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On the Self-Dual Representations of Finite Groups of Lie Type

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CONTENTS.

1. INTRODUCTION

It is a classical theorem in the representation theory of semisimple algebraic groups (or, of compact connected Lie groups) that there exists an element h in the center of such a group of order ≤ 2 which acts by 1 in an irreducible self-dual representation if and only if the representation is orthogonal, cf. Lemma 79 in [10]; in particular, any self-dual representation of an adjoint semisimple group is orthogonal. It is the purpose of this article to prove such a theorem in the context of finite groups of Lie type, and also to provide a counterexample to such a possibility in some cases as for instance $SL(4n+2, \mathrm{F}_q)$ for $q \equiv 3 \bmod 4$ treated in Section 8. The results in this article depend in an essential way on the uniqueness of Whittaker models, and work only for representations which have Whit-

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taker models (called *generic* representations). There is a lot of literature on the calculation of Schur indices of finite groups of Lie type as the author found out by searching through *Mathscinet*. Out of the many articles, we quote [2, 3, 6, 7, 8] as containing results especially close to our own. Most of the results proved in this article for generic representations are actually known for *all* representations of classical groups except for the case of $SL(n, F_q)$, cf. [2, 3, 6]. Our methods are more uniform and perhaps more transparent. The counterexample we construct seems not to have been noticed. The question remains whether the results proved here for generic representations remain true without the genericity hypothesis. The works [2, 3, 6] for classical groups suggest that the answer may be yes.

2. PRELIMINARIES

Let G be a connected reductive algebraic group defined over a finite field F_q. By Lang's theorem, G contains a Borel subgroup B defined over F_q. Let T_s be a maximal split torus in B and let T be a maximal torus in B containing T_s and which is defined over F_q. Let U be the unipotent radical of B. One can decompose U by the adjoint action of T_s and get the root spaces U_α defined over F_q, and the notion of simple root spaces. These root spaces are also invariant under T and define an F_q rational representation of T on these U_α. Denote by T_α the image of T under the representation of T defined on U_α. In this way we get a mapping Φ: $T \to \prod T_\alpha$, the product taken over the simple roots α in U. The kernel of Φ is the center of G which is denoted by Z. The mapping Φ plays an important role in this article. In each T_α there is the element -1_α which operates on the root space U_α by multiplication by -1. Consider the element $\prod(-1_\alpha) \in \prod T_\alpha$, the product taken over simple roots in U. The methods of this article work as long as this element in $\prod T_\alpha$ is the image of an element in $T(F_q)$ under Φ.

A character ψ: $U(F_q) \to C^*$ is called nondegenerate if its restriction to all the simple root subspaces of U with respect to T_s is nontrivial, and its restriction to all nonsimple root spaces is trivial. We fix such a nondegenerate character ψ on the unipotent radical of a Borel subgroup in all that follows.

The following basic theorem was proved by Gelfand and Graev for $G = SL_n$, and was proved by Steinberg in general in [10, Theorem 49].

THEOREM 1. *Let π be an irreducible representation of $G(F_q)$. Then π has an at most one-dimensional subspace on which $U(F_q)$ acts via the nondegenerate character ψ.*

3. THE MAIN LEMMA

Our analysis of self-dual representations depends on the following lemma. This lemma was used in [9] several times without explicitly stating it in this generality.

LEMMA 1. *Let H be a subgroup of a finite group G. Let s be an element of G which normalizes H and whose square belongs to the center of G. Let ψ: $H \to \mathbb{C}^*$ be a one-dimensional representation of H which is taken to its inverse by the inner conjugation action of s on H. Let π be an irreducible representation of G in which the character ψ of H appears with multiplicity* 1. *Then if π is self-dual, it is orthogonal if and only if the element s^2 belonging to the center of G operates by* 1 *on π.*

Proof. Fix a nondegenerate bilinear form B on the vector space V underlying the representation π. Let v_0 be a vector in V such that $h \cdot v_0 = \psi(h) v_0$ for all h belonging to H. Because s normalizes H and takes ψ to its inverse,

$$hsv_0 = \psi^{-1}(h) sv_0.$$

Assume $\psi^{-1} \neq \psi$. In this case, v_0 and sv_0 are linearly independent isotropic vectors which generate a two-dimensional nondegenerate subspace of V. The nondegenerate bilinear form B on V is symmetric if and only if its restriction to this two-dimensional subspace is symmetric. Because v_0 and sv_0 are isotropic vectors, $B(v_0, sv_0)$ must be nonzero. We have

$$B(v_0, sv_0) = B\left(sv_0, s^2 v_0\right).$$

This implies that B is symmetric if and only if s^2 acts by 1.

If the character ψ is of order 2, then the one-dimensional subspace on which H operates via ψ is a nondegenerate subspace of V, forcing the bilinear form to be symmetric.

Remark. If ψ is of order 2 in the preceding lemma, then there is no need to consider the element s (or, one could take it to be the identity element), and in the presence of such a character ψ of multiplicity 1 in a self-dual representation π, π is forced to be orthogonal.

QUESTION. *Is the Lemma* 1 *true for p-adic groups, where one works with linear forms on π which transform under H by a character instead of vectors in π transforming under H by a character? If the lemma holds in p-adic groups, we are again able to give a criterion as to when a self-dual generic representation is orthogonal or symplectic depending on the action of an element in the center of the group of order* ≤ 2. *Such a question was asked for p-adic groups by Serre, cf.* [4, *the question on* p. 938].

4. THE MAIN THEOREM

THEOREM 2. *Let G be a connected reductive algebraic group defined over a finite field F_q. Let Z be the center of G. Let $B = TU$ be a Borel subgroup of G defined over F_q. Let $s \in T(F_q)$ be such that it operates by -1 on all the simple root spaces of U. (Such an s may or may not exist). Then $t = s^2$ belongs to $Z(F_q)$, and $t = s^2$ acts on an irreducible, generic, self-dual representation by 1 if and only if the representation is orthogonal.*

Proof. The proof of this theorem is a trivial consequence of Lemma 1 for the subgroup $H = U(F_q)$ of $G(F_q)$ and ψ is a nondegenerate character of $U(F_q)$. The inner conjugation action of s on H takes ψ to ψ^{-1}, and therefore Lemma 1 yields the theorem.

As a corollary to the previous theorem, we obtain the following.

THEOREM 3. *Let G be a connected reductive algebraic group defined over a finite field F_q. Assume that either*

(a) *the center of G is connected,*

or,

(b) *the center of G is of odd cardinality.*

Then there exists an element t in $G(F_q)$ belonging to the center of G such that an irreducible, generic, self-dual representation of $G(F_q)$ is orthogonal if and only if t acts by 1 on the representation space.

Proof. We have an exact sequence of algebraic groups,

$$1 \to Z \to T \xrightarrow{\Phi} \prod T_\alpha \to 1.$$

Taking F_q rational points, we have

$$1 \to Z(F_q) \to T(F_q) \to \prod T_\alpha(F_q) \to H^1(\mathrm{Gal}, Z) \to 1.$$

By hypothesis, either Z is connected, in which case by Lang's theorem $H^1(\mathrm{Gal}, Z) = 0$, or Z is of odd cardinality, in which case $H^1(\mathrm{Gal}, Z)$ has no elements of order 2. If we let $A[2]$ denote the elements of order a power of 2 in any abelian group A, we have an exact sequence,

$$1 \to Z(F_q)[2] \to T(F_q)[2] \to \prod T_\alpha(F_q)[2] \to 1.$$

It follows that there exists an element s in $T(F_q)$ whose image in $\prod T_\alpha(F_q)$ is $\prod(-1_\alpha)$. Clearly, $s^2 \in Z(F_q)$, and it follows from the previous lemma that $t = s^2$ acts by 1 on an irreducible, self-dual, generic representation π if and only if π is orthogonal.

Remark. In the following cases the center of G is of odd order, cf. [10, p. 227].

(a) G adjoint group.

(b) F_q of characteristic 2.

(c) G of type $A_{2n}, E_6, E_8, F_4, G_2$.

5. SELF-DUAL REPRESENTATIONS FOR $GL(n, \mathrm{F}_q)$

Observe that for $GL(n, \mathrm{F}_q)$, $B = T \cdot U$ with T the diagonal subgroup and U the strictly upper triangular subgroup of $GL(n, \mathrm{F}_q)$, one can take s in Theorem 3 to be

$$\begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix}.$$

Therefore $s^2 = 1$. It follows from Theorem 3 that any self-dual generic representation of $GL(n, \mathrm{F}_q)$ is orthogonal. In the case of $GL(n, \mathrm{F}_q)$ we can actually prove a theorem for all the irreducible representations.

THEOREM 4. *Any irreducible self-dual representation of $GL(n, \mathrm{F}_q)$ is orthogonal.*

Proof. We recall that given any irreducible representation π of $GL(n, \mathrm{F}_q)$, there exists a partition of n as $n = a_1 n_1 + \cdots + a_r n_r$, irreducible cuspidal representations π_i of $GL(n_i, \mathrm{F}_q)$, such that π is contained in $I(a_1\pi_1, \dots, a_r\pi_r)$, the representation of $GL(n, \mathrm{F}_q)$ obtained by parabolic induction of the representation,

$$\pi_1 \times \cdots \times \pi_1 \times \pi_2 \times \cdots \times \pi_2 \times \cdots \ \pi_r \times \cdots \times \pi_r,$$

(the representation π_i is repeated a_i times) of the Levi component,

$$GL(n_1, \mathrm{F}_q) \times \cdots \times GL(n_1, \mathrm{F}_q) \times GL(n_2, \mathrm{F}_q) \times \cdots \times GL(n_2, \mathrm{F}_q) \times \cdots \times GL(n_r, \mathrm{F}_q) \times \cdots \times GL(n_r, \mathrm{F}_q),$$

(the factor $GL(n_i, \mathrm{F}_q)$ is repeated a_i times) of the standard parabolic in $GL(n, \mathrm{F}_q)$ with this as the Levi component. Moreover, the partition of n as $n = a_1 n_1 + \cdots + a_r n_r$, and the representations π_i are unique up to permutation. This implies that if π is self-dual, the set of representations π_i is invariant under the involution $\pi_i \to \pi_i^*$, and the multiplicity associated to π_i and π_i^* is the same. There is a representation of the product of symmetric groups $\mathscr{S}_{a_1} \times \mathscr{S}_{a_2} \times \cdots \times \mathscr{S}_{a_r}$ via intertwining operators on $I(a_1\pi_1, \dots, a_r\pi_r)$ commuting with the $GL(n, \mathrm{F}_q)$ action such that as a representation space for $GL(n, \mathrm{F}_q) \times (\mathscr{S}_{a_1} \times \mathscr{S}_{a_2} \times \cdots \times \mathscr{S}_{a_r})$, $I(a_1\pi_1, \dots, a_r\pi_r)$ decomposes as

$$\sum V_i \otimes W_i,$$

where the W_i are the irreducible representations of

$$\mathscr{S}_{a_1} \times \mathscr{S}_{a_2} \times \cdots \times \mathscr{S}_{a_r},$$

and the V_i are irreducible representations of $GL(n, \mathrm{F}_q)$. The proof now follows from the well-known fact that the representations of a symmetric group are defined over Q, orthogonality of cuspidal self-dual representation of $GL(m, \mathrm{F}_q)$, combined with the following lemma. (The following lemma is used to prove that if π is self-dual, then the induced representation $I(a_1\pi_1, \dots, a_r\pi_r)$ is defined over R, and therefore by the foregoing multiplicity 1 decomposition, each irreducible component, and so π, is also defined over R which is equivalent to orthogonality of π.)

LEMMA 2. *For any cuspidal representation π of $GL(n, \mathrm{F}_q)$, the representation $I(\pi, \pi^*)$ of $GL(2n, \mathrm{F}_q)$ obtained by parabolic induction of the representation $\pi \times \pi^*$ of the Levi subgroup $GL(n, \mathrm{F}_q) \times GL(n, \mathrm{F}_q)$ of the standard parabolic in $GL(2n, \mathrm{F}_q)$ with this as the Levi component is defined over the field of real numbers.*

Proof. We have not been able to find a "pure thought" proof of this lemma. Here is one argument anyway. Look at the restriction of $I(\pi, \pi^*)$ to the subgroup $GL(n, \mathrm{F}_{q^2}) \subset GL(2n, \mathrm{F}_q)$. Clearly $I(\pi, \pi^*)$ contains

$$\mathrm{Ind}_{GL(n, \mathrm{F}_q)}^{GL(n, \mathrm{F}_{q^2})} \pi \otimes \pi^*,$$

which in turn contains the trivial representation of $GL(n, \mathrm{F}_{q^2})$ with multiplicity 1. By a calculation of double cosets, $GL(n, \mathrm{F}_{q^2}) \backslash GL(2n, \mathrm{F}_q) / P(n, n)$, where $P(n, n)$ is the standard parabolic in $GL(2n, \mathrm{F}_q)$ with Levi $GL(n, \mathrm{F}_q) \times GL(n, \mathrm{F}_q)$, one can see that it is the unique copy of the trivial representation of $GL(n, \mathrm{F}_{q^2})$. Therefore, because $I(\pi, \pi^*)$ is clearly self-dual, it is orthogonal, and therefore it is defined over R.

6. CALCULATION OF THE ELEMENT s AND CONSEQUENCES

$SL(4n, \mathrm{F}_q)$: In this case the diagonal matrix,

$$s = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix}$$

belongs to $SL(4n, \mathrm{F}_q)$, and operates by -1 on all the simple root spaces. Because $s^2 = 1$, we conclude that for $SL(4n, \mathrm{F}_q)$, all the self-dual, generic representations are orthogonal.

$SL(4n+2, \mathrm{F}_q)$, $q \equiv 1 \bmod 4$: In this case the diagonal matrix,

$$s = \begin{pmatrix} i & & & & & \\ & -i & & & & \\ & & i & & & \\ & & & -i & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix}$$

belongs to $SL(4n+2, \mathrm{F}_q)$ and operates by -1 on all the simple root spaces. Because $s^2 = -1$, we conclude that a self-dual, generic representation of $SL(4n+2, \mathrm{F}_q)$, $q \equiv 1 \bmod 4$ is orthogonal if and only if the element -1 in the center of $SL(4n+2, \mathrm{F}_q)$ operates by 1.

$SO(2n, \mathrm{F}_q)$: The simple roots are $\{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$. It is clear that in this parametrization, the element $s = (1, -1, 1, \cdots)$ (n entries) operates by -1 on all the simple root spaces. Because $s^2 = 1$, any irreducible, self-dual, generic representation of $SO(2n, \mathrm{F}_q)$ is orthogonal.

$SO(2n+1, \mathrm{F}_q)$: This is an adjoint group, therefore any irreducible, self-dual, generic representation of $SO(2n+1, \mathrm{F}_q)$ is orthogonal.

$Sp(2n, \mathrm{F}_q)$, $q \equiv 1 \bmod 4$: The simple roots are $\{e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n\}$. If $q \equiv 1 \bmod 4$, the element $s = (i, -i, i, -i, \cdots)$ (n entries) operates by -1 on all the simple root spaces. Because $s^2 = -1$, any irreducible, self-dual, generic representation of $Sp(2n, \mathrm{F}_q)$ for $q \equiv 1 \bmod 4$ is orthogonal if and only if the element -1 in the center of $Sp(2n, \mathrm{F}_q)$ operates by 1. We take up the case of $Sp(2n, \mathrm{F}_q)$ when $q \equiv 3 \bmod 4$ in the next section by a method which works for all finite fields.

7. SYMPLECTIC GROUPS

In this section we prove that an irreducible, generic, self-dual representation of a symplectic group is orthogonal if and only if the element -1 in its center operates by 1 on the representation. The proof of such a result is not a direct consequence of Theorems 2 and 3 but instead we need a slight modification using the symplectic similitude groups.

For a vector space V of dimension $2n$ over F_q equipped with a nondegenerate alternating bilinear form B, let $GSp(V)$ denote the subgroup of $GL(V)$ which preserves the bilinear form B up to scaling,

$$GSp(V) = \{g \in GL(V) | B(gv_1, gv_2) = \lambda_g B(v_1, v_2),\ \lambda_g \in \mathrm{F}_q^* \text{ for all } v_1, v_2 \in V\}.$$

The mapping $g \to \lambda_g$ defines a homomorphism λ: $GSp(V) \to \mathrm{G}_m$. Note that there is an inclusion of G_m in $GSp(V)$ as the subgroup of scalar matrices, and therefore for any irreducible representation of $GSp(V)$ one obtains a character, to be denoted by ω_π and to be called the central character of π, by which this central subgroup operates on π.

It is a classical result that if g_0 is a fixed automorphism of V with the property that $B(g_0 v, g_0 w) = -B(v, w)$ for all $v, w \in V$, then for any $g \in Sp(V)$, g^{-1} is conjugate to $g_0 g g_0^{-1}$ in $Sp(v)$. In the following proposition we state a generalization of this classical result to the similitude group $GSp(V)$ which is essential to us. A proof of such a generalization can be given following the proof of the result for $Sp(V)$ given in [5].

PROPOSITION 1. *Let $GSp(V)$ be the symplectic similitude group over an arbitrary field together with the homomorphism λ: $GSp(V) \to \mathrm{G}_m$ as in the previous text. Then for any g in $GSp(V)$, g is conjugate to $\lambda(g) \cdot g^{-1}$ where $\lambda(g)$ denotes the element of G_m sitting inside $GSp(V)$ via the natural inclusion $\mathrm{G}_m \hookrightarrow GSp(V)$.*

COROLLARY. *For any irreducible representation π of $GSp(V)$ for a symplectic space V over a finite or p-adic field,*

$$\pi^* \cong \pi \otimes \omega_\pi^{-1},$$

where one abuses notation to also denote by ω_π the one-dimensional representation of $GSp(V)$ obtained by composing ω_π (the central character of π) and the map λ: $GSp(V) \to \mathrm{G}_m$.

Remark. The preceding corollary is a well-known result for $GL(2)$ of finite and p-adic fields.

In the proof of the following proposition characterizing orthogonal and symplectic representations of a symplectic group, we need to look at the root space decomposition for the symplectic group explicitly. For this purpose, suppose that the symplectic form on the $2n$-dimensional vector space V on which $Sp(2n, \mathrm{F}_q)$ acts, is given by

$$X_1 \wedge Y_1 + X_2 \wedge Y_2 + \cdots + X_n \wedge Y_n.$$

Let B be the Borel subgroup in $Sp(2n, \mathrm{F}_q)$ stabilizing the isotropic flag,

$$(X_1) \subset (X_1, X_2) \subset \cdots \subset (X_1, X_2, \dots, X_n).$$

This Borel subgroup stabilizes the complete flag

$$\begin{aligned}(X_1) \subset (X_1, X_2) \subset \cdots \subset (X_1, \dots, X_n) \subset (X_1, \dots, X_n, Y_n) \\ \subset (X_1, \dots, X_n, Y_n, Y_{n-1}) \subset \cdots \subset (X_1, \dots, X_n, Y_n, \dots, Y_1),\end{aligned}$$

and realizes this as the subgroup of the group of upper triangular matrices, and the subgroup of diagonal elements in $Sp(2n, \mathrm{F}_q)$ as the maximal torus; the simple roots are

$$\frac{t_1}{t_2}, \frac{t_2}{t_3}, \dots, \frac{t_{n-1}}{t_n}, t_n^2,$$

in the standard parametrization,

$$(t_1, \dots, t_n)(X_1, \dots, X_n, Y_n, \dots, Y_1) = \left(t_1 X_1, \dots, t_n X_n, t_n^{-1} Y_n, \dots, t_1^{-1} Y_1\right).$$

PROPOSITION 2. *An irreducible self-dual generic representation π of $Sp(2n, \mathrm{F}_q)$ is orthogonal if and only if the element $-1 \in Sp(2n, \mathrm{F}_q)$ acts trivially on π.*

Proof. Let $\hat{\pi}$ be an irreducible representation of $GSp(2n, \mathrm{F}_q)$ containing the representation π of $Sp(2n, \mathrm{F}_q)$. From the previous proposition we have

$$\hat{\pi}^* \cong \hat{\pi} \otimes \omega_{\hat{\pi}}^{-1}.$$

It follows that there is a bilinear form,

$$B \colon \hat{\pi} \times \hat{\pi} \to \mathrm{C},$$

such that

$$B(gv_1, gv_2) = \omega_{\hat{\pi}}(\lambda_g) B(v_1, v_2).$$

The bilinear form B with the foregoing properties is unique up to scalar multiples. Therefore B is either symmetric or skew-symmetric.

It is easy to see that the automorphism $s \in GSp(2n, \mathrm{F}_q)$ given by

$$s = \begin{cases} X_i \to (-1)^{i+1} X_i, \\ Y_i \to (-1)^i Y_i \end{cases}$$

normalizes the Borel subgroup B introduced just before this proposition, and acts by -1 on all the simple root subspaces of the unipotent radical U of such a Borel. We also note that $\lambda_s = -1$. Because π is assumed to be generic, there exists a $v_0 \in \pi$ such that $nv_0 = \psi(n)v_0$ for all n in $U(\mathrm{F}_q)$ for a nondegenerate character ψ.

We have

$$B(sv_0, s^2 v_0) = \omega_{\hat{\pi}}(\lambda_s) B(v_0, sv_0).$$

Because $\lambda_s = -1$, this simplifies to

$$B(sv_0, v_0) = \omega_{\hat{\pi}}(-1) B(v_0, sv_0).$$

This implies that B is symmetric or skew-symmetric depending on whether $\omega_{\hat{\pi}}(-1) = 1$ or -1.

Next we observe that the restriction of $\hat{\pi}$ to $Sp(V)$ is multiplicity free as follows from the uniqueness of the Whittaker model (for $GSp(V)$!). Therefore if π is self-dual, the restriction of B to π is nondegenerate, and is symmetric or skew-symmetric depending on whether $\pi_{\hat{\pi}}(-1) = 1$, or -1, proving the proposition.

8. SELF-DUAL REPRESENTATIONS FOR $SL(n, \mathrm{F}_q)$

We proved in Section 4 that any irreducible self-dual generic representation of $SL(2n+1, \mathrm{F}_q)$ and $SL(4n, \mathrm{F}_q)$ is orthogonal, and that a representation of $SL(4n+2, \mathrm{F}_q)$ for $q \equiv 1 \bmod 4$ is orthogonal if and only if the element -1 in its center acts by 1. These results together with what we proved in the last section about representations of symplectic groups might suggest that an irreducible, generic, self-dual representation of $SL(4n+2, \mathrm{F}_q)$ is orthogonal if and only if the element -1 in its center acts by 1. However, we see in the next section that this is not the case. In this section we prove some positive results in this direction.

THEOREM 5. *A cuspidal irreducible self-dual representation of* $SL(4n+2, \mathrm{F}_q)$ *is orthogonal if and only if the element* -1 *in its center acts by* 1.

The proof of this theorem follows exactly the same lines as the proof of the corresponding theorem for $Sp(2n, \mathrm{F}_q)$, and depends on the next theorem. We omit the proof of the previous theorem except to say that one takes the matrix s to execute the proof, the matrix,

$$s = \begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \cdot & \\ & & & & & \cdot \end{pmatrix}.$$

THEOREM 6. *Let V be an irreducible cuspidal representation of $GL(n, \mathrm{F}_q)$, $n > 2$ such that for a character μ of F_q^*, $V \cong V^* \otimes \mu$. Then n must be even $n = 2m$, and $\mu^m = \omega_V$. Here ω_V denotes the central character of V, and we are doing the usual abuse of notation to identify characters of F_q^* and $GL(2m, \mathrm{F}_q)$.*

Proof. From the isomorphism, $V \cong V^* \otimes \mu$, it follows that $\omega_V^2 = \mu^{2m}$. So, the main part of the theorem fixes the square root of this equation.

We recall that cuspidal representations of $GL(n, \mathrm{F}_q)$ are parametrized by characters of $\mathrm{F}_{q^n}^*$ which are not fixed by any nontrivial element in the Galois group of F_{q^n} over F_q. Two characters of $\mathrm{F}_{q^n}^*$ give rise to isomorphic representations of $GL(n, \mathrm{F}_q)$ if and only if they are Galois conjugate. This correspondence is equivariant under twisting and taking duals. Therefore if the representation V is associated to a character χ of $\mathrm{F}_{q^n}^*$, the isomorphism $V \cong V^* \otimes \mu$ implies that there is an element τ in the Galois group of F_{q^n} over F_q such that

$$\chi(x)\chi(\tau x) = \mu(\mathrm{Nm}\, x), \qquad (*)$$

for all $x \in \mathrm{F}_{q^n}^*$ where Nm denotes the norm mapping from $\mathrm{F}_{q^n}^*$ to F_q^*. Replacing x by τx in this equation, we find that

$$\chi(x) = \chi(\tau^2 x).$$

Because the character χ is not fixed by any nontrivial element of the Galois group, this means that $\tau^2 = 1$. So, either $\tau = 1$, or τ is of order 2. If τ is of order 2 then n is even equal to $2m$, and we let E_1 by the subfield of F_{q^n} fixed by τ. The equation $\chi(x \cdot \tau\chi) = \mu(\mathrm{Nm}\, x)$ together with the surjectivity of the norm mapping from $\mathrm{F}_{q^n}^*$ to E_1^* implies that $\chi|_{E_1^*} = \mu(\mathrm{Norm})$, where the norm this time is from E_1^* to F_q^*. Therefore $\chi|_{\mathrm{F}_q^*} = \mu^m$, completing the proof of the theorem in this case. If τ in Eq. $(*)$ is trivial, we find that $\chi^2(x) = \mu(\mathrm{Nm}\, x)$, and therefore $\chi^2(x) = \chi^2(\eta x)$, for any η in the Galois group. Because there is a unique character of order 2 on $\mathrm{F}_{q^n}^*$, we find that n must be ≤ 2 which is omitted from the theorem, completing the proof of the theorem.

Remark. Since $GL(n, F_q)/SL(n, F_q)$ is a commutative group, if V is an irreducible representation of $GL(n, F_q)$ containing an irreducible self-dual representation of $SL(n, F_q)$, then $V \cong V^* \otimes \mu$ for a character μ of F_q^*.

9. THE COUNTEREXAMPLE FOR $SL(6, \mathrm{F}_q)$, $q \equiv 3 \bmod 4$

In this section we construct an irreducible generic self-dual representation of $SL(6, \mathrm{F}_q)$ for $q \equiv 3 \bmod 4$ which is symplectic (resp., orthogonal) even though -1 in its center operates trivially (resp., nontrivially) on the representation. The idea for the construction is to use a cuspidal representation for $GL(2, \mathrm{F}_q)$ for which the conclusion of Theorem 6 does not hold well. This does not yield a counterexample for $SL(2, \mathrm{F}_q)$ as the restriction to $SL(2, \mathrm{F}_q)$ of such a representation is not irreducible. However one can use this representation on a part of Levi in a parabolic in $GL(4n + 2, \mathrm{F}_q)$ and parabolic induction to construct a representation of $GL(4n + 2, \mathrm{F}_q)$ ($n \geq 1$) whose restriction to $SL(4n + 2, \mathrm{F}_q)$ is irreducible.

Let π_1 be a representation of $GL(2, \mathrm{F}_q)$ such that $\pi_1 \cong \pi_1 \otimes \omega$ for the unique nontrivial quadratic character ω of F_q^*. Such a representation π_1 is known to exist; these are exactly the representations which when restricted to $SL(2, \mathrm{F}_q)$ are sums of 2 irreducible representations (which are dual to each other if $q \equiv 3 \bmod 4$). We take two other cuspidal representations π_2, π_3 of $GL(2, \mathrm{F}_q)$ such that the central character of π_2 and π_3 is the same as the central character of π_1 multiplied by the unique quadratic character ω of F_q^*,

$$\omega_{\pi_2} = \omega_{\pi_3} = \omega \cdot \omega_{\pi_1}.$$

We choose π_2, π_3 in such a manner that the triple (π_1, π_2, π_3) is not invariant under any nontrivial permutation. In that case the representation $I(\pi_1, \pi_2, \pi_3)$ of $GL(6, \mathrm{F}_q)$ parabolically induced from the Levi subgroup $GL(2, \mathrm{F}_q) \times GL(2, \mathrm{F}_q) \times GL(2, \mathrm{F}_q)$ of $GL(6, \mathrm{F}_q)$ with this as the Levi subgroup is an irreducible representation of $GL(6, \mathrm{F}_q)$. Moreover, using $\pi_1 \cong \pi_1 \otimes \omega$ we have

$$\begin{aligned} I(\pi_1, \pi_2, \pi_3)^* &\cong I(\pi_1^*, \pi_2^*, \pi_3^*) \cong I\left(\pi_1 \otimes \omega_{\pi_1}^{-1}, \pi_2 \otimes \omega_{\pi_2}^{-1}, \pi_3 \otimes \omega_{\pi_3}^{-1}\right) \\ &\cong I(\pi_1, \pi_2, \pi_3) \otimes \omega^{-1}\omega_{\pi_1}^{-1}. \end{aligned}$$

So we have the isomorphism, $I(\pi_1, \pi_2, \pi_3) \cong I(\pi_1, \pi_2, \pi_3)^* \otimes \omega\omega_{\pi_1}$. On the other hand, the central character of $I(\pi_1, \pi_2, \pi_3)$ is $\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} = \omega_{\pi_1}^3$. Because $(\omega \cdot \omega_{\pi_1})^3$ is not equal to $(\omega_{\pi_1})^3$, we therefore see that the conclusion of Theorem 6 goes wrong. We finally need to note the following simple lemma, cf. Lemma 2.1 in [1], which is used to prove irreducibility of $I(\pi_1, \pi_2, \pi_3)$ restricted to $SL(6, \mathrm{F}_q)$.

LEMMA 3. *For an irreducible representation π of $GL(n, \mathrm{F}_q)$, the cardinality of the set of characters ν of F_q^* such that $\pi \otimes \nu \cong \pi$ is equal to the number of irreducible representations in π when restricted to $SL(n, \mathrm{F}_q)$. In particular an irreducible representation π of $GL(n, \mathrm{F}_q)$ remains irreducible when restricted to $SL(n, \mathrm{F}_q)$ if and only if there are no nontrivial characters ν of F_q^* such that $\pi \otimes \nu$ is isomorphic to π.*

Remark. In the earlier lemma we used the fact that any irreducible representation of $GL(n, \mathrm{F}_q)$ decomposes with multiplicity 1 when restricted to $SL(n, \mathrm{F}_q)$. This follows because $SL(n, \mathrm{F}_q)$ is normal in $GL(n, \mathrm{F}_q)$ and the quotient is a cyclic group.

From this lemma it follows that one can choose π_2 and π_3 such that $I(\pi_1, \pi_2, \pi_3)$ remains irreducible when restricted to $SL(6, \mathrm{F}_q)$. From the isomorphism, $I(\pi_1, \pi_2, \pi_3)^* \cong I(\pi_1, \pi_2, \pi_3) \otimes \omega^{-1}\omega_{\pi_1}^{-1}$ we find that the restriction of $I(\pi_1, \pi_2, \pi_3)$ to $SL(6, \mathrm{F}_q)$ is self-dual. Thus $I(\pi_1, \pi_2, \pi_3)$ restricted to $SL(6, \mathrm{F}_q)$ is an irreducible, self-dual, generic representation of $SL(6, \mathrm{F}_q)$ for which (arguing as in the proof of Proposition 2) the element -1 acts trivially on the representation but the representation is symplectic, or -1 does not act trivially but the representation is orthogonal, and both the possibilities can be ensured depending on the value of $\omega_{\pi_1}(-1)$.

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