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The Space of Degenerate Whittaker Models for General Linear Groups over a Finite Field

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1 Introduction

Let $G = \underline{G}(\mathbb{F})$ be the $\mathbb{F}$-rational points of a reductive algebraic group $\underline{G}$ over a finite field $\mathbb{F}$. Let $\underline{P} = \underline{MN}$ be the Levi decomposition of a parabolic subgroup $\underline{P}$ of $\underline{G}$ defined over $\mathbb{F}$. We denote the corresponding decomposition of $\mathbb{F}$-rational points as $P = MN$. Let π be any irreducible finite-dimensional complex representation of G, and let ψ be any irreducible representation of N. The sum, call it $\pi_{N,\psi}$, of all irreducible representations of N inside π, on which N operates via ψ, is a representation space of M_ψ, which is the subgroup of M consisting of those elements in M which leave the isomorphism class of ψ invariant under the inner conjugation action of M on N. Since the representation theory of groups such as G is now fairly well understood, it seems like an interesting question to understand for which irreducible representations π, $\pi_{N,\psi}$ is nonzero, and then to understand the structure of $\pi_{N,\psi}$ as a module for M_ψ.

The questions most studied in this context is when $\underline{P}$ is a Borel subgroup of $\underline{G}$ and when one takes a *nondegenerate* character on N. By a theorem due to Gel′fand and Graev for GL_n which was generalised for arbitrary reductive algebraic groups by Steinberg, one knows that the dimension of $\pi_{N,\psi}$ is at most 1. This theorem and the study of representations π with $\pi_{N,\psi} \neq 0$, called *generic representations*, play an important part in many questions in representation theory.

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There is some work by Kawanaka [K] for more general parabolics, but these questions seem largely not touched upon in the literature so far. In this paper, we make a detailed analysis of one very special case.

Let $G = GL_{2n}(\mathbb{F})$, where $\mathbb{F}$ is a finite field. Let P be the (n, n) parabolic in G with Levi subgroup $GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$ and with unipotent radical $N = M_n(\mathbb{F})$. Let ψ_0 be a nontrivial additive character $\psi_0 : \mathbb{F} \to \mathbb{C}^*$. Let $\psi(X) = \psi_0(\mathrm{tr}\, X)$ be the additive character on $N = M_n(\mathbb{F})$. Let π be an irreducible admissible representation of G. Let $\pi_{N,\psi}$ be the largest subspace of π on which N operates via ψ. Since $\mathrm{tr}(gXg^{-1}) = \mathrm{tr}(X)$, it follows that $\pi_{N,\psi}$ is a representation space for $H = \Delta\, GL_n(\mathbb{F}) \hookrightarrow GL_n(\mathbb{F}) \times GL_n(\mathbb{F})$. The space $\pi_{N,\psi}$ is referred to as the space of degenerate Whittaker models, or sometimes also as the twisted Jacquet functor of the representation π. The space of linear forms on $\pi_{N,\psi}$ is the same as the space of linear forms on π on which $N = M_n(\mathbb{F})$ operates via ψ, generalising the notion of Whittaker models in the case of $GL_2(\mathbb{F})$. The term degenerate is used as in [MW], as these linear functionals have invariance property for the unipotent radical of a nonminimal parabolic.

The aim of this work is to calculate $\pi_{N,\psi}$ as a representation space for $GL_n(\mathbb{F})$. We begin with the statement of the main theorem of this paper.

Theorem 1. Let π be a cuspidal representation of $GL_{2n}(\mathbb{F})$ obtained from a character θ of $\mathbb{F}_{2n}^*$. Then

$$\pi_{N,\psi} = \mathrm{Ind}_{\mathbb{F}_n^*}^{GL_n(\mathbb{F})} \left(\theta|_{\mathbb{F}_n^*}\right). \qquad \square$$

The proof of this theorem is done by brute force. We prove by an explicit calculation that the characters of the twisted Jacquet functor $\pi_{N,\psi}$ and the induced representation $\mathrm{Ind}_{\mathbb{F}_n^*}^{GL_n(\mathbb{F})}(\theta|_{\mathbb{F}_n^*})$ at an arbitrary element of $GL_n(\mathbb{F})$ are the same. Therefore, the two representations are isomorphic.

We note that although we restrict ourselves to finite fields in this paper, the study of $\pi_{N,\psi}$ is specially relevant to p-adic fields and automorphic forms where it is connected to Fourier expansion and has indeed been studied by many authors in the p-adic context. Most of these works in the p-adic context prove a multiplicity one theorem about $\pi_{N,\psi}$ as a module for M_ψ without getting a fuller understanding of $\pi_{N,\psi}$ and use them for developing a theory of L-functions. We refer the reader to the book of Ginzburg, Piatetski-Shapiro, and Rallis [GPSR] for one such context. In a recent work, E. M. Baruch and S. Rallis have proved a multiplicity one theorem in the p-adic case (cf. [BR] for one special case). They work with $G = Sp(n)$, P the "Klingen parabolic" whose Levi subgroup has $Sp(n-1)$ for its semisimple part, and a Heisenberg group for its unipotent radical.

They prove that the maximal quotient of an irreducible representation of $\mathrm{Sp}(n)$ on which this Heisenberg group operates by the oscillator representation has a multiplicity one property for $\mathrm{Sp}(n-1)$.

Since the representation theory of finite groups of Lie type is much better understood than that of p-adic groups, and since these theories are closely related, it is clear that an understanding of $\pi_{N,\psi}$ in the case of finite fields will throw some light on analogous questions in the p-adic case. Indeed, this was also the motivation behind this work that arose because of the need to understand $\pi_{N,\psi}$ in the p-adic case for GL_{2n}, which plays an important role in the work of [PR].

2 Preliminaries

In this paper, $\mathbb{F}$ denotes a finite field with q elements. We review the representation theory of $\mathrm{GL}_m(\mathbb{F})$ due to J. A. Green. According to Green, cuspidal representations of $\mathrm{GL}_m(\mathbb{F})$, from which all the other irreducible representations of $\mathrm{GL}_m(\mathbb{F})$ are obtained via the process of parabolic induction, are associated to *regular* characters of $\mathbb{F}_m^*$, where $\mathbb{F}_m$ is the unique field extension of degree m of $\mathbb{F}$. A character χ of $\mathbb{F}_m^*$ is called regular if, under the action of the Galois group of $\mathbb{F}_m$ over $\mathbb{F}$, χ gives rise to m distinct characters of $\mathbb{F}_m^*$. Two regular characters of $\mathbb{F}_m^*$ give rise to the same cuspidal representation if and only if one is obtained from the other by the action of an element in the Galois group.

We denote the representation of $\mathrm{GL}_m(\mathbb{F})$ associated to a regular character θ of $\mathbb{F}_m^*$ by π_θ and the character of the representation π_θ by Θ_θ.

There is an embedding of $\mathbb{F}_m$ inside $M_m(\mathbb{F})$ as algebras which is unique up to inner conjugation by $\mathrm{GL}_m(\mathbb{F})$. This way, every element of $\mathbb{F}_m^*$ gives rise to a well-defined conjugacy class in $\mathrm{GL}_m(\mathbb{F})$. The conjugacy classes in $\mathrm{GL}_m(\mathbb{F})$, which are so obtained from elements of $\mathbb{F}_m^*$, are said to be associated to $\mathbb{F}_m^*$. In particular, if an element of $\mathbb{F}_m^*$ belongs to a proper subfield, then the associated element in $\mathrm{GL}_m(\mathbb{F})$ will look like a direct sum of matrices, with the same matrix in each block.

We summarise the information about the character Θ_θ in the following theorem. We refer to the paper of S. I. Gel'fand [Ge] for the statement of this theorem in this explicit form, which is originally due to Green [G]. (See also the paper of Springer and Zelevinsky [SZe].)

Theorem 2 [G]**.** Let Θ_θ be the character of a cuspidal representation π_θ of $\mathrm{GL}_m(\mathbb{F})$ associated to a regular character θ of $\mathbb{F}_m^*$. Let $g = s \cdot u$ be the Jordan decomposition of an element g in $\mathrm{GL}_m(\mathbb{F})$. If $\Theta_\theta(g) \neq 0$, then the semisimple element s must come from $\mathbb{F}_m^*$. Suppose that s comes from $\mathbb{F}_m^*$. Let z be an eigenvalue of s in $\mathbb{F}_m$, and let t be the dimension of the kernel of $(g - z)$ over $\mathbb{F}_m$. Then

$$\Theta_\theta(s \cdot u) = (-1)^{m-1}\left[\sum \theta(s^\alpha)\right](1-q')(1-q'^2)\cdots(1-q'^{t-1}),$$

where q' is the cardinality of the field generated by z over $\mathbb{F}$, and the summation is over the various distinct Galois conjugates of s. □

3 Character of the induced representation

From the well-known result about the character of an induced representation, we have the following lemma, whose proof is omitted.

Lemma 1. For a character θ of $\mathbb{F}_n^*$, the character Θ_{Ind}, of the induced representation

$$\mathrm{Ind}_{\mathbb{F}_n^*}^{\mathrm{GL}_n(\mathbb{F})}(\theta)$$

at an element s of $\mathbb{F}_n^*$ which generates an extension of $\mathbb{F}$ of degree d, is given by

$$\begin{aligned}\Theta_{\mathrm{Ind}}(s) &= \frac{1}{(q^n-1)}\sum_{g\in \mathrm{GL}_n(\mathbb{F})}\theta(g^{-1}sg)\\ &= \frac{|\mathrm{GL}_{d'}(\mathbb{F}_d)|}{(q^n-1)}\left[\sum_\alpha \theta(s^\alpha)\right],\end{aligned}$$

where $d' = n/d$. In the first sum, we have followed the standard convention of putting $\theta(x) = 0$ if $x \notin \mathbb{F}_n^*$. The second sum is over the different Galois conjugates of s, thought of as an element of $\mathbb{F}_d^*$. The value of the character Θ_{Ind} at an element of $\mathrm{GL}_n(\mathbb{F})$ which does not come from $\mathbb{F}_n^*$ is zero. □

4 Some linear algebra

To calculate the character of the twisted Jacquet functor, we need to calculate the number of $(n\times n)$-matrices over $\mathbb{F}$ of a given rank and of a given trace. First, we fix some notation for this purpose. We fix a set of basis vectors $\{e_1,\ldots,e_{m+k}\}$ for $\mathbb{F}^{m+k}$.

Let $Y_{m,k}^\alpha$ denote the number of $((m+k)\times(m+k))$-matrices over $\mathbb{F}$ of rank k and trace α. Let $X_{m,k}^\alpha$ denote the number of $((m+k)\times(m+k))$-matrices over $\mathbb{F}$ of rank k and trace α which have a fixed m-dimensional subspace of $\mathbb{F}^{m+k}$ in its kernel which, without loss of generality, we take to be $\{e_1,\ldots,e_m\}$. Let $X_{m,k}$ denote the number of $((m+k)\times(m+k))$-matrices over $\mathbb{F}$ of rank k. In this notation, we have suppressed the cardinality of $\mathbb{F}$. When not specifying the cardinality of $\mathbb{F}$ might lead to confusion, we use $X^{\alpha}_{m,k}(q)$. We often use without explicitly mentioning the fact that $Y^{\alpha}_{m,k} = Y^{\beta}_{m,k}$ if $\alpha\beta \neq 0$; similarly, $X^{\alpha}_{m,k} = X^{\beta}_{m,k}$ if $\alpha\beta \neq 0$.

Clearly, the rank k endomorphisms of $\mathbb{F}^{m+k}$ with kernel $\{e_1, \ldots, e_m\}$ are in bijective correspondence with injective maps of the vector space $\{e_{m+1}, \ldots, e_{m+k}\}$ into $\mathbb{F}^{m+k}$. Our calculation of $X^{\alpha}_{m,k}$, etc., depends on a recursive relation we find between these and the corresponding objects for index $(m, k-1)$. An injective map from $\{e_{m+1}, \ldots, e_{m+k}\}$ into $\mathbb{F}^{m+k}$ is built from an injective map of $\{e_{m+1}, \ldots, e_{m+k-1}\}$ into $\mathbb{F}^{m+k}$, plus a condition that the image of e_{m+k} should not belong to the image of $\{e_{m+1}, \ldots, e_{m+k-1}\}$. We count the number of endomorphisms in $X^{0}_{m,k}$. There are two possibilities for such endomorphisms.

Case 1. The image of the subspace $\{e_{m+1}, \ldots, e_{m+k-1}\}$ is contained in $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+k-1}\}$.

Case 2. The image of the subspace $\{e_{m+1}, \ldots, e_{m+k-1}\}$ is not contained in $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+k-1}\}$.

In Case 1, there are two subcases to consider.

Case 1_a. The endomorphism induced on the $(m+k-1)$-dimensional subspace $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+k-1}\}$ has trace zero.

Case 1_b. The endomorphism induced on the $(m+k-1)$-dimensional subspace $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+k-1}\}$ does not have trace zero.

In Case 1_a, the number of possibilities for the image of e_{m+k} is $(q^{m+k-1} - q^{k-1})$.

In Case 1_b, the number of possibilities for the image of e_{m+k} is q^{m+k-1}.

In Case 2, the image of e_{m+k} is not to belong to the $(k-1)$-dimensional subspace of $\mathbb{F}^{m+k}$, which is the image of $\{e_{m+1}, \ldots, e_{m+k-1}\}$, but should lie in the hyperplane defined by $\ell_k(v) = \text{constant}$, where

$$\ell_k(e_{m+k}) = 1,$$
$$\ell_k(e_i) = 0, \quad i \neq m+k.$$

Thus, having chosen the images of $e_{m+1}, \ldots, e_{m+k-1}$, this gives $(q^{n-1} - q^{k-2})$ number of possibilities for the image of e_{m+k}.

Adding all the contributions, we have the recursion relation

$$X^{0}_{m,k} = X^{0}_{m,k-1}\left(q^{m+k-1} - q^{k-1}\right) + \left(X_{m,k-1} - X^{0}_{m,k-1}\right)q^{m+k-1}$$

$$+\left[\frac{X_{m,k}}{q^{m+k}-q^{k-1}}-X_{m,k-1}\right]\left(q^{m+k-1}-q^{k-2}\right),$$

where the three terms correspond to Cases 1_a, 1_b, and 2, respectively.

Simplifying, we have

$$X^0_{m,k}=-X^0_{m,k-1}q^{k-1}+X_{m,k-1}q^{k-2}+q^{-1}X_{m,k},$$

or

$$\left(X_{m,k}-qX^0_{m,k}\right)=-q^{k-1}\left(X_{m,k-1}-qX^0_{m,k-1}\right).$$

Since $X_{m,k}=X^0_{m,k}+(q-1)X^1_{m,k}$, we find

$$\left(X^1_{m,k}-X^0_{m,k}\right)=-q^{k-1}\left(X^1_{m,k-1}-X^0_{m,k-1}\right).$$

Iterating this recursion relation, we find

$$\begin{aligned}X^1_{m,k}-X^0_{m,k}&=(-1)^k q^{k(k-1)/2}\left(X^1_{m,0}-X^0_{m,0}\right)\\&=(-1)^{k-1}q^{k(k-1)/2},\end{aligned}$$

where we have used the fact that $X^1_{m,0}=0$, and $X^0_{m,0}=1$.

We have proved the following lemma.

Lemma 2. Let $Y^\alpha_{m,k}$ denote the number of rank k endomorphisms of $\mathbb{F}^{m+k}$ with trace α. Then

$$Y^1_{m,k}-Y^0_{m,k}=(-1)^{k-1}q^{k(k-1)/2}\left|\mathrm{Gr}(m+k,m)\right|,$$

where $|\mathrm{Gr}(m+k,m)|$ denotes the number of m-dimensional subspaces in $\mathbb{F}^{m+k}$. □

5 Calculation of the dimension of the twisted Jacquet functor

It is clear that the dimension of the twisted Jacquet functor of a representation π of $\mathrm{GL}_{2n}(\mathbb{F})$ is given by

$$\dim(\pi_{N,\psi})=\frac{1}{q^{n^2}}\sum_{X\in M_n(\mathbb{F})}\Theta_\pi\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix}\bar{\psi}(X),$$

where $\bar{\psi}$ denotes the complex conjugate of ψ.

Since the number of Jordan blocks in a unipotent matrix u is the dimension of the kernel of $(u-1)$, it is easily seen that the number of Jordan blocks in $\left(\begin{smallmatrix}1 & X\\ 0 & 1\end{smallmatrix}\right)$ is equal to $2n-\text{rank}(X)$. Therefore, using Lemma 2, using Green's theorem giving the value of the character of the cuspidal representation π at the unipotent element $\left(\begin{smallmatrix}1 & X\\ 0 & 1\end{smallmatrix}\right)$, and using the fact that $\sum_{x\neq 0}\bar{\psi}_0(x)=-1$, we have

$$\begin{aligned}\dim(\pi_{N,\psi}) &= \frac{1}{q^{n^2}}\sum_{i=0}^{n}(-1)^i(q-1)\cdots\left(q^{2n-i-1}-1\right)\left[Y^0_{n-i,i}-Y^1_{n-i,i}\right]\\ &= \frac{1}{q^{n^2}}\sum_{i=0}^{n}q^{i(i-1)/2}(q-1)\cdots\left(q^{2n-i-1}-1\right)\left|\text{Gr}(n,n-i)\right|\\ &= \frac{1}{q^{n^2}}\sum_{i=0}^{n}q^{i(i-1)/2}(q-1)\cdots\left(q^{2n-i-1}-1\right)\frac{(q^n-1)\cdots\left(q^{n-i+1}-1\right)}{(q^i-1)\cdots(q-1)},\end{aligned}$$

where $|\text{Gr}(n,n-i)|$ denotes the cardinality of the Grassmanian of $(n-i)$-planes in $\mathbb{F}^n$. Since the cardinality of $\text{GL}_n(\mathbb{F})/\mathbb{F}_n^*$ is $(q^n-q)\cdots(q^n-q^{n-1})$, the following lemma proves that the representations $V_{N,\psi}$ and $\text{Ind}_{\mathbb{F}_n^*}^{\text{GL}_n(\mathbb{F})}(\theta|_{\mathbb{F}_n^*})$ have the same dimension. The author is indebted to Dr. Heng Huat Chan for supplying the proof of the following lemma.

Lemma 3. We have

$$\begin{aligned}&(q^n-q)\cdots\left(q^n-q^{n-1}\right)\\ &\quad= \frac{1}{q^{n^2}}\sum_{i=0}^{n}q^{i(i-1)/2}(q-1)\cdots\left(q^{2n-i-1}-1\right)\frac{(q^n-1)\cdots\left(q^{n-i+1}-1\right)}{(q^i-1)\cdots(q-1)}.\end{aligned}$$ □

Proof. Set

$$(a;q)_n=\prod_{k=1}^{n}\left(1-aq^{k-1}\right)=(1-a)(1-aq)\cdots\left(1-aq^{n-1}\right).$$

Then the identity needing to be proved in the lemma becomes

$$(-1)^{n-1}q^{n(n-1)/2}(q;q)_{n-1}=\frac{1}{q^{n^2}}\sum_{i=0}^{n}(-1)^{i-1}q^{i(i-1)/2}\frac{(q^n;q)_{n-i}(q;q)_{n-1}}{(q;q)_i(q;q)_{n-i}}(q;q)_n,$$

or

$$q^{(3n-1)n/2}=\sum_{i=0}^{n}(-1)^{n-i}q^{i(i-1)/2}\frac{(q^n;q)_{n-i}(q;q)_n}{(q;q)_i(q;q)_{n-i}}.$$

Replacing $n-i$ by k, this is the same as writing

$$\begin{aligned} q^{(3n-1)n/2} &= \sum_{k=0}^{n} (-1)^k q^{(n-k)(n-k-1)/2} \frac{(q^n;q)_k (q;q)_n}{(q;q)_{n-k}(q;q)_k} \\ &= \sum_{k=0}^{n} (-1)^k q^{(n^2-n)/2} q^{(k^2-k)/2} q^k q^{-kn} \frac{(q^n;q)_k (q;q)_n}{(q;q)_{n-k}(q;q)_k}. \end{aligned}$$

It can be shown that (cf. [GR, Appendix 1, (I.12)])

$$(-1)^k q^{(k^2-k)/2-kn} \frac{(q;q)_n}{(q;q)_{n-k}} = (q^{-n};q)_k.$$

This implies that the identity needing to be proved is

$$q^{n^2} = \sum_{k=0}^{n} q^k \frac{(q^{-n};q)_k (q^n;q)_k}{(q;q)_k}.$$

Define the following q-analogue of the corresponding hypergeometric series

$${}_2\phi_1(a,b,c;q;Z) = \sum_{k=0}^{\infty} \frac{(a;q)_k (b:q)_k}{(c;q)_k} \frac{Z^k}{(q,q)_k}.$$

In this notation, the identity we need to prove reduces to the elegant identity

$$q^{n^2} = {}_2\phi_1(q^{-n}, q^n, 0; q; q).$$

However, we have the following result in the hypergeometric series (cf. [GR, formula (1.5.3)])

$${}_2\phi_1(q^{-n}, b, c; q; q) = \frac{\left(\frac{c}{b};q\right)_n}{(c;q)_n} b^n.$$

Putting $b=q^n$ and $c=0$, the previous identity and, hence, the lemma is proved. ■

6 Proof of the main theorem

We prove Theorem 1 by proving that the character of the representation $\pi_{N,\psi}$ and $\mathrm{Ind}_{\mathbb{F}_n^*}^{\mathrm{GL}_n(\mathbb{F})}(\theta|_{\mathbb{F}_n^*})$ at any element g in $\mathrm{GL}_n(\mathbb{F})$ is the same. We divide the proof into two cases depending on whether the element g is not semisimple or is semisimple. In the first case, because of Lemma 1, we prove that the character of $\pi_{N,\psi}$ at any non-semisimple element is zero.

We make the general remark that by Theorem 2, due to Green, about character of a cuspidal representation of $GL_m(\mathbb{F})$, the character of $\pi_{N,\psi}$ is zero at any element g with Jordan decomposition $g = s \cdot u$, where the semisimple element s does not come from a subfield of $\mathbb{F}_n$. (Equivalently, not all the eigenvalues of g are conjugate under the Galois action of $\mathbb{F}_n$ over $\mathbb{F}$.) By Lemma 1, the character of $\mathrm{Ind}_{\mathbb{F}_n^*}^{GL_n(\mathbb{F})}(\theta|_{\mathbb{F}_n^*})$ is also zero at all the elements $g = su$, where the semisimple element s does not come from a subfield of $\mathbb{F}_n$. Therefore, in the proof of Theorem 1, we always assume that we are looking at an element $g = su$, whose semisimple part s comes from a subfield of $\mathbb{F}_n$.

6.1 Character calculation at a non-semisimple element

Lemma 4. The character of the twisted Jacquet functor at an element $g = su$, where u is a nontrivial unipotent element, is zero. □

Proof. Although we could consider the case of arbitrary s from the outset, it is helpful to first consider the case when $s = 1$; so we assume that $g = u$ is a nontrivial unipotent element in $GL_n(\mathbb{F})$. The character of $\pi_{N,\psi}$, to be denoted by $\Theta_{N,\psi}$, is given by

$$\Theta_{N,\psi}(u) = \frac{1}{q^{n^2}} \sum_{X \in M_n(\mathbb{F})} \Theta_\pi \left[\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \right] \bar{\psi}(X)$$

$$= \frac{1}{q^{n^2}} \sum_{X \in M_n(\mathbb{F})} \Theta_\pi \begin{bmatrix} u & X \\ 0 & u \end{bmatrix} \bar{\psi}(u^{-1}X).$$

Therefore, from Green's theorem about the character of a cuspidal representation of $GL_{2n}(\mathbb{F})$ at a unipotent element, the calculation of the character of the twisted Jacquet functor depends on understanding the number of Jordan blocks of the unipotent matrix

$$h = \begin{pmatrix} u & X \\ 0 & u \end{pmatrix},$$

as X varies in $M_n(\mathbb{F})$ with a given trace for $u^{-1}X$. For this purpose, it is convenient to represent the matrix

$$h = \begin{pmatrix} u & X \\ 0 & u \end{pmatrix}$$

as an endomorphism of the vector space $V \oplus V$, which leaves the first copy of V inside $V \oplus V$

stable. Let $\{e_1, \ldots, e_n\}$ be a basis for the first copy of V inside $V \oplus V$, and let $\{f_1, \ldots, f_n\}$ be the corresponding basis for the second copy of V inside $V \oplus V$.

Assume without loss of generality that $\{e_1, \ldots, e_m\}$ is a basis for the kernel of $(u-1)$. In particular, this means that the Jordan decomposition of u has m blocks.

Clearly, $(h-1)$ takes $V \oplus \{f_1, \ldots, f_m\}$ inside V and is the largest subspace of $V \oplus V$ containing V which $(h-1)$ takes into V. In particular, the kernel of $(h-1)$ is contained in $V \oplus \{f_1, \ldots, f_m\}$. Suppose that $(h-1)V = W$, which is a subspace of V of codimension m. The dimension of the kernel of $(h-1)$ is determined by the dimension of the image of $(h-1)$ acting on $V \oplus \{f_1, \ldots, f_m\}$, which in turn is determined by the intersection of W with the image of $\{f_1, \ldots, f_m\}$ under X. Thus, if $m < n$, which is the case as u is a nontrivial unipotent, the number of Jordan blocks in g depends only on the restriction of the action of X to the proper subspace $\{f_1, \ldots, f_m\}$. This means that the number of matrices X, with a given trace $\mathrm{tr}(u^{-1}X) = a$ giving rise to the matrix

$$h = \begin{pmatrix} u & X \\ 0 & u \end{pmatrix}$$

of k blocks, does not depend on the value of a. We elaborate this point further. Note that if u^{-1} takes basis elements e_i to f_i, then the trace of Xu^{-1} is the sum of coefficients of e_i in the expansion of Xf_i using the basis e_i. Thus, $\mathrm{tr}(Xu^{-1})$ takes all values equally as often as X ranges over all the endomorphisms of the underlying vector space with a fixed restriction to a proper subspace. This proves that the character of the twisted Jacquet functor is zero at any nontrivial unipotent element.

We now carry out the proof when the element s in the Jordan decomposition of $g = su$ is an arbitrary element of $\mathbb{F}_n^*$. For this, we need the following lemma.

Lemma 5. Let

$$h = \begin{pmatrix} g & X \\ 0 & g \end{pmatrix}$$

be an endomorphism of $V \oplus V$. Let λ be an eigenvalue of g over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$, and let $W' = \ker(g - \lambda)$. Then the number of Jordan blocks of h over $\overline{\mathbb{F}}$ corresponding to the eigenvalue λ is $2 \dim W' - \dim(XW') + \dim\{XW' \cap \mathrm{Im}(g - \lambda)\}$. □

Proof. From the earlier argument, we have

$$\dim \ker(h - \lambda) = \dim V + \dim W'$$

$$- \left[\dim \mathrm{Im}(g - \lambda) + \dim(XW') - \dim \left\{ XW' \cap \mathrm{Im}(g - \lambda) \right\} \right]$$
$$= 2 \dim W' - \dim(XW') + \dim \left\{ XW' \cap \mathrm{Im}(g - \lambda) \right\}. \qquad \blacksquare$$

Corollary. The number of Jordan blocks of h corresponding to the eigenvalue λ depends only on the restriction of X to $W' = \ker(g - \lambda)$. $\square$

The following lemma summarises some elementary information from *Galois descent* that we need. We omit the proof of this lemma.

Lemma 6. Let L be a finite Galois extension of a field K. Let V be a finite-dimensional vector space over K. Let W be a subspace of the vector space $V \otimes_K L$ over L. Assume that the various Galois conjugates of W have $V \otimes_K L$ as their direct sum. Then there is a canonical isomorphism

$$\mathrm{Hom}_K(V, V) \cong \mathrm{Hom}_L \left(W, V \otimes_K L \right).$$

The isomorphism of $V \otimes_K L$ with the direct sum of the conjugates of W gives, in particular, a homomorphism from $V \otimes_K L$ to W. Thus, composing with the above isomorphism of $\mathrm{Hom}_K(V, V)$ with $\mathrm{Hom}_L(W, V \otimes_K L)$, we get a mapping from $\mathrm{Hom}_K(V, V)$ to $\mathrm{Hom}_L(W, W)$. If an endomorphism $\phi \in \mathrm{Hom}_K(V, V)$ goes to $\phi' \in \mathrm{Hom}_L(W, W)$, then $\mathrm{tr}(\phi) = \mathrm{tr}_{L/K}(\mathrm{tr}\, \phi')$. $\square$

Suppose that g is not semisimple. Let $\mathbb{F}_d$ be the extension of $\mathbb{F}$ generated by an eigenvalue, say λ, of g. Let W be the generalised eigenspace of g with eigenvalue λ, that is, the maximal subspace of $V \otimes_{\mathbb{F}} \mathbb{F}_d$ on which $(g - \lambda)$ is a nilpotent endomorphism. Let W' be the λ eigenspace of g in $V \otimes_{\mathbb{F}} \mathbb{F}_d$. Since g is not semisimple, W' is a proper subspace of W. Since the semisimple part of g comes from $\mathbb{F}_d^*$, W and its Galois conjugates form a direct sum decomposition of $V \otimes_{\mathbb{F}} \mathbb{F}_d$. Therefore, by Lemma 6, $\mathrm{Hom}_{\mathbb{F}}(V, V)$ can be identified to $\mathrm{Hom}_{\mathbb{F}_d}(W, V \otimes \mathbb{F}_d)$. By corollary to Lemma 5, the number of Jordan blocks of

$$h = \begin{pmatrix} g & X \\ 0 & g \end{pmatrix}$$

depends only on the restriction of the homomorphism in $\mathrm{Hom}_{\mathbb{F}_d}(W, V \otimes \mathbb{F}_d)$, corresponding to the homomorphism $X \in \mathrm{Hom}(V, V)$ to the proper subspace W'. Clearly, the set of elements in $\mathrm{Hom}_{\mathbb{F}_d}(W, V \otimes \mathbb{F}_d)$ with a given restriction on W' can be identified to an affine space. Under this identification, the trace of the projected map to $\mathrm{Hom}_{\mathbb{F}_d}(W, W)$ becomes a nonconstant affine map with values in $\mathbb{F}_d$. In particular, the number of homomorphisms $X \in \mathrm{Hom}(V, V)$ such that the restriction of the corresponding homomorphism in $\mathrm{Hom}(V\otimes\mathbb{F}_d, V\otimes\mathbb{F}_d)$ to W' is a given homomorphism, and the trace of $g^{-1}X$ is a given element a of $\mathbb{F}_d$, is independent of a. Since

$$\sum_{x\in\mathbb{F}} \psi_0(x) = 0,$$

this completes the proof of Lemma 4. ■

6.2 Character calculation at a semisimple element

We follow the notation of the previous section, but we now assume that g is semisimple. Let $W'' = \mathrm{Im}(g-\lambda)$. The space W'' is a direct summand of W inside $V\otimes\mathbb{F}_d$. The dimension of W over $\mathbb{F}_d$ is $d' = n/d$, and the dimension of W'' is $n - d'$. The endomorphism X of V, thought of as a homomorphism of W into $V\otimes\mathbb{F}_d$, gives rise to a linear map $\widetilde{X}$ from W into $(V\otimes\mathbb{F}_d)/W'' = W$. Clearly,

$$\begin{aligned}\mathrm{rank}\left(\widetilde{X}\right) &= \dim\left(XW + W''\right) - \dim W'' \\ &= \dim(XW) - \dim\left(XW\cap W'\right).\end{aligned}$$

It follows from Lemma 5 that the number of Jordan blocks for the endomorphism h for any given eigenvalue is $2\dim W - \mathrm{rank}(\widetilde{X})$.

The number of linear maps from W to $V\otimes\mathbb{F}_d$, which, when projected to $(V\otimes\mathbb{F}_d)/W''$, gives rise to a fixed linear map, is equal to the cardinality of $\mathrm{Hom}_{\mathbb{F}_d}(W, W'')$, which is the same as $q^{(\dim W\cdot\dim W'')d} = q^{n(n-d')}$. It follows that the number of linear maps from W to $V\otimes\mathbb{F}_d$, which, when projected to $V\otimes\mathbb{F}_d/W''\cong W$, gives rise to a linear map from W to W of rank i, and trace $\alpha\in\mathbb{F}_d$ is equal to

$$Y^{\alpha}_{n/d-i,i}\left(q^d\right)\cdot q^{n(n-d')}.$$

We now observe that g preserves W and acts on it by λ. From Lemma 6, it follows that

$$\mathrm{tr}(Xg^{-1}) = \mathrm{tr}_{\mathbb{F}_d/\mathbb{F}}\left[\lambda^{-1}\,\mathrm{tr}\left(\widetilde{X}\right)\right].$$

Finally, we are ready to calculate the character of the twisted Jacquet functor $\pi_{N,\psi}$ at the element g, which (using Green's theorem) is given by

$$\Theta_{N,\psi}(g) = q^{-n^2}\sum_{X\in M_n(\mathbb{F})}\Theta_\pi\begin{pmatrix} g & X\\ 0 & g\end{pmatrix}\bar{\psi}(Xg^{-1})$$

$$= (-1)^{2n-1}\Big[\sum \theta(g^{\alpha})\Big]\cdot\sum_i (1-q')\cdots\big(1-q'^{2d'-i-1}\big)\cdot q^{-nd'}$$
$$\times\left[\sum_{\beta} Y^{\beta}_{d'-i,i}\big(q^d\big)\bar{\psi}_0\big(\operatorname{tr}_{\mathbb{F}_d/\mathbb{F}}\{\beta\lambda^{-1}\}\big)\right].$$

The second equality arises by changing the sum over $X \in M_n(\mathbb{F})$ to the sum over $\mathrm{Hom}(W, V\otimes\mathbb{F}_d)$ whose projection to $\mathrm{Hom}(W,W)$ is of rank i and trace β, and then summed over i, β using Green's theorem.

We have

$$\sum_{\beta} Y^{\beta}_{d'-i,i}\big(q^d\big)\bar{\psi}_0\big(\operatorname{tr}_{\mathbb{F}_d/\mathbb{F}}\{\beta\lambda^{-1}\}\big)$$
$$= Y^{0}_{d'-i,i}\big(q^d\big) + Y^{1}_{d'-i,i}\big(q^d\big)\left[\sum_{\beta\neq 0}\bar{\psi}_0\big(\operatorname{tr}_{\mathbb{F}_d/\mathbb{F}}\beta\lambda^{-1}\big)\right]$$
$$= Y^{0}_{d'-i,i}\big(q^d\big) - Y^{1}_{d'-i,i}\big(q^d\big)$$
$$= (-1)^i q^{di(i-1)/2}\big|\mathrm{Gr}(d',i)(\mathbb{F}_d)\big|,$$

where we have used Lemma 2 to arrive at the last equality. Substituting this in the earlier equation, we get

$$\Theta_{N,\psi}(g)$$
$$= \Big[\sum\theta(g^{\alpha})\Big]\cdot\sum_i\,(q'-1)\cdots\big(q'^{2d'-i-1}-1\big)\cdot q^{-nd'}q^{di(i-1)/2}\cdot\big|\mathrm{Gr}(d',i)(\mathbb{F}_d)\big|.$$

Using Lemma 3 with n replaced by $d' = n/d$ and q replaced by q^d, we find that

$$\Theta_{N,\psi}(g) = \frac{\big|\mathrm{GL}_{d'}(\mathbb{F}_d)\big|}{\big(q^n-1\big)}\Big[\sum\theta(g^{\alpha})\Big] = \Theta_{\mathrm{Ind}}(g).$$

The proof of Theorem 1 is complete. ■

7 Generalised Steinberg representations for finite fields

Our main theorem can be used to calculate the twisted Jacquet functor for some other representations besides cuspidal representations.

We recall that if π is a cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$, where $\mathbb{F}$ is a finite field, then the principal series representation $\mathrm{Ps}(\pi,\pi)$ of $\mathrm{GL}_{2n}(\mathbb{F})$, which is obtained by parabolic induction of the representation $\pi\otimes\pi$ of $\mathrm{GL}_n(\mathbb{F})\times\mathrm{GL}_n(\mathbb{F})$, which is the Levi subgroup of the (n, n) parabolic $P(n, n)$, is not irreducible and has two irreducible components each of which appear with multiplicity 1 (cf. [SZ]). The component with larger dimension is called the generalised Steinberg representation and is denoted by $\mathrm{St}(\pi)$, and the component with smaller dimension is called the generalised trivial representation and is denoted by $\mathrm{Sp}(\pi)$.

The following theorem is a consequence of Lusztig's work, the so-called Jordan decomposition for characters (cf. [L] or [DM, Theorem 13.23]), by which the proof of the following theorem reduces to a theorem about GL_2.

Theorem 3. Let π be a cuspidal representation of $\mathrm{GL}_n(\mathbb{F})$ associated to a character θ_0 of $\mathbb{F}_n^*$. Let θ be the character of $\mathbb{F}_{2n}^*$ obtained from θ_0 by composing with the norm mapping to $\mathbb{F}_n^*$. Define the class function Θ_θ on $\mathrm{GL}_{2n}(\mathbb{F})$ as in Theorem 2, due to Green. Then the class function Θ_θ is the difference of the characters of the generalised Steinberg representation of $\mathrm{GL}_{2n}(\mathbb{F})$ associated to π and the generalised trivial representation $\mathrm{Sp}(\pi)$ of $\mathrm{GL}_{2n}(\mathbb{F})$ associated to π. □

We observe that the proof of Theorem 1 does not use the cuspidal property of the representation under consideration and works for any class function Θ_θ associated by Theorem 2 to a character θ of $\mathbb{F}_{2n}^*$. It works in particular for θ, which is obtained from a character θ_0 of $\mathbb{F}_n^*$ via the norm mapping. It follows that

$$\mathrm{St}(\pi)_{N,\psi} - \mathrm{Sp}(\pi)_{N,\psi} = \mathrm{Ind}_{\mathbb{F}_n^*}^{\mathrm{GL}_n(\mathbb{F})} \left(\theta|_{\mathbb{F}_n^*}\right).$$

Since the sum of $\mathrm{St}(\pi)$ and $\mathrm{Sp}(\pi)$ is the principal series representation $\mathrm{Ps}(\pi, \pi)$ of $\mathrm{GL}_{2n}(\mathbb{F})$, its twisted Jacquet functor can be calculated by the orbit method of Mackey. This allows the calculation of the twisted Jacquet functor of generalised Steinberg and generalised trivial representation of $\mathrm{GL}_{2n}(\mathbb{F})$.

We apply the Mackey theory to calculate the Jacquet functor of a principal series representation in the next section, where we have for simplicity taken the case of $\mathrm{GL}_4(\mathbb{F})$ only. Although similar in spirit, we have not carried out the computation in the case of $\mathrm{GL}_{2n}(\mathbb{F})$, which does not seem to have the simple expression we have in Theorem 4.

8 Principal series

Let π_1 and π_2 be irreducible representations of $\mathrm{GL}_2(\mathbb{F})$. Denote by $\mathrm{Ps}(\pi_1, \pi_2)$ the principal series representation of $\mathrm{GL}_4(\mathbb{F})$ induced from the $(2, 2)$ parabolic with Levi subgroup $\mathrm{GL}_2(\mathbb{F}) \times \mathrm{GL}_2(\mathbb{F})$. In this section, we calculate the twisted Jacquet functor of $\mathrm{Ps}(\pi_1, \pi_2)$.

Theorem 4. The twisted Jacquet functor of $\mathrm{Ps}(\pi_1, \pi_2)$, where π_1 and π_2 are irreducible representations of $\mathrm{GL}_2(\mathbb{F})$ (neither of which is 1-dimensional) and with central characters ω_1 and ω_2, is

$$\pi_1 \otimes \pi_2 \oplus \mathrm{Ps}(\omega_1, \omega_2),$$

where $\mathrm{Ps}(\omega_1, \omega_2)$ is the principal series representation of $\mathrm{GL}_2(\mathbb{F})$ induced from the character (ω_1, ω_2) of $\mathbb{F}^* \times \mathbb{F}^*$. □

Proof. Let P denote the $(2, 2)$ parabolic stabilising the 2-dimensional subspace $\{e_1, e_2\}$ of the 4-dimensional space $\{e_1, e_2, e_3, e_4\}$. The set $\mathrm{GL}_4(\mathbb{F})/P$ can be identified to the set of 2-dimensional subspaces of $\{e_1, e_2, e_3, e_4\}$; two elements of $\mathrm{GL}_4(\mathbb{F})/P$ are in the same orbit of P if and only if the corresponding subspaces intersect $\{e_1, e_2\}$ in the same dimensional subspaces of $\{e_1, e_2\}$. It follows that there are three orbits of P on $\mathrm{GL}_4(\mathbb{F})/P$ corresponding to the dimension of intersection 0, 1, 2.

Denote by ω the automorphism that takes e_1 to e_3, e_2 to e_4, e_3 to e_1, and e_4 to e_2. Also, denote by ω_{23} the automorphism that takes e_1 to e_1, e_2 to e_3, e_3 to e_2, and e_4 to e_4. It follows that we have the decomposition

$$\mathrm{GL}_4(\mathbb{F}) = P \coprod P\omega_{23}P \coprod P\omega P.$$

By Mackey theory, the restriction of $\mathrm{Ps}(\pi_1, \pi_2)$ to P is

$$(\pi_1 \otimes \pi_2) \oplus \mathrm{Ind}^{P}_{P \cap \omega_{23} P \omega_{23}} (\pi_1 \otimes \pi_2) \oplus \mathrm{Ind}^{P}_{P \cap wPw} (\pi_1 \otimes \pi_2),$$

where the first summand $\pi_1 \otimes \pi_2$ is a representation of P on which N operates trivially. Therefore, this summand does not contribute to the twisted Jacquet functor. Since $P \cap \omega P \omega = \mathrm{GL}_2(\mathbb{F}) \times \mathrm{GL}_2(\mathbb{F})$, it is easy to see that

$$\mathrm{Ind}^{P}_{\mathrm{GL}_2(\mathbb{F}) \times \mathrm{GL}_2(\mathbb{F})} (\pi_1 \otimes \pi_2) \cong \pi_1 \otimes \pi_2 \otimes \mathbb{C}[M_2(\mathbb{F})]$$

as a representation space for $N = M_2(\mathbb{F})$. From this isomorphism, it is easy to see that the twisted Jacquet functor of $\mathrm{Ind}^{P}_{\mathrm{GL}_2(\mathbb{F}) \times \mathrm{GL}_2(\mathbb{F})} (\pi_1 \otimes \pi_2)$ is $\pi_1 \otimes \pi_2$ as a representation space for $\mathrm{GL}_2(\mathbb{F})$. Finally, we calculate the twisted Jacquet functor of $\mathrm{Ind}^{P}_{P \cap \omega_{23} P \omega_{23}} (\pi_1 \otimes \pi_2)$. For this, we first calculate $P \cap \omega_{23} P \omega_{23}$. We note that since P is the stabiliser of $\{e_1, e_2\}$, $\omega_{23} P \omega_{23}$ is the stabiliser of the 2-dimensional subspace $\{e_1, e_3\}$. Therefore, $P \cap \omega_{23} P \omega_{23}$ is the stabiliser of the pair of planes $\{e_1, e_2\}$ and $\{e_1, e_3\}$. It follows that $P \cap \omega_{23} P \omega_{23}$ is exactly the set of matrices of the form

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & 0 & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix}.$$

It is easy to see that

$$\omega_{23} \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & 0 & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \omega_{23} = \begin{pmatrix} x_{11} & x_{13} & x_{12} & x_{14} \\ 0 & x_{33} & 0 & x_{34} \\ 0 & 0 & x_{22} & x_{24} \\ 0 & 0 & 0 & x_{44} \end{pmatrix}.$$

We note that in the induced representation, $\mathrm{Ind}_{P \cap \omega_{23} P \omega_{23}}^{P} (\pi_1 \otimes \pi_2)$, $\pi_1 \otimes \pi_2$ is considered as a representation space of $P \cap \omega_{23} P \omega_{23}$ via the inclusion of

$$P \cap \omega_{23} P \omega_{23} \hookrightarrow P$$

by $x \to \omega_{23} x \omega_{23}$.

Observe that since π_1 is not 1-dimensional, the representation π_1 has exactly a 1-dimensional subspace spanned by a vector, say ν_1, on which the upper-triangular unipotent matrices operate via the character ψ. Similarly, we find a vector ν_2 in π_2. Therefore, recalling the expression for $\omega_{23} P \omega_{23}$ given earlier, the set of matrices of the form

$$\begin{pmatrix} x_{11} & x_{13} & x_{12} & x_{14} \\ 0 & x_{22} & 0 & x_{34} \\ 0 & 0 & x_{11} & x_{24} \\ 0 & 0 & 0 & x_{22} \end{pmatrix}$$

operate on the vector $\nu_1 \otimes \nu_2$ in $\pi_1 \otimes \pi_2$ by

$$\omega_1(x_{11})\omega_2(x_{22})\psi(x_{12})\psi(x_{34}),$$

from which it is easy to see (by recalling the definition of induced representations) that the twisted Jacquet functor of $\mathrm{Ind}_{P \cap \omega_{23} P \omega_{23}}^{P} (\pi_1 \otimes \pi_2)$ is $\mathrm{Ps}(\omega_1, \omega_2)$. This completes the proof of the theorem. ∎

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