Distinguished representations for quadratic extensions of finite fields

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1 Introduction

Let G be a semisimple, simply-connected, algebraic group over an algebraically closed field $\bar{k}$. Let σ be an automorphism of G of order 2. It is a theorem due to Steinberg, cf. [St], that $H = G^\sigma = \{g \in G | \sigma(g) = g\}$ is a connected reductive algebraic subgroup of G. We have the following theorem due to Helgason when $\bar{k} = \mathbf{C}$, cf. [H], and due to Springer in general, cf. [S].

Theorem 1 *1. H has an open orbit on the flag variety of G.*

2. If V is an irreducible representation of G, then V has an atmost 1 dimensional space of H-fixed vectors.

3. A finite dimensional irreducible representation V with highest weight $\lambda : B \rightarrow k^*$ (B a Borel subgroup of G) has an H-invariant vector if and only if for the open orbit of H on the flag variety of G written as $H/B_H \hookrightarrow G/B$, the character λ is trivial on $B_H = B \cap H$.

It is natural to ask if such a theorem holds also for complex representations of finite groups of Lie type where instead of the parametrization of algebraic representations used in the above theorem in terms of the highest weight, we will have to use some other parametrization.

G. Lusztig has in [L1] investigated this problem in which he looks at the representations $R_{T,\theta}$, the Deligne-Lusztig representation associated to a character θ of a maximal torus T. The following theorem is due to him. This is a special case of a more general theorem due to him.

Theorem 2 *Let G be a simply-connected semisimple algebraic group. Let σ be an automorphism of order 2 of G with H as its group of fixed points. Let T be a maximal torus of G on which σ operates via inversion, i.e. $\sigma(t) = t^{-1}$ for all $t \in T$. For a character θ of $T(k)$, the representation $R_{T,\theta}$ has a $H(k)$ invariant vector if and only if θ restricted to $T(k) \cap H(k)$ is trivial, and in which case the dimension of the space of $H(k)$-invariant vectors in $R_{T,\theta}$ is $|T(k) \cap H(k)|$. (We actually take $\pm R_{T,\theta}$ where the sign is so chosen that it is a genuine representation for θ generic.)*

For the case of $G = GL_n$, the theorem of Lusztig has been made explicit in a very beautiful paper of Henderson, cf. [H].

The aim of this paper is to investigate a rather special case of this general problem in which the involution σ arises from the non-trivial Galois automorphism of a quadratic extension K of a finite field k. More precisely, for a reductive algebraic group G over a finite field k, and K a quadratic extension of k, we determine which irreducible representations of $G(K)$ have a $G(k)$-invariant vector. (Note that $G(k)$ is the set of fixed points of the automorphism σ of $G(K)$ induced from the Galois automorphism of K over k.)

The main theorem of this paper was proved in an earlier paper of the author where since the author was simultaneously dealing with both p-adic and the finite field case, the author had not given much details in the finite field case. So this work is essentially an elaboration of [P] for the case of finite fields. We refer to Lusztig [L2] for a more recent and much more complete work.

Before we come to the main theorem of this paper, we make the following definition.

Stable Representation : For any finite field K, and any algebraic group G over K, a representation π of $G(K)$ is called stable if the value of the character of π is the same at any two elements of $G(K)$ which are conjugate as elements in $G(\bar{K})$ where $\bar{K}$ denotes the algebraic closure of K.

Here is the main theorem of this paper.

Theorem 3 *Let G be a connected algebraic group over a finite field k, and K the quadratic extension of k. An irreducible stable representation π of $G(K)$ has a $G(k)$-invariant vector if and only if*

$$\pi^\sigma \cong \pi^*,$$

where for a representation π of $G(K)$, π^σ denotes the representation of $G(K)$ obtained using the automorphism σ of $G(K)$ which comes from the Galois automorphism of K over k. If π is a stable representation of $G(K)$ with $\pi^\sigma \cong \pi^*$, the dimension of the space of $G(k)$-invariant vectors in π is 1.

Proof : Let E be the semi-direct product of $G(K)$ with $\mathbb{Z}/2$ such that the non-trivial element σ of $\mathbb{Z}/2$ operates on $G(K)$ via the automorphism of $G(K)$ obtained from the non-trivial automorphism of K over k.

Given the representation π of $G(K)$ acting on a vector space V, $\pi \otimes \pi^\sigma$ is the representation of $G(K)$ on $V \otimes V$ on which an element $g \in G(K)$ acts as

$$g \cdot (v_1 \otimes v_2) = gv_1 \otimes g^\sigma v_2.$$

This representation extends naturally to a representation of E, again denoted by $\pi \otimes \pi^\sigma$ by defining

$$\sigma(v_1 \otimes v_2) = v_2 \otimes v_1.$$

The proof of the theorem will be accomplished by looking at the seemingly unrelated problem of understanding when this representation $\pi \otimes \pi^\sigma$ of E has an E-invariant vector.

We denote the character of the representation $\pi \otimes \pi^\sigma$ of E by $\Theta_{\pi\otimes\pi^\sigma}(x)$, and the character of π by $\Theta_\pi(x)$.

We will calculate the dimension of the space of E-invariant vectors using the following expression for it,

$$\frac{1}{|E|} \cdot \sum_{x \in E} \Theta_{\pi\otimes\pi^\sigma}(x).$$

Since $E = G(K) \cup G(K) \cdot \sigma$, we need to calculate the character of the representation $\pi \otimes \pi^\sigma$ at the element $g \cdot \sigma$. The following lemma achieves this.

Lemma 1 $\Theta_{\pi\otimes\pi^\sigma}(g \cdot \sigma) = \Theta_\pi(g \cdot g^\sigma)$.

Proof : Let $\{e_1, \cdots, e_n\}$ be a basis of the vector space underlying π. The action of $\pi(g)$ will be denoted by the matrix $\pi(g)_{ij}$ with respect to this basis. The action of $g \cdot \sigma$ at the basis element $e_i \otimes e_j$ of $\pi \otimes \pi^\sigma$ is given by

$$\begin{aligned}
(g \cdot \sigma)(e_i \otimes e_j) &= g(e_j \otimes e_i) \\
&= \sum_{k,l} \pi(g)_{jk} e_k \otimes \pi(g^\sigma)_{il} e_l \\
&= \sum_{k,l} \pi(g)_{jk} \cdot \pi(g^\sigma)_{il} e_k \otimes e_l
\end{aligned}$$

It follows that

$$\begin{aligned}\Theta_{\pi\otimes\pi^\sigma}(g\cdot\sigma) &= \sum_{j,i}\pi(g)_{ji}\cdot\pi(g^\sigma)_{ij} \\ &= \sum_j \pi(g\cdot g^\sigma)_{jj} \\ &= \Theta_\pi(g\cdot g^\sigma).\end{aligned}$$

Following is the main lemma of this work which simplifies the sum of the character of the representation $\pi\otimes\pi^\sigma$ of E at the elements of the non-trivial coset of $G(K)$ in E.

Lemma 2

$$\frac{1}{|G(K)|}\sum_{g\in G(K)}\Theta_\pi(g\cdot g^\sigma) = \frac{1}{|G(k)|}\sum_{g\in G(k)}\Theta_\pi(g).$$

2 The norm map

The proof of lemma 2 depends on understanding the map $g\to g\cdot g^\sigma$. Closely related map has been studied in the theory of base change, and it is that map which has nice properties. We will first define that map, to be called the *norm map*, and deduce some of its properties and then say how our map is related to the norm map. We refer to the paper of Digne [D] for these properties for which we have provided below detailed proofs.

Denote by $\bar{k}$ an algebraic closure of k, and by σ, again, the mapping $x\to x^q$ where q is the cardinality of k, and also the corresponding map on $G(\bar{k})$ for any algebraic group G over k.

It is a consequence of Lang's theorem that every element x of $G(\bar{k})$ can be written as

$$x = y\cdot\sigma(y^{-1}),$$

for some element $y\in G(\bar{k})$. Define the norm mapping,

$$\mathrm{Nm}(x) = y^{-1}\sigma^2(y).$$

Define two elements x_1 and x_2 of $G(\bar{k})$ (or of $G(K)$) to be σ-conjugate if there exists an element $t\in G(\bar{k})$ (or, $G(K)$) such that

$$tx_1\sigma(t^{-1}) = x_2.$$

We will check in what follows that the norm map is a well-defined mapping from the set of σ-conjugacy classes of elements of $G(K)$ to the set of conjugacy classes of elements of $G(k)$.

It is clear that in writing an element $x \in G(K)$ as

$$x = y \cdot \sigma(y^{-1}),$$

the non-uniqueness in y arises only through replacement of y by $y \cdot t$ for a $t \in G(k)$. Changing y to $y \cdot t$ changes $y^{-1}\sigma^2(y)$ to $t^{-1}[y^{-1}\sigma^2(y)]t$, and therefore $\mathrm{Nm}(x)$ is a well-defined element up to conjugation by $G(k)$.

Next we would like to note that if $x \in G(K)$ with $x = y \cdot \sigma(y^{-1})$, then $\mathrm{Nm}(x) = y^{-1}\sigma^2(y)$ belongs to $G(k)$. To check this, we only need to see if

$$\begin{aligned} \sigma(y^{-1}\sigma^2 y) &= y^{-1}\sigma^2 y \\ \text{or,} \quad \sigma y^{-1}\sigma^3 y &= y^{-1}\sigma^2 y, \\ \text{or,} \quad \sigma^3 y \sigma^2 y^{-1} &= (\sigma y)y^{-1}, \\ \text{or,} \quad \sigma^2[(\sigma y)y^{-1}] &= (\sigma y)y^{-1}. \end{aligned}$$

As $x = y\sigma y^{-1} = [(\sigma y)y^{-1}]^{-1}$ belongs to $G(K)$, $(\sigma y)y^{-1}$ is invariant under σ^2 which proves that $y^{-1}\sigma^2 y$ belongs to $G(k)$.

This calculation not only proves that $\mathrm{Nm}(x) = y^{-1}\sigma^2 y$ belongs to $G(k)$ but also proves that if $y^{-1}\sigma^2 y$ belongs to $G(k)$, then $x = y\sigma y^{-1}$ belongs to $G(K)$. By Lang's theorem, every element of $G(\bar{k})$ can also be written as $y^{-1}\sigma^2 y$, this, therefore, implies that $x \to \mathrm{Nm}(x)$ is a surjective map from $G(K)$ to $G(k)$ conjugacy classes in $G(k)$.

We check that the norm mapping takes σ-conjugacy classes in $G(K)$ to conjugacy classes in $G(k)$. This follows from noting that if $x = y\sigma y^{-1} \in G(K)$, and $t \in G(K)$, then

$$tx\sigma(t^{-1}) = ty\sigma(y^{-1})\sigma(t^{-1}) = ty\sigma(ty)^{-1}.$$

Therefore, $\mathrm{Nm}(tx\sigma(t^{-1})) = y^{-1}t^{-1}\sigma^2(ty)$. Since $\sigma^2(t) = t$ as $t \in G(K)$, σ-conjugate elements of $G(K)$ go surjectively onto conjugacy classes in $G(k)$.

Finally, we need to understand which elements of $G(K)$ have the same norm (as a conjugacy class in $G(k)$). If $x_1 = y_1\sigma y_1^{-1}$, and $x_2 = y_2\sigma y_2^{-1}$ are two elements of $G(K)$ with $\mathrm{Nm}(x_1) = \mathrm{Nm}(x_2)$, then we have,

$$\begin{aligned} y_1^{-1}\sigma^2 y_1 &= y_2^{-1}\sigma^2 y_2 \\ \text{or,} \quad y_2 y_1^{-1} &= \sigma^2(y_2 y_1^{-1}). \end{aligned}$$

This means that $y_2 y_1^{-1} \in G(K)$, say, $y_2 = ty_1$, for some $t \in G(K)$. Therefore,

$$\begin{aligned} x_2 &= y_2\sigma(y_2^{-1}) \\ &= ty_1\sigma(y_1^{-1})\sigma(t^{-1}) \\ &= tx_1\sigma(t^{-1}), \end{aligned}$$

proving that x_2 and x_1 are σ-conjugate. We summarise our analysis so far of the norm map in the following lemma.

Lemma 3 *For an element $x \in G(K)$ written as $x = y\sigma(y^{-1})$, define $\mathrm{Nm}(x) = y^{-1}\sigma^2(y)$. This defines a bijective correspondence between σ-conjugacy classes of $G(K)$ onto conjugacy classes of $G(k)$.*

We use this lemma to prove the following lemma.

Lemma 4 *For any class function f on $G(k)$,*

$$\frac{1}{|G(K)|}\sum_{g\in G(K)} f(\mathrm{Nm}(g)) = \frac{1}{|G(k)|}\sum_{g\in G(k)} f(g).$$

Proof : It suffices to prove that for an element $x \in G(K)$, the cardinality of the σ-centraliser of x in $G(K)$ is the same as the cardinality of centraliser of $\mathrm{Nm}(x)$ in $G(k)$.

An element $t \in G(K)$ belongs to the σ-centraliser of $x = y\sigma(y^{-1})$ if and only if,

$$\begin{aligned} ty\sigma(y^{-1})\sigma(t^{-1}) &= y\sigma(y^{-1}), \\ \text{or,} \quad y^{-1}ty &= \sigma(y^{-1})\sigma(t)\sigma(y), \\ \text{or,} \quad y^{-1}ty &\in G(k), \\ \text{or,} \quad t &\in yG(k)y^{-1} \cap G(K). \end{aligned}$$

On the other hand, $t \in G(k)$ commutes with $\mathrm{Nm}(x) = y^{-1}\sigma^2(y)$ if and only if $ty^{-1}(\sigma^2 y)t^{-1} = y^{-1}\sigma^2 y$. This holds if and only if

$$\begin{aligned} yty^{-1} &= (\sigma^2 y)(\sigma^2 t)(\sigma^2 y^{-1}) \\ \text{or,} \quad yty^{-1} &\in G(K), \\ \text{or,} \quad t &\in G(k) \cap y^{-1}G(K)y. \end{aligned}$$

It is clear that

$$|G(k) \cap y^{-1}G(K)y| = |yG(k)y^{-1} \cap G(K)|$$

completing the proof of the lemma.

To relate the norm map $\mathrm{Nm}(x)$ to the map which comes up in our analysis of characters, note that for $x = y\sigma y^{-1}$, $\mathrm{Nm}(x) = y^{-1}\sigma^2 y$, and $x \cdot x^\sigma = y \cdot \sigma^2 y^{-1}$. It follows that $(x \cdot x^\sigma)^{-1} = \sigma^2(y) \cdot y^{-1}$ and $\mathrm{Nm}(x)$ are conjugate in $G(\bar{k})$. Therefore if f is any function on $G(K)$ which takes the same value on any two conjugacy classes in $G(K)$ which become the same in $G(\bar{k})$,

$$\begin{aligned}
\frac{1}{|G(K)|} \sum_{x \in G(K)} f(x \cdot x^\sigma) &= \frac{1}{|G(K)|} \sum_{x \in G(K)} f([\mathrm{Nm}(x)]^{-1}) \\
&= \frac{1}{|G(k)|} \sum_{x \in G(k)} f(x)
\end{aligned}$$

Lemma 5 *If f is any function on $G(K)$ which takes the same value on any two conjugacy classes in $G(K)$ which become the same in $G(\bar{k})$,*

$$\frac{1}{|G(K)|} \sum_{x \in G(K)} f(x \cdot x^\sigma) = \frac{1}{|G(k)|} \sum_{x \in G(k)} f(x).$$

Applying this lemma to $f = \Theta_\pi$, we get lemma 2.

3 Proof of the main theorem

We are now ready to prove the main theorem. Using lemma 2, we find that the dimension of the space of E-invariant vectors in $\pi \otimes \pi^\sigma$ is

$$\begin{aligned}
&\frac{1}{2|G(K)|} \left[\sum_{G(K)} \Theta_\pi(x)\Theta_{\pi^\sigma}(x) + \sum_{x \in G(K)} \Theta_\pi(x \cdot x^\sigma) \right] \\
= \; &\frac{1}{2|G(K)|} \sum_{x \in G(K)} \Theta_\pi(x)\Theta_{\pi^\sigma}(x) + \frac{1}{2|G(k)|} \sum_{x \in G(k)} \Theta_\pi(x)
\end{aligned}$$

Let

$$I_1 = \frac{1}{2|G(K)|} \sum_{x \in G(K)} \Theta_\pi(x) \Theta_{\pi^\sigma}(x),$$

and let

$$I_2 = \frac{1}{2|G(k)|} \sum_{x \in G(k)} \Theta_\pi(x).$$

Since $\pi \otimes \pi^\sigma$ can have atmost 1 dimensional space of $G(K)$-invariant vectors, $\pi \otimes \pi^\sigma$ can have atmost 1 dimensional space of E-invariant vectors. This implies that $I_1 + I_2$ can be either 0 or 1, and that if $I_1 + I_2 = 1$, then $I_1 = 1/2$.

Therefore if $I_1 + I_2 = 1$, $I_1 = 1/2$. $I_2 = 1/2$, and in particular, $I_1 = I_2$. On the other hand, if $I_1 + I_2 = 0$, then since both I_1 and I_2 are non-negative (being $1/2$ times the dimensions of certain vector spaces), it follows that $I_1 = I_2 = 0$.

So, in all cases $I_1 = I_2$, proving the theorem.

We note the following corollary to our proof of the theorem

Corollary 1 *If $\pi^* \cong \pi^\sigma$, then the unique $G(K)$-invariant vector in $\pi \otimes \pi^\sigma$ is invariant under E, or, which is the same thing, invariant under the involution $v_1 \otimes v_2 \to v_2 \otimes v_1$.*

4 A counter-example

In this section we construct a representation of $SL_2(\mathbb{F}_{q^2})$ for which the space of $SL_2(\mathbb{F}_q)$-invariant vectors is 2 dimensional. This shows that theorem 3 is not true for all representations of $SL_2(\mathbb{F}_{q^2})$. This example was mentioned in [P] but without giving any details.

We will have to fix some notation for this purpose. For characters χ_1, χ_2 of $\mathbb{F}_{q^2}^*$ with values in $\mathbf{C}^*$, let $Ps(\chi_1, \chi_2)$ denote the corresponding principal series representation of $GL_2(\mathbb{F}_{q^2})$. The action of $GL_2(\mathbb{F}_q)$ on $\mathbf{P}^1(\mathbb{F}_{q^2})$ has two orbits; one of which is $\mathbf{P}^1(\mathbb{F}_q)$, and the other its complement which is isomorphic to $GL_2(\mathbb{F}_q)/\mathbb{F}_{q^2}^*$; the stabiliser of this orbit at any point can be identified to $\mathbb{F}_{q^2}^*$ which is included inside a Borel subgroup of $GL_2(\mathbb{F}_{q^2})$ via $x \to (x, \bar{x})$.

¿From Mackey's theory of restriction of an induced representation, it follows that the restriction of the principal series representation $Ps(\chi_1, \chi_2)$ of $GL_2(\mathbb{F}_{q^2})$ to $GL_2(\mathbb{F}_q)$ is,

$$Ps(\chi_1|_{\mathbb{F}_q^*}, \chi_2|_{\mathbb{F}_q^*}) \oplus \mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2(\mathbb{F}_q)} \chi_1 \bar{\chi}_2,$$

where $\bar{\chi}_2 = \chi_2(\bar{x})$, and $Ps(\chi_1|_{\mathbb{F}_q^*}, \chi_2|_{\mathbb{F}_q^*})$ denotes the principal series of $GL_2(\mathbb{F}_q)$ induced from the restriction of (χ_1, χ_2) to $\mathbb{F}_q^*$.

Suppose $q \equiv 1 \bmod 4$ so that $\mathbb{F}_q^*$ has a character χ_0 of order 4. Let $\chi_1 = \mathrm{Nm} \circ \chi_0$, and $\chi_2 = \chi_1^{-1}$ where Nm denotes the norm mapping from $\mathbb{F}_{q^2}^*$ to $\mathbb{F}_q^*$. The restriction $Ps(\chi_1, \chi_2)$ to $GL_2(\mathbb{F}_q)$ is therefore,

$$\begin{aligned} & Ps(\chi_0^2, \chi_0^{-2}) \oplus \mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2(\mathbb{F}_q)} \chi_o(x\bar{x})\chi_0^{-1}(x\bar{x}) \\ = \quad & Ps(\chi_0^2, \chi_0^2) \oplus \mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2(\mathbb{F}_q)} \mathbf{1}. \end{aligned}$$

Since every 1 dimensional representation of $GL_2(\mathbb{F}_q)$ is obtained via composition with the determinant map of a character of $\mathbb{F}_q^*$: $GL_2(\mathbb{F}_q) \xrightarrow{\det} \mathbb{F}_q^* \xrightarrow{\chi} \mathbf{C}^*$, it follows that $\mathrm{Ind}_{\mathbb{F}_{q^2}^*}^{GL_2(\mathbb{F}_q)} \mathbf{1}$ does not contain any 1 dimensional representation of $GL_2(\mathbb{F}_q)$ besides the trivial representation. The principal series representation $Ps(\chi_0^2, \chi_0^2)$ contains the 1 dimensional representation $GL_2(\mathbb{F}_q) \xrightarrow{\det} \mathbb{F}_q^* \xrightarrow{\chi_0^2} \mathbf{C}^*$.

Since the space of $SL_2(\mathbb{F}_q)$-invariant vectors in $Ps(\chi_1, \chi_2)$ is a $GL_2(\mathbb{F}_q)$-module on which $SL_2(\mathbb{F}_q)$ acts trivially, it is clear that the space of $SL_2(\mathbb{F}_q)$-invariant vectors in $Ps(\chi_1, \chi_2)$ is exactly the space on which $GL_2(\mathbb{F}_q)$ operates via sum of characters. It follows that the space of $SL_2(\mathbb{F}_q)$-invariant vectors is a 2-dimensional vector space for the principal series $V = Ps(\mathrm{Nm} \circ \chi_0, \mathrm{Nm} \circ \chi_0^{-1})$. The representation V is a sum of 2 irreducible $SL_2(\mathbb{F}_{q^2})$ representations, $V = I_1 \oplus I_2$. (If it were irreducible, it will contradict theorem 3!) Since $GL_2(\mathbb{F}_{q^2})/SL_2(\mathbb{F}_{q^2})$ is a cyclic group, it can be seen that $I_1 \not\cong I_2$. If we could prove that the dimension of the space of $SL_2(\mathbb{F}_q)$-invariant vectors in these two representations is not the same then the only possibility for dimensions of I_1 and I_2 is 0 and 2 (or, 2 and 0).

It is clear from our construction that $V = I_1 \oplus I_2$ is a representation of $SL_2(\mathbb{F}_{q^2})/\pm 1$. We note that I_1 and I_2 are conjugate by an element of $GL_2(\mathbb{F}_{q^2})$. It follows that the only elements of $SL_2(\mathbb{F}_{q^2})/\pm 1$ on which I_1 and I_2 could have different character values is $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ where $t \in \mathbb{F}_{q^2}^*/(\mathbb{F}_{q^2}^{*2})$. Since $I_1 \not\cong I_2$, they must indeed have different character values at *all* these elements.

It follows from the expression

$$\frac{2}{|SL_2(\mathbb{F}_q)|} \sum_{x \in SL_2(\mathbb{F}_q)/\pm 1} \Theta_\pi(x)$$

for the dimension of $SL_2(\mathbb{F}_q)$-invariant vectors in a representation π, that the corresponding expressions for I_1 and I_2 are the same except for the unipotent elements $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ where $t \in \mathbb{F}_q^*$ which form a single conjugacy class in $SL_2(\mathbb{F}_{q^2})$. This implies that the dimension of $SL_2(\mathbb{F}_q)$ invariant vectors in I_1 and I_2 is different.

Remark : The two irreducible components I_1 and I_2 are distinguished by which characters of the unipotent group $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ where $t \in \mathbb{F}_{q^2}$ appear. If $\chi(x)$ is a character of $\mathbb{F}_{q^2}$, then any other character of $\mathbb{F}_{q^2}$ is $\chi(tx)$ for $t \in \mathbb{F}_{q^2}$. If $\chi(x)$ is a character of the unipotent group appearing in I_1, it is easy to see that all the character $\chi(t^2x)$ for $t \in \mathbb{F}_{q^2}^*$ also appear in I_1. From this it is easy to see that $I_i^* \cong I_i^\sigma$ for $i = 1, 2$ (as this property is true for the principal series in which I_1 and I_2 lie). Therefore we also have a representation π of $SL_2(\mathbb{F}_{q^2})$ with $\pi^* \cong \pi^\sigma$ but without a $SL_2(\mathbb{F}_{q^2})$-invariant vector.

5 Shintani transform

In this section we study in some detail the map $x^{-1}x^{[q]} \to x^{[q]}x^{-1}$ on $G(\mathbb{F}_q)$ which has played an important role in this paper. Here $x \to x^{[q]}$ denotes the Frobenius map on $G(\bar{\mathbb{F}}_q)$.

Observe that if

$$x^{-1}x^{[q]} = x_1^{-1}x_1^{[q]},$$

then

$$(x_1x^{-1}) = (x_1x^{-1})^{[q]}.$$

Therefore, $x_1 = tx$ for $t \in G(\mathbb{F}_q)$, and $x_1^{[q]}x_1^{-1} = t(x^{[q]}x^{-1})t^{-1}$. Thus the mapping $x^{-1}x^{[q]} \to x^{[q]}x^{-1}$ restricted to $G(\mathbb{F}_q)$ is well defined as a mapping on conjugacy classes in $G(\mathbb{F}_q)$. We call this mapping the Shintani transform $Sh : G(\mathbb{F}_q) \to G(\mathbb{F}_q)$.

The following lemma is clear.

Lemma 6 *If $\phi : G' \to G$ is a homomorphism of connected algebraic groups defined over $\mathbb{F}_q$, then the following diagram commutes.*

$$\begin{array}{ccc} G'(\mathbb{F}_q) & \xrightarrow{\phi} & G(\mathbb{F}_q) \\ {\scriptstyle Sh}\downarrow & & \downarrow{\scriptstyle Sh} \\ G'(\mathbb{F}_q) & \xrightarrow{\phi} & G(\mathbb{F}_q). \end{array}$$

Now note that any semisimple element in $G(\mathbb{F}_q)$, G a connected reductive algebraic group over $\mathbb{F}_q$, is contained in a maximal torus defined over $\mathbb{F}_q$. Since the Shintani transform is trivial on an abelian group, we deduce that the Shintani transform is trivial on any semisimple element of $G(\mathbb{F}_q)$. Further, by the Jacobson-Morozov theorem, any unipotent element in $G(\mathbb{F}_q)$ belongs to a 1-parameter unipotent subgroup of $G(\mathbb{F}_q)$ defined over $\mathbb{F}_q$ (for large primes), and hence Shintani transform is trivial for large primes for both semisimple and unipotent elements. Thus Jordan decomposition implies that the Shintani descent is non-trivial perhaps only on those elements x in $G(\mathbb{F}_q)$ with Jordan decomposition $x = su$ for which the centraliser of u is disconnected.

6 Stability for Unipotent groups

Our theorem 3 can be used for any connected algebraic groups, in particular for unipotent algebraic groups. It appears that for most unipotent groups, any irreducible representation is stable. However, we have not found any general theorems in this direction. We will thus content ourselves with some examples in the form of the following lemmas. Observe that because of Lang's theorem, if the centraliser of any element of an algebraic group G over a finite field $\mathbb{F}_q$ is connected, then any two elements of the group $G(\mathbb{F}_q)$ which are conjugate in $G(\bar{\mathbb{F}}_q)$ are already conjugate in the group $G(\mathbb{F}_q)$. Thus any representation of such a group will be stable in our sense.

Lemma 7 *Let U be a unipotent algebraic group over a finite field $\mathbb{F}_q$ with Lie algebra $\mathfrak{U}$. Assume that the exponential map gives an isomorphism from $\mathfrak{U}$ to U. Then the centraliser of any element in U is connected.*

Proof: Let g be an element of U, and $h = \exp(X)$ be an element of U for an element X of $\mathfrak{U}$. Clearly h commutes with g if and only if $hgh^{-1} = g$, i.e., if and only if $Ad(g)(X) = X$, where $Ad(g)$ represents the adjoint action of U on $\mathfrak{U}$. Since the space of solutions of $Ad(g)(X) = X$ is a vector subspace of $\mathfrak{U}$, it is connected. Hence the centraliser of g in U which is the image under the exponential map (which is a polynomial map for unipotent groups) of this connected space is connected.

Lemma 8 *Let U_n be the unipotent algebraic group of upper triangular unipotent matrices of size n over any field k. Then the centraliser of any element in U_n is connected.*

Proof : Let $\mathfrak{U}_n$ be the Lie algebra of U_n consisting of all upper triangular matrices with zeros on the diagonal. Clearly $X \to 1 + X$ gives a bijection from $\mathfrak{U}_n$ to U_n. If $g(1 + X)g^{-1} = (1 + X)$, then clearly $gXg^{-1} = X$. Thus the set of matrices $1 + X$ commuting with g have a vector space structure, and hence form a connected set.

Lemma 9 *Let V_n be a maximal unipotent subgroup of an orthogonal or symplectic group, say G, of size n over a finite field $\mathbb{F}_q$ of characteristic not 2. Then the centraliser of any element in V_n is connected.*

Proof : We assume without loss of generality that V_n is contained in the group of upper triangular matrices, thus V_n is the subgroup of the group of upper triangular matrices which belong to G.

An orthogonal or symplectic group is defined as the subgroup of GL_n with

$$gg' = 1$$

where $g \to g'$ is an involution on GL_n, i.e. an anti-automorphism of order 2 on GL_n. The Lie algebra of such a group is defined as the set of matrices X with $X + X' = 0$.

Let $\mathfrak{V}_n$ be the Lie algebra of V_n consisting of all upper triangular matrices with zeros on the diagonal.

Clearly the Cayley transform $X \to (1 + X)(1 - X)^{-1}$ gives a bijection from $\mathfrak{V}_n$ to V_n. The equation $g(1 + X)(1 - X)^{-1}g^{-1} = (1 + X)(1 - X)^{-1}$ can be simplified to give $gXg^{-1} = X$. Thus the set of matrices $(1 + X)(1 - X)^{-1}$ commuting with g have a vector space structure, and hence form a connected set.

Remark : What matters for this section is the existence of an isomorphism between a unipotent algebraic group and its Lie algebra which is equivariant under the adjoint action of the group. If such an isomorphism existed, then the centralser of any element in the unipotent group would be connected. Several instances of such an isomorphism are used above to conclude this.

7 Weakly-stable representation

The definition of stable representation introduced in the Introduction of this paper appears to be much too strong, and it is not so clear that there are many irreducible representations of a group $G(\mathbb{F}_q)$ with G reductive algebraic, which are stable if G is not GL_n or U_n. However, we note that for the proof of Lemma 2, all that is necessary is that

$$\Theta_\pi(g) = \Theta_\pi(Sh(g)),$$

which is usually much weaker than our requirement for a stable representation. We define a representation π to be weakly-stable if its character Θ_π satisfies,

$$\Theta_\pi(g) = \Theta_\pi(Sh(g)).$$

The following important result appears as Corollary 2.5, chapter 4, of [DM].

Theorem 4 *Any linear combination of Deligne-Lusztig representation is weakly-stable.*

This gives us the following corollary.

Corollary 2 *Let G be a connected reductive group over a finite field k, and K the quadratic extension of k. Suppose π is an irreducible representation of $G(K)$ which is a sum of Deligne-Lusztig representations. Then π has a $G(k)$-invariant vector if and only if*

$$\pi^\sigma \cong \pi^*,$$

where for a representation π of $G(K)$, π^σ denotes the representation of $G(K)$ obtained using the automorphism σ of $G(K)$ which comes from the Galois automorphism of K over k. If π has the property that $\pi^\sigma \cong \pi^*$, the dimension of the space of $G(k)$-invariant vectors in π is 1.

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