. Since T is anisotropic, the image of T lands inside the product of norm 1 tori. □

3. Schanuel's conjecture implies conjecture 2. In this section we prove that our conjecture 2 about closures in Euclidean topology of finitely generated subgroups in general tori is a consequence of Schanuel's conjecture.

We should, however, add that although our approach in this paper is via Schanuel's conjecture, there is a possibility that there may be a simpler proof for conjecture 2, just by using the more primitive methods of Geometry of Numbers, as we are dealing with a rather specific number theoretic context.

We begin with the statement of Schanuel's conjecture, which is one of the most outstanding problems in transcendental number theory. In the statement of this conjecture, as well as everywhere else in the paper, one means by $\log A$, for a complex number A to be *any* complex number B such that $\exp(B) = A$.

Conjecture 4 (Schanuel's Conjecture). If $\alpha_1, \ldots, \alpha_n$ are algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$, then $\log \alpha_1, \ldots, \log \alpha_n$ are algebraically independent over $\mathbb{Q}$.

We will need the following lemma about number fields.

LEMMA 4. *Let L be a number field that is Galois over $\mathbb{Q}$. Enumerate the elements of the Galois group G of L over $\mathbb{Q}$ as $\sigma_1 = 1, \sigma_2, \ldots, \sigma_d$. For an element z of L^*, denote the various (not necessarily distinct) Galois conjugates of z by $z_1 = z, z_2 = \sigma_2(z), \ldots, z_d = \sigma_d(z)$. Let x be an element L^*. Then there is a nonzero integer m and a unit ϵ in (the ring of integers of) L, such that whenever $x_1^{n_1} x_2^{n_2} \ldots x_d^{n_d}$ is a unit in (the ring of integers of) L for a d-tuple $(n_1, n_2, \ldots, n_d)$ inside $\mathbb{Z}^d$, $(x^m\epsilon)_1^{n_1}(x^m\epsilon)_2^{n_2} \ldots (x^m\epsilon)_d^{n_d} = 1$. The integer m can be taken to be the order of the class group of L times the degree of L over $\mathbb{Q}$, hence can be chosen to be independent of x.*

Proof. Write the (fractional) ideal generated by x as a product of prime ideals:

$$(x) = \wp_1^{m_1} \cdots \wp_r^{m_r}.$$

(We will assume that if a certain prime $\wp_i$ occurs in the above decomposition, so does any Galois conjugate of it, with exponent perhaps 0.) Let k be an integer such that each of the ideals $\wp_i^k$ is a principal ideal generated by, say ϖ_i. If H_i denotes the subgroup of the Galois group that fixes the prime ideal $\wp_i$, then the elements of H_i will take ϖ_i into itself up to a unit: $h(\varpi_i) = h_i \cdot \varpi_i$. Clearly $h \to h_i$ is a 1-cocycle on H_i with values in the group of units of L^*. Since H_i is a finite group, a finite power of the cocycle becomes a coboundary, i.e., there is a positive integer r, and a unit ν_i such that

$$h(\varpi_i^r) = h_i^r \cdot \varpi_i^r = h(\nu_i)\nu_i^{-1}\varpi_i^r.$$

It follows that $\nu_i^{-1}\varpi_i^r$ is invariant under H_i. Thus we can choose generators π_i of the principal ideals $\wp_i^{rk}$ in such a way that if an element of the Galois group takes one such ideal into another such, then the same holds for the generators (and not just only up to units). From the equality of ideals, $(x^{rk}) = (\pi_1^{m_1}\pi_2^{m_2}\cdots\pi_r^{m_r})$, there is a unit ϵ with, $x^{rk}\epsilon = \pi_1^{m_1}\pi_2^{m_2}\cdots\pi_r^{m_r}$. Now observe that if a product of certain elements of L^* with any two *distinct* elements coprime (such as π_i's) is a unit, then the product is in fact 1 (and is in some sense the "empty product"). From this it follows that $(x^{kr}\epsilon)_1^{n_1}(x^{kr}\epsilon)_2^{n_2}\cdots(x^{kr}\epsilon)_d^{n_d} = 1$. Finally, the proof given here works with the choice of $m = rk$ to be the product of the order of the class group of L and the degree of L over $\mathbb{Q}$. □

COROLLARY 1 (OF THE PROOF). *With the notation as in the lemma, given elements* $\{x^{(1)}, x^{(2)}, \ldots, x^{(n)}\}$ *in* L^*, *there is an integer* m *and units* ϵ_i *in the ring of integers of* L *such that for* $y^{(i)} = (x^{(i)})^m\epsilon_i$, *the subgroup of* L^* *generated by* $\sigma_j(y^{(i)})$ *does not contain any unit of the ring of integers of* L *other than 1.*

THEOREM 3. *Schanuel's conjecture implies conjecture 2.*

Proof. As already noted in Remark 2, it suffices to prove Conjecture 2 for cyclic subgroups. Furthermore, because of Lemma 3, we can assume that the anisotropic torus S is the product of the norm 1 tori: $S = \prod_{i=1}^m (R_{K_i/\mathbb{Q}}\mathbb{G}_m)^1$. We will further assume (by enlarging the anisotropic torus which does not affect the conclusion regarding closures) that the fields K_i are Galois over $\mathbb{Q}$, and by taking the compositum, we assume that the fields K_i are the same, say L, a Galois extension of degree $d + 1$ over $\mathbb{Q}$, which we will assume to be totally real. The case when L has complex places is very similar, although notationally more cumbersome.

For an element $x^{(i)} \in L$, we let $x_j^{(i)}$, $j \in \{1, 2, \ldots, d+1\}$, denote the various Galois conjugates of $x^{(i)}$.

Let $x = (x^{(1)}, x^{(2)}, \ldots, x^{(m)}) \in S = \prod_{i=1}^m (R_{L/\mathbb{Q}}\mathbb{G}_m)^1$. Replacing $x^{(i)}$ by $y^{(i)} = (x^{(i)})^m\epsilon_i$ as in Corollary 1, we can assume that $y = (y^{(1)}, y^{(2)}, \ldots, y^{(m)})$ is such that the group generated by the various Galois conjugates $y_j^{(i)}$ intersects the units in the ring of integers of L in identity alone.

We note that a general algebraic character of L^* is given by $z \to \prod_j (z_j)^{n_j}$. Denote by A the group of characters χ of $S = \prod_{i=1}^m (R_{L/\mathbb{Q}}\mathbb{G}_m)^1$ such that $\chi(y) = 1$.

Let the rank of the abelian group A be $dm - k$. Therefore the subgroup of L^* generated by $\{y_j^{(i)}\}$ is a free abelian group of rank k.

We will use the homomorphism with finite kernel:

$$\begin{array}{ccc} [L \otimes \mathbb{R}]^1 & \rightarrow & \mathbb{R}^d \\ (x_1, \dots, x_{d+1}) & \rightarrow & (\log|x_1|, \dots, \log|x_d|), \end{array}$$

under which (by Dirichlet unit theorem) the group of units of the ring of integers of L of norm 1, $\mathcal{O}_L^{*1}$ goes to a lattice Λ in $\mathbb{R}^d$ with $\mathbb{R}^d/\Lambda$ compact.

Taking the direct sum of this homomorphism m number of times, we get a homomorphism from $S(\mathbb{R})$ to $\mathbb{R}^{dm}/\Lambda^m$, whose kernel is an arithmetic group in $S(\mathbb{R})$, to be denoted by $S(\mathbb{Z})$. We will denote the image of $y = (y^{(1)}, y^{(2)}, \dots, y^{(m)})$ in $\mathbb{R}^{dm}$ as $(\log(|y_1^{(1)}|), \dots, \log(|y_d^{(1)}|), \dots, \log(|y_1^{(m)}|), \dots, \log(|y_d^{(m)}|))$. By Lemma 1($c$), to prove this theorem it suffices to prove that the rank of the matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1d} & \cdots & a_{m1} & \cdots & a_{md} \\ \ell(1)_{11} & \cdots & \ell(1)_{1d} & \cdots & \ell(1)_{m1} & \cdots & \ell(1)_{md} \\ . & \cdots & . & \cdots & . & \cdots & . \\ . & \cdots & . & \cdots & . & \cdots & . \\ . & \cdots & . & \cdots & . & \cdots & . \\ \ell(k-1)_{11} & \cdots & \ell(k-1)_{1d} & \cdots & \ell(k-1)_{m1} & \cdots & \ell(k-1)_{md} \end{pmatrix}$$

is k where $(a_{11}, \dots, a_{1d}, \dots, a_{m1}, \dots, a_{md}) = (r \log(|y_1^{(1)}|) + \ell_{11}, \dots, r \log(|y_d^{(1)}|) + \ell_{1d}, \dots, r \log(|y_1^{(m)}|) + \ell_{m1}, \dots, r \log(|y_d^{(m)}|)) + \ell_{md})$, r is a non-zero integer, $(\ell_{11}, \dots, \ell_{1,d}, \dots, \ell_{m1}, \dots, \ell_{md})$ is a vector in Λ^m, and where the 2nd to kth rows of this matrix represent $(k-1)$ $\mathbb{Z}$-linearly independent vectors in Λ^m.

Since the rank of the matrix

$$B = \begin{pmatrix} \ell(1)_{11} & \cdots & \ell(1)_{1d} & \cdots & \ell(1)_{m1} & \cdots & \ell(1)_{md} \\ . & \cdots & . & \cdots & . & & \cdots \\ . & \cdots & . & \cdots & . & & \cdots \\ . & \cdots & . & \cdots & . & & \cdots \\ \ell(k-1)_{11} & \cdots & \ell(k-1)_{1d} & \cdots & \ell(k-1)_{m1} & \cdots & \ell(k-1)_{md} \end{pmatrix}$$

is $(k-1)$, there is a $(k-1) \times (k-1)$ submatrix with nonzero determinant. After re-indexing the coordinates in $\mathbb{R}^{dm}$, we assume that the first $(k-1) \times (k-1)$ submatrix of B has rank $(k-1)$, i.e., has nonzero determinant.

Since the rank of the group generated by $\{y_j^{(i)}\}$ is k, there is at least one index, say $y_{j_0}^{(i_0)}$, such that no power of it belongs to the group generated by the $y's$ corresponding to the first $(k-1)$ entries in the first row of A (after re-indexing introduced above). Denote these $y's$ as $y_1, y_2, \dots, y_{k-1}$, and the corresponding ℓ's as $\ell_1, \ell_2, \dots, \ell_{k-1}$. Also, denote $y_{j_0}^{(i_0)}$ as y_k.

Let C be the $k \times k$ submatrix of A comprised of the first $(k-1)$ columns of A, and the kth column corresponding to $y_{j_0}^{(i_0)}$. We want to prove that $\det(C) \neq 0$.

Clearly $\det(C) = \sum_{i=1}^{k} [r \log(|y_i|) + \ell_i] \det(L_i)$, where L_i is a $(k-1) \times (k-1)$ matrix consisting of log of units in L, and ℓ_i are also log of units in L.

It follows from Schanuel's conjecture and our hypothesis that no (nonzero) power of y_k belongs to the group generated by y_i, $i = 1, \ldots, k-1$, that $\log(|y_k|)$ is algebraically independent over the subfield of $\mathbb{C}$ generated by log of algebraic units and the $\log(|y_i|)$, $i = 1, \ldots, k-1$.

By our assumption, the first $(k-1) \times (k-1)$ submatrix of B has nonzero determinant, which is $\det(L_k)$, hence $\det(C) = \sum [r \log(|y_i|) + \ell_i] \det(L_i)$ is nonzero (by algebraic independence of the kth term from the rest). □

4. Counter-example to a more general question. It is natural to ask if an analogue of Conjecture 2 can be made more generally. The general question is about the algebraicity of the connected component of identity of the closure in Euclidean topology of a finitely generated subgroup of $\mathbb{C}^{*n}$ with algebraic coordinates, where $\mathbb{C}^{*n}$ is considered as the $2n$-dimensional torus defined over $\mathbb{R}$ as the Weil restriction of scalars $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m^n$. For example, can one drop the condition on the torus S in Conjecture 2 being anisotropic over $\mathbb{Q}$, and instead of taking $S(\mathbb{Z})$, which is a cocompact discrete subgroup of $S(\mathbb{R})$ if S is anisotropic, take any cocompact discrete subgroup Γ of $S(\mathbb{R})$ contained in $S(\mathbb{Q})$? A simple counter-example shows that this is not possible, shattering any hope for a simple answer to the general question above.

To construct the counter-example, take $S = \mathbb{G}_m^3$, the 3-dimensional split torus over $\mathbb{Q}$. The principle behind the counter-example is the well-known observation that although the determinant of a skew-symmetric $n \times n$ matrix consisting of logarithm of algebraic numbers is 0 if n is odd, the rows and columns could be linearly independent over $\mathbb{Q}$, such as for the matrix:

$$\begin{pmatrix} 0 & \log 2 & \log 3 \\ -\log 2 & 0 & \log 5 \\ -\log 3 & -\log 5 & 0 \end{pmatrix}.$$

Since the determinant of the following matrix is nonzero,

$$\begin{pmatrix} 0 & \log 2 & \log 3 \\ -\log 2 & 0 & \log 5 \\ \log 7 & 0 & \log 2 \end{pmatrix},$$

it follows that the subgroup of $\mathbb{R}^{*3}$ generated by the elements, $x = (1, 2, 3)$, $y = (1/2, 1, 5)$, $z = (7, 1, 2)$, is a discrete cocompact subgroup Γ of $\mathbb{R}^{*3}$. However, the closure of the image of the cyclic group generated by the image of $w = (3, 5, 1)$ in $\Gamma\backslash\mathbb{R}^{*3}$ is a 2-dimensional topological torus (this follows from Lemma, 2(b)), which does not arise from an algebraic subtorus of $\mathbb{R}^{*3}$, as is easy to see.

Remark. The connected component of identity of the closure of a finitely generated subgroup of algebraic numbers of $\mathbb{C}^*$ is $\{1, \mathbb{R}^+, \mathbb{S}^1, \mathbb{C}^*\}$. We refer to Theorem 1.10, p. 56 of [W3] for a proof of this assuming Schanuel's conjecture. (The subtlety lies in proving that algebraic points cannot be dense on a closed connected subgroup of $\mathbb{C}^*$ besides those mentioned above.)

5. Non-abelian analogue. It seems very natural to extend the scope of Conjecture 2 by replacing the anisotropic torus T by a general algebraic group G over $\mathbb{Q}$.

We recall that by a theorem due to Borel and Harish-Chandra, for a reductive algebraic group G over $\mathbb{Q}$, which is anisotropic over $\mathbb{Q}$, $G(\mathbb{Z})\backslash G(\mathbb{R})$ is compact. We would like to suggest the analogue of Conjecture 2 to assert that the closure of the image in $G(\mathbb{Z})\backslash G(\mathbb{R})$ of a finitely generated subgroup F of $G(\bar{\mathbb{Q}}_{\mathbb{R}})$ (where $\bar{\mathbb{Q}}_{\mathbb{R}}$ is the subfield of algebraic numbers in $\mathbb{R}$) is of the form $\Gamma_H\backslash H(\mathbb{R})$ for an algebraic subgroup H of G defined over $\mathbb{Q}$ with $\Gamma_H = H(\mathbb{R}) \cap G(\mathbb{Z})$. (Note that if the image of a subgroup $H(\mathbb{R})$ in $G(\mathbb{Z})\backslash G(\mathbb{R})$ is closed, hence compact, then $\Gamma_H = H(\mathbb{R}) \cap G(\mathbb{Z})$ is a cocompact lattice in $H(\mathbb{R})$, and hence if H is algebraic, it is defined over $\mathbb{Q}$ by the Borel density theorem.) Notice that we have not proved even for a torus (even after assuming Schanuel's conjecture), a theorem in this generality as we have always restricted ourselves to finitely generated subgroups F of the torus which are contained in the group of $\mathbb{Q}$-rational points. This seems to have been necessary for the proof of Conjecture 2 given here.

We remark that our suggested analogue contains a consequence of M. Ratner's theorem (the proof of the so-called Raghunathan conjecture) as observed by Dani and Raghunathan, cf. Cor. 4.9 in [V] in a very special case. It states that if a semi-simple group G over $\mathbb{R}$ (with $G(\mathbb{R})$ noncompact) has two distinct $\mathbb{Q}$ structures, with corresponding lattices Γ_1 and Γ_2, then $\Gamma_1 \cdot \Gamma_2$ is dense in $G(\mathbb{R})$ (in the Euclidean topology).

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