Symplectic root numbers of two-dimensional Galois representations: an interpretation

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Let F be $\mathbb{R}$ or a non-archimedean local field of odd residual characteristic, $\overline{F}$ a separable algebraic closure, $\mathrm{Gal}(\overline{F}/F)$ (resp. W_F) the absolute Galois group (resp. Weil group) of F, and $\sigma : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}(V)$ a continuous representation on a complex vector space V of dimension n. Denote by $L(\sigma, s)$ and $\varepsilon(\sigma, s)$ respectively the associated $L-$ function and $\varepsilon-$ factor ([T]); they are also defined for virtual representations (and for representations of W_F). The *root number* of (σ, V) is defined to be

$$W(\sigma) \;=\; \varepsilon(\sigma, 1/2).$$

It is independent of all choices if V has dimension zero and determinant 1, and satisfies the identity $W(\sigma)W(\sigma^\vee) = 1$, where $(\sigma^\vee, V^\vee)$ signifies the dual representation of (σ, V). In particular we have

$$\sigma \text{ self} - \text{dual} \implies W(\sigma) = \pm 1.$$

Determination of the sign of $W(\sigma)$ is a basic problem. When σ is given by the restriction of a global representation ρ, $W(\sigma)$ is a factor of the global root number $W(\rho)$, whose sign gives information on the vanishing of $L(\rho, s)$ at the critical center $s = 1/2$.

Assume from now on that (σ, V) is self-dual. Then there exists a non-degenerate bilinear form B on V which is invariant under $\sigma(\mathrm{Gal}(\overline{F}/F))$. One says that (σ, V) is *orthogonal* (resp. *symplectic*) if B is symmetric (resp. alternating); exactly one of these possibilities occurs when V is irreducible.

Suppose (σ, V) is a virtual sum of orthogonal representations. Then one has the associated Stiefel-Whitney classes $w_i(\sigma)$ in $H^i(F, \mathbb{Z}/2)$. Let $\tilde{w}_i(\sigma)$ denote 1 (resp. -1) if $w_i(\sigma)$ is trivial (resp. non-trivial). If σ is a genuine (orthogonal) representation with determinant 1, then $w_2(\sigma)$ is simply the class in $H^2(F, \mathbb{Z}/2)$ of the extension of $\mathrm{Gal}(\overline{F}/F)$ by $\{\pm 1\}$ obtained by pulling back via σ the extension of $\mathrm{SO}(V)$ defined by its double cover, namely the spin group of V; in this case, $\tilde{w}_2(\sigma) = 1$ iff σ lifts to a representation of $\mathrm{Gal}(\overline{F}/F)$ into $\mathrm{Spin}(V)$. One has the following

Theorem. *(Deligne [D1]) Let (σ, V) be orthogonal of determinant 1 and dimension 0 (in the Grothendieck group). Then*

$$W(\sigma) \;=\; \tilde{w}_2(\sigma).$$

Since $Sp(n, \mathbb{C})$ is simply connected, this raises the question of how one could understand symplectic representations. Our idea is to use the local Langlands correspondence to attach suitable *orthogonal* representations of certain compact groups and study their Stiefel-Whitney numbers. In order to state our result (for $n = 2$), we denote by D the unique quaternion division algebra over F and recall (cf. [Ku] + [JL]) that every irreducible two-dimensional representation (σ, V) of W_F corresponds to a finite dimensional (irreducible) $\mathbb{C}-$ representation (π, X) of D^*, such that the root numbers (and conductors) of σ and π coincide. Moreover, the central character ω_π of π identifies with the character of F^* attached to the determinant of σ by class field theory. It may be seen that π is self-dual whenever σ is. We first establish the following key

Proposition A. *For every irreducible two-dimensional symplectic representation* (σ, V)*, the associated representation* (π, X) *of* D^* *is orthogonal.*

Note that $\det(\sigma)$ is trivial when σ is symplectic, and so π factors through a representation of the compact group D^*/F^*. Given any virtual (orthogonal) representation (ρ, Y) of a closed subgroup G of D^*/F^*, one can associate $w_2(\rho) \in H^2(G, \mathbb{Z}/2)$ and $\tilde{w}_2(\rho) \in \{\pm 1\}$ as above. When F is non-archimedean of residue field $\mathbb{F}_q$, $q = 2m+1$, denote by $\mathfrak{f}(\pi)$ the (exponent of the) conductor of π, and define $s(\pi)$ to be $m\mathfrak{f}(\pi)/2$ (resp. 0) when $\mathfrak{f}(\pi)$ is even (resp. odd).

Theorem B. *Let* F *be a non-archimedean local field of odd residual characteristic. Let* (σ, V), (σ', V') *be irreducible, continuous two-dimensional symplectic representations of* W_F*, and let* (π, X), (π', X') *be the associated representations of* D^*/F^**. Assume that* $\det(\pi) = \det(\pi')$ *and that* $s(\pi) \equiv s(\pi')$ *(mod 2). Then*

$$W(\sigma \ominus \sigma') = \tilde{w}_2(\pi \ominus \pi').$$

Note that this gives in particular an interpretation of the way the root number of σ changes when twisted by quadratic characters. It should also be remarked that $s(\pi) \equiv s(\pi') \pmod 2$ when $\pi \ominus \pi'$ is, for example, of dimension 0 and determinant 1 (see §5), and also when σ and σ' are both attached to characters of ramified quadratic extensions. See Prop. 5.4 for a variant describing $W(\pi \ominus \pi')$ assuming only that $\det(\pi \ominus \pi') = 1$. The archimedean case is treated in Proposition 5.5. The global implication of our results is not yet clear.

Our method is to analyze the behavior of the irreducible representations of D^* when restricted the various toric subgroups T. More explicitly we consider, for each (π, X), the representation $\tilde{\pi} : D^*/F^* \to SO(X \oplus \mathbb{C})$ defined by $g \to (\pi(g), det(\pi(g)))$, and establish criteria (in §3 and §4) for $\tilde{\pi}|_{T/F^*}$ to lift to $\mathrm{Spin}(X \oplus \mathbb{C})$. This leads to the following

Theorem C. *Let* F *be non-archimedean of residue field* $\mathbb{F}_q$, q *odd, and let* ω *denote the unique non-trivial quadratic character of* $\mathbb{F}_q^*$*. Let* π *be an irreducible representation of* D^*/F^* *with values in* $O(X)$ *attached to a character* χ *of the multiplicative group of a quadratic extension* K *of* F*. Then the associated representation* $\tilde{\pi}$ *lifts to* $\mathrm{Spin}(X \oplus \mathbb{C})$ *if and only if* $\omega(-2) = -1$ *and* $\varepsilon(\pi) = \omega(-1)$ *if* K *is ramified and* $\mathfrak{f}(\pi) = 2f+1$*, and* $\omega(-1)^{f-1} = -1$ *and* $\varepsilon(\pi) = -1$ *if* K *is unramified and the conductor of* $\mathfrak{f}(\pi) = 2f$*.*

The calculations underlying the proof yield an explicit formula (see §4) for $\tilde{w}_2(\tilde{\pi})$ in terms of $W(\pi)$ and other extraneous factors, which simplify when we consider $\pi \ominus \pi'$.

In §6 we indicate a geometric approach based on the cohomology of the Drinfeld coverings of $p-$ adic upper half spaces and show how to deduce Proposition A for $F = \mathbb{Q}_p$ from this point of view.

We end the introduction with the following conjecture for any non-archimedean local field F: For any $n \geq 1$, let D^* be the multiplicative group of a division algebra D over F of dimension n^2, and let $\sigma \mapsto \pi$ be the correspondence predicted by the local Langlands conjecture [La]. Then, whenever σ is self-dual and symplectic, π is orthogonal.

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1. Preliminaries

Let F be $\mathbb{R}$ or a non-archimedean local field of odd residual characetristic. If K/F is a quadratic extension, we will denote by $\omega_{K/F}$ the quadratic character of F^* given by class field theory. When F is non-archimedean, let $\mathfrak{O}_F$ denote the ring of integers of F, ϖ $(= \varpi_F)$ a fixed uniformizer, q the cardinality of the residue field, and ω the unique non-trivial quadratic character of $\mathbb{F}_q^*$. In this case, the Weil group W_F is the subgroup of Gal$(\overline{F}/F)$ consisting of automorphisms τ which induce an integral power of the Frobenius $\phi_q : x \mapsto x^q$ on $\overline{\mathbb{F}}_q$; it is thus a non-trivial extension of $\{\phi_q^n\} \simeq \mathbb{Z}$ by the inertia group I_F and has dense image in Gal$(\overline{F}/F)$. When $F = \mathbb{R}$, $W_{\mathbb{R}}$ can be realized as $\mathbb{C}^* \cup j\mathbb{C}^*$, where j satisfies $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for all $z \in \mathbb{C}^*$; it is the unique non-trivial extension of Gal$(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$ by $\mathbb{C}^*$.

Fix a non-trivial (unitary) character $\psi = \psi_F$ of the additive group of F and a Haar measure dx on F^+. We refer to the articles of Deligne ([D2]) and Tate ([T]) for the definition and the basic properties of the epsilon factors $\varepsilon(\sigma, \psi, dx, s)$ of representations (σ, V) of Gal$(\overline{F}/F)$ (or W_F). We note that when V has dimension zero and determinant 1, $\varepsilon(\sigma, \psi, dx, s)$ is independent of (ψ, dx), and we will simply write $\varepsilon(\sigma, s)$. To avoid ambiguity in general, we will take ψ to have conductor $\mathfrak{O}_F$, and normalize dx to be the self-dual measure relative to ψ. Thus our epsilon factors are those defined by Langlands. For self-dual representations, we will set: $W(\sigma) = W(\sigma, \psi) = \varepsilon(\sigma, \psi, 1/2)$.

If η is a representation of Gal$(\overline{F}/F)$ (or W_F), let $\mathfrak{f}(\eta)$ denote the exponent of the conductor of η (resp. 0) if F is non-archimedean (resp. archimedean).

The following three basic results on local constants will be used in our calculations.

Proposition 1.1. *([D2]) Let F be non-archimedean, σ a representation of W_F of dimension n, and μ a (quasi-) character of F^* ($\simeq W_F^{\mathrm{ab}}$). If either μ or σ is unramified, we have*

$$W(\sigma \otimes \mu) \,=\, \det(\sigma)(\varpi^{\mathfrak{f}(\mu)})\mu(\varpi^{\mathfrak{f}(\sigma)})W(\sigma)W(\mu)^n.$$

Theorem 1.2. *([D2, Lemma 4.1.6]) Let F be non-archimedean, and let α, β be two (quasi-) characters of F^* such that $\mathfrak{f}(\alpha) \geq 2\,\mathfrak{f}(\beta)$. Choose $y \in F$ as* *follows: If $\mathfrak{f}(\alpha)$ is positive, let y be such that $\alpha(1+x) = \psi(xy)$ for all $x \in F$ with $\operatorname{val}(x) \geq \frac{1}{2}\,\mathfrak{f}(\alpha)$; if the conductor of α is 0, let $y = \varpi^{-cond(\psi)}$. Then*

$$W(\alpha\beta, \psi) \ = \ \beta^{-1}(y) W(\alpha, \psi) \ .$$

Theorem 1.3. *(Frohlich-Queyrut [F-Q, Theorem 3]) Let K be a separable quadratic extension of a local field F, and let ψ_K be the additive character of K defined by $\psi_K(x) = \psi(\operatorname{tr} x)$. Then for any character χ of K^* which is trivial on F^*, and any $x_0 \in K^*$ with $\operatorname{tr}(x_0) = 0$,*

$$W(\chi, \psi_K) \ = \ \chi(x_0) \ .$$

Let (σ, V) be an irreducible representation of $W_{\mathbb{R}}$. Then it is easy to see it must be of dimension 1 or 2; in the latter case there exists a (quasi) character $\chi : \mathbb{C}^* \to \mathbb{C}^*$ such that $\chi(z) \neq \chi(\overline{z})$ and $\sigma \simeq \operatorname{Ind}_{\mathbb{C}^*}^{W_{\mathbb{R}}}(\chi)$ with $\det(\sigma) = \omega_{\mathbb{C}/\mathbb{R}}(\chi|_{\mathbb{C}^*})$.

When F is non-archimedean of odd residual characteristic, every irreducible *two-dimensional* representation σ of W_F has a similar description. It is the pull back via $W_F \to W_{K/F}$, for some quadratic extension K/F, of an induced representation $Ind_{K^*}^{W_{K/F}}(\chi)$, where χ is a (quasi) character of K^* such that $\chi \neq \chi \circ \rho$, ρ being the non-trivial automorphism of K/F.

Lemma 1.4. *Let F be non-archimedean, and let K/F be a ramified quadratic extension. Suppose χ is a character of K^* which restricts to $\omega_{K/F}$ on F^*. Then either $\mathfrak{f}(\chi) = 2m$, with m a positive integer, or $\mathfrak{f}(\chi) = 1$ and the restriction of χ to $\mathfrak{O}_K^*$ is the inflation $\tilde{\omega}_{K/F}^0$ of the character of $(\mathfrak{O}_K/\varpi_K\mathfrak{O}_K)^*$ defined by $\omega_{K/F}$ via its natural identification with $(\mathfrak{O}_F/\varpi\mathfrak{O}_F)^*$.*

To see this, let $\tilde{\omega}_{K/F}$ denote either of the (two possible) extensions of $\tilde{\omega}_{K/F}^0$ to K^*, and consider $\mu := \chi/\tilde{\omega}_{K/F}$. Then μ defines a character of $K^*/F^*(1 + \varpi_K^{\mathfrak{f}(\chi)}\mathfrak{O}_K)$. The assertion is clear if $\mathfrak{f}(\chi) = 1$, and when $\mathfrak{f}(\chi > 1$, it is a consequence of the fact that $F^*(1 + \varpi_K^{2n}\mathfrak{O}_K) \ = \ F^*(1 + \varpi_K^{2n+1}\mathfrak{O}_K)$ for all positive n.

Let D be the unique quaternion division algebra over (the local field) F with reduced norm Nrd: $D^* \ \to \ F^*$. For every irreducible representation (π, X) of D^*, one knows (cf. [JL]) how to associate an epsilon factor $\varepsilon(\pi, \psi, s)$. One sets $W(\pi, \psi) \ = \ \varepsilon(\pi, \psi, 1/2)$. Denote by $\mathfrak{f}(\pi)$ the exponent of the conductor of π, and by ω_π the central character of π. If μ is a character of F^*, we will write $\pi \otimes \mu$ for the representation $\pi \otimes (\mu \circ \mathrm{Nrd})$. If $\pi^\vee$ denotes the contragredient (dual) of π, then it is isomorphic to $\pi \otimes \omega_\pi^{-1}$. When π is self-dual, $W(\pi, \psi)$ is ± 1 and ω_π is quadratic. The key theorem below follows by combining the main results of [J-L] and [K].

Theorem 1.5. *There is a bijection $\sigma \ \to \ \pi \ = \ \pi(\sigma)$ of the set of equivalence classes of continuous, irreducible $2-$ dimensional $\mathbb{C}-$ representations of W_F onto the set of equivalence classes of continuous, irreducible $\mathbb{C}-$ representations of D^* of dimension > 1 (resp. ≥ 1) if F is non-archimedean (resp. $F = \mathbb{R}$). This bijection satisfies*

(1) $det(\sigma) \ = \ \omega_\pi$;
(2) $\mathfrak{f}(\sigma) \ = \ \mathfrak{f}(\pi)$;
(3) $\pi(\sigma^\vee) \ \simeq \ \pi(\sigma)^\vee$; *and*
(4) $\varepsilon(\sigma \otimes \mu, \psi, s) \ = \ \varepsilon(\pi \otimes \mu, \psi, s)$,

for all characters μ *of* F^* *.*

Note that under this bijection, σ is self-dual iff π is. Moreover, if π is the representation of D^* associated to a σ induced by a character χ of a separable quadratic extension K/F , then $\omega_\pi = \chi|_{F^*}\omega_{K/F}$.

The following Proposition summarizes the information we need about the characters of irreducible representations π of D^*/F^*, for F non-archimedean. (There is a similar, but simpler, formula for $F = \mathbb{R}$, which we will not need; our treatment of that case will be more direct.) It can be deduced by combining the explicit character formulae for irreducible admissible representations π' of $PGL(2, F)$ (see [Si], p.50-51) with the fact (see [J-L], Prop.15.5) that there is an injection $\pi \to \pi'$ of the irreducible representations of D^*/F^* into the discrete series of $PGL(2, F)$ preserving epsilon factors such that the characters of π and π' agree on the elliptic tori up to sign.

Proposition 1.6. *Let* F *be non-archimedean,* K/F *a quadratic extension and* $\pi = \pi_\chi$ *be the representation of* D^*/F^* *attached to a character* χ *of* K^**. Then we have the following table*

K/F	$\mathfrak{f}(\chi)$	$\dim(\pi)$	$\mathfrak{f}(\pi)$
unramified	f	$2q^{f-1}$	$2f$
ramified	$2f$	$(q+1)q^{f-1}$	$2f+1$

Let L *be any quadratic extension of* F *, and* x *the unique element of* L^*/F^* *of order 2. Denote by* Θ_π *the character of* π*. Then we have:*

(1) *If* $L \neq K$, $\Theta_\pi(x) = 0.$
(2) *If* $L = K$ *and* K/F *unramified,* $\Theta_\pi(x) = (-1)^{f+1}2\chi(x)$
(3) *If* $L = K$ *and* K/F *ramified,*

$$\Theta_\pi(x) \; = \; -2G_\chi\omega(2)\omega(-1)^{f-1}\chi(x),$$

where

$$G_\chi \; = \; \frac{1}{\sqrt{q}} \sum_{x\in(\mathfrak{O}_F/\varpi_F)^*} \chi(1+\varpi_K^{2f-1}x)\omega(x).$$

We will also need the following result on the toric restriction of representations of D^* .

Proposition 1.7. *Let* F *be non-archimedean,* K/F *a (separable) quadratic extension,* σ *an irreducible symplectic representation of* W_F *of dimension* 2*, and* π *the associated (irreducible) representation of* D^*/F^**. Then*

(1) *The restriction of* π *to* K^* *is multiplicity free;*
(2) *For a character* χ *of* K^*/F^* *to occur in* $\pi|_{K^*}$ *, it is necessary and sufficient that*

$$W(\sigma|_{W_K} \otimes \chi^{-1}, \psi \circ tr_{K/F}) \; = \; -1.$$

For a proof of part (1), see [P1], Remark 3.5, and for part (2) (Tunnell's formula), see [Tu].

We conclude this section by recalling some basic facts about Stiefel-Whitney classes. For any compact group G, let $\mathfrak{C}(G,\mathbb{R})$ denote the category of *real* representations of G. A continuous representation σ of G on a finite-dimensional vector space V is real iff it is realizable over $\mathbb{R}$. It is easy to see that a self-dual σ has a real character, and it is realizable over $\mathbb{R}$ iff it is orthogonal. Denote by $R(G,\mathbb{R})$ the Grothendieck group of virtual representations in $\mathfrak{C}(G,\mathbb{R})$, and by $H^*(G,\mathbb{Z}/2)$ the $\mathbb{Z}/2-$ cohomology ring $\oplus_{i\geq 0} H^i(G,\mathbb{Z}/2)$ (with G acting trivially on $\mathbb{Z}/2$). Then there is a Stiefel-Whitney homomorphism of groups (see [De1])

$$w_* : R(G.\mathbb{R}) \longrightarrow H^*(G,\mathbb{Z}/2)^\times,$$

which sends σ to $\sum_{i\geq 0} w_i(\sigma)$, with $w_0(\sigma) = 1$ and $w_1(\sigma)$ being the image of det under the isomorphism $\mathrm{Hom}(G,\pm 1) \simeq H^1(G,\mathbb{Z}/2)$. By construction, if σ_1, σ_2 are virtual sums of real representations, then we have

$$w_2(\sigma_1 \oplus \sigma_2) = w_2(\sigma_1) + w_2(\sigma_2) + w_1(\sigma_1) \cup w_1(\sigma_2). \tag{1.8}$$

As in the introduction, we let $\tilde{w}_i(\sigma)$ to be 1 or -1 according as $w_i(\sigma)$ is trivial or not.

If σ is a genuine real representation with trivial determinant, then $w_2(\sigma)$ is the class of the extension of G by $\{\pm 1\}$ obtained by pulling back via σ the short exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \to \mathrm{SO(V)} \to 1,$$

where $\mathrm{Spin}(V)$ is the Spin group of V, the non-trivial double cover of the special orthogonal group SO(V); thus $\tilde{w}_2(\sigma)$ is trivial (in this case) iff σ can be lifted to a representation of G into the spin group.

Finally, we define, for any orthogonal (σ, V), a homomorphism

$$\tilde{\sigma} : G \longrightarrow \mathrm{SO}(V \oplus \mathbb{C})$$

by $g \to (\sigma(g), \det(\sigma(g)))$.

2. Orthogonality of π

In view of the remark following Proposition A, it is a consequence of

Proposition 2.1. *Every finite dimensional representation of D^*/F^* is orthogonal.*

Proof. If $x \mapsto \bar{x}$ denote the canonical anti-automorphism of D^* such that $x \cdot \bar{x} = Nrd(x)$ where $Nrd(x)$ is the reduced norm of x, then as an element of D^*/F^*, $\bar{x} = x^{-1}$. By Skolem-Noether theorem, x and $\bar{x}$ are conjugate, and therefore x is conjugate to x^{-1} in D^*/F^*. This implies that every representation of D^*/F^* is self-dual.

When $F = \mathbb{R}$, D^*/F^* is isomorphic to SO (3), and any irreducible representation π of this group is odd dimensional. (See Remark 2.5 below for an explicit description.) Clearly, every self-dual representation of odd dimension must be orthogonal. So we may (and we will) assume henceforth that F is non-archimedean.

We can find a quadratic extension E of F such that the trivial character of E^* appears in X. In fact we have

Lemma 2.2. *Let π be associated to a character χ on a quadratic extension K of F, so that the corresponding representation σ of W_F is $\mathrm{Ind}_{W_K}^{W_F}(\chi)$. Then if K is ramified, π contains the trivial character of the unramified quadratic extension, while if K is unramified, then π contains the trivial character of any ramified quadratic extension.*

Indeed, for any quadratic extension E/F, one knows that by part (2) of Proposition 1.7, the trivial representation of E^* occurs iff we have

$$W(\sigma|_{W_E}, \psi \circ tr_{E/F}) = W(\pi,\psi)W(\pi\otimes\omega_{E/F},\psi)\omega_{E/F}(-1) = -1. \tag{2.3}$$

When $E = K$, $\pi\otimes\omega_{E/F} \simeq \pi$. Since $W(\sigma,\psi) = \pm 1$, the trivial character of K^* occurs in π iff $\omega_{K/F}(-1) = -1$, which happens iff K/F is ramified and $q \equiv 3$ modulo 4. If K/F is ramified, but with $q \equiv 1$ modulo 4, take E to be the unramified quadratic extension. Then $\omega_{E/F}$ is an unramified character, and by Prop.1.1 (and Theorem 1.5),

$$W(\pi\otimes\omega_{E/F},\psi) = \omega_{E/F}(\mathfrak{f}(\pi))W(\pi)W(\omega_{E/F})^2 = -W(\pi). \tag{2.4}$$

The second equality comes from the fact (cf. Proposition 1.6) that $\mathfrak{f}(\pi)$ is odd when K/F is ramified. Combining (2.4) with (2.3), we see that the trivial character of E^* occurs in π.

Finally, let K be unramified. Take E to be either of the two ramified extensions of F. Then we claim that

$$W(\pi\otimes\omega_{E/F}) = -\omega_{E/F}(-1)W(\pi). \tag{2.5}$$

Indeed, if π corresponds to the representation $\sigma = \mathrm{Ind}_{W_K}^{W_F}(\chi)$, then the additivity and inductivity in dimension zero of the epsilon factors gives $W(\pi\otimes\omega_{E/F},\psi)/W(\pi,\psi) = W(\chi\mu,\psi_K)/W(\chi,\psi_K)$, where $\mu = \omega_{E/F}\circ N_{K/F}$ is the unique nontrivial quadratic character of K^*/F^*. Let ν be the unramified character of K^* taking the value -1 on any uniformizing parameter ϖ_K. Then by construction, ν restricts to $\omega_{K/F}$ on restriction to F^*; so do the characters χ and $\chi\mu$ as the determinant of $\mathrm{Ind}_{W_K}^{W_F}(\chi)$ is trivial. On the other hand, for any character β of K^*, we have by Proposition 1.1: $W(\beta\nu,\psi_K) = \nu(\mathfrak{f}(\beta))W(\beta,\psi_K)W(\nu,\psi_K)$, where $\psi_K = \psi\circ tr_{K/F}$. This gives $W(\chi\mu,\psi_K)/W(\chi,\psi_K) = W(\chi\mu\nu,\psi_K)/W(\chi\nu,\psi_K)$, which, by Theorem 1.3, equals $\chi\mu\nu(t)/\chi\nu(t) = \mu(t)$, where t is the unique element of order 2 in K^*/F^*. Using the fact that we may represent t in K^* by the square-root of a unit u in $\mathfrak{O}_F^*$ which is a non-square in the residue field $\mathbb{F}_q$, one sees that $\mu(t)$ is -1 (resp. 1) when q is 1 (resp. 3) modulo 4, which is the negative of the value of $\omega_{E/F}$ at -1. Hence the claim. The Lemma now follows by combining (2.3) and (2.5).

Let E/F be as in the Lemma. Then, since the restriction of π to E^* is multiplicity-free (see Proposition 1.7, part (1)), the eigenspace of X corresponding to the trivial character of E^* must in particular be one-dimensional. The unique non-degenerate bilinear form on X must be non-zero on this one-dimensional subspace, and therefore the bilinear form must be symmetric, whence the Proposition. □

Remark 2.6. It should be noted that self-dual representations π of D^* not factoring through D^*/F^* need not be orthogonal. To see this let $F = \mathbb{R}$ and $\pi = \rho \otimes \det(\rho)^{-1/2}$, where ρ is the standard two-dimensional $\mathbb{C}-$ representation of D^*. Then the symmetric square of π is irreducible.

The proof of Proposition 2.1 shows more generally that a self-dual irreducible representation (η, Y) of a group G must be orthogonal if we can find a subgroup H such that the restriction to H is completely reducible and contains the trivial representation of H with multiplicity one. One gets the following

Proposition 2.7. *Every irreducible, admissible, self-dual, generic representation (η, Y) of $GL(n, F)$, F non-archimedean, is orthogonal for any $n \geq 1$.*

Indeed, the theory of new vectors for generic representations of $GL(n, F)$ (cf. [J-PS-S]) gives the existence of an open compact subgroup C such that the space of $C-$ invariant vectors in Y is one-dimensional. The restriction of η to C is completely reducible by admissibility.

Note that since every discrete series representation is generic, this Proposition applies in particular to any representation of $GL(2, F)$ associated to an irreducible representation (π, X) of D^* by the Jacquet-Langlands correspondence.

3. Criteria for liftability

Let $\Sigma(F)$ denote the set of quadratic extensions of F in $\overline{F}$. The object of this section is to show that for any virtual sum σ of orthogonal representations of D^*/F^*, the second Stiefel-Whitney number $\tilde{w}_2(\sigma)$ is 1 iff it is so when restricted to K^*/F^*, for every $K \in \Sigma(F)$. In fact we have

Proposition 3.1. *The natural homomorphism (given by restriction)*

$$H^2(D^*/F^*, \mathbb{Z}/2) \longrightarrow \bigoplus_{K \in \Sigma(F)} H^2(K^*/F^*, \mathbb{Z}/2)$$

is injective.

Proof. First consider the case $F = \mathbb{R}$. One has a natural isomorphism $H^2(D^*/\mathbb{R}^*, \mathbb{Z}/2) \simeq \mathrm{Hom}(\pi_1(D^*/\mathbb{R}^*), \mathbb{Z}/2)$. Hence it suffices to establish the surjectivity at the fundamental group level, i.e., $\pi_1(\mathbb{C}^*/\mathbb{R}^*) \to\!\!\to \pi_1(D^*/\mathbb{R}^*)$. This is clear via the identifications of $\mathbb{C}^*/\mathbb{R}^*$ and $D^*/\mathbb{R}^*$ with SO (2) and SO (3) respectively.

Now let F be non-archimedean, and denote by U_D^1 the image in D^*/F^* of the first congruence subgroup of D^* under the standard filtration. Then since the residue characteristic of F is odd, $H^i(U_D^1, \mathbb{Z}/2) = 0$ if $i > 0$. It follows that $H^2(D^*/F^*, \mathbb{Z}/2) = H^2(D^*/F^*U_D^1, \mathbb{Z}/2)$. Now $D^*/F^*U_D^1$ is a dihedral group defined by the extension

$$0 \to \mathbb{F}_{q^2}^*/\mathbb{F}_q^* \to D^*/F^*U_D^1 \to \mathbb{Z}/2 \to 0 ,$$

where $\mathbb{F}_q$ is the residue field of F.

Let D_r be the quotient of $D^*/F^*U_D^1$ by its maximal subgroup of odd order. Then $H^2(D^*/F^*, \mathbb{Z}/2)$ is the same as $H^2(D_r, \mathbb{Z}/2)$, and D_r is a dihedral $2-$ group given by

$$0 \to \mathbb{Z}/2^r \to D_r \to \mathbb{Z}/2 \to 0 .$$

Clearly $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \subseteq D_r$, and one sees that $H^2(D_r, \mathbb{Z}/2) \cong H^2(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2)$, and an element of $H^2(\mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2)$ is zero if and only if its restriction to all the three $\mathbb{Z}/2$'s in $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ is zero. These three $\mathbb{Z}/2$'s come from the three distinct quadratic extensions, whence the proposition.

Lemma 3.2. *Let $SO(2n+1, \mathbb{C})$ correspond to the quadratic form $q = x_1x_2 + \cdots + x_{2n-1}x_{2n} + x_{2n+1}^2$, and T the associated maximal torus. For characters $(\chi_1, \ldots, \chi_n)$ of an abelian group G , let ϕ be the representation of G with values in $SO(2n+1, \mathbb{C})$ given by $x \mapsto (\chi_1(x), \chi_1^{-1}(x), \chi_2(x), \chi_2^{-1}(x), \ldots, \chi_n(x), \chi_n^{-1}(x), 1)$. Then the representation ϕ of G lifts to $\mathrm{Spin}(2n+1, \mathbb{C})$ if and only if $\prod_{i=1}^{n} \chi_i = \mu^2$ for some character μ of G , i.e. if and only if $\prod_{i=1}^{n} \chi_i$ is trivial on the subgroup $G[2] = \{g \in G \mid 2g = 1\}$.*

Proof. The assertion is a direct consequence of the fact that the spin covering of $SO(2n+1, \mathbb{C})$ when restricted to the maximal torus $T = \{(z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}, 1) \mid z_i \in \mathbb{C}^*\}$ is the two-fold cover of T obtained by attaching $\sqrt{\prod z_i}$.

4. Formulae for toric restrictions

In this section F will always be non-archimedean, and (π, X) an irreducible representation of D^*, trivial on F^*, of dimension > 1 and character Θ_π .

Let $E = F(x)$ be a separable quadratic extension. with $x^2 \in F^*$. Clearly $x = x(E)$ is the unique element of E^*/F^* of order 2.

By Skolem-Noether theorem, there is an element g in D^*/F^* such that the inner conjugation action of g on K^* preserves K^* and induces the non-trivial Galois action on it. It follows that whenever a character μ of K^* appears in π , so does μ^{-1} . The (multiplicity-free) restriction of π to E^* may then be decomposed as

$$X \;=\; \sum_{\mu \in S} \mu \oplus \sum_{\mu \in S} \mu^{-1} \oplus a \cdot 1 \oplus b \cdot \nu \qquad (i)$$

where $a = a(E)$ and $b = b(E)$ are integers $0 \le a, b \le 1$, $\nu = \nu(E)$ is the unique character of E^*/F^* of order 2, and $S = S(E)$ a finite set of characters of E^*/F^* of order ≥ 3 . Since the dimension of X is even (cf. Prop. 1.6), $a(E) = b(E)$. Recall that $\nu(x) = -1$ except when E is a quadratic unramified extension of F with $q \equiv 3 \mod (4)$, in which case $\nu(x) = 1$.

Lemma 4.1. *Let $s = s(E)$ denote the number of characters μ in $S(E)$ which take the value -1 on x. Define $\delta_q = \delta_q(E)$ to be 1 if E is unramified with q congruent to 3 modulo 4 , and to be 0 otherwise. Then*

$$\dim(X) \;=\; 4s + 2a(1 - 2\delta_q) + \Theta_\pi(x). \qquad (ii)$$

Moreover, $a(E)$ is > 0 in exactly the following cases:

1. *E is unramified and π is not associated to E;*
2. *E is ramified and π is unramified;*
3. *E is ramified, π is associated to E , and q is congruent to 3 modulo 4 ; and*
4. *E is ramified, π is ramified, but not associated to E, and $q \equiv 1 \,(\mathrm{mod}\, 4)$.*

Proof. Let $r = r(E)$ denote the number of characters μ in $S(E)$ which take the value 1 at x. (Since x has order 2 in E^*/F^*, every character of E^* trivial on F^* is ± 1 at x.) Evaluating the trace of $\pi|_{E^*/F^*}$ at 1 and x respectively, we get

$$\dim(X) = 2(r+s) + 2a$$

and

$$\Theta_\pi(x) = 2(r-s) + 2a\delta_q,$$

whence the assertion on the dimension of X.

Since $a = a(E)$ is the multiplicity of the trivial representation in the restriction of π to E^*, the occurrance of cases (1), (2) and (3) is contained in Lemma 2.2. Suppose we are in case (4) with E ramified, q congruent to 1 modulo 4, and π associated to a character χ of the other ramified quadratic extension L, say. Then $\omega_{E/F}(-1) = 1$, and we have to show that $W(\pi) = -W(\pi \otimes \omega_{E/F})$. Noting that the pull back of $\omega_{E/F}$ to L^* is the unique character $\nu = \nu(L)$ of order 2, which is unramified, and that $\mathfrak{f}(\chi)$ is even (see Prop. 1.6), we get $W(\pi \otimes \omega_{E/F}, \psi)/W(\pi, \psi) = W(\mathrm{Ind}_{W_L}^{W_F}(\chi\nu \ominus \chi), \psi_L) = W(\chi\nu, \psi_L)/W(\chi, \psi_L) = W(\nu, \psi_L) = \nu(x(L))$, which is indeed -1. It is also now evident that there are no further cases when the trivial representation occurs. □

Proposition 4.2. *Let π be an irreducible representation of D^*/F^* with values in $O(X)$ associated to a quadratic extension K of F, and let L be a quadratic extension of F different from K. Let f denote $\mathfrak{f}(\pi)$. Then the restriction to L^*/F^* of the associated representation $\tilde{\pi}$ with values in $SO(X \oplus \mathbb{C})$ lifts to* $\mathrm{Spin}(X \oplus \mathbb{C})$ *if and only if $\omega(-2) = -1$ if K is a ramified extension, and $\omega(-1)^{f-1} = -1$ if K is the unramified extension.*

Proof. By Lemma 3.2, the restriction of π to L^*/F^* lifts to $\mathrm{Spin}(X \oplus \mathbb{C})$ if and only if

$$\left(\nu^a \cdot \prod_{\mu \in X} \mu\right)(x) = 1 . \qquad (iii)$$

As x has order 2 in L^*/F^*, all the characters of L^*/F^* take the value ± 1 on x. Let r be the number of characters μ from X such that $\mu(x) = 1$, and let s be the number of characters μ from X such that $\mu(x) = -1$. By Proposition 1.4, the character of π at x is zero. Therefore we get (by Lemma 4.1)

$$\mathrm{d}im(X) = 4s + 2(a - \delta_q) . \qquad (iv)$$

We also have

$$\tilde{w}_2(\tilde{\pi}|_{L^*/F^*}) = (-1)^s \nu(x)^a . \qquad (v)$$

Proposition 4.3. *We have the following table for $L \neq K$ and $q = 2m+1$:*

K/F	$\tilde{w}_2(\tilde{\pi}\|_{L^*/F^*})$
unramified	$(-1)^{1+m(f-1)}$
ramified	$(-1)^{[1+\frac{m}{2}]}$

where $[t]$ *denotes, for any* $t \in \mathbb{R}$, *the integral part of* t.

Proof. By Proposition 1.6, the dimension of π is $2q^{f-1}$ (resp. $(q+1)q^{f-1}$) when K is unramified (resp. ramified), where f (resp. $2f$) denotes $\mathfrak{f}(\chi)$. Let us first consider the unramified case. Then δ_q is zero as L is ramified, and by equation (iv) above, $4s + 2a = 2q^{f-1}$. Since $a \in \{0,1\}$, we must have $a = 1$. (This also follows from Lemma 4.1.) Moreover,

$$s \;=\; \frac{1}{2}((2m+1)^{f-1} - 1) \;=\; \frac{1}{2}\sum_{j=1}^{f-1}\binom{f-1}{j}(2m)^j \;\equiv\; m(f-1) \pmod 2$$

which yields the assertion, thanks to equation (v).

Now let K/F be ramified. Then we know by Lemma 4.1 that a is 1 if either L is ramified or if L is ramified with $q = 2m+1$ congruent to 1 modulo 4; it is 0 otherwise. Thus when m is even (resp. odd), $a(1-\delta_q)$ is 1 (resp. 0), and by equation (iv), s is congruent modulo 2 to $\frac{m}{2}$ (resp. $\frac{m+1}{2}$). Furthermore, since $a = 0$ when m is odd, $\nu(x)^a = 1$ in that case. Applying (v) we see that $\tilde{w}_2(\pi|_{L^*/K^*})$ equals $(-1)^{\frac{m}{2}+1}$ (resp. $(-1)^{\frac{m+1}{2}}$) when m is even (resp. odd). This finishes the proof of Proposition 4.3.

Proposition 4.2 follows directly from this once we note that $\omega(-1) = (-1)^m$ and that $\omega(-2) = 1$ iff $q = 2m+1$ is congruent modulo 8 to 5 or 7.

We next consider the lifting problem for a representation π of D^*/F^* associated to a quadratic field K when restricted back to K^*/F^*. In this case the obstruction to lifting is related to the epsilon factor of π.

Proposition 4.4. *Let* π *be an irreducible representation of* D^*/F^* *with values in* $O(X)$ *associated to a character* χ *of* K^*, *where* K *is a (separable) quadratic extension of* F. *Then the restriction to* K^*/F^* *of the associated representation* $\tilde{\pi}$ *with values in* $SO(X\oplus\mathbb{C})$ *lifts to* $\mathrm{Spin}(X\oplus\mathbb{C})$ *if and only if* $\varepsilon(\pi) = -\omega(2)$ *if* K *is ramified and the conductor of the representation* π *is* $2f+1$, *and* $\varepsilon(\pi) = \omega(-1)^{f-1}$ *if* K *is unramified and the conductor of* π *is* $2f$.

Proof. The reasoning here is very similar to that above, and the assertion is an immediate consequence of the following

Proposition 4.5. *Write* $q = 2m+1$. *Then we have the following table:*

K/F	$\tilde{w}_2(\tilde{\pi}\|_{K^*/F^*})$
unramified	$(-1)^{m(f-1)+\frac{1}{2}(1-W(\pi))}$
ramified	$(-1)^{[(m+1)/2]+\frac{1}{2}(1+W(\pi))}$

Proof. Again, let x denote the unique element of order 2 in K^*/F^*. When K is ramified, x is the image of a uniformizing parameter ϖ_K such that $\varpi = \varpi_K^2 \in F^*$. Recall that by Proposition 1.4, we have

$$\Theta_\pi(x) \;=\; -2G_\chi\omega(2)\omega(-1)^{f-1}\chi(x),$$

when K/F is ramified, and

$$\Theta_\pi(x) \;=\; (-1)^{f-1}2\chi(x),$$

when K/F is unramified. Substituting the formula for $\dim(X)$, we get by Lemma 4.1,

$$4s + 2a \;=\; (q+1)q^{f-1} + 2G_\chi \omega(2)\omega(-1)^{f-1}\chi(x) \qquad (vi)$$

when K/F is ramified, and

$$4s \;=\; 2q^{f-1} + (-1)^{f-1}2\chi(x) \qquad (vii)$$

when K/F is unramified.

To establish Proposition 4.5, we now need to relate the character value at x to the epsilon factor. To begin, since the representation σ of W_F associated to π is induced from the character χ of K^*, the additivity of epsilon factors gives us the following:

$$\varepsilon(\pi) \;=\; \varepsilon(\mathrm{Ind}_{K^*}^{W_F}\chi, \psi_F) \;=\; \varepsilon(\mathrm{Ind}_{K^*}^{W_F}(\chi-1), \psi_F)\cdot\varepsilon(\mathrm{Ind}_{K^*}^{W_F}1, \psi_F)$$

We get by the inductivity (in dimension 0) of epsilon factors,

$$W(\pi,\psi) \;=\; W(\chi,\psi_K)W(\omega_{K/F},\psi). \qquad (viii)$$

Choose a (quasi) character β of K^* which extends $\omega_{K/F}$. Then $\chi\beta$ is trivial on F^*, and we get by Theorems 1.2 and 1.3,

$$W(\pi,\psi) \;=\; W(\chi\beta,\psi_K)\beta(y)W(\omega_{K/F},\psi) \;=\; (\chi\beta)(x)\beta(y)W(\omega_{K/F},\psi), \qquad (ix)$$

where y is the element of K^* with the property:

$$\chi\beta(1+x) \;=\; \psi_K(xy) \qquad \text{for all} \quad x \quad \text{with} \quad \mathrm{val}\,(x) \geq \frac{1}{2}\mathfrak{f}(\chi).$$

Lemma 4.6. *We have the following table:*

K/F	$W(\pi)$
unramified	$(-1)^f\chi(x)$
ramified	$\omega(2)\omega(-1)^{f+1}G_\chi\chi(x)$

Proof of Lemma. First we make the choice of β explicit as follows. When K is unramified, take β to be the unramified character of K^* which takes the value -1 at any uniformizer (say ϖ_K) of K. When K is ramified, take β to be the character $\tilde{\omega}_{K/F}$ defined in the proof of Lemma 1.4. Note that $\mathfrak{f}(\beta)$ is 0 when K is unramified, and equals 1 when K is ramified.

First consider the unramified case. Then, since $\omega_{K/F}$ is unramified, our choice of ψ in §1 implies that $W(\omega_{K/F})$ is trivial, and thus (iii) gives $W(\pi,\psi) = W(\chi,\psi_K)$. Moreover, since β is unramified, we get (by Prop.1.1 or by using (ix))

$$W(\chi\beta,\psi_K) \;=\; W(\chi,\psi_K)\beta(\varpi_K^f)W(\beta,\psi_K),$$

which equals $(-1)^fW(\chi,\psi_K)$ as $\beta(\varpi_K) = -1$. Thus $W(\pi,\psi) = (-1)^fW(\chi\beta,\psi_K)$. But the theorem of Frohlich and Queyrut (Thm.1.3) says that

$W(\chi\beta, \psi_K)$ is $\chi\beta(x)$. Clearly, $\beta(x) = 1$ as x is a unit and β unramified. The assertion of the Lemma follows for K/F unramified.

Next we consider the ramified case, which is more subtle. Let y be as in equation (ix). Then we can write: $y = \varpi_K^{-(2f+1)} a_0(\chi) +$ higher order terms. It follows that

$$\chi(1 + \varpi_K^{2f-1} x) \ = \ \psi_K(\varpi_F^{-1} a_0(\chi) x) \ .$$

From the definition of epsilon factors ([T]),

$$\sum_{x \in (\mathfrak{O}_F/\varpi)^*} \omega(x) \psi_F(\varpi_F^{-1} x) \ = \ \sqrt{q}\, \omega_{K/F}(\varpi) W(\omega_{K/F}, \psi) \ ,$$

and therefore

$$\sum_{x \in (\mathfrak{O}_F/\varpi)^*} \omega(x) \chi(1 + \varpi_K^{2f-1} x) \ = \ \sqrt{q}\, \omega_{K/F}(2 a_0(\chi) \varpi_F) W(\omega_{K/F}, \psi) \ .$$

Comparing with the definition of G_χ, we get

$$G_\chi \ = \ \omega_{K/F}(2 a_0(\chi) \varpi_F) W(\omega_{K/F}, \psi_F) \ .$$

Thus by equation (ix), $W(\pi, \psi)$ is given by

$$(\chi\beta)(\varpi_K) \beta(\varpi_K^{-(2f+1)} a_0(\chi)) W(\omega_{K/F}, \psi) \ = \ \chi(\varpi_K) \omega_{K/F}(\varpi_F^{-f} a_0(\chi)) W(\omega_{K/F}, \psi),$$

which equals $\chi(\varpi_K)\omega(2)\omega(-1)^{f+1} G_\chi$ as $\omega_{K/F}(\varpi_F) = \omega(-1)$. Hence the Lemma. $\square$.

Proof of Prop.4.5 (contd.) Combining this Lemma with equation (vi), we get, for K/F ramified,

$$4(s+a) \ = \ (q+1) q^{f-1} + 2a + 2W(\pi, \psi) \ . \qquad (viii)$$

As K is ramified, $\nu(x) = -1$, and we have (by (v)) $\tilde{w}_2(\tilde{\pi}|_{K^*/F^*}) = (-1)^{s+a}$. We know by Lemma 4.1 that, for K ramified, m is even iff $a = 0$. Suppose $m = 2k$, for some $k \in \mathbb{N}$. Then

$$s + a \ = \ k(4k+1)^{f-1} + \frac{1}{2}\{ \sum_{j=0}^{f-1} \binom{f-1}{j} (4k)^j \ + \ W(\pi) \},$$

which is congruent modulo 2 to $k + \frac{1}{2}(1 + W(\pi))$. If $m = 2k-1$, then we see that

$$s + a \ = \ k(4k-1)^{f-1} + \frac{1}{2}(1 + W(\pi)),$$

which is congruent modulo 2 to $k + \frac{1}{2}(1 + W(\pi))$.

This proves Proposition 4.5 in the ramified case. Suppose K is unramified. Then we have by equation (vii) and Lemma 4.6,

$$2s \ = \ q^{f-1} - 2W(\pi).$$

Setting $q = 2m+1$, we get

$$s \ = \ \frac{1}{2} \sum_{j=1}^{f-1} \binom{f-1}{j} (2m)^j \ + \ (1 - W(\pi)),$$

which is congruent modulo 2 to $(f-1)m + \frac{1}{2}(1 - W(\pi))$. Applying equation (iii) and the fact that $a = 0$ (cf. Lemma 4.1), we see that $\tilde{w}_2(\pi|_{K^*/F^*})$ equals $W(\pi)$ or $-W(\pi)$ depending on whether $(f-1)m$ is even or odd. $\square$.

5. The main result

Propositions 4.2 and 4.4 can now be combined to give Theorem C.

We now begin the proof of Theorem B. First note that, since the virtual representation $\pi \ominus \pi'$ has by hypothesis determinant 1, we have by (1.8),

$$w_2(\tilde{\pi}) \;=\; w_2(\tilde{\pi} \ominus \tilde{\pi}') + w_2(\tilde{\pi}') \;=\; w_2(\pi \ominus \pi') + w_2(\tilde{\pi}').$$

This implies, by Proposition 3.1, that

$$\tilde{w_2}(\pi \ominus \pi') = 1 \quad \text{IFF} \quad w_2(\tilde{\pi})|_{E^*} = w_2(\tilde{\pi}')|_{E^*}, \quad \forall\, E \in \Sigma(F). \tag{5.1}$$

Lemma 5.2. *Let F be non-archimedean and π an irreducible representation of D^* of dimension >1 and trivial central character. Then we have*

$$\det(\pi) \;=\; \omega_{L/F} \circ Nrd,$$

where L is a separable quadratic extension chosen as follows: $L = K$ if K/F unramified or if F is non-archimedean with $q \equiv 1$ modulo 4. If F is non-archimedean with $q \equiv 3$ modulo 4 and K/F ramified, L is the other ramified quadratic extension. In particular, any such π is associated to a unique (separable) quadratic extension.

Proof. Since the kernel of Nrd is the commutator subgroup of D^*, we can write $\det(\pi)$ as $\mu \circ Nrd$, for a character μ of F^*. Since π is self-dual, its determinant has order dividing 2, and by class field theory, μ is either trivial or $\omega_{E/F}$, for a quadratic extension E/F. Let K be a (separable) quadratic extension of F such that π is associated to a character χ of K^*. Let us first consider the non-archimedean case. For any $E \in \Sigma(F)$, the decomposition of (π, X) given by (i) (of §4) implies that

$$\det(\pi|_{E^*/F^*}) \;=\; 1 \quad \text{IFF} \quad a(E) \;=\; 0,$$

in which case μ is trivial on the norm subgroup NE^*. Suppose K is unramified or ramified with q congruent to 1 modulo 4. Then by Lemma 4.1, $a(K) = 0$, and $a(L) = 1$ for either of the remaining extensions L in $\Sigma(F)$. So μ is trivial on NK^*, but not on NL^*. Hence $\mu = \omega_{K/F}$. Similarly, when K is ramified with q congruent to 3 modulo 4, $a(E) = 0$ for the other ramified quadratic extension E, and $a(L) = 1$ for L different from E. Thus $\beta = \omega_{E/F}$ as claimed.

The expression for the determinant immediately gives the assertion about the uniqueness of K given such a π. (This can also be seen directly by first showing that if π is associated to more than one quadratic extension of F, then it is associated to all the three quadratic extensions, which contradicts the formula for $\dim(\pi)$ (Proposition 1.6). Note that π can be associated to more than one extension if either ω_π is non-trivial or if the residual characteristic is 2.) $\square$

Let π, π' be as in Theorem B. Then, since $\det(\pi)$ equals $\det(\pi')$, we see by the Lemma above that there is a unique $K \in \Sigma(F)$ such that π and π' are both associated to characters χ and χ' of K^*.

Combining Propositions 4.3 and 4.5, we get

Proposition 5.3. *Let $\pi,\ \pi'$ be as above, associated to characters $\chi,\ \chi'$ of a quadratic extension K/F. Then we have the following table for F non-archimedean with residue field of $q = 2m+1$ elements, and for $L \neq K$ quadratic over F :*

K/F	$\tilde{w}_2(\tilde{\pi}\|_{K^*/F^*})/\tilde{w}_2(\tilde{\pi}'\|_{K^*/F^*})$	$\tilde{w}_2(\tilde{\pi}\|_{L^*/F^*})/\tilde{w}_2(\tilde{\pi}'\|_{L^*/F^*})$
unramified	$(-1)^{m(\mathfrak{f}(\chi)-\mathfrak{f}(\chi'))+\frac{1}{2}(W(\pi)-W(\pi'))}$	$(-1)^{m(\mathfrak{f}(\chi)-\mathfrak{f}(\chi'))}$
ramified	$(-1)^{\frac{1}{2}(W(\pi)-W(\pi'))}$	1

Proof. The assertion is an immediate consequence of Propositions 4.3 and 4.5. □

When K is ramified, then $\mathfrak{f}(\pi)$ and $\mathfrak{f}(\pi')$ are both odd (see Prop.1.5), and $s(\pi) = s(\pi') = 0$. When K is unramified, $\mathfrak{f}(\pi) = 2\mathfrak{f}(\chi)$ and $\mathfrak{f}(\pi') = 2\mathfrak{f}(\chi')$; thus $s(\pi) = m\mathfrak{f}(\chi)$ and $s(\pi') = m\mathfrak{f}(\chi')$. Since $s(\pi)$ and $s(\pi')$ are assumed to have the same parity, Theorem B now follows (for F non-archimedean) by appealing to (5.1). Note also that, when $\pi \ominus \pi'$ has determinant 1, the conductors of π and π' are both even or both odd as they are associated to the same quadratic extension K; when they are even, then K must be unramified. If moreover, $\pi \ominus \pi'$ has dimension zero (modulo 2), then as claimed in the introduction, $s(\pi)$ and $s(\pi')$ have the same parity. □

An immediate consequence of Prop. 5.3 and (5.1) is the following variant of Theorem B:

Proposition 5.4. *Let F be a non-archimedean local field of odd residual characteristic. Suppose $\sigma,\ \sigma'$ are two-dimensional, irreducible, symplectic representations of W_F such that the associated representations $\pi,\ \pi'$ of D^* have the same determinant. Then*

$$W(\sigma \ominus \sigma') \,=\, (-1)^{s(\pi)-s(\pi')} \prod_{E \in \Sigma(F)} \tilde{w}_2((\pi \ominus \pi')|_{E^*/F^*}).$$

It remains to treat the case $F = \mathbb{R}$. As noted before, $D^*/\mathbb{R}^*$ identifies with SO(3), and its irreducible representations are parametrized by their dimension, which must be odd. For every integer $k \geq 0$, the unique irreducible π_{2k+1}, say, of dimension $2k+1$ corresponds to the symmetric $2k-$th power representation of the standard representation of SU(2). It is easy to see that the image of π_{2k+1} lands in SO$(2k+1)$. Define

$$\lambda : \mathbb{Z}/4 \times \mathbb{Z}/4 \to \{\pm 1\}$$

by sending (a,b) to 1 iff either (a,b) or (b,a) is of one of the following types: (i) (a,a); (ii) $(1,2)$; and (iii) $(3,0)$. If $k,\ \ell$ are integers with images $\overline{k},\ \overline{\ell}$ in $\mathbb{Z}/4$, we will write $\lambda(k,\ell)$ for $\lambda(\overline{k},\overline{\ell})$.

Proposition 5.5. *Let $\sigma,\ \sigma'$ be irreducible, $2-$dimensional, symplectic representations of $W_\mathbb{R}$ such that the associated representations $\pi,\ \pi'$ of D^* have dimensions $2k+1,\ 2\ell+1$ respectively. Then we have*

$$W(\sigma \ominus \sigma') \,=\, \lambda(k,\ell)\, \tilde{w}_2(\pi \ominus \pi').$$

Proof. Every irreducible two-dimensional self-dual representation of $W_{\mathbb{R}}$ is of the form $\sigma_m = Ind_{\mathbb{C}^*}^{W_{\mathbb{R}}}(\chi_m)$, with $\chi_m(z) = (z/|z|)^m$, for $m \geq 1$. Since $\det(\sigma_m) = \mathrm{sgn}^{m+1}$, it is symplectic iff m is odd. The Langlands correspondence pairs σ_{2k+1} with π_{2k+1}. So we get the following, by the additivity and inductivity in dimension zero of epsilon factors:

$$W(\sigma_{2k+1} \ominus \sigma_{2\ell+1}, \psi) \,=\, W(\chi_{2k+1}, \psi_{\mathbb{C}})W(\chi_{2\ell+1}, \psi_{\mathbb{C}}).$$

But for any m, $W(\chi_m, \psi_{\mathbb{C}})$ equals i^{-m} (cf.[T]). So we get

$$\varepsilon(\pi \ominus \pi') \,=\, (-1)^{k-\ell}.$$

On the other hand, we know by Proposition 3.1 that π_{2k+1} lifts to the spin group iff its restriction to $\mathbb{C}^*$ does. It is easy to see that $\pi_{2k+1}|_{\mathbb{C}^*}$ can be decomposed as $z^{2k} \oplus z^{2k-1}\overline{z} \oplus ... \oplus z\overline{z}^{2k-1} \oplus \overline{z}^{2k}$. Applying Lemma 3.2 and noting that i defines the unique non-trivial element of $(\mathbb{C}^*/\mathbb{R}^*)[2]$, we see that $\tilde{w}_2(\pi_{2k+1})$ is trivial iff k is 0 or 3 modulo 4. The Proposition now follows easily. □

6. A geometric approach

In this section we will indicate a geometric approach to prove Proposition A, which incidentally works for even residual characteristic as well. First we need some preliminaries. Fix a *non-archimedean* local field F of characteristic zero, with ring of integers $\mathfrak{O}$, uniformizer ϖ, residue field $\mathbb{F}_q$ and a separable algebraic closure $\overline{F}$. Denote by F^{ur} maximal unramified extension in $\overline{F}$ with completion $\hat{F}^{ur} \subset \hat{\overline{F}}$. Let $\mathfrak{O}^{ur} \subset \overline{\mathfrak{O}}$ and $\hat{\mathfrak{O}}^{ur} \subset \hat{\overline{\mathfrak{O}}}$ be the corresponding inclusions of rigs of integers. For every $n \geq 1$, let Ω_F^n denote the complement of the union of all the rational hyperplanes in the projective space $\mathbb{P}_F^{n-1}$, equipped with the rigid analytic structure defined by Drinfeld, and X the universal family of formal groups associated to the corresponding formal $\mathfrak{O}-$ scheme $\hat{\Omega}_F^n$. For every $m \geq 1$, the ϖ^m- division subgroup Γ_m of X define rigid étale covers $\hat{\Gamma}_m \,=\, \Gamma_m \otimes_{\hat{\mathfrak{O}}^{ur}} \hat{F}^{ur}$ of $\Omega_F^n \otimes_F F^{ur}$ (see [Ca]). Set

$$\Sigma^{n,m} \,=\, \hat{\Gamma}_m - \hat{\Gamma}_{m-1}$$

and

$$\Sigma_0^{n,m} \,=\, \mathrm{Res}'_{\hat{F}^{ur}/F}(\Sigma^{n,m}),$$

where Res' denotes the descent to F of the disjoint union of $\Sigma^{n,m} \otimes_{\hat{F}^{ur},\tau} \hat{F}^{ur}$, with τ running over all the integral powers of the Frobenius ϕ_q. Let Σ_0^n be the projective limit of $\{\Sigma_0^{n,m} | \, m \geq 1\}$.

Fix a prime ℓ. Denoting by H^* the rigid étale cohomology (cf. V. Berkovich, "Étale cohomology for non-archimedean analytic spaces", to appear in Publ. Math. IHES), we set:

$$H^{n-1} \,=\, H^{n-1}(\Sigma_o^{n,m} \otimes_F \hat{\overline{F}}, \overline{\mathbb{Q}}_\ell)).$$

This space admits simultaneous commuting actions of W_F, $\mathrm{GL}\,(n, F)$ and of D^*, where D is the unique division algebra of dimension n^2 over F of invariant $1/n$, but the action of $\mathrm{GL}\,(n, F)$ is not smooth. In fact, H^{n-1} is the linear dual of an admissible representation of $\mathrm{GL}(n,F)$. Let us define H^{n-1}_{adm} to be the admissible subspace of H^{n-1} (for the action of $\mathrm{GL}(n,F)$). It decomposes as a direct sum

$$H^{n-1}_{\mathrm{adm}} \simeq \oplus m_{\sigma\otimes\pi'^{\vee}\otimes\pi} \quad \sigma\otimes\pi'^{\vee}\otimes\pi,$$

where σ, π' and π run over certain irreducible representations of W_F, $\mathrm{GL}(n,F)$ and D^* respectively, with $m_{\sigma\otimes\pi'^{\vee}\otimes\pi}$ denoting the multiplicity of $\sigma\otimes\pi'^{\vee}\otimes\pi$. The expectation (see [Ca], §3.3) is that this should give a geometric model of the local Langlands conjecture giving rise to a trijection $\sigma\mapsto\pi'\mapsto\pi$, at least when restricted to *supercuspidal* π'. In particular, $m_{\sigma\otimes\pi'^{\vee}\otimes\pi}$ should be 1. This conjecture is known to be true for $n=2$ by Carayol ([Ca]) (once we take care to use H^1_{adm} instead of H^1). The facts on the rigid cohomology assumed in [Ca] have now been provided by the work of Berkovich (loc. cit.). One actually knows by M. Harris ("Supercuspidal representations in the cohomology of Drinfeld upper upper half spaces; elaboration of Carayol's program", preprint) that for any n, every supercuspidal representation of $\mathrm{GL}(n,F)$ occurs in the linear dual of H^{n-1}. There is also now a purely local proof due to Faltings (The trace formula and Drinfeld's upper half plane, preprint) of the conjecture for $n=2$, giving a description of the rigid H^1 with compact supports of $\Sigma^n_o\otimes_F\hat{\overline{F}}$, where all the supercuspidals occur.

Now we give a second proof of Proposition A, which says that an irreducible representation π of D^* is orthogonal when the corresponding representation σ of W_F is symplectic. Indeed, fix such a pair (σ,π) and consider also the (supercuspidal) representation π' of $\mathrm{GL}(2,F)$ given by the Jacquet-Langlands correspondence. Then $\pi'\simeq\pi'^{\vee}$, and $\sigma\otimes\pi'\otimes\pi$ occurs in H^1_{adm} by the results above. Moreover, thanks to Berkovich (loc. cit, §6.3), one has Poincaré duality relating H^j with H^{2-j}_c, which gives us a a non-degenerate pairing defined by cup product:

$$<,>: H^1\times H^1_c\to H^2_c.$$

Carayol's description of the connected components in [Ca], §4.3, describes H^2_c as a module under $W_F\times$ $\mathrm{GL}(2,F)\times D^*$. Moreover, the pairing $<,>$ can be seen to be equivariant for the action of this product group. Consequently we find using the self-duality of σ, π and π', multiplicity-freeness of H^1_{adm}, and the fact that supercuspidals do not intertwine with other representations, that $<,>$ defines a non-degenerate bilinear form B on (the space of) $\sigma\otimes\pi'\otimes\pi$ with values in a one-dimensional subspace of $H^2(\Sigma^n_o\otimes_F\hat{\overline{F}},\overline{\mathbb{Q}}_\ell)$, on which all three groups act by the trivial representation, as $\det(\sigma)$ is trivial. It is evident that this invariant form B must be skew-symmetric; thus the tensor product $\sigma\otimes\pi'\otimes\pi$ is symplectic. But we have already proved (see Proposition 2.7) that every supercuspidal, even generic, self-dual representation of $\mathrm{GL}(2,F)$ is orthogonal. This implies that π must be orthogonal, as σ and $\sigma\otimes\otimes\pi'\otimes\pi$ are both symplectic.

For general $n>2$, note that σ can be symplectic only when n is even, in which case the cup product pairing on the self-dual part of H^{n-1}_{adm} will still be skew-symmetric. So once one knows the truth of the conjectural decomposition of H^{n-1}_{adm}, our reasoning above will show that the corresponding representation π of D^* is orthogonal, which justifies the conjecture made at the end of the introduction.

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