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Tate Cycles on a Product of Two Hilbert Modular Surfaces

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1. INTRODUCTION

Let X be a smooth projective variety defined over a number field K. Let L be a finite extension of K and denote by $Z_j(X; L)$ the free abelian group generated by codimension j subvarieties of X which are defined over L. There is a cycle class map with values in the ℓ-adic cohomology,

$$Z_j(X; L) \otimes \mathbb{Q}_\ell \to H_\ell^{2j}(\bar{X})(j)^{\mathrm{Gal}(\bar{\mathbb{Q}}/L)}.$$

A conjecture of Tate asserts that this map is surjective. It has been shown to hold in a number of nontrivial cases. Let us denote by

$$\mathrm{T}a_\ell(X; L) = H_\ell^{2j}(\bar{X})(j)^{\mathrm{Gal}(\bar{\mathbb{Q}}/L)}$$

the space of Tate cycles on X defined over L and by

$$\mathrm{T}a_\ell(X) = \bigcup_L \mathrm{T}a_\ell(X; L)$$

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the space of all Tate cycles on X. The aim of this paper is to describe all the Tate cycles on the product of two Hilbert modular surfaces in terms of automorphic representations (or, Hilbert modular forms) including the exact determination of their fields of definition. We have been unable to say if all of these Tate cycles come from algebraic cycles.

Let F be a real quadratic field and let S denote a Hilbert modular surface corresponding to this field. Thus, $S = S_K$ is a surface defined over $\mathbb{Q}$ which is the smooth toroidal compactification of an open surface S^o (cf. [HLR]) which satisfies

$$S^o(\mathbb{C}) = \mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})/K_\infty K$$

where $\mathrm{G} = \mathrm{R}_{F/\mathbb{Q}} \mathrm{G}L_{2/F}$, K is a compact open subgroup of $\mathrm{G}(\mathbb{A}_f)$ and $K_\infty = K_\infty^c Z$ where K_∞^c is the connected component of the identity of a maximal compact subgroup of $\mathrm{G}L_2(F \otimes \mathbb{R})$,

$$K_\infty^c \cong SO_2(\mathbb{R}) \times SO_2(\mathbb{R}),$$

and Z is the center of $GL_2(F \otimes \mathbb{R})$.

Let F_1 and F_2 be two real quadratic fields and let S_1 and S_2 denote corresponding Hilbert modular surfaces with respect to K_1 and K_2 (respectively). In this paper we show that $\mathrm{T}a_\ell(S_1 \times S_2)$ is spanned by $\mathrm{T}a_\ell(S_1) \otimes \mathrm{T}a_\ell(S_2)$ and certain additional codimension 2 cycles which we shall construct. Since Tate cycles on a Hilbert modular surface are known to be algebraic ([HLR], [MR], [K], [O]), the Tate conjecture for $S_1 \times S_2$ is therefore equivalent to proving the algebraicity of these additional cycles.

2. PRELIMINARIES ON THE TENSOR PRODUCT OF TWO DIMENSIONAL REPRESENTATIONS

In this section we prove several results about the tensor product of two 2 dimensional representations. All the results proved are of a rather elementary nature and are surely well-known but for lack of suitable reference, we have included all the proofs. The results of this section are for finite dimensional complex representations of a general group G.

We define an irreducible 2 dimensional complex representation π of a group G to be *dihedral* if there exists a normal subgroup N of index 2 in G and a character $\chi: N \to \mathbb{C}^*$ such that π is obtained from inducing the character χ of N to G.

THEOREM 2.1. *The tensor product of two 2 dimensional irreducible complex representations of a group G is reducible only if either both the representations are dihedral, or they are a twist of each other by a character.*

The proof of this theorem will be completed in several steps which we break in the following lemmas and propositions. Some of these will be of independent interest to us in later sections.

LEMMA 2.2. *Let V be a finite dimensional complex vector space with a quadratic form Q. Let π_1 and π_2 be two representations of a group G into $O(V, Q)$ (resp. $GO(V, Q)$) which become equivalent in $GL(V)$. Suppose that Q is the unique quadratic form on V, up to scaling, which is preserved by π_1 (and therefore π_2). Then π_1 is equivalent to π_2 in $O(V, Q)$ (resp. $GO(V, Q)$).*

Proof. Writing ${}^tX$ for the transpose of a matrix X, we have

$$\pi_1(x)\, Q\, {}^t\pi_1(x) = q_1(x)\, Q,$$

$$\pi_2(x)\, Q\, {}^t\pi_2(x) = q_2(x)\, Q.$$

Since π_1 and π_2 are equivalent, let A be an element in $GL(V)$ such that $\pi_2(x) = A\pi_1(x)\, A^{-1}$. From $\pi_1(x)\, Q\, {}^t\pi_1(x) = q_1(x)\, Q$ we get

$$A\pi_1(x)\, A^{-1} A Q\, {}^tA\, {}^tA^{-1}\, {}^t\pi_1(x)\, {}^tA = q_1(x)\, A Q\, {}^tA.$$

Since $\pi_2(x) = A\pi_1(x)\, A^{-1}$ preserves a unique quadratic form up to scaling, $AQ\, {}^tA = \lambda Q$ for some constant λ, completing the proof.

LEMMA 2.3. *Let V be a 4-dimensional representation of a group G such that for a unique quadratic form (up to scalars), the representation of G lands inside $GO(V)$. Suppose that there exists 2-dimensional G-modules V_1, V_2, W_1, W_2 with $V \cong V_1 \otimes V_2$, and also $V \cong W_1 \otimes W_2$. Then there exists a character χ of G and $i \in \{1, 2\}$ such that $V_1 \cong W_i \otimes \chi$ and $V_2 \cong W_j \otimes \chi^{-1}$ for $j \neq i$.*

Proof. Taking the tensor product of 2 two dimensional representations, we get a 4 dimensional representation together with a quadratic form (as the product of two symplectic forms is orthogonal) left invariant by the representation up to similitudes. Since the mapping from $GL(2) \times GL(2)$ lands inside the connected component $GSO(4, \mathbb{C})$ of $GO(4, \mathbb{C})$, our 4 dimensional representation takes values in $GSO(4, \mathbb{C})$. We have the exact sequence

$$0 \to \mathbb{C}^* \to GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \to GSO(4, \mathbb{C}) \to 0.$$

It follows that if the representation of G inside $GO(V, \mathbb{C})$ is written as $V_1 \otimes V_2$ and also as $W_1 \otimes W_2$, then these correspond to two ways of lifting the representation of G into $GSO(V, \mathbb{C})$ to $GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$. Since the kernel of the mapping of $GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$ to $GSO(V, \mathbb{C})$ is central, the ambiguity in such a lifting is by a character into $\mathbb{C}^*$. By the previous lemma, the representation V is well defined up to conjugacy inside $GO(V, \mathbb{C})$. Observe that the inner conjugation action of an element of $GO(V, \mathbb{C})$ which does not lie in $GSO(V, \mathbb{C})$ lifts to an action on $GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$ which permutes the two factors. This concludes the proof of this lemma.

Remark. That the previous lemma is not true without some hypothesis is shown by the following example. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8. It has a unique irreducible representation ρ of dimension 2. Then it is easy to see that $\rho \otimes \rho \cong \{1 \oplus \alpha\} \otimes \{1 \oplus \beta\}$, where α and β are any two distinct non-trivial characters of G.

LEMMA 2.4. *Let σ be an irreducible 2-dimensional representation of a group G. If for a non-trivial character χ of G, $\sigma \cong \sigma \otimes \chi$, then χ is of order 2 and σ is induced from a character on the kernel of χ.*

Proof. Since $\det(\sigma \otimes \chi) = \det(\sigma)\, \chi^2 = \det(\sigma)$, $\chi^2 = 1$. Let H be the kernel of χ. Let $A \in \mathrm{Hom}_G(\sigma, \sigma \otimes \chi)$ be a non-zero element. Since $\chi^2 = 1$, A^2 is an intertwining operator from σ to itself. After scaling A, we can therefore assume that $A^2 = 1$. Since $\chi \neq 1$, A is not the identity map. (We think of A as an endomorphism of the vector space underlying σ.) This implies that the eigenspaces of A with eigenvalues ± 1 are non-zero, and invariant under H. The action of H on these define two characters of H either of which induce to give the representation σ of G.

COROLLARY 2.5. *If the tensor product $\sigma_1 \otimes \sigma_2$ of two 2-dimensional irreducible representations of a group G contains two characters then both σ_1 and σ_2 are induced from a subgroup H of G of index 2.*

Proof. Write $\sigma_1 \otimes \sigma_2 = \chi_1 \oplus \chi_2 \oplus \tau$. By Schur's lemma, $\chi_1 \neq \chi_2$, and we have $\sigma_1 \cong \sigma_2^* \otimes \chi_1$, and $\sigma_1 \cong \sigma_2^* \otimes \chi_2$ for distinct characters χ_1 and χ_2. This implies that σ_1 and σ_2 are dihedral.

COROLLARY 2.6. *$Sym^2(\sigma)$ is reducible if and only if σ is dihedral.*

Proof. Since $\sigma \otimes \sigma = Sym^2(\sigma) \oplus \Lambda^2(\sigma)$, if $Sym^2(\sigma)$ is reducible, $\sigma \otimes \sigma$ must contain two characters of G, and we are done by the previous corollary.

LEMMA 2.7. *Let G be a group, N a subgroup of index 2, and H a subgroup of index 2 of N. Let σ be a 2-dimensional irreducible representation of G which is a sum of 2 characters when restricted to H. Then the representation σ of G is dihedral.*

Proof. We thank the referee for this proof which is slightly simpler than our earlier proof. Let χ_1 and χ_2 be the two characters of H appearing in the restriction of σ to H. We have

$$Sym^2(\sigma)|_H = \chi_1^2 \oplus \chi_1\chi_2 \oplus \chi_2^2.$$

The action of N permutes the three eigenspaces for H in $Sym^2(\sigma)|_H$. If all three eigenspaces are invariant, then

$$Sym^2(\sigma)|_N = \lambda_1 \oplus \lambda_2 \oplus \lambda_3$$

for extensions of the characters of H to N. The action of G permutes these eigenspaces and must preserve at least one. Hence σ is dihedral by Corollary 2.6. Otherwise, the action of N preserves one eigenspace of H and permutes the other two, so that

$$Sym^2(\sigma)|_N = \lambda \oplus \tau$$

for a character λ (extending the character of H on the fixed eigenspace) and a two dimensional irreducible representation τ of N. Since N is normal in G, the action of G must preserve the λ eigenline of N. Thus again σ is dihedral by Corollary 2.6.

PROPOSITION 2.8. *Let σ_1 and σ_2 be two 2-dimensional irreducible representations of a group G. Then if $\sigma_1 \otimes \sigma_2$ is a sum of two 2-dimensional irreducible representations then both σ_1 and σ_2 are dihedral representations.*

Proof. Suppose that $\sigma_1 \otimes \sigma_2 = \tau_1 \oplus \tau_2$ with both τ_1 and τ_2 irreducible. Since both σ_1 and σ_2 are two dimensional representations of G, σ_1 and σ_2 preserve alternating forms up to scaling on a two dimensional vector space. Taking the tensor product of the alternating forms, we get a quadratic form on the vector space underlying $\sigma_1 \otimes \sigma_2$ which the representation $\sigma_1 \otimes \sigma_2$ of G preserves up to scaling.

Define the nullity of a quadratic space to be the dimension of the maximal subspace which is perpendicular to the whole space under the corresponding bilinear form. The maximal null space is invariant under the similitude group. Since τ_1 and τ_2 are irreducible representations, the nullity of the quadratic form on $\sigma_1 \otimes \sigma_2$ restricted to the subspaces underlying τ_1 and τ_2 must be either 0 or 2. If the nullity of the quadratic form on $\sigma_1 \otimes \sigma_2$ restricted to τ_1 is 2, i.e., the quadratic form restricted to τ_1 is identically zero, the associated bilinear form $\tau_1 \times \tau_2 \to \mathbb{C}$ must be non-degenerate. This implies that as representations of G, $\tau_2^* \cong \tau_1 \otimes \chi$ for a character χ of G. Therefore $\tau_1 \oplus \tau_2 \cong \tau_1 \otimes [1 \oplus (\det \tau_1)^{-1}\chi^{-1}]$. If τ_1 is not a dihedral representation, there is up to scaling a unique quadratic form on $\tau_1 \oplus \tau_2$ which is left invariant up to scaling by G. This implies by Lemma 2.3 that one of σ_1 or σ_2 is reducible, contrary to our assumption. Therefore τ_1 and τ_2 must be dihedral representations of G. It is easy to see that this is also the case when both τ_1 and τ_2 are non-degenerate subspaces. Therefore in all cases if $\sigma_1 \otimes \sigma_2 = \tau_1 \oplus \tau_2$ with both τ_1 and τ_2 irreducible, the representations corresponding to τ_1 and τ_2 land inside $GO(2)$. Since $GSO(2)$ is of index 2 inside $GO(2)$, the representations τ_1 and τ_2 define subgroups H_1 and H_2 of G of index 2. However $H_1 = H_2$ as the representation $\sigma_1 \otimes \sigma_2 = \tau_1 \oplus \tau_2$ is inside $GSO(4)$. Since $GSO(2)$ is abelian, we find from Corollary 2.5 combined with Lemma 2.7 that the representations σ_1 and σ_2 are dihedral.

Proposition 2.8 completes the proof of the theorem at the beginning of the section. We next note the following lemma.

LEMMA 2.9. *For 2 dimensional irreducible non-dihedral representations σ_1 and σ_2 of a group G, $Sym^2\,\sigma_1 \cong Sym^2\,\sigma_2$ if and only if $\sigma_1 \cong \sigma_2 \otimes \chi$ for a quadratic character χ of G.*

Proof. Taking the determinant of $Sym^2\,\sigma_1$ and $Sym^2\,\sigma_2$, we find that $(\det \sigma_1)^3 = (\det \sigma_2)^3$. The vector space underlying $Sym^2\,\sigma_1$ has a quadratic form on it which is preserved by G up to a scalar. Because $Sym^2\,\sigma_1$ is irreducible, such a quadratic form is unique up to a scalar. The similitude factor for the action of G on such a quadratic form is $(\det \sigma_1)^2$. Therefore from the isomorphism of $Sym^2\,\sigma_1$ with $Sym^2\,\sigma_2$, $(\det \sigma_1)^2 = (\det \sigma_2)^2$. Combining this with the earlier identity $(\det \sigma_1)^3 = (\det \sigma_2)^3$, we get that $\det \sigma_1 = \det \sigma_2$. Therefore $\det \sigma_1^{-1}\,Sym^2\,\sigma_1 \cong \det \sigma_2^{-1}\,Sym^2\,\sigma_2$. Or, $Ad(\sigma_1) \cong Ad(\sigma_2)$. Therefore $\sigma_1 \cong \sigma_2 \otimes \chi$ for a character χ of G of order 2.

Remark 2.10. More generally, exactly the same argument as above yields that if $Sym^2\,\sigma_1 \cong Sym^2\,\sigma_2 \otimes \mu$ for 2 dimensional irreducible non-dihedral representations σ_1 and σ_2 of a group G, and a character μ of G, then $\sigma_1 \cong \sigma_2 \otimes \chi$ for a character χ of G with $\mu = \chi^2$.

3. TENSOR INDUCTION

From the work of many mathematicians starting with the pioneering work due to Eichler and Shimura which was refined by Deligne, Langlands and Carayol, and which culminated in the work of Blasius and Rogawski [BR], and R. Taylor [T], one knows that to a cohomological cuspidal automorphic form π on $GL(2)$ of a totally real number field k, there is a 2-dimensional ℓ-adic representation σ_π of $Gal(\bar{\mathbb{Q}}/k)$ with the same L-function as π. If the degree of k over $\mathbb{Q}$ is d, then the automorphic representation π contributes a 2^d dimensional ℓ-adic representation of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ to the dth cohomology of the corresponding Hilbert modular variety. The process of going from a 2-dimensional representation of $Gal(\bar{\mathbb{Q}}/k)$ to the 2^d-dimensional representation of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ is a general one which we review now.

Given any finite dimensional representation V of dimension n of a subgroup H of index d of any group G, there is a representation of G denoted by $M(V)$ of dimension n^d which is called *tensor induction or multiplicative induction.* We will not recall the definition of $M(V)$ here but refer the reader to [C-R]. However we note that if H is a normal subgroup of G then the representation $M(V)$ of G when restricted to H is the tensor product of the various conjugates of V under the action of G/H. The representation $M(V)$ has the following properties.

(1) $M(V_1 \otimes V_2) \cong M(V_1) \otimes M(V_2)$.

(2) $M(V)^* \cong M(V^*)$

(3) $M(\chi)$ for a character χ of H is the transfer of χ to G, i.e., it is the composite of χ under the transfer map $G/[G, G] \to H/[H, H]$.

We also recall that for an extension K of a local or global field k, the transfer map from the Weil group W_k to the subgroup W_K is given by the inclusion of the idele group of k into that of K.

The tensor induction has the property that if H is a normal subgroup of G, then

$$M(V) \cong M(V^g)$$

for the conjugation by any element g of G on any representation V of H.

Finally, for our purposes, if H is a subgroup of G of index 2 and V is a representation of G, then

$$M(V|_H) \cong Sym^2(V) \oplus \omega_{G/H} \Lambda^2 V \tag{3.1}$$

where $\omega_{G/H}$ is the non-trivial character of G trivial on H.

4. COHOMOLOGY OF A HILBERT MODULAR SURFACE

Let S_K be a Hilbert modular surface associated to a real quadratic field F, and a compact open subgroup $K \subset GL_2(\mathbb{A}_f \otimes_{\mathbb{Q}} F)$. We have the decomposition ([HLR], Section 5)

$$H^2_\ell(S_K) = \mathbb{H}^2_\ell(\bar{S}_K) \oplus H^2_\ell(S^\infty_K)$$

where $\bar{S}_K$ is the Baily-Borel compactification of S_K^o, $\mathbb{H}$ denotes intersection cohomology and S_K^∞ denotes the divisor at infinity such that $S_K = S_K^o \cup S_K^\infty$. The action of the Hecke algebra induces a decomposition of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ modules

$$\mathbb{H}_\ell^2(\bar{S}_K) = \sum_\pi H_\ell^2(\pi) \otimes \pi_f^K.$$

The cuspidal automorphic representations π of $GL(2)$ over F which appear in the above decomposition have the discrete series representation of $PGL(2, F \otimes \mathbb{R})$ of highest weight ± 2 for the component at infinity; in particular, the central character of such π is trivial at infinity; π_f^K denotes the K-invariants in the finite part of the automorphic representation π which is $\pi = \pi_\infty \otimes \pi_f$.

It will be convenient to consider the direct limit of $\mathbb{H}_\ell^2(\bar{S}_K)$ as K shrinks to the identity. Define

$$\mathbb{H}_\ell^2(\bar{S}) = \varinjlim_K \mathbb{H}_\ell^2(\bar{S}_K).$$

We call $\mathbb{H}_\ell^2(\bar{S})$ the 2nd (intersection) cohomology of the Hilbert modular surface associated to the real quadratic field F. It has the decomposition,

$$\mathbb{H}_\ell^2(\bar{S}) = \sum_\pi H_\ell^2(\pi) \otimes \pi_f.$$

For representations π appearing in the above decomposition, it follows from Blasius and Rogawski [BR], and independently by Taylor [T], that there is a representation σ_π of $Gal(\bar{\mathbb{Q}}/F)$ of dimension 2 with the property that

$$L(\sigma_\pi, s) = L(\pi, s)$$

where the L function on the right is the standard degree 2 L-function associated to the automorphic representation π of $GL(2, F)$. Automorphic representations π for which there exists a Galois representation σ_π with the above equality of L-functions will be called automorphic representations with associated Galois representations in this paper.

We have $H_\ell^2(\pi)(1) = M(\sigma_\pi)$ as $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ representations where $M(\sigma_\pi)$ is the 4 dimensional representation of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ obtained from the representation σ_π of $Gal(\bar{\mathbb{Q}}/F)$ by the process of tensor induction of last section. In particular, the representation $H_\ell^2(\pi)(1)$ restricted to $Gal(\bar{\mathbb{Q}}/F)$ is $\sigma_\pi \otimes \sigma_\pi^\tau$ where τ is the non-trivial element of the Galois group of F over $\mathbb{Q}$ operating by conjugation on σ_π.

A cuspidal representation π of $GL(2)$ of a number field K is called of CM type if there exists a character ε of the idele class group of K of order 2 such that $\pi \otimes \varepsilon \cong \pi$. If π has a Galois representation σ_π associated to it, then π is of CM type if and only if σ_π is a dihedral representation.

The work of Harder, Langlands, Rapoport describes the Tate classes in the 2nd cohomology of a Hilbert modular surface. We review some of their work here. We begin with the following proposition.

PROPOSITION 4.1. *Suppose that π is a cuspidal, non-CM automorphic representation of $GL(2)$ over a number field K. Suppose K is a quadratic extension k with τ as the Galois automorphism of K over k. If $\pi^\tau \cong \pi \otimes \chi$ for a Grössencharacter χ of K, then χ is trivial when restricted to the ideles of k.*

Proof. This proposition is a simple consequence of Theorem 11.5.2 of Rogawski's book [Ro]. By looking at the central characters of the two sides of the isomorphism $\pi^\tau \cong \pi \otimes \chi$, we have $\omega_\pi^\tau = \omega_\pi \cdot \chi^2$. Also, applying τ to the isomorphism $\pi^\tau \cong \pi \otimes \chi$, we have $\pi \cong \pi^\tau \otimes \chi^\tau$, and therefore $\pi \cong \pi \otimes \chi \cdot \chi^\tau$. Since π does not have CM, this implies that $\chi \cdot \chi^\tau = 1$. Combining $\omega_\pi^\tau = \omega_\pi \chi^2$, and $\chi \cdot \chi^\tau = 1$, we have $(\omega_\pi \chi)^\tau = \omega_\pi \chi$. Therefore there exists a character μ of J_K such that $\mu\mu^\tau = \omega_\pi \cdot \chi$.

Taking the duals in the isomorphism $\pi^\tau \cong \pi \otimes \chi$, we have

$$(\pi^*)^\tau \cong \pi \cdot (\omega_\pi \chi)^{-1} = \pi \cdot (\mu\mu^\tau)^{-1}.$$

Therefore,

$$(\pi\mu^{-1})^{*\tau} \cong \pi\mu^{-1}.$$

Let ε be the automorphism of $GL(2, K)$ given by $g \to ({}^t g^{-1})^\tau$. For automorphic representations π of $GL(2, K)$, $\pi \circ \varepsilon \cong \pi^{*\tau}$, and therefore

$$(\pi\mu^{-1}) \circ \varepsilon \cong \pi\mu^{-1}.$$

Now, Theorem 11.5.2 of [Ro] gives

$$\omega_{\pi\mu^{-1}}|_{J_k} = 1.$$

From $\omega_\pi \chi = \mu\mu^\tau$, we have $(\omega_\pi \cdot \chi)|_{J_k} = \mu^2$. Therefore,

$$\omega_{\pi\mu^{-1}}|_{J_k} = (\omega_\pi \cdot \mu^{-2})|_{J_k} = \chi^{-1}|_{J_k} = 1,$$

completing the proof of the lemma.

LEMMA 4.2. *Let π_1 and π_2 be two cuspidal representations of $GL(2)$ over a real quadratic field F which have associated representations σ_{π_1} and σ_{π_2} of $Gal(\overline{\mathbb{Q}}/F)$. Suppose that π_1 and π_2 and hence σ_{π_1} and σ_{π_2} are both non-CM, and that σ_{π_1} and σ_{π_2} are twists of each other over an extension of F, then σ_{π_1} and σ_{π_2} are twists of each other over F.*

Proof. Since π_i are non-CM, σ_{π_i} remain irreducible and non-dihedral over any number field by Satz 4.5.4 of [HLR]. Since σ_{π_1} and σ_{π_2} are twists of each other over an extension which we can assume to be Galois over F, we find that $\sigma_{\pi_1} \otimes \sigma^*_{\pi_2}$ contains a 1-dimensional representation when restricted to a normal extension of F. However, by the non-CM hypothesis and Corollary 2.5, $\sigma_{\pi_1} \otimes \sigma^*_{\pi_2}$ cannot contain more than one 1-dimensional representation. This implies that the corresponding vector in $\sigma_{\pi_1} \otimes \sigma^*_{\pi_2}$ must be invariant under $Gal(\bar{\mathbb{Q}}/F)$, i.e., σ_{π_1} and σ_{π_2} are twists of each other over F.

PROPOSITION 4.3. *Suppose that π is a cuspidal, non-CM automorphic representation of $GL(2)$ over a real quadratic field F which has associated to it a Galois representation σ_π. Then, for a number field k, the representation $M(\sigma_\pi)$ of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ contains a vector on which $Gal(\bar{\mathbb{Q}}/k)$ acts trivially if and only if there exists a cuspidal automorphic representation π_0 of $GL(2)$ over $\mathbb{Q}$ such that*

$$\pi = BC^F_{\mathbb{Q}}(\pi_0) \otimes \chi$$

with $\omega_{\pi_0} \cdot \omega_{F/\mathbb{Q}} \cdot \chi|_{J_{\mathbb{Q}}}$ trivial on the image of the ideles of k inside the ideles of $\mathbb{Q}$ under the norm mapping. Here ω_{π_0} is the central character of π_0 and $\omega_{F/\mathbb{Q}}$ is the character of the ideles of $\mathbb{Q}$ associated by class field theory to the quadratic extension F, and $BC^F_{\mathbb{Q}}(\pi_0)$ denotes the base change of the automorphic representation π_0 of $GL(2, \mathbb{Q})$ to $GL(2, F)$.

Proof. If $M(\sigma_\pi)$ has a vector on which $Gal(\bar{\mathbb{Q}}/k)$ acts trivially, then in particular $\sigma_\pi \otimes \sigma^\tau_\pi$ has a vector on which $Gal(\bar{\mathbb{Q}}/kF)$ acts trivially. This implies that the representations σ_π and σ^τ_π are twists of each other over kF, and therefore over F by Lemma 4.2. This implies that

$$\sigma^\tau_\pi \cong \sigma_\pi \otimes \alpha$$

for some Grössencharacter α of $Gal(\bar{\mathbb{Q}}/F)$ which is trivial on the ideles of $\mathbb{Q}$ by Proposition 4.1. Therefore α can be written as $\alpha = \chi^\tau/\chi$. So.

$$(\sigma_\pi \otimes \chi^{-1})^\tau \cong \sigma_\pi \otimes \chi^{-1}.$$

Therefore $\sigma_\pi \otimes \chi^{-1}$ can be written as a base change, i.e., there exists π_0 automorphic on $GL(2)$ over $\mathbb{Q}$ without CM with $\sigma_\pi \otimes \chi^{-1} = \sigma_{\pi_0}|_F$, or $\pi = BC^F_{\mathbb{Q}}(\pi_0) \otimes \chi$ (see Remark 4.4 below).

Therefore,

$$M(\sigma_\pi) \cong M(\sigma_{\pi_0}|_{Gal(\bar{\mathbb{Q}}/F)}) \otimes \chi|_{J_{\mathbb{Q}}}.$$

So, by property (3.1) of tensor induction,

$$M(\sigma_\pi) \cong (Sym^2\, \sigma_{\pi_0} \oplus \omega_{\pi_0} \cdot \omega_{F/\mathbb{Q}}) \otimes \chi|_{J_{\mathbb{Q}}}.$$

It follows that $M(\sigma_\pi)$ contains a vector on which $Gal(\bar{\mathbb{Q}}/k)$ operates trivially if and only if $\omega_{\pi_0} \cdot \omega_{F/\mathbb{Q}} \cdot \chi|_{J_\mathbb{Q}} = 1$ on the image of the ideles of k inside the ideles of $\mathbb{Q}$ under the norm mapping.

Remark 4.4. Let L be a cyclic extension of a number field K. Let π be an automorphic representation of GL_2 over L which has an associated ℓ-adic representation σ_π of $Gal(\bar{\mathbb{Q}}/L)$. If σ_π can be extended to an ℓ-adic representation σ'_π of $Gal(\bar{\mathbb{Q}}/K)$, then clearly π itself can be obtained as a base change of an automorphic representation, say π_0, of GL_2 over K. However, it is not clear that one can choose π_0 such that its L-function is that of σ'_π in general. If, however, π_0 has an ℓ-adic representation attached to it, then this is possible as is easily seen.

The next theorem due to Harder, Langlands and Rapaport, and which is a consequence of Proposition 4.3, gives a complete parametrization of Tate classes on a Hilbert modular surface coming from non-CM automorphic forms.

THEOREM 4.5. *Suppose that π is a cuspidal non-CM automorphic representation of $GL(2)$ over a real quadratic field F. Assume that π contributes to the 2nd cohomology of the corresponding Hilbert modular surface. Then this contribution of π to the 2nd cohomology of the Hilbert modular surface contains a Tate class if and only if $\pi \cong BC_\mathbb{Q}^F(\pi_0) \otimes \chi$ for a cuspidal automorphic representation π_0 of $GL(2)$ over $\mathbb{Q}$, and a Grössencharacter χ of F such that $\omega_{\pi_0} \cdot \omega_{F/\mathbb{Q}} \cdot \chi|_{J_\mathbb{Q}}$ is of finite order. This finite order character defines an abelian extension of $\mathbb{Q}$ which is the field of definition of the corresponding Tate class.*

5. TATE CLASSES ON THE PRODUCT IN THE NON-CM CASE

Since the first cohomology of a Hilbert modular surface vanishes, the essential component of the Tate cycles which are contained in $H^4_\ell(S_1 \times S_2)(2)$ is

$$H^2_\ell(S_1)(1) \otimes H^2_\ell(S_2)(1).$$

Decomposing this $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-module according to the action of the Hecke algebra, we are reduced to considering

$$H^2_\ell(\pi_1)(1) \otimes H^2_\ell(\pi_2)(1)$$

for certain cuspidal automorphic forms π_i on $GL(2, F_i)$.

THEOREM 5.1. *Assume that π_1 and π_2 are cuspidal, non-CM automorphic forms on GL(2) over real quadratic fields F_1 and F_2 respectively. Assume that π_i contributes to the 2nd cohomology of the corresponding Hilbert modular surfaces. Then $H^2_\ell(\pi_1)(1) \otimes H^2_\ell(\pi_2)(1)$ has a Tate class over a number field k which does not come as a product of Tate classes from individual factors if and only if either*

(i) *$F_1 = F_2$, and π_1 and π_2 are twists of each other, say $\pi_1 = \pi_2 \otimes \chi$ with the property that*

$$(\omega_{\pi_2}\chi)\,|_{J_{\mathbb{Q}}}$$

is a finite order character of the idele class group of $\mathbb{Q}$ which is trivial on the image of J_k inside $J_{\mathbb{Q}}$ under the norm mapping;

or,

(ii) *π_1 and π_2 are, up to twist by characters, base change from $\mathbb{Q}$ to F_1 and F_2 respectively of the same cuspidal representation on $\mathbb{Q}$, say*

$$\pi_1 \cong BC^{F_1}_{\mathbb{Q}}(\Pi) \otimes \nu_1$$

$$\pi_2 \cong BC^{F_2}_{\mathbb{Q}}(\Pi^*) \otimes \nu_2,$$

for a cuspidal representation Π on GL(2) over $\mathbb{Q}$, and Grössencharacters ν_1 and ν_2 of F_1 and F_2 respectively, with the property that

$$(\nu_1 \nu_2)\,|_{J_{\mathbb{Q}}}$$

is a finite order character of the idele class group of $\mathbb{Q}$ which is trivial on the image of J_k inside $J_{\mathbb{Q}}$ under the norm mapping.

Proof. Assume that there is a Tate class in $H^2_\ell(\pi_1)(1) \otimes H^2_\ell(\pi_2)(1)$ defined over k; then in particular, $H^2_\ell(\pi_1)(1) \otimes H^2_\ell(\pi_2)(1)$ has a one-dimensional subspace invariant under $Gal(\bar{\mathbb{Q}}/kF_1F_2)$. By hypothesis π_i are non-CM and therefore σ_{π_i} remain irreducible, non-dihedral over any number field. Let θ_1 be an element of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ which restricts to the non-trivial automorphism of F_1, and if $F_1 \neq F_2$, it restricts to the trivial automorphism of F_2; define θ_2 similarly. From Section 2 we know that $\sigma_{\pi_i} \otimes \sigma^{\theta_i}_{\pi_i}$ restricted to kF_1F_2 is either irreducible, or is the sum of a one dimensional representation and an irreducible 3 dimensional representation. If $\sigma_{\pi_i} \otimes \sigma^{\theta_i}_{\pi_i}$ contains a 1 dimensional representation when restricted to kF_1F_2, it follows from Lemma 4.2 that $\sigma^{\theta_i}_{\pi_i} \cong \sigma_{\pi_i} \otimes \chi_i$ over F_i for certain characters χ_i of F_i. By Lemma 4.1, χ_i are trivial on the ideles of $\mathbb{Q}$, and therefore $\pi_i = BC^{F_i}_{\mathbb{Q}}(\Pi_i) \otimes \mu_i$.

On the other hand if $\sigma_{\pi_i} \otimes \sigma_{\pi_i}^{\theta_i}$ is irreducible for $i = 1, 2$, then under the hypothesis that $H_\ell^2(\pi_1)(1) \otimes H_\ell^2(\pi_2)(1)$ has a Tate cycle over k, we have the isomorphism of $Gal(\bar{\mathbb{Q}}/kF_1F_2)$ modules

$$\sigma_{\pi_1} \otimes \sigma_{\pi_1}^{\theta_1} \cong (\sigma_{\pi_2} \otimes \sigma_{\pi_2}^{\theta_2})^*.$$

By Lemmas 2.3 and 4.2, we assume without loss of generality that there is an isomorphism of $Gal(\bar{\mathbb{Q}}/F_1F_2)$-modules

$$\sigma_{\pi_1} \cong \sigma_{\pi_2} \otimes \chi$$

for some Grössencharacter χ of F_1F_2.

Assume that $F_1 \neq F_2$. Since θ_1 operates trivially on F_2, it operates trivially on the representation σ_{π_2} restricted to $Gal(\bar{\mathbb{Q}}/F_1F_2)$. Therefore applying θ_1 to the isomorphism $\sigma_{\pi_1} \cong \sigma_{\pi_2} \otimes \chi$, we find that σ_{π_1} and $\sigma_{\pi_1}^{\theta_1}$ are twists of each other when restricted to F_1F_2, and therefore σ_{π_1} and $\sigma_{\pi_1}^{\theta_1}$ are twists of each other over F_1 contradicting the irreducibility of $\sigma_{\pi_i} \otimes \sigma_{\pi_i}^{\theta_i}$.

It follows that if $H_\ell^2(\pi_1)(1) \otimes H_\ell^2(\pi_2)(1)$ has a Tate cycle over a number field k and either $F_1 \neq F_2$, or if $F_1 = F_2$, π_1 and π_2 are not twists of each other, then there are automorphic forms Π_1 and Π_2 for $GL(2)$ over $\mathbb{Q}$, and Grössencharacters μ_1 and μ_2 of F_1 and F_2 respectively, such that

$$\pi_1 \cong BC_{\mathbb{Q}}^{F_1}(\Pi_1) \otimes \mu_1$$
$$\pi_2 \cong BC_{\mathbb{Q}}^{F_2}(\Pi_2) \otimes \mu_2.$$

From the property (3.1) of the tensor induction,

$$H_\ell^2(\pi_1)(1) = M(\sigma_{\pi_1}) \cong (Sym^2\, \sigma_{\Pi_1} \oplus \det \sigma_{\Pi_1} \cdot \omega_{F_1/\mathbb{Q}}) \otimes \mu_1|_{J_{\mathbb{Q}}},$$
$$H_\ell^2(\pi_2)(1) = M(\sigma_{\pi_2}) \cong (Sym^2\, \sigma_{\Pi_2} \oplus \det \sigma_{\Pi_2} \cdot \omega_{F_2/\mathbb{Q}}) \otimes \mu_2|_{J_{\mathbb{Q}}}.$$

Therefore, $H_\ell^2(\pi_1)(1) \otimes H_\ell^2(\pi_2)(1)$ contains a Tate cycle over k which is not a product of Tate cycles on individual factors if there is an isomorphism of $Gal(\bar{\mathbb{Q}}/k)$ modules

$$\mu_1 \otimes Sym^2\, \sigma_{\Pi_1} \cong (\mu_2 \otimes Sym^2\, \sigma_{\Pi_2})^* \tag{5.1}$$

It follows from Remark 2.10 that if the symmetric squares of two non-dihedral representations differ by a character, the representations themselves differ by a character. So, we can assume that there is a cusp form Π on $GL(2)$ of $\mathbb{Q}$, and Grössencharacters ν_1 and ν_2 of F_1 and F_2 respectively, such that

$$\pi_1 \cong BC_{\mathbb{Q}}^{F_1}(\Pi) \otimes \nu_1$$
$$\pi_2 \cong BC_{\mathbb{Q}}^{F_2}(\Pi^*) \otimes \nu_2.$$

Now the condition in (5.1) for the existence of a Tate class in $H^2_\ell(\pi_1)(1)\otimes H^2_\ell(\pi_2)(1)$ over k translates into the condition that the character

$$(\nu_1\nu_2)|_{J_{\mathbb{Q}}}$$

of the idele class group of $\mathbb{Q}$ is trivial on the image of J_k in $J_{\mathbb{Q}}$ under the norm mapping.

Finally, if $F_1=F_2$, and $\pi_1=\pi_2\otimes\chi$, then

$$M(\sigma_{\pi_1})\otimes M(\sigma_{\pi_2})=M(\chi\cdot\omega_{\pi_2}\oplus\chi\cdot Sym^2\,\sigma_{\pi_2}).$$

Since π_2 is assumed to be non-CM, $M(\chi\cdot\omega_{\pi_2}\oplus\chi\cdot Sym^2\,\sigma_{\pi_2})$ contains a Tate class, i.e., an invariant of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, if and only if either $M(\chi\cdot\omega_{\pi_2})$, or $M(\chi\cdot Sym^2\,\sigma_{\pi_2})$ contains a Tate class. The analysis of when $M(\chi\cdot Sym^2\,\sigma_{\pi_2})$ contains a Tate class is exactly as in the case $F_1\neq F_2$. The extra Tate classes in the case $F_1=F_2$ come from $M(\chi\cdot\omega_{\pi_2})$ which from property (3) of the tensor induction in Section 3, contains a Tate class if and only if the character $(\omega_{\pi_2}\cdot\chi)|_{J_{\mathbb{Q}}}$ of the idele class group of $\mathbb{Q}$ is trivial on the image of J_k in $J_{\mathbb{Q}}$ under the norm mapping, proving the theorem.

Corollary 5.2. *All the Tate cycles arising out of non-CM forms on a product of two Hilbert modular surfaces are defined over abelian extensions of* $\mathbb{Q}$.

The Tate cycles constructed in case (ii) of the previous theorem will be called special Tate cycles, and as mentioned in the introduction, we have not been able to find algebraic cycles corresponding to them.

Definition 5.3. Let Π be a cusp form on $GL(2)$ over $\mathbb{Q}$, μ_1, μ_2 Grössencharacters on F_1, and F_2 respectively, and $\pi_1=BC_{\mathbb{Q}}^{F_1}(\Pi)\otimes\mu_1$, and $\pi_2=BC_{\mathbb{Q}}^{F_2}(\Pi^*)\otimes\mu_2$ be cusp forms on $GL(2)$ over F_1 and F_2 respectively. Assume that $(\mu_1\mu_2)|_{J_{\mathbb{Q}}}$ is a finite order character. Then in $H^2_\ell(\pi_1)(1)\otimes H^2_\ell(\pi_2)(1)$ there is a Tate cycle, called *special Tate Cycle*. It is defined over an abelian extension of $\mathbb{Q}$ corresponding to this finite order character.

Remark 5.4. Assume that the cuspidal automorphic representations π_i of $GL(2)$ over F_i are base change of cuspidal automorphic representations Π_i on $GL(2)$ over $\mathbb{Q}$. The special classes occur in $H^2(\pi_1)\otimes H^2(\pi_2)$. From the fundamental work of Oda [O], we know that,

$$H^2(\pi_1)\otimes H^2(\pi_2)\simeq H^1(\Pi_1)^{\otimes 2}\otimes H^1(\Pi_2)^{\otimes 2}$$

where Π_1 and Π_2 contribute to the cohomology of Abelian varieties A_1 and A_2 (say). These abelian varieties belong to a family for which one knows that all Tate cycles are algebraic. Indeed, A_i has multiplication by a field

E_i satisfying $dim\ A_i = [E_i : \mathbb{Q}]$. In [M1] it was proved that for such abelian varieties which do not have complex multiplication, the ring of Tate cycles is algebraic and generated by the classes of divisors. If A_i has complex multiplication, by a result of Shimura, it is isogenous over $\bar{\mathbb{Q}}$ to a power of an elliptic curve. It is easy to show that if one of the A_i has complex multiplication, then so does the other. Hence, over $\bar{\mathbb{Q}}$, $A_1 \times A_2$ is isogenous to a product of elliptic curves. It is well-known that for such an abelian variety, all Tate cycles are algebraic. (See [M2], for a proof.) Therefore if the isomorphism $H^2(\pi_i) \simeq H^1(\Pi_i)^{\otimes 2}$ is induced from an algebraic cycle, the Tate cycles that we construct will be algebraic.

Remark 5.5. Our construction of special cycles is very general and seems to be yet another example of a modular construction of Tate cycles which has no apparent geometric realization. An example of a modular construction (unrelated to ours) which *has* been proved algebraic can be found in [EG].

6. TATE CLASSES IN THE CM-CASE

Suppose that π_1 is of CM-type. Then over a sufficiently large field, $H^2_\ell(\pi_1)$ decomposes as a sum of four one-dimensional representations. From this it is easy to see that if there is a Tate class in $H^2_\ell(\pi_1)(1) \otimes H^2_\ell(\pi_2)(1)$ which is not the product of Tate classes on individual factors, then π_2 must also be a CM form.

Hence we assume in this section that π_1 and π_2 are CM cuspidal automorphic representations on $GL(2)$ over F_1 and F_2 respectively. Let us write $\pi_i = \mathrm{Ind}(\psi_i)$ for Grössencharacters ψ_i of an imaginary quadratic extension M_i of F_i. We denote by $V_\ell(\psi_i)$ the ℓ-adic representation of $Gal(\bar{\mathbb{Q}}/M_i)$ associated to ψ_i. The ℓ-adic representation σ_{π_i} is equal to $Ind^{F_i}_{M_i} V_\ell(\psi_i)$, and therefore the restriction of σ_{π_i} to M_i is $V_\ell(\psi_i) \oplus V_\ell(\bar{\psi}_i)$ where $\bar{\psi}_i$ is the complex conjugate of ψ_i. If θ_i denotes an element of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ which is non-trivial when restricted to F_i, then $\sigma^{\theta_i}_{\pi_i}$ is induced from a Grössencharacter ψ'_i on M'_i ($M'_i = \theta_i(M_i)$) obtained from the Grössencharacter ψ_i on M_i by "transport de structure." Over the field $\tilde{M}_1 = M_1 M'_1$, we have the decomposition

$$H^2_\ell(\pi_1)(1) \simeq (V_\ell(\psi_1) \oplus V_\ell(\bar{\psi}_1)) \otimes (V_\ell(\psi'_1) \oplus V_\ell(\bar{\psi}'_1)),$$
$$\simeq V_\ell(\psi_1\psi'_1) \oplus V_\ell(\psi_1\bar{\psi}'_1) \oplus V_\ell(\bar{\psi}_1\psi'_1) \oplus V_\ell(\bar{\psi}_1\bar{\psi}'_1).$$

We note that the restriction of a Grössencharacter to a field extension corresponds to composition with the norm mapping. So, in the above decomposition over $\tilde{M}_1$, we have abused notation to denote $\psi_1\psi'_1$, for instance, for the Grössencharacter on $\tilde{M}_1$ which is the product of two Grössencharacters on $\tilde{M}_1$ which are obtained from ψ_1 on M_1 and ψ'_1 on M'_1 via composition with the norm mapping.

For a number field F with normal closure $\tilde{F}$, set $G = Gal(\tilde{F}/\mathbb{Q})$ and $H = Gal(\tilde{F}/F)$. We note that the infinity type of a Grössencharacter on F is the same as an integer valued function on G/H (as G/H can be identified to the set of embeddings of F into $\mathbb{C}$). The advantage of this notation for us is that the infinity type of a Grössencharacter ψ on F thought of as a function on G/H when thought of as a function on G gives the infinity type of the Grössencharacter on $\tilde{F}$ obtained from ψ by composing with the norm mapping from $\tilde{F}$ to F. We also recall that if a Grössencharacter χ of a CM extension K of a totally real field F contributes to the cohomology of the corresponding Hilbert modular variety, then the infinity type of χ is a set of embeddings of K in $\mathbb{C}$ whose restriction to F is precisely the set of embeddings of F into $\mathbb{C}$.

We will use the following lemma several times in the proof of the next proposition.

LEMMA 6.1. (1) *Let f_1, f_2 be two functions on a group G right invariant under subgroups H_1, H_2 of G. If the function $f_1 + f_2$ on G is invariant under the right action of the subgroup H generated by H_1 and H_2 inside G, then f_1 and f_2 are also invariant under the right action of H.*

(2) *Let f_1, f_2, f_3 be three functions on a group G right invariant respectively under subgroups H_1, H_2, H_3 of G. Assume that $f_1 + f_2 = f_3$. If the inner conjugation action of H_1 leaves H_2 and H_3 invariant, and if H_1 is a finite group which is contained in the subgroup generated by H_2 and H_3, then f_2 is invariant under the subgroup generated by H_1 and H_2.*

Proof. We only prove part (2) as part (1) is rather trivial. We have for any $g \in G$, and $h \in H_1$

$$f_1(g) + f_2(g) = f_3(g),$$

$$f_1(gh) + f_2(gh) = f_3(gh).$$

Therefore for any $g \in G$ and $h \in H_1$ we have,

$$f_2(g) - f_2(gh) = f_3(g) - f_3(gh).$$

Since H_1 leaves H_2 and H_3 invariant under the inner conjugation action, this implies that the function $f_2(g) - f_2(gh)$ is invariant under H_2 and H_3, and therefore under H_1. Since H_1 is a finite group, this implies that this function must be identically zero, i.e., $f_2(g)$ is invariant under H_1.

PROPOSITION 6.2. *If $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ supports a Tate class which does not come as the tensor product of Tate classes on individual factors, then the M_i are biquadratic extensions of $\mathbb{Q}$ with $M_1 \cap M_2 = M$, a quadratic imaginary extension of $\mathbb{Q}$. Moreover, the infinity type of the Grössencharacters ψ_1 and ψ_2 are invariant under the Galois automorphism of M_1 over M and of M_2 over M.*

Proof. The field M_i is either Galois over $\mathbb{Q}$, or its normal closure $\tilde{M}_i$ is of degree 8 over $\mathbb{Q}$ with Galois group the dihedral group $D_8 = \{x^4 = 1, y^2 = 1, yxy^{-1} = x^{-1}\}$. We know that $H^2_\ell(\pi_i)$ is a sum of 4 Grössencharacters on $\tilde{M}_i$. Therefore $H^2_\ell(\pi_1) \otimes H^2_\ell(\pi_2)$ is a sum of 16 Grössencharacters on $\tilde{M}_1\tilde{M}_2$. If this is to contain a Tate cycle, then the product of a Grössencharacter χ_1 appearing in $H^2_\ell(\pi_1)$ and a Grössencharacter χ_2 appearing in $H^2_\ell(\pi_2)$ must have the constant infinity type on $\tilde{M}_1\tilde{M}_2$. By Lemma 6.1(1) applied to $G = Gal(\tilde{M}_1\tilde{M}_2/\mathbb{Q})$, $H_1 = Gal(\tilde{M}_1\tilde{M}_2/\tilde{M}_1)$, $H_2 = Gal(\tilde{M}_1\tilde{M}_2/\tilde{M}_2)$, f_1 the infinity type of χ_1, f_2 the infinity type of χ_2, we find that the infinity type of the Grössencharacter χ_1 is invariant under $Gal(\tilde{M}_1/\tilde{M}_1 \cap \tilde{M}_2)$. If $\tilde{M}_1 \cap \tilde{M}_2 = \mathbb{Q}$, then χ_1 and χ_2 themselves correspond to Tate classes, and we need not consider this case. If $\tilde{M}_1 \cap \tilde{M}_2 \neq \mathbb{Q}$, we will need to consider several cases depending on this intersection. The Grössencharacter χ_1 is itself the product of two characters ψ_1 and ψ'_1 (or, ψ_1 and $\bar{\psi}'_1$ etc.). We will apply Lemma 6.1(2) to this situation to deduce some properties of ψ_1 and ψ_2 which will complete the proof. We will prove the proposition assuming $F_1 \neq F_2$.

If $\tilde{M}_1 \cap \tilde{M}_2 = M$, a quadratic extension of $\mathbb{Q}$, then if $M_1 = \tilde{M}_1$, and $M_2 = \tilde{M}_2$, both M_1 and M_2 are biquadratic extensions $M_1 = F_1 M$, and $M_2 = F_2 M$, and since M_i are CM extensions of F_i, M is a quadratic imaginary field. By Lemma 6.1(1), the infinity type of the Grössencharacters χ_1 on M_1 and χ_2 on M_2 are the pull back of the infinity type of Grössencharacters on M. From this it is easy to see that the infinity type of the Grössencharacters ψ_1 and ψ_2 are pull back of Grössencharacters on M.

If $\tilde{M}_1 \neq M_1$, and $\tilde{M}_1 \cap \tilde{M}_2 = M$ a quadratic extension of $\mathbb{Q}$, let M'_1 be the image of M_1 under an element of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ such that $M'_1 \neq M_1$. Let σ (resp. τ) be the nontrivial element of $Gal(\tilde{M}_1/\mathbb{Q})$ stabilising M_1 (resp. M'_1). We claim that σ does not act trivially on M, because otherwise τ which is a conjugate of σ will also act trivially on M. This would imply that M is contained in F_1 which is not possible. By Lemma 6.1(2) applied to $G = Gal(\tilde{M}_1/\mathbb{Q})$, $H_1 = \langle \sigma \rangle$, $H_2 = \langle \tau \rangle$, $H_3 = Gal(\tilde{M}_1/M)$, and $f_1 =$ infinity type of ψ_1, $f_2 =$ infinity type of ψ'_1, $f_3 =$ infinity type of χ_1, we find that the infinity type of ψ_1 is the pull back of an infinity type from F_1. Such infinity types do not correspond to cohomological representations.

If $\tilde{M}_1 \cap \tilde{M}_2 = M$ is of degree 4 over $\mathbb{Q}$, then neither M_1 nor M_2 is Galois over $\mathbb{Q}$, and M must be the biquadratic field F_1F_2. An application of

Lemma 6.1(1) implies that the infinity type of the Grössencharacter χ_1 on $\tilde{M}_1$ is pullback from an infinity type on M. Application now of Lemma 6.1(2) to $G = Gal(\tilde{M}_1/\mathbb{Q})$, $H_1 = \langle \sigma \rangle$, $H_2 = \langle \tau \rangle$, $H_3 = Gal(\tilde{M}_1/M)$, and the same functions as in the last paragraph, implies that the infinity type of the Grössencharacter ψ_1 restricted to $\tilde{M}_1$, and therefore the infinity type of ψ_1 is pull back from F_1. Again this is not allowed as we are considering cohomological representations only. This completes the proof of the proposition.

Since any two Grössencharacters with the same infinity type differ by a finite order character, and since we can construct a Grössencharacter of a CM number field with a given infinity type (with obvious constraints arising out of Dirichlet unit theorem: $n_\sigma + n_{\bar{\sigma}}$, a constant independent of σ), the previous proposition implies the following theorem.

THEOREM 6.3. *If π_1 and π_2 are CM forms on $GL(2)$ over F_1 and F_2 respectively such that $H^2_\ell(\pi_1)(1) \otimes H^2_\ell(\pi_2)(1)$ contains a Tate cycle which does not come as the product of Tate cycles from individual factors, then π_1 and π_2 come from Grössencharacters ψ_1 and ψ_2 on biquadratic fields $M_1 = MF_1$ and $M_2 = MF_2$ where M is an imaginary quadratic extension of $\mathbb{Q}$. Moreover the Grössencharacters ψ_1 and ψ_2 are obtained up to finite order characters on the idele class group of M_1 and M_2 respectively, from Grössencharacters ϕ_1 and ϕ_2 of M via the norm mapping, where ϕ_1 corresponds to an embedding of M into $\mathbb{C}$, and ϕ_2 also corresponds to an embedding of M into $\mathbb{C}$. Conversely, such a construction gives rise to a Tate cycle. The dimension of the Tate cycles in $H^2_\ell(\pi_1)(1) \otimes H^2_\ell(\pi_2)(1)$ is 6 of which a 4 dimensional subspace is spanned by the tensor product of Tate cycles on individual factors.*

Remark 6.4. The automorphic form π_1 (resp. π_2) of $GL(2)$ over F_1 (resp. F_2) in the above theorem is not in general the base change of an automorphic form on $GL(2)$ over $\mathbb{Q}$ even after twisting by a Grössencharacter on F_1 (resp. F_2) unlike in the non-CM case earlier. The Tate cycles in this theorem are not in general defined over abelian extensions of $\mathbb{Q}$ again unlike the non-CM case.

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