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Lectures on Tate's thesis

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1 The Adelic Language

A finite extension K of $\mathbb{Q}$ is called a number field. Let $\mathcal{O}_K$ denote the ring of integers in K. Then $\mathcal{O}_K$ is what is called a Dedekind domain, and has in particular, unique factorization for ideals, i.e., any ideal $\mathfrak{a}$ in $\mathcal{O}_K$ can be uniquely written as $\mathfrak{a} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$ where $\mathfrak{p}_i$ are nonzero prime ideals in $\mathcal{O}_K$. Therefore for any $x \in \mathcal{O}_K$, one can write

$$(x) = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}.$$

Define $v_{\mathfrak{p}}(x)$ to be the exponent of $\mathfrak{p}$ in the prime ideal decomposition of (x). This can be extended to a homomorphism $v_{\mathfrak{p}} : K^* \to \mathbb{Z}$. Let $N\mathfrak{p}$ be the cardinality of the residue field, $\mathcal{O}_K/\mathfrak{p}$, of $\mathfrak{p}$. Define,

$$||x||_{\mathfrak{p}} = (N\mathfrak{p})^{-v_{\mathfrak{p}}(x)}.$$

The completion of K with respect to the metric $d_{\mathfrak{p}}(x, y) = ||x - y||_{\mathfrak{p}}$ is a field denoted $K_{\mathfrak{p}}$, containing $\mathcal{O}_{\mathfrak{p}}$, the completion of $\mathcal{O}_K$, as a maximal compact subring.

The $v_{\mathfrak{p}}$'s as $\mathfrak{p}$ runs over the set of nonzero prime ideals gives rise to what are called non-Archimedean valuations. Besides these, one also considers Archimedean valuations which can be taken to be embeddings $i_v : K \to \mathbb{R}, \mathbb{C}$ as the case may be except that when $i_v(K)$ is not contained in $\mathbb{R}$, then $\imath_v$ and $\bar{i}_v$ are considered to be equivalent valuations.

One can induce a metric on K via embeddings $i_v : K \to \mathbb{R}, \mathbb{C}$, and the completion of K with respect to these embeddings is nothing but $\mathbb{R}, \mathbb{C}$.

Define the Adele ring $\mathbb{A}_K$ as the subring of $\prod_v K_v$, product taken over all valuations of K, consisting of tuples (x_v) such that x_v belongs to the maximal compact subring $\mathcal{O}_v$ for almost all valuations v. Let K_∞ be the product of K_v taken over all the Archimedean valuations. One can define a topology on $\mathbb{A}_K$ by declaring $K_\infty \times \prod_{\mathfrak{p}<\infty} \mathcal{O}_\mathfrak{p}$ to be an open subset with its natural topology.

This gives $\mathbb{A}_K$ the structure of a locally compact topological ring in which K sits as a discrete subgroup.

Lemma 1 *$\mathbb{A}_K/K$ is compact.*

Proof : This is a consequence of Chinese remainder theorem which can be used to prove that $K_\infty \times \prod_{\mathfrak{p}<\infty} \mathcal{O}_\mathfrak{p}$ surjects onto $\mathbb{A}_K/K$, and then the lemma follows from the compactness of $K_\infty/\mathcal{O}_K$. □

Next we come to the notion of ideles, denoted by $\mathbb{A}_K^*$ which consists of the subgroup of invertible elements of the ring $\mathbb{A}_K$. It consists of elements $x = (x_v) \in \prod_v K_v^*$ such that $x_v \in \mathcal{O}_v^*$ for all but finitely many places v of K. We declare $K_\infty^* \times \prod_{\mathfrak{p}<\infty} \mathcal{O}_\mathfrak{p}^*$ to be an open subset of $\mathbb{A}_K^*$ with its natural topology. This gives $\mathbb{A}_K^*$ the structure of a locally compact abelian group. It is easy to see that K^* sits naturally via 'diagonal' embedding inside $\mathbb{A}_K^*$ as a discrete subgroup.

Define a homomorphism

$$|\cdot| : \mathbb{A}_K^* \to \mathbb{R}^*,$$

by

$$|(x_v)| = \prod_v |x_v|,$$

which makes sense as $|x_v| = 1$ for almost all v. Let $\mathbb{A}_K^1 \subset \mathbb{A}_K^*$ be the kernel of $|\cdot|$.

The following theorem is a reformulation of two of the most basic theorems in Algebraic Number Theory, the finiteness of the class group, and the Dirichlet unit theorem, in the Adelic language.

Theorem 1
1. *$K^* \hookrightarrow \mathbb{A}_K^1$, and sits as a discrete subgroup of $\mathbb{A}_K^1$.*
2. *$\mathbb{A}_K^1/K^*$ is compact.*

□

Next we come to the notion of a Grössencharacter which are nothing but characters of the group $\mathbb{A}_K^*/K^*$, i.e., a Grössencharacter is a character $\chi : \mathbb{A}^*/K^* \to \mathbb{C}^*$.

Remark about convention.

1. A character on a topological group is a continuous homomorphism $\chi : G \to \mathbb{C}^*$.

2. A unitary character is a homomorphism $\chi : G \to \mathbb{S}^1$.

A character $\chi_v : K_v^* \to \mathbb{C}^*$ is called unramified if $\chi|_{\mathcal{O}_v^*} \equiv 1$. Define the local L-function $L(s, \chi_v)$ at non-Archimedean place v by

1. $L(s, \chi_v) = \frac{1}{(1 - \frac{\chi_v(\pi_v)}{Nv^s})}$ if χ_v is unramified, in which case $\chi_v(\pi_v)$ is independent of the choice of π_v with $v(\pi_v) = 1$, called a uniformizing parameter.

2. $L(s, \chi_v) = 1$ if χ_v is ramified.

Because of the way topology is defined on $\mathbb{A}_K^*$, and because $\mathbb{C}^*$ has no subgroups in a small neighborhood of identity, it easily follows that for a character $\chi = \prod_v \chi_v : \mathbb{A}_K^* \to \mathbb{C}^*$, χ_v is unramified for all but finitely many places v of K. For a character $\chi = \prod_v \chi_v : \mathbb{A}_K^* \to \mathbb{C}^*$, define the global L function

$$L(s, \chi) = \prod_v L(s, \chi_v)$$

where the product is taken only over finite places. (Later we will extend this definition by taking a product including all the places at infinity too.)

The aim of these lectures is to give a proof of the following theorem proved first by Hecke, for which Tate in his thesis gave an elegant proof based on Harmonic Analysis on the Adeles, and which has been generalised in many ways both to define and analyse global L-functions attached to (Automorphic) representations of larger adelic groups, such as $GL_n(\mathbb{A}_K)$.

Theorem 2 *$L(s, \chi)$ defined initially for $re(s)$ large has analytic continuation to all of $\mathbb{C}$ with possibly a pole at $s = 1$ (which happens if and only if $\chi = 1$), and no where else. Further $L(s, \chi)$ satisfies a functional equation.*

□

Remark : It is possible to define Grössencharacters in the classical language too, which is often used in some literature, so we give it here. Let I denote the free abelian group on the set of nonzero prime ideals. For any ideal $\mathfrak{a}$, let $I(\mathfrak{a})$ denote the subgroup of ideals generated by primes $\mathfrak{p}$ such that $\mathfrak{p}$ is coprime to $\mathfrak{a}$. A character $\chi : I(\mathfrak{a}) \to \mathbb{C}^*$ is said to be a Grössencharacter if

$$\chi((a)) = \prod_{v|\infty} \chi_v(a_v),$$

for all $a \in K^*$ such that $(a-1) \in \mathfrak{a}$.

2 Recalling Fourier Analysis

We recall that a locally compact group has a Haar measure which is unique up to scaling. The existence and uniqueness of Haar measure on totally disconnected groups, those arising from non-Archimedean valuations, is a straightforward matter.

Given a locally compact abelian group G, let $\hat{G} = \{\chi : G \to \mathbb{S}^1\}$. Clearly $\hat{G}$ is a group. It can be given what is called the *compact-open* topology, to make it a locally compact group. The group $\hat{G}$ comes equipped with the natural bilinear form $G \times \hat{G} \to \mathbb{S}^1$ which is a perfect pairing, and in particular gives an identification of closed subgroups of G with quotients of $\hat{G}$.

Given $f \in L^1(G)$, one can define $\hat{f} : \hat{G} \to \mathbb{C}$ by

$$\hat{f}(\chi) = \int_G f(g)\chi^{-1}(g)dg.$$

For appropriate choice of Haar measures on G and $\hat{G}$, one has the Fourier inversion theorem

$$\hat{\hat{f}}(x) = f(-x).$$

If $B : G \times G \to \mathbb{S}^1$ is a perfect pairing, then any character on G is of the form $x \to B(x,g)$ for some $g \in G$, so $G \cong \hat{G}$; such groups G are called self-dual. For such groups, Fourier transform can be considered to be a function on G, and there is a unique choice of Haar measure (given the bilinear form $B : G \times G \to \mathbb{S}^1$) such that the Fourier inversion holds.

Examples :

1. K_v is self-dual: For this fix a nontrivial character $\psi : K_v \to \mathbb{S}^1$. Then $B : K_v \times K_v \to \mathbb{S}^1$ defined by $B(x,y) = \psi(xy)$ is a perfect pairing.

2. $\mathbb{A}_K$ is self-dual.

3 Local Zeta functions

Observe that for a non-Archimedean local field K_v^*, $K_v^* = \mathcal{O}_v^* \times \{\pi_v\}^{\mathbb{Z}}$. Define characters $\omega_s(x) = |x|^s$ for $s \in \mathbb{C}$. This gives the set of characters the structure of a Riemann Surface which is a disjoint union of connected Riemann surfaces $M_\chi = \{\chi\omega_s | s \in \mathbb{C}\}$.

Given any character χ on K_v^*, there is a unique real number t such that $\chi\omega_t$ is a unitary character. The real number $-t$ is called the exponent of χ.

Define $\mathcal{S}(k_v)$ in the usual way for Archimedean fields $\mathbb{R}$ and $\mathbb{C}$, consisting of functions which together with all their derivatives, decay at infinity even when multiplied by any polynomial. For non-Archimedean fields, define $\mathcal{S}(k_v)$ to be the space of locally constant, compactly supported functions.

It will be important for us to note that $\mathcal{S}(k_v)$ is left stable by the Fourier transform.

Zeta function : Given $f \in \mathcal{S}(k)$, define the local zeta function

$$Z(f, chi, s) = \int_{k^*} f(x)\chi(x)|x|^s d^*x.$$

This is called a local zeta function on k; it is initially defined for $re(s)$ sufficiently large, in fact for $re(s) > 0$ if χ is unitary, but turns out to have analytic continuation and functional equation.

An Example : We calculate $Z(f, \chi, s)$ where f is the characteristic function of $\mathcal{O}$, and χ is an unramified character. We will fix aHaar measure on k^* such that the volume of $\mathcal{O}^*$ is 1.

$$\begin{aligned} Z(f, \chi, s) &= \int_{k^*} f(x)\chi(x)|x|^s d^*x \\ &= \int_{\mathcal{O}} \chi(x)|x|^s d^*x \\ &= \sum_{n=0}^{\infty} \int_{\pi^n\mathcal{O} - \pi^{n+1}\mathcal{O}} \chi(x)|x|^s d^*x \\ &= \sum_{n=0}^{\infty} \chi(\pi)^n q^{-ns} \\ &= \frac{1}{1 - \frac{\chi(\pi)}{q^s}}. \end{aligned}$$

We next observe that if for χ unramified we defined,

$$Z_0(f,\chi,s) = \int_{k^*} [f(x) - f(\pi^{-1}x)]\chi(x)|x|^s d^*x$$

then,

$$\begin{aligned} Z_0(f,\chi,s) &= \int_{k^*} [f(x) - f(\pi^{-1}x)]\chi(x)|x|^s d^*x \\ &= \int_{k^*} f(x)\chi(x)|x|^s d^*x - \frac{\chi(\pi)}{q^s}\int_{k^*} f(x)\chi(x)|x|^s d^*x \\ &= (1 - \frac{\chi(\pi)}{q^s})Z(f,\chi,s) \\ &= \frac{Z(f,\chi,s)}{L(\chi,s)} \end{aligned}$$

Since $f(x)-f(\pi^{-1}x)$ is a compactly suported locally constant function on k^*, $Z_0(f,\chi,s)$ is a finite Laurent polynomial in q^s, i.e., $Z_0(f,\chi,s) \in \mathbb{C}[q^s, q^{-s}]$.

For χ ramified, note that

$$Z(f,\chi,s) = \int_{k^* - \pi^n\mathcal{O}} f(x)\chi(x)|x|^s d^*x,$$

for all n sufficiently large. Therefore in this case, $Z(f,\chi,s) \in \mathbb{C}[q^s, q^{-s}]$.

Remark :

1. From the foregoing, we find that in all cases,

   $$\frac{Z(f,\chi,s)}{L(\chi,s)}$$

   is an entire function which in fact belongs to $\mathbb{C}[q^s, q^{-s}]$.

2. Furthermore, it is easy to see that there is a choice of a function $f \in \mathcal{S}(k)$ such that $Z(f,\chi,s) = L(\chi,s)$.

We now turn our attention to the functional equation satisfied by such zeta functions. The approach we take was suggested by A. Weil, for which we follow the treatment given by Kudla in [Ku].

Let $\mathcal{D}(k)$ denote the space of distributions, which are just the dual space of the space $\mathcal{S}(k)$ for k non-Archimedean. Note that k^* operates on $\mathcal{S}(k)$, and therefore also on $\mathcal{D}(k)$. The following lemma, for which we omit details, is simple to check using the following exact sequence,

$$0 \to \mathcal{S}(k^*) \to \mathcal{S}(k) \to \mathbb{C} \to 0.$$

Lemma 2

1. *For any character χ of k^*, the χ-eigenspace in $\mathcal{D}(k)$ is of dimension 1.*
2. *$f \to \frac{Z(f,\chi,s)}{L(\chi,s)}$ is a distribution which belongs to $\chi\omega_s$ eigenspace.*
3. *$f \to \hat{f}$ is an isomorphism of $\mathcal{S}(k)$ onto itself, and takes χ-eigenspace to $\chi^{-1}\omega$ eigenspace.*

Corollary 1

$$\frac{Z(\hat{f},\chi^{-1},1-s)}{L(\chi^{-1},1-s)} = \epsilon(\chi,s)\frac{Z(f,\chi,s)}{L(\chi,s)},$$

for a certain $\epsilon(\chi,s)$ which is of the form aq^{ns} for some $n \in \mathbb{Z}$.

Proof. As already observed, $\frac{Z(f,\chi,s)}{L(\chi,s)} \in \mathbb{C}[q^s,q^{-s}]$, therefore since there is an f with $\frac{Z(f,\chi,s)}{L(\chi,s)} = 1$, $\epsilon(\chi,s) \in \mathbb{C}[q^s,q^{-s}]$. Arguing with $\hat{f}$ instead of f, we find that $\epsilon(\chi,s)$ is in fact a unit in $\mathbb{C}[q^s,q^{-s}]$, therefore of the form aq^{ns}.

4 Archimedean Theory

We first need to define $L(\chi,s)$ in the Archimedean case. We note that the definitions are so made that $L(\chi w_t,s) = L(\chi,s+t)$, and therefore it suffices to define $L(\chi,s)$ for χ an equivalence class of the relation defined by $\{\chi\omega_t\}$.

We begin with $k = \mathbb{R}$, in which case there are exactly two equivalence classes of character, which are defined below together with their L-functions.

1. $\chi = 1$. In this case $L(1,s) = \pi^{-s/2}\Gamma(s/2)$.
2. $\chi = \epsilon$, the sign character $\mathbb{R}^* \to \pm 1$. In this case $L(\epsilon,s) = L(1,s+1) = \pi^{-(s+1)/2}\Gamma([s+1]/2)$.

Now for $\mathbb{C}$, we note that the equivalence classes of characters on $\mathbb{C}^*$ are represented by $\chi_n(re^{i\theta}) = e^{in\theta}$. Define $L(\chi_n,s)$ by

$$L(\chi_n,s) = (2\pi)^{1-s}\Gamma(s+|n|/2).$$

Lemma 3 *1.* $f \to \frac{Z(f,\chi,s)}{L(\chi,s)}$ *is a distribution which represents an entire function of* $s \in \mathbb{C}$.

2. *There exists an explicit choice of functions* f_χ *in* $\mathcal{S}(k)$ *such that* $Z(f_\chi, \chi, s) = L(\chi, s)$, *and such that* $\hat{f}_\chi = c(\chi) f_{\chi^{-1}}$ *for a certain constant* $c(\chi)$ *which is a 4th root of unity.*

3.
$$\frac{Z(\hat{f}_\chi, \chi^{-1}, 1-s)}{L(\chi^{-1}, 1-s)} = \epsilon(\chi, s) \frac{Z(f_\chi, \chi, s)}{L(\chi, s)},$$
for a certain constant $\epsilon(\chi, s)$ *which is in fact a 4th root of unity.*

Remark : Part 1. follows from integration by parts. The explicit functions f_χ satisfying the properties in 2. are given in Tate's thesis. Given 2., 3. is obvious.

5 Global Zeta function

We begin by noting that if a group G is the restricted product of groups G_v with given compact open subgroups H_v, then one can define a Haar measure on G by taking the product of Haar measures on G_v chosen so that H_v has volume 1.

Define the Schwartz space $\mathcal{S}(\mathbb{A}_K)$ to be the tensor product of the Schwatz space of K_∞ with the space of locally constant compactly supported functions on $K^f = \prod_{v<\infty} K_v$.

Global Zeta function : Given $f \in \mathcal{S}(\mathbb{A}_K)$, and a Grössencharacter $\chi : \mathbb{A}_K^*/K^* \to \mathbb{C}^*$, define the global zeta function

$$Z(f, \chi, s) = \int_{\mathbb{A}_K^*} f(x)\chi(x)|x|^s d^*x.$$

This is called a global zeta function on $\mathbb{A}_K$; it is initially defined for $re(s)$ sufficiently large, infact for $re(s) > 1$ if χ is unitary (by the same reason why the Euler product $\prod_p \frac{1}{1-\frac{1}{p^s}}$ converges for $re(s) > 1$, and hence similar products over places of a number field). It has analytic continuation and functional equation as follows.

Theorem 3 *(a) $Z(f,\chi,s)$ has analytic continuation to all of $\mathbb{C}$.*
(b)$Z(f,\chi,s) = Z(\hat{f},\chi^{-1},1-s)$.

Corollary 2 *Analytic continuation and Functional equation for $\Lambda(\chi,s)$ which is the product of $L(\chi,s)$ defined earlier using the finite places, together with the places at infinity for which the L-factors have now been defined.*

Proof : Let $f = \prod_v f_v$ be a function in $\mathcal{S}(\mathbb{A}_K)$ such that $Z(f_v,\chi_v,s) = L(\chi_v,s)$ for all places v of K. We know that this is possible both at finite and infinite places. Thus the analytic continuation of $Z(f,\chi,s)$ implies the analytic continuation of $\Lambda(\chi,s)$.

Choosing a finite set of places including all the places at infinity and all the places where χ_v or the field is ramified, we write the functional equation

$$Z(f,\chi,s) = Z(\hat{f},\chi^{-1},1-s)$$

as

$$\prod_{v\in S}\frac{Z(f_v,\chi_v,s)}{L(\chi_v,s)}\Lambda(\chi,s) = \prod_{v\in S}\frac{Z_v(\hat{f}_V,\chi_v^{-1},1-s)}{L(\chi_v^{-1},1-s)}\Lambda(\chi^{-1},1-s).$$

This implies that,

$$\Lambda(\chi,s) = \prod_{v\in S}\epsilon(\chi_v,s)\Lambda(\chi^{-1},1-s).$$

It remains to prove the analytic continuation and functional equation for the zeta functions. But for this we begin by recalling the Poisson summation formula.

6 Poisson summation formula

Let G be a self-dual locally compact abelian group, given with a non-degenerate bilinear form $B : G\times G \to \mathbb{S}^1$. Let Γ be a discrete subgroup of G such that $\Gamma^\perp = \Gamma$. Then for a suitable space of functions (for applications we have in mind, it will suffice to take $G = \mathbb{A}_K$, $\Gamma = K$, and the space of functions to be $\mathcal{S}(\mathbb{A}_K)$), we have

$$\sum_{\gamma\in\Gamma} f(\gamma) = \sum_{\gamma\in\Gamma}\hat{f}(\gamma).$$

The proof of this follows by applying the Fourier inversion formula to the function

$$F(g) = \sum_{\gamma \in \Gamma} f(g\gamma),$$

on G/Γ.

Corollary 3 *For $f \in \mathcal{S}(\mathbb{A}_K)$,*

$$\sum_{\gamma \in K} f(\gamma) = \sum_{\gamma \in K} \hat{f}(\gamma).$$

Corollary 4 *For $f \in \mathcal{S}(\mathbb{A}_K)$,*

$$\sum_{\gamma \in K} f(a\gamma) = \frac{1}{||a||} \sum_{\gamma \in K} \hat{f}(\frac{\gamma}{a}).$$

7 Analytic Continuation, Functional Equation

We now come to the main theorem of Tate's thesis which is about analytic continuation and functions equation of zeta functions.

Theorem 4 *$Z(f, \chi, s)$ has analytic continuation and functional equation.*

Proof : For simplicity of exposition, we give the proof only in the case when χ restricted to $\mathbb{A}_K^1$ is nontrivial. When it is trivial, a minor modification is needed to the proof (and to the results). We will assume without loss of generality that χ is a unitary character.

We begin by writing the integral defining the zeta function as a sum of two integrals:

$$\begin{aligned} Z(f, \chi, s) &= \int_{\mathbb{A}_K^*} f(x)\chi(x)|x|^s d^*x \\ &= \int_{|x| \leq 1} f(x)\chi(x)|x|^s d^*x + \int_{|x| \geq 1} f(x)\chi(x)|x|^s d^*x. \end{aligned}$$

We note that the term, $\int_{|x|\geq 1} f(x)\chi(x)|x|^s d^*x$ has analytic continuation as an entire function of $s \in \mathbb{C}$. This follows for $re(s) > 1$ as in this region the integral is absolutely convergent, and since for $re(s) \leq 1$, the value of the integrand decreases in absolute value, it is all the more convergent in this region too, and hence represents an entire function. In our next lemma, we prove that

$$\int_{|x|\leq 1} f(x)\chi(x)|x|^s d^*x = \int_{|x|\geq 1} \hat{f}(x)\chi^{-1}(x)|x|^{1-s}d^*x,$$

which will then yield,

$$Z(f,\chi,s) = \int_{|x|\geq 1} \hat{f}(x)\chi^{-1}(x)|x|^{1-s}d^*x + \int_{|x|\geq 1} f(x)\chi(x)|x|^s d^*x,$$

which gives both analytic continuation and the functional equation.

It thus suffices to prove the following lemma.

Lemma 4 *For $re(s) > 1$,*

$$\int_{|x|\leq 1} f(x)\chi(x)|x|^s d^*x = \int_{|x|\geq 1} \hat{f}(x)\chi^{-1}(x)|x|^{1-s}d^*x.$$

Proof : Observe that for a fundamental domain E in $\mathbb{A}_K^1$ for the action of K^* on $\mathbb{A}_K^1$, and for any $t > 0$, a real number,

$$\begin{aligned}
\int_{\mathbb{A}_K^1} f(tb)\chi(tb)|tb|^s d^*b &= \sum_{\gamma\in K^*}\int_E f(\gamma tb)\chi(\gamma tb)|\gamma tb|^s d^*b \\
&= \int_E [\sum_{\gamma\in K^*} f(\gamma tb)]\chi(tb)t^s d^*b \\
&= \int_E [\sum_{\gamma\in K} f(\gamma tb)]\chi(tb)t^s d^*b \\
&= \int_E [\frac{1}{t}\sum_{\gamma\in K} \hat{f}(\frac{\gamma}{tb})]\chi(tb)t^s d^*b \\
&= \int_{\mathbb{A}_K^1} [\frac{1}{t}\hat{f}(\frac{1}{tb})]\chi(tb)t^s d^*b.
\end{aligned}$$

(We have twice used the fact that the integral of a nontrivial character on a compact group is 0.) Therefore,

$$\int_{t\leq 1}\int_{\mathbb{A}_K^1} f(tb)\chi(tb)|tb|^s d^*bd^*t = \int_{t\geq 1}\int_{\mathbb{A}_K^1} [\hat{f}(tb)]\chi^{-1}(tb)t^{1-s}d^*bd^*t,$$

completing the proof of the lemma, and hence that of the theorem.

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