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On an Archimedean analogue of Tate's conjecture

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Abstract

We consider an Archimedean analogue of Tate's conjecture, and verify the conjecture in the examples of isospectral Riemann surfaces constructed by Vignéras and Sunada. We prove a simple lemma in group theory which lies at the heart of T. Sunada's theorem about isospectral manifolds.

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The aim of this short note is to formulate a conjecture about isospectral Riemann surfaces, which can be considered as an Archimedean analogue of Tate's conjecture, and to verify the conjecture in the examples of isospectral Riemann surfaces constructed by Vignéras and Sunada. We first prove a simple lemma in group theory which lies at the heart of T. Sunada's theorem, cf. [S], about isospectral manifolds (i.e., Riemannian manifolds for which the eigenvalues of the Laplacian, counted with multiplicity, are the same). The lemma has many other applications to similar problems.

We begin with the group theoretic lemma. Let G be a finite group. For a G -module M , denote by M^G the submodule of invariants of G .

Lemma 1. *Suppose that G is a finite group with subgroups H_1 and H_2 such that each conjugacy class in G intersects H_1 and H_2 in equal number of elements. Assume that V is a representation space of G over a field k of characteristic zero. Then there exists an*

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isomorphism $i: V^{H_1} \rightarrow V^{H_2}$, commuting with the action of any endomorphism Δ of V which commutes with the action of G on V , i.e. the following diagram commutes:

$$\begin{array}{ccc} V^{H_1} & \xrightarrow{i} & V^{H_2} \\ \Delta \downarrow & & \downarrow \Delta \\ V^{H_1} & \xrightarrow{i} & V^{H_2} \end{array}$$

Further if $k = \mathbb{C}$ and the action of G on V is unitary, the isomorphism can be chosen to be unitary.

Proof. The hypothesis on the subgroups H_1 and H_2 of G implies that the character of the left regular representation of G on the space of functions on G/H_1 with values in k , $k[G/H_1]$, is the same as the character of the representation of G on $k[G/H_2]$. Hence these representations are isomorphic (over $k!$). The isomorphism is well-known to exist by character theory over an algebraically closed field containing k , and hence the two representations are isomorphic over k too by a general result in group representations: two representations of a group G over a field k which become isomorphic over a field extension of k , are isomorphic over k (cf. page 110 of the article of Atiyah and Wall in the book edited by J.W.S. Cassels and A. Fröhlich on ‘Algebraic Number Theory’). The article of Atiyah and Wall deals with $k = \mathbb{Q}$ which is all that is needed by us, however the same proof works for any infinite field, and can be proved for finite fields too by a simple application of Lang’s theorem.

By Frobenius reciprocity, for any representation V of G , there is an isomorphism of V^H with $[V \otimes k[G/H]]^G$ in which we send a vector $v \in V^H$ to $\sum gv \otimes e_g$, the sum running over the distinct elements of G/H , denoted by e_g .

The isomorphism of V^H with $[V \otimes k[G/H]]^G$ for $H = H_1, H_2$, taken together with a G -equivariant isomorphism $S: k[G/H_1] \rightarrow k[G/H_2]$, gives an isomorphism of V^{H_1} with V^{H_2} :

$$V^{H_1} \rightarrow [V \otimes k[G/H_1]]^G \rightarrow [V \otimes k[G/H_2]]^G \rightarrow V^{H_2}.$$

Any endomorphism of V which commutes with the action of G on V preserves the invariant subspaces V^{H_1} and V^{H_2} . It is clear that the isomorphism between V^{H_1} and V^{H_2} constructed here commutes with such endomorphisms.

It remains to check the unitarity. Let V be a unitary G -module. Since $\mathbb{C}[G/H_1]$ and $\mathbb{C}[G/H_2]$ are isomorphic as G -modules, we can choose a unitary isomorphism with the natural unitary structures on $\mathbb{C}[G/H_1]$ and $\mathbb{C}[G/H_2]$. This is a general fact: if there are two unitary structures on a complex representation V of a group G , then there exists a G -invariant intertwining operator between the two unitary structures; we omit the proof here. For each $g \in G$, the elements $gv \otimes e_g$ are mutually orthogonal, each one of norm equal to norm of v , where e_g denotes the characteristic function of the coset space gH in G/H . Hence, it follows that the isomorphism of V^H with $[V \otimes \mathbb{C}[G/H]]^G$ in which we send a vector $v \in V^H$ to $\frac{1}{\sqrt{|G/H|}} \sum_{g \in G/H} gv \otimes e_g$ is unitary,

if $V \otimes \mathbb{C}[G/H]$ is given the usual unitary structure for a tensor product. Thus if V is a unitary G -module, we obtain a unitary isomorphism from V^{H_1} to V^{H_2} . \square

Corollary 1 (Sunada’s theorem). *Suppose that G is a finite group with subgroups H_1 and H_2 such that each conjugacy class in G intersects H_1 and H_2 in equal number of elements. Suppose that X is a Riemannian manifold on which G acts by isometries and freely (i.e., $gx = x$ implies $g = e$). Then the quotient of X by H_1 and H_2 with the induced metrics are isospectral manifolds.*

Proof. Let V be the space of \mathcal{C}^∞ functions on X . This is a representation of G such that V^{H_i} is the space of \mathcal{C}^∞ functions on the quotient of X by H^i . The Laplacian on X commutes with the G -action, and when restricted to functions in V^{H_i} , it corresponds to the Laplacian on \mathcal{C}^∞ functions on the quotient of X by H_i . The proof of the corollary hence follows from the lemma. \square

Corollary 2. *Let H be a Lie group, $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma$ discrete subgroups of G with*

$$\Gamma_0 \subset \Gamma_i \subset \Gamma, \quad \text{for } i = 1, 2$$

such that Γ_0 is a normal subgroup of finite index in Γ with the property that the subgroups Γ_i/Γ_0 of Γ/Γ_0 for $i = 1, 2$ intersect each conjugacy class in Γ/Γ_0 in equal number of elements. Then $L^2(\Gamma_1 \backslash H)$ and $L^2(\Gamma_2 \backslash H)$ are isomorphic as H -modules.

Proof. The proof of the corollary follows from Lemma 1 applied to $V = L^2(\Gamma_0 \backslash H)$, with $G = \Gamma_0 \backslash \Gamma$, $H_1 = \Gamma_0 \backslash \Gamma_1$ and $H_2 = \Gamma_0 \backslash \Gamma_2$. We note here that a similar corollary is also available in the work of Vignéras, [V, Corollary 5]. \square

Corollary 3. *Let X be a projective algebraic variety over a field k together with an action of a finite group G on X over k . Let \bar{k} be a separable closure of k , and \bar{X} denote X base changed to \bar{k} . Let H_1 and H_2 be two subgroups of G which intersect each conjugacy class in G in equal number of elements. Then for any i , $H_{\text{ét}}^i(\bar{X}, \mathbb{Q}_\ell)^{H_1}$ and $H_{\text{ét}}^i(\bar{X}, \mathbb{Q}_\ell)^{H_2}$ are isomorphic as $\text{Gal}(\bar{k}/k)$ -modules. Equivalently, for any i , $H_{\text{ét}}^i(\bar{X}/H_1, \mathbb{Q}_\ell)$ and $H_{\text{ét}}^i(\bar{X}/H_2, \mathbb{Q}_\ell)$ are isomorphic as $\text{Gal}(\bar{k}/k)$ -modules. Hence for k a finite field, or a number field, the L -functions associated to $H_{\text{ét}}^i(\bar{X}/H_1, \mathbb{Q}_\ell)$ and $H_{\text{ét}}^i(\bar{X}/H_2, \mathbb{Q}_\ell)$ are the same.*

Corollary 4. *Let X be a projective algebraic curve over a field k with an action of a finite group G on X over k . Let H_1 and H_2 be two subgroups of G which intersect each conjugacy class in G in equal number of elements. Then the Jacobians of the curves X/H_1 and X/H_2 are isogenous over k .*

Proof. It can be seen, cf. lemma below, that the Jacobian J_i of X/H_i is isogenous to the connected component of the H_i fixed points of the Jacobian J of X . Hence once

again, the map

$$J^{H_1} \rightarrow J^{H_2}$$

defined by $x \rightarrow \sum \Phi(g)g \cdot x$ provides the necessary isogeny from J_1 to J_2 . (Here Φ is an integral valued function on G corresponding to an isomorphism of $\mathbb{Q}[G/H_1]$ with $\mathbb{Q}[G/H_2]$, interpreted as a function on the double coset space $H_2 \backslash G/H_1$.) \square

The following lemma is well known. However, not finding an appropriate reference, we have included a proof here.

Lemma 2. *Let X be a projective algebraic curve over a field k with an action of a finite group H on X over k . Then the Jacobian of X/H is isogenous to the connected component of the H fixed points of the Jacobian J of X .*

Proof. Let n be the order of H . We can assume that k is algebraically closed. We will prove that multiplication by n^2 takes the group of H -fixed points on J to the natural image of the Jacobian of X/H into the Jacobian of X , proving the lemma. Let $k(X)$ denote the function field of X , and $\text{Div}^0(X)$, the divisors of degree 0 on X . We have the exact sequence of H -modules,

$$0 \rightarrow k(X)^*/k^* \rightarrow \text{Div}^0(X) \rightarrow \text{Pic}^0(X) \rightarrow 0.$$

Since $H^1(H, A)$ is annihilated by n for any H -module A , we have that any H -invariant element in $\text{Pic}^0(X)$, when multiplied by n comes from an H -invariant divisor on X . But any H -invariant divisor on X multiplied by n comes from X/H , completing the proof of the lemma. \square

Remark 1. Many examples of triples $(G; H_1, H_2)$ as above have been known for a very long time. Some of these are given in Sunada’s paper; see also Serre’s book, *Linear Representations of finite groups*, Section 13.2, Exercises 5 and 6.

Remark 2. Corollary 3 specialises to give examples of (non-conjugate) number fields with the same zeta function. The extension of this phenomenon to geometric context was Sunada’s original motivation for his paper.

Corollary 4 lends support to the following conjecture, which can be considered as an Archimedean analogue of a corollary of Tate’s conjecture and proved by Faltings [F], that if the Hasse–Weil zeta functions of two curves defined over a number field K are equal, then the Weil restriction of scalars from K to \mathbb{Q} of the corresponding Jacobians are isogenous.

Conjecture. *Suppose that X and Y are complete algebraic curves defined over a number field K . Suppose that the compact Riemann surfaces associated to X and Y , via an embedding of K into \mathbb{C} , are isospectral with respect to the Kähler metric of constant*

curvature -1 . Then the Hasse–Weil zeta functions of X and Y are the same over a finite extension L of K , and hence the Weil restriction of scalars from L to \mathbb{Q} of the corresponding Jacobians are isogenous.

Remark 3. Let X and X' be two Shimura curves defined using quaternion division algebras over totally real number field. It is known by a result of A. W. Reid (Duke J. 65 (1992)), that if the Shimura curves are isospectral then the underlying quaternion algebras are isomorphic.

We briefly recall the construction of isospectral surfaces by Vignéras [V], and verify the above conjecture. Let K be a totally real number field, and let \mathbb{H} be a quaternion division algebra over K which is ramified at all but one of the real places v_0 . Suppose that \mathcal{O} and \mathcal{O}' are two maximal orders in \mathbb{H} , which are not conjugate by any \mathbb{Q} -automorphism of \mathbb{H} . Let Γ (resp. Γ') be the group of elements of reduced norm 1 in \mathcal{O} (resp. \mathcal{O}') modulo the group ± 1 . Projecting to the v_0 -component, Γ and Γ' give rise to co-compact lattices in $\mathrm{PSL}(2, \mathbb{R})$. Let X and X' be the corresponding Riemann surfaces. Under further technical conditions Vignéras shows that X and X' are isospectral but not isometric. By Theorem 2.5 of Shimura [Sh], the curves X and X' are defined over a certain number field K , and that X is isomorphic to a Galois conjugate of X' . This implies in particular that the Hasse–Weil zeta functions of X and X' are the same over some number field.

Note added in proof: We do not know if the following much stronger form of the above conjecture is true: If two Riemann surfaces (not necessarily defined over a number field) are isospectral, then the Jacobian of one is isogenous to a conjugate of the other by an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, where σ preserves the spectrum of the corresponding Riemann surface. More specifically, for Riemann surfaces defined over \mathbb{Q} , we do not know if isospectral implies isogeny of the Jacobians.

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