Representation theory of $GL(n)$ over non-Archimedean local fields

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1 Introduction

The aim of these notes is to give an elementary introduction to representation theory of p-adic groups via the basic example of $GL(n)$. We have emphasised those topics which are relevant to the theory of automorphic forms. To keep these notes to a reasonable size, we have omitted many proofs which would have required a lot more preparation. We have given references throughout the notes either to original or more authoritative sources. Most of the omitted proofs can be found in the fundamental papers of Bernstein-Zelevinsky [2], Bushnell-Kutzko [5], or Casselman's unpublished notes [7] which any serious student of the subject will have to refer sooner or later.

Representation theory of p-adic groups is a very active area of reasearch. One of the main driving forces in the development of the subject is the Langlands program. One aspect of this program, called the local Langlands corrspondence, implies a very intimate connection between representation theory of p-adic groups with the representation theory of the Galois group of the p-adic field. This conjecture of Langlands has recently been proved by Harris and Taylor and also by Henniart for the case of $GL(n)$. We have given an introduction to Langlands' conjecture for $GL(n)$ as well as some representation theory of Galois groups in the last two sections of these notes.

These notes are based on the lectures given by the first author in the workshop 'Automorphic Forms on GL(n)' organized by Professors G. Harder and M.S. Raghunathan at ICTP, Trieste in August 2000. The authors would like to thank the organizers for inviting them for this nice conference; the European commission for financial support; Professor L. Gottsche for having taken care of all the details of our stay as well as the conference so well.

2 Generalities on representations

Let G denote a locally compact totally disconnected topological group. In the terminology of [2] such a group is called an l-group. For these notes the fundamental example of an l-group is the group $GL_n(F)$ of invertible $n \times n$ matrices with entries in a non-Archimedean local field F.

By a *smooth representation* (π, V) of G we mean a group homomorphism of G into the group of automorphisms of a complex vector space V such that for every vector $v \in V$, the stabilizer of v in G, given by $\mathrm{stab}_G(v) = \{g \in G : \pi(g)v = v\}$, is open. The space V is called the *representation space* of π.

By an *admissible representation* (π, V) of G we mean a smooth representation (π, V) such that for any open compact subgroup K of G, the invariants in V under K, denoted V^K, is finite dimensional.

Given a smooth representation (π, V) of G, a subspace W of V is said to be stable or invariant under G if for every $w \in W$ and every $g \in G$ we have $\pi(g)w \in W$. A smooth representation (π, V) of G is said to be *irreducible* if the only G stable subspaces of V are (0) and V.

If (π, V) and (π', V') are two (smooth) representations of G then by $\mathrm{Hom}_G(\pi, \pi')$ we denote the space of G intertwining operators from V into V', i.e., $\mathrm{Hom}_G(\pi, \pi')$ is the space of all linear maps $f : V \to V'$ such that $f(\pi(g)v) = \pi'(g)f(v)$ for all $v \in V$ and all $g \in G$.

Let (π, V) be a smooth representation of G. Let V^* denote the space of linear functionals on V, i.e., $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. There is an obvious action π^* of G on V^* given by $(\pi^*(g)f)(v) = f(\pi(g^{-1})v)$ for $g \in G$, $v \in V$ and $f \in V^*$. However this representation is in general not smooth. Let $V^\vee$ denote the subspace of V^* which consists of those linear functionals in V^* whose stabilizers are open in G under the above action. It is easily seen that $V^\vee$ is stabilized by G and this representation denoted $(\pi^\vee, V^\vee)$ is called the *contragredient representation* of (π, V).

The following is a basic lemma in representation theory.

Lemma 2.1 (Schur's Lemma) *Let (π, V) be a smooth irreducible admissible representation of an l-group G. Then the dimension of* $\mathrm{Hom}_G(\pi, \pi)$ *is one.*

Proof : Let $A \in \mathrm{Hom}_G(\pi, \pi)$. Let K be any open compact subgroup of G. Then A takes the space of K-fixed vectors V^K to itself. Choose a K such that $V^K \neq (0)$. Since V^K is a finite dimensional complex vector space, there exists an eigenvector for the action of A on V^K, say, $0 \neq v \in V^K$ with $Av = \lambda v$. It follows that the kernel of $(A - \lambda \mathbf{1}_V)$ is a nonzero G invariant subspace of V. Irreducibility of the representation implies that $A = \lambda \mathbf{1}_V$. □

Corollary 2.2 *Any irreducible admissible representation of an abelian l-group is one dimensional.*

Corollary 2.3 *On an irreducible admissible representation (π, V) of an l-group G the centre Z of G , operates via a character ω_π, i.e., for all $z \in Z$ we have $\pi(z) = \omega_\pi(z)\mathbf{1}_V$. (This character ω_π is called the* central character *of π.)*

Exercise 2.4 (Dixmier's lemma) Let V be a complex vector space of countable dimension. Let Λ be a collection of endomorphisms of V acting irreducibly on V. Let T be an endomorphism of V which commutes with every element of Λ. Then prove that T acts as a scalar on V. (*Hint.* Think of V as a module over the field of rational functions in one variable $\mathbb{C}(X)$ with X acting on V via T and use the fact that $\left\{\frac{1}{X-\lambda} | \lambda \in \mathbb{C}\right\}$ is an uncountable set consisting of linearly independent elements in the $\mathbb{C}$-vector space $\mathbb{C}(X)$.) Deduce that if G is an l-group which is a countable union of compact sets and (π, V) is an irreducible smooth representation of G then the dimension of $\mathrm{Hom}_G(\pi, \pi)$ is one.

One of the most basic ways of constructing representations is by the process of induction. Before we describe this process we need a small digression on Haar measures.

Let G be an l-group. Let $d_l x$ be a left Haar measure on G. So in particular $d_l(ax) = d_l(x)$ for all $a, x \in G$. For any $g \in G$ the measure $d_l(xg)$ (where x is the variable) is again a left Haar measure. By uniqueness of Haar measures there is a positive real number $\Delta_G(g)$ such that $d_l(xg) = \Delta_G(g) d_l(x)$. It is easily checked that $g \mapsto \Delta_G(g)$ is a continuous group homomorphism and is called the *modular character* of G. Further $d_r(x) := \Delta_G(x)^{-1} d_l(x)$ is a right Haar measure. A left Haar measure is right invariant if and only if the modular character is trivial and in this case G is said to be *unimodular*.

Example 2.5 Let F be a non-Archimedean local field. The group $G = GL_n(F)$ is unimodular. (In general a reductive p-adic group is unimodular.) This may be seen as follows. Note that Δ_G is trivial on $[G, G]$ as the range of Δ_G is abelian. Further by the defining relation Δ_G is trivial on the centre Z. (Both these remarks are true for any group.) Hence Δ_G is trivial on $Z[G, G]$. Observe that $Z[G, G]$ is of finite index in G and since positive reals admits no non-trivial finite subgroups we get that Δ_G is trivial.

Exercise 2.6 Let B be the subgroup of all upper triangular matrices in $G = GL_n(F)$. Then show that B is not unimodular as follows. Let d^*y be a Haar measure on F^* and let dx be a Haar measure on F^+. The normalized absolute value on F^* is defined by $d(ax) = |a|_F dx$ for all $a \in F^*$ and $x \in F$.

For $b \in B$ given by

$$b = \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & . & \\ & & & y_n \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & x_{i,j} \\ & & . & \\ & & & 1 \end{pmatrix},$$

show that a left Haar measure on B is given by the formula :

$$db = \prod_i d^*y_i \prod_{1 \le i < j \le n} dx_{i,j}.$$

Deduce that the modular character Δ_B of B is given by :

$$\Delta_B(\mathrm{diag}(a_1, ..., a_n)) = |a_1|_F^{1-n} |a_2|_F^{3-n} ... |a_n|_F^{n-1}.$$

Let $\delta_B = \Delta_B^{-1}$. Show that

$$\delta_B(b) = |\det(\mathrm{Ad}(b)|_{\mathrm{Lie}(U)})|_F$$

where $\mathrm{Ad}(b)|_{\mathrm{Lie}(U)}$ is the adjoint representation of B on the Lie algebra $\mathrm{Lie}(U)$ of U which is the unipotent radical of B. (In some places δ_B is used as the definition of the modular character, see for e.g., [1].)

For any l-group G, let Δ_G denote the modular character of G. Let B be a closed subgroup of G. Let $\Delta = \Delta_G|_B \cdot \Delta_B^{-1}$. Let (σ, W) be a smooth representation of B. To this is associated the *(normalized) induced representation* $\mathrm{Ind}_B^G(\sigma)$ whose representation space is :

$$\left\{ f : G \to W \;\middle|\; \begin{array}{ll} (1) & f(bg) = \Delta(b)^{1/2}\sigma(b)f(g),\ \forall b \in B,\ \forall g \in G \\ (2) & f(hu) = f(h) \text{ for all } u \in U_f \text{ an open set in } G \end{array} \right\}.$$

The group G acts on this space by right translation, i.e., given $x \in G$ and $f \in \mathrm{Ind}_B^G(\sigma)$ we have $(x \cdot f)(g) = f(gx)$ for all $g \in G$.

We remark here that throughout these notes we deal with normalized induction. By normalizing we mean the $\Delta^{1/2}$ factor appearing in (1) above. This is done so that unitary representations are taken to unitary representations under induction. This simplifies some formulae like that which describes the contragredient of an induced representation and complicates some other formulae like Frobenius reciprocity.

This induced representation admits a subrepresentation which is also another 'induced' representation called that obtained by *compact induction.* This is denoted as $\mathrm{ind}_B^G(\sigma)$ whose representation space is given by

$$\mathrm{ind}_B^G(\sigma) = \{f \in \mathrm{Ind}_B^G(\sigma) : f \text{ is compactly supported modulo } B\}.$$

The following theorem summarises the basic properties of induced representations.

Theorem 2.7 *Let G be an l-group and let B be a closed subgroup of G.*

1. *Both Ind_B^G and ind_B^G are exact functors from the category of smooth representations of B to the category of smooth representations of G.*

2. *Both the induction functors are transitive,i.e., if C is a closed subgroup of B then $\mathrm{Ind}_B^G(\mathrm{Ind}_C^B) = \mathrm{Ind}_C^G$. A similar relation holds for compact induction.*

3. *Let σ be a smooth representation of B. Then $\mathrm{Ind}_B^G(\sigma)^\vee = \mathrm{ind}_B^G(\sigma^\vee)$.*

4. *(Frobenius Reciprocity) Let π be a smooth representation of G and σ be a smooth representation of B.*

   (a) $\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G(\sigma)) = \mathrm{Hom}_B(\pi|_B, \Delta^{1/2}\sigma)$.

   (b) $\mathrm{Hom}_G(\mathrm{ind}_B^G(\sigma), \pi) = \mathrm{Hom}_B(\Delta^{-1/2}\sigma, (\pi^\vee|_B)^\vee)$.

   Both the above identifications are functorial in π and σ.

The reader may refer paragraphs 2.25, 2.28 and 2.29 of [2] for a proof of the above theorem.

We recall some basic facts now about taking invariants under open compact subgroups. Let C be an open compact subgroup of an l-group G. Let $\mathcal{H}(G, C)$ be the space of compactly supported bi-C-invariant functions on G. This forms an algebra under convolution which is given by :

$$(f * g)(x) = \int_G f(xy^{-1})g(y)\, dy$$

with the identity element being $\mathrm{vol}(C)^{-1}e_C$ where e_C is the characteristic function of C as a subset of G.

Let (π, V) be a smooth admissible representation of G then the space of invariants V^C is a finite dimensional module for $\mathcal{H}(G, C)$. The following proposition is not difficult to prove. The reader is urged to try and fix a proof of this. (See paragraph 2.10 of [2].)

Proposition 2.8 *Let the notations be as above.*

1. *The functor $(\pi, V) \mapsto (\pi^C, V^C)$ is an exact additive functor from the category of smooth admissible representations of G to the category of finite dimensional modules of the algebra $\mathcal{H}(G, C)$.*

2. *Let (π, V) be an admissible representation of G such that $V^C \neq (0)$. Then (π, V) is irreducible if and only if (π^C, V^C) is an irreducible $\mathcal{H}(G, C)$ module.*

3. *Let (π_1, V_1) and (π_2, V_2) be two irreducible admissible representations of G such that both V_1^C and V_2^C are non-zero. We have $\pi_1 \simeq \pi_2$ as G representations if and only if $\pi_1^C \simeq \pi_2^C$ as $\mathcal{H}(G, C)$ modules.*

3 Preliminaries on $GL_n(F)$

In this section we begin the study of the group $G = GL_n(F)$ and its representations. (Unless otherwise mentioned, from now on G will denote the group $GL_n(F)$.) We begin by describing this group and some of its subgroups which are relevant for these notes. The emphasis is on various decompositions of G with respect to these subgroups.

Let F be a non-Archimedean local field. Let $\mathcal{O}_F$ be its ring of integers whose unique maximal ideal is $\mathfrak{P}_F$. Let ϖ_F be a uniformizer for F, i.e., $\mathfrak{P}_F = \varpi_F \mathcal{O}_F$. Let q_F be the cardinality of the residue field $k_F = \mathcal{O}_F / \mathfrak{P}_F$.

The group $G = GL_n(F)$ is the group of all invertible $n \times n$ matrices with entries in F. A more invariant description is that it is the group of all invertible linear transformations of an n dimensional F vector space V and in this case it is denoted $GL(V)$. If we fix a basis of V then we can identify $GL(V)$ with $GL_n(F)$.

We let B denote the subgroup of G consisting of all upper triangular matrices. Let T denote the subgroup of all diagonal matrices and let U denote the subgroup of all upper triangular unipotent matrices. Note that T normalizes U and B is the semi-direct product TU. This B is called the *standard Borel subgroup* with U being its *unipotent radical* and T is called the diagonal *torus*. We let U^- to be the subgroup of all lower triangular unipotent matrices.

Let W, called the *Weyl group* of G, denote the group $N_G(T)/T$ where $N_G(T)$ is the normalizer in G of the torus T. It is an easy exercise to check that

$N_G(T)$ is the subgroup of all monomial matrices and so W can be identified with S_n the symmetric group on n letters. We will usually denote a diagonal matrix in T as $\mathrm{diag}(a_1, ..., a_n)$.

We now introduce the Borel subgroup and more generally the parabolic subgroups in a more invariant manner. Let V be an n dimensional F vector space. Define a flag in V to be a *strictly* increasing sequence of subspaces

$$W_\bullet = \{W_0 \subset W_1 \subset \cdots W_m = V\}.$$

The subgroup of $GL(V)$ which stabilizes the flag $W_\bullet$, i.e., with the property that $gW_i = W_i$ for all i is called a *parabolic subgroup* of G associated to the flag $W_\bullet$.

If $\{v_1, ..., v_n\}$ is a basis of V then the stabilizer of the flags of the form

$$W_\bullet = \{(v_1) \subset (v_1, v_2) \subset \cdots (v_1, v_2, ..., v_n) = V\}$$

is called a Borel subgroup. It can be seen that $GL(V)$ operates transitively on the set of such flags, and hence the stabilizer of any two such (maximal) flags are conjugate under $GL(V)$.

If

$$W_\bullet = \{W_0 \subset W_1 \subset \cdots W_m = V\},$$

then inside the associated parabolic subgroup P, there exists the normal subgroup N consisting of those elements which operate trivially on W_{i+1}/W_i for $0 \leq i \leq m-1$. The subgroup N is called the unipotent radical of P. It can be seen that there is a semi direct product decomposition $P = MN$ with

$$M = \prod_{i=0}^{m-1} GL(W_{i+1}/W_i).$$

The decomposition $P = MN$ is called a *Levi decomposition* of P with N the *unipotent radical*, and M a *Levi subgroup* of P.

We now introduce some of the open compact subgroups of G which will be relevant to us. We let $K = GL_n(\mathcal{O}_F)$ denote the subgroup of elements G with entries in $\mathcal{O}_F$ and whose determinant is a unit in $\mathcal{O}_F$. This is an open compact subgroup of G. The following exercise contains most of the basic properties of K.

Exercise 3.1 Let V be an n dimesional F vector space. Let L be a lattice in V, i.e., an $\mathcal{O}_F$ submodule of rank n. Show that $\mathrm{stab}_G(L)$ is an open compact subgroup of $G = GL(V)$. If C is any open compact subgroup then show that there is a lattice L such that $C \subset \mathrm{stab}_G(L)$. Deduce that up to conjugacy K is the unique maximal open compact subgroup of $GL_n(F)$.

For every integer $m \geq 1$, the map $\mathcal{O}_F \to \mathcal{O}_F/\mathfrak{P}_F^m$ induces a map $K \to GL_n(\mathcal{O}_F/\mathfrak{P}_F^m)$. The kernel of this map, denoted K_m is called the *principal congruence subgroup of level* m. We also define K_0 to be K. For all $m \geq 1$, we have

$$K_m = \{g \in GL_n(\mathcal{O}_F) : g - \mathbf{1}_n \in \varpi_F^m M_n(\mathcal{O}_F)\}.$$

Note that K_m is an open compact subgroup of G and gives a basis of neighbourhoods at the identity. Hence G is an l-group, i.e., a locally compact totally disconnected topological group.

We are now in a position to state the main decomposition theorems which will be of use to us later in the study of representations of G.

Theorem 3.2 (Bruhat decomposition)

$$G = \coprod_{w \in W} BwB = \coprod_{w \in W} BwU.$$

Proof: This is an elementary exercise in basic linear algebra involving row reduction (on the left) and column reduction (on the right) by elementary operations. Since the rank of an element in G is n we end up with a monomial matrix with such operations and absorbing scalars into $T \subset B$ we end up with an element in W. We leave the details to the reader.

The disjointness of the union requires more work and is not obvious. We refer the reader to Theorem 8.3.8 of [15]. □

Theorem 3.3 (Cartan decomposition) *Let* $A = \{\mathrm{diag}(\varpi_F^{m_1}, ..., \varpi_F^{m_n}) : m_i \in \mathbb{Z},\ \ m_1 \leq m_2 \leq \cdots \leq m_n\}$.

$$G = \coprod_{a \in A} K \cdot a \cdot K$$

Proof : Let $g \in G$. After fixing a basis for an n dimensional F vector space V we identify $GL(V)$ with $GL_n(F)$. Let L be the standard lattice in V corresponding to this basis. Let L_1 be the lattice $g(L)$. Let r be the least integer such that $\varpi_F^r L$ is contained in L_1 and let $L_2 = \varpi_F^r L$.

The proof of the Cartan decomposition falls out from applying the structure theory of finitely generated torsion modules over principal ideal domains. In our context we would apply this to the torsion $\mathcal{O}_F$ module L_1/L_2. We urge the reader to fill in the details. $\square$

Theorem 3.4 (Iwasawa decomposition)

$$G = K \cdot B$$

Proof : We sketch a proof for this decomposition for $GL_2(F)$. The proof for $GL_n(F)$ uses induction on n and the same matrix manipulations as in the $GL_2(F)$ case is used in reducing from n to $n-1$.

Assume that $G = GL_2(F)$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. If $a = 0$ then write g as $w(w^{-1}g) \in K \cdot B$. If $a \neq 0$ and if $a^{-1}c \in \mathcal{O}_F$ then

$$g = \begin{pmatrix} 1 & 0 \\ ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in K \cdot B.$$

If $a^{-1}c \notin \mathcal{O}_F$ then replace g by kg where

$$k = \begin{pmatrix} 1 & 1 + \varpi_F^r \\ 1 & \varpi_F^r \end{pmatrix} \in K.$$

We may choose r large enough such that the (modified) a and c satisfy the property $a^{-1}c \in \mathcal{O}_F$ which gets us to the previous case and we may proceed as before. $\square$

Theorem 3.5 (Iwahori factorization) *For $m \geq 1$,*

$$K_m = (K_m \cap U^-) \cdot (K_m \cap T) \cdot (K_m \cap U).$$

Proof : This is an easy exercise in row and column reduction. See paragraph 3.11 of [2]. $\square$

We are now in a position to begin the study of representations of $G = GL_n(F)$. *The purpose of these notes is to give a detailed account of irreducible representations of G with emphasis on those topics which are relevant to the theory of automorphic forms.*

The first point to notice is that most representations we deal with are infinite dimensional. We leave this as the following exercise.

Exercise 3.6 1. Show that the derived group of $GL_n(F)$ is $SL_n(F)$, the subgroup of determinant one elements. Any character χ of F^* gives a character $g \mapsto \chi(\det(g))$ of $G = GL_n(F)$ and every character of G looks like this.

2. Show that a finite dimensional smooth irreducible representation of G is one dimensional and hence is of the form $g \mapsto \chi(\det(g))$.

Note that any representation π of G can be *twisted by a character* χ denoted as $\pi \otimes \chi$ whose representation space is the same as π and is given by $(\pi \otimes \chi)(g) = \chi(\det(g))\pi(g)$. It is trivial to see that π is irreducible if and only if $\pi \otimes \chi$ is irreducible and that $\omega_{\pi\otimes\chi} = \omega_\pi \chi^n$ as characters of the centre $Z \simeq F^*$. For any complex number s we will denote $\pi(s)$ to be the representation $\pi \otimes |\cdot|_F^s$.

Even if one is primarily interested in irreducible representations, some natural constructions such as parabolic induction defined in the next section, force us to also consider representations which are not irreducible. However they still would have certain finiteness properties. We end this section with a quick view into these finiteness statements.

Let (π, V) be a smooth representation of G. We call π to be a *finitely generated* representation if there exist finitely many vectors $v_1, ..., v_m$ in V such that the smallest G invariant subspace of V containing these vectors is V itself. We say that π has *finite length* if there is a sequence of G invariant subspaces

$$(0) = V_0 \subset V_1 \subset \cdots \subset V_m = V$$

such that each succesive quotient V_{i+1}/V_i for $0 \le i \le m-1$ is an irreducible representation of G.

The following theorem is a non-trivial theorem with a fairly long history and various mathematician's work has gone into proving various parts of it (especially Harish-Chandra, R. Howe, H. Jacquet and I.N. Bernstein). (See Theorem 4.1 of [2]. The introduction of this same paper has some relevant history.)

Theorem 3.7 *Let (π, V) be a smooth representation of $G = GL_n(F)$. Then the following are equivalent :*

1. *π has finite length.*

2. *π is admissible and is finitely generated.*

Exercise 3.8

1. Prove the above theorem for $G = GL_1(F) = F^*$.
2. Show that the sum of all characters of F^* on which the uniformizer ϖ_F acts trivially is an admissible representation which is not of finite length.
3. Let f be the characteristic function of $\mathcal{O}_F^\times$ thought of as a subset of F^*. Let V be the representation of F^* generated by f inside the regular representation of F^* on $C_c^\infty(F^*)$. Show that V gives an example of a smooth finitely generated representation which is not of finite length.

4 Parabolic induction

One important way to construct representations of G is by the process of parabolic induction.

Let $P = MN$ be a parabolic subgroup of $G = GL_n(F)$. Recall that P is the stabilizer of some flag and its unipotent radical is the subgroup which acts trivially on all successive quotients. More concretely, for a partition $n = n_1 + n_2 + \cdots + n_k$ of n, let $P = P(n_1, n_2, ..., n_k)$ be the *standard parabolic subgroup* given by block upper triangular matrices :

$$P = \left\{ \begin{pmatrix} g_1 & * & * & * \\ & g_2 & * & * \\ & & \cdot & \cdot \\ & & & g_k \end{pmatrix} : g_i \in GL_{n_i}(F) \right\}.$$

The unipotent radical of P is the block upper triangular unipotent matrices given by :

$$N_P = N = \left\{ \begin{pmatrix} \mathbf{1}_{n_1} & * & * & * \\ & \mathbf{1}_{n_2} & * & * \\ & & \cdot & \cdot \\ & & & \mathbf{1}_{n_k} \end{pmatrix} \right\}$$

and the Levi subgroup of P is the block diagonal subgroup :

$$M_P = M = \left\{ \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \cdot & \\ & & & g_k \end{pmatrix} : g_i \in GL_{n_i}(F) \right\} \simeq \prod_{i=1}^{k} GL_{n_i}(F).$$

Let (ρ, W) be a smooth representation of M. Since M is the quotient of P by N we can inflate ρ to a representation of P (sometimes referred to as 'extending it trivially across N') also denoted ρ. Now we can consider $\mathrm{Ind}_P^G(\rho)$. (See § 2.) We say that this representation is *parabolically induced* from M to G. To recall, $\mathrm{Ind}_P^G(\rho)$ consists of all locally constant functions $f : G \to V_\rho$ such that

$$f(pg) = \delta_P(p)^{1/2}\rho(p)f(g)$$

where $\delta_P(p) = |\det(\mathrm{Ad}(p)|_{\mathrm{Lie}(N)})|_F$. (See Exercise 2.6.) The following theorem contains the basic properties of parabolic induction.

Theorem 4.1 *Let $P = MN$ be a parabolic subgroup of $G = GL_n(F)$. Let (ρ, W) be a smooth representation of M.*

1. *The functor $\rho \mapsto \mathrm{Ind}_P^G(\rho)$ is an exact additive functor from the category of smooth representations of M to the category of smooth representations of G.*
2. $\mathrm{Ind}_P^G(\rho) = \mathrm{ind}_P^G(\rho)$.
3. $\mathrm{Ind}_P^G(\rho^\vee) \simeq \mathrm{Ind}_P^G(\rho)^\vee$.
4. *If ρ is unitary then so is $\mathrm{Ind}_P^G(\rho)$.*
5. *If ρ is admissible then so is $\mathrm{Ind}_P^G(\rho)$.*
6. *If ρ is finitely generated then so is $\mathrm{Ind}_P^G(\rho)$.*

Proof : General properties of induction stated in Theorem 2.7 gives (1). Iwasawa decomposition (Theorem 3.4) implies (2). Theorem 2.7 and (2) imply (3).

For a proof of (4) observe that given an M invariant unitary structure (see Definition 7.1) on W we can cook up a G invariant unitary structure on $\mathrm{Ind}_P^G(\rho)$ by integrating functions against each other which is justified by (2). We urge the reader to fill in the details.

We now prove (5). It suffices to prove the space of vectors fixed by K_m (for any m) is finite dimensional where K_m is the principal congruence subgroup of level m.

Let $f \in \mathrm{Ind}_P^G(\rho)$ which is fixed by K_m. Note that the Iwasawa decomposition (Theorem 3.4) gives that $P\backslash G/K_m$ is a finite set. Let $g_1, ..., g_r$ be a set of representatives for this double coset decomposition. We may and shall choose these elements to be in K. The function f is completely determined by its values on the elements g_i. Since we have $f(mg_ik) = \delta_P^{1/2}(m)\rho(m)f(g_i)$ for all $m \in M$ and $k \in K_m$ we get that $f(g_i)$ is fixed by $M \cap (g_iK_mg_i^{-1}) = M \cap K_m$ which is simply the principal congruence subgroup of level m for M. Hence each of the $f(g_i)$ takes values in a finite dimensional space by admissibility of ρ which implies that $\mathrm{Ind}_P^G(\rho)^{K_m}$ is finite dimensional.

Now for the proof of (6) we give the argument to show that the space of locally constant functions on $P\backslash G$ is a finitely generated representation of G. The argument for general parabolically induced representations is similar. To prove that the space of locally constant functions on $P\backslash G$ is finitely generated, it is sufficient to treat the case when P is the Borel subgroup B of the group of upper triangular matrices. (We do this because the essence of the argument is already seen in the case of $B\backslash G$. It is however true that a subrepresentation of a finitely generated representation of G is itself finitely generated. See Theorem 4.19 of [2].)

The proof of finite generation of the space of locally constant functions on $B\backslash G$ depends on the Iwahori factorization (Theorem 3.5). Recall that if K_m is a principal congruence subgroup of level m then we have

$$K_m = (K_m \cap U^-) \cdot (K_m \cap T) \cdot (K_m \cap U).$$

We note that there are elements in T which shrink $K_m^- = K_m \cap U^-$ to the identity. For example if we take the matrix

$$\mu = \mathrm{diag}(1, \varpi_F, \varpi_F^2, ..., \varpi_F^{n-1})$$

then the powers of μ have the property that they shrink K_m^- to the identity, i.e., $\lim_{i\to\infty}\mu^{-i}K_m^-\mu^i = \{1\}$.

Let χ_X denote the characteristic function of a subset X of a certain ambient space. Look at the translates of $\chi_{B\cdot K_m^-}$ by the powers μ^{-i}. This will give us,

$$\mu^{-i} \cdot \chi_{BK_m^-} = \chi_{BK_m^-\mu^i} = \chi_{B\mu^{-i}K_m^-\mu^i}$$

Therefore translating $\chi_{BK_m^-}$ by μ^{-i}, we get the characteristic function of BC for arbitrarily small open compact subgroups C. These characteristic functions together with their G translates clearly span all the locally constant functions on $B\backslash G$, completing the proof of (6). $\square$

We would like to emphasize that even if ρ is irreducible the representation $\mathrm{Ind}_P^G(\rho)$ is in general not irreducible. However the above theorem assures us, using Theorem 3.7, that it is of finite length. In general, understanding when the induced representation is irreducible is an extremely important one and this is well understood for G. We will return to this point in the section on Langlands classification (§ 8). One instance of this is given in the next example.

Example 4.2 (Principal series) Let $\chi_1, \chi_2, ..., \chi_n$ be n characters of F^*. Let χ be the character of B associated to these characters, i.e.,

$$\chi\left(\begin{pmatrix} a_1 & * & * & * \\ & a_2 & * & * \\ & & \cdot & \cdot \\ & & & a_n \end{pmatrix}\right) = \chi_1(a_1)\chi_2(a_2)...\chi_n(a_n).$$

The representation $\pi(\chi) = \pi(\chi_1, ..., \chi_n)$ of G obtained by parabolically inducing χ to G is called a *principal series representation* of G. It turns out that $\pi(\chi)$ is reducible if and only if there exist $i \neq j$ such that $\chi_i = \chi_j|\cdot|_F$ where $|\cdot|_F$ is the normalized multiplicative absolute value on F. In particular if $\chi_1, ..., \chi_n$ are all unitary then $\pi(\chi)$ is a unitary irreducible representation.

5 Jacquet functors

Parabolic induction constructs representations of $GL_n(F)$ from representations of its Levi subgroups. There is a dual procedure, more precisely, an adjoint functor, which constructs representations of Levi subgroups from representations of $GL_n(F)$. The importance and basic properties of this construction was noted by Jacquet for the $GL_n(F)$ case, which was generalized to all reductive groups by Harish-Chandra.

Definition 5.1 Let $P = MN$ be the Levi decomposition of a parabolic subgroup P of $G = GL_n(F)$. For a smooth representation (ρ, V) of P, define ρ_N to be the largest quotient of ρ on which N operates trivially. Let $\rho(N) = V(N) = \{n \cdot v - v | n \in N, v \in V\}$. Then

$$\rho_N = V/V(N).$$

This ρ_N is called the *Jacquet functor of* ρ. If ρ is a smooth representation of G then the Jacquet functor ρ_N of ρ is just that of ρ restricted to P.

It is easily seen that the Jacquet functor ρ_N is a representation for M. This is seen by noting that M normalizes N. There is an easy and important characterization of this subspace $V(N)$ and this is given in the following lemma.

Lemma 5.2 *For a smooth representation V of N, $V(N)$ is exactly the space of vectors $v \in V$ such that*

$$\int_{K_N} n \cdot v \, dn = 0,$$

where K_N is an open compact subgroup of N and dn is a Haar measure on N. (The integral is actually a finite sum.)

Proof : The main property of N used in this lemma is that it is a union of open compact subgroups. Clearly the integral of the vectors of the form $n \cdot v - v$ on an open compact subgroup of N containing n is zero. On the other hand if the integral equals zero, then $v \in V(N)$ as it reduces to a similar conclusion about finite groups, which is easy to see.

□

The following theorem contains most of the basic properties of Jacquet functors. The reader is urged to compare this with Theorem 4.1.

Theorem 5.3 *Let (π, V) be a smooth representation of G. Let $P = MN$ be a parabolic subgroup of G. Then*

1. *The Jacquet functor $V \to V_N$ is an exact additive functor from the category of smooth representations of G to the category of smooth representations of M.*

2. *(Transitivity) Let $Q \subset P$ be standard parabolic subgroups of G with Levi decompositions $P = M_P N_P$ and $Q = M_Q N_Q$. Hence $M_Q \subset M_P$, $N_P \subset N_Q$, and $M_Q(N_Q \cap M_P)$ is a parabolic subgroup of M_P with M_Q as a Levi subgroup and $N_Q \cap M_P$ as the unipotent radical, and we have $(\pi_{N_P})_{N_Q \cap M_P} \simeq \pi_{N_Q}$.*

3. *(Frobenius Reciprocity) For a smooth representation σ of M,*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\sigma)) \simeq \mathrm{Hom}_M(\pi_N, \sigma\delta_P^{1/2}).$$

4. *If π is finitely generated then so is π_N.*

5. *If π is admissible then so is π_N.*

Proof : For the proof of (1) it suffices to prove that $V_1 \cap V(N) = V_1(N)$ which is clear from the previous lemma. We urge the reader to fix proofs of (2) and (3). For (2) note that $N_Q = N_P(N_Q \cap M_P)$. For (3) it is an exercise in modifying the proof of the usual Frobenius reciprocity while using the definitions of parabolic induction and Jacquet functors. An easy application of Iwasawa decomposition (Theorem 3.4) gives (4).

Statement (5) is originally due to Jacquet. We sketch an argument below as it is important application of Iwahori factorization. Let $P = P(n_1, ..., n_k)$ be a standard parabolic subgroup and let $P = MN$ be its Levi decomposition. The principal congruence subgroup K_m of level m admits an Iwahori factorization which looks like

$$K_m = K_m^- K_m^0 K_m^+$$

where $K_m^+ = K_m \cap N$ and $K_m^0 = K_m \cap M$ and $K_m^- = K_m \cap N^-$ where $P^- = MN^-$ is the opposite parabolic subgroup obtained by simply taking transposes of all elements in P.

Let $A : V \to V_N$ be the canonical projection. Under this projection we show that

$$A(V^{K_m}) = (V_N)^{K_m^0}.$$

Clearly, using admissiblity of π, this would prove admissiblity of π_N because K_m^0 is the principal congruence subgroup of level m of M and these form a basis of neighbourhoods at the identity for M.

Since A is a P equivariant map, we get that $A(V^{K_m}) \subset (V_N)^{K_m^0}$. Let $\bar{v} \in (V_N)^{K_m^0}$. Choose $v \in V$ such that $A(v) = \bar{v}$. For any $z \in Z_M$, the centre of M, given by

$$z = \begin{pmatrix} \varpi_F^{m_1} \mathbf{1}_{n_1} & & & \\ & \varpi_F^{m_2} \mathbf{1}_{n_2} & & \\ & & . & \\ & & & \varpi_F^{m_k} \mathbf{1}_{n_k} \end{pmatrix}$$

let $t(z) = \max_{i,j} |m_i - m_j|$. Let $v_1 = z^{-1} v$. Then $A(v_1) = A(z^{-1} v) = z^{-1} A(v) = z^{-1} \bar{v}$. Since z is central in M, $A(v_1)$ is also fixed by K_m^0.

Choose $t(z) \gg 0$ such that $zK_m^- z^{-1} \subset \mathrm{stab}_G(v)$. Hence K_m^- will fix $v_1 = z^{-1}v$. To summarize, v_1 is fixed by K_m^- and $A(v_1)$ is fixed by K_m^0 and K_m^+. (The latter because N acts trivially on V_N.)

Choose a Haar measure on G such that $\mathrm{vol}(K_m \cap P) = 1$. Let $v_2 = \int_{K_m} \pi(k)v_1 \, dk$. It is easy to check that $A(v_2) = z^{-1}\bar{v}$. The upshot is that given $\bar{v} \in (V_N)^{K_m^0}$ there exists $z \in Z_M$ such that $z^{-1}\bar{v} \in A(V^{K_m})$.

Now if $\bar{v_1}, ..., \bar{v_r}$ are any linearly independent vectors in $(V_N)^{K_m^0}$ then there exists $z \in Z_M$ such that $z^{-1}\bar{v_1}, ..., z^{-1}\bar{v}$ are in $A(V^{K_m})$ and are also linearly independent. This implies that the dimension of $(V_N)^{K_m^0}$ is bounded above by that of $A(V^{K_m})$ and hence they must be equal. □

We now compute the Jacquet functor of the principal series representation introduced in Example 4.2. As a notational convenience for a smooth representation π of finite length of any l-group H, we denote by π_{ss} the semi-simplification of π, i.e., if $\pi = \pi_0 \supset \pi_1 \supset \cdots \supset \pi_n = \{0\}$ with each π_i/π_{i+1} irreducible, then $\pi_{ss} = \oplus_{i=0}^{n-1}(\pi_i/\pi_{i+1})$. By the Jordan-Holder theorem, π_{ss} is independent of the filtration $\pi = \pi_0 \supset \pi_1 \supset \cdots \supset \pi_n = \{0\}$.

Theorem 5.4 (Jacquet functor for principal series) *Let π be the principal series representation $\pi(\chi) = \mathrm{Ind}_B^{GL_n(F)}(\chi)$. Then the Jacquet functor of π with respect to the Borel subgroup $B = TU$ is given as a module for T as*

$$(\pi_U)_{\mathrm{ss}} \simeq \sum_{w \in W} \chi^w \delta_B^{1/2}$$

where χ^w denotes the character of the torus obtained by twisting χ by the element w in the Weyl group W, i.e $\chi^w(t) = \chi(w(t))$.

Proof : The representation space of π can be thought of as a certain space of "functions on G/B twisted by the character χ"; more precisely, π can be thought of as the space of locally constant functions on G/B with values in a sheaf $\mathcal{E}_\chi$ obtained from the character χ of the Borel subgroup B.

If Y is a closed subspace of a topological space X "of the kind that we are considering here", e.g. locally closed subspaces of the flag variety, then there is an exact sequence,

$$0 \to C_c^\infty(X - Y, \mathcal{E}_\chi|_{X-Y}) \to C_c^\infty(X, \mathcal{E}_\chi) \to C_c^\infty(Y, \mathcal{E}_\chi|_Y) \to 0.$$

It follows that Mackey's theory (originally for finite groups) about restriction of an induced representation to a subgroup holds good for p-adic groups too. Hence using the Bruhat decomposition $GL(n) = \coprod_{w\in W} BwB$, and denoting $B\cap wBw^{-1}$ to be $T\cdot U_w$, we have

$$\begin{aligned}
(\mathrm{Res}_B \mathrm{Ind}_B^G(\chi))_{\mathrm{ss}} &= \sum_{w\in W} \mathrm{ind}_{B\cap wBw^{-1}}^B((\chi\delta_B^{1/2})^w) \\
&= \sum_{w\in W} \mathrm{ind}_{T\cdot U_w}^B((\chi\delta_B^{1/2})^w) \\
&= \sum_{w\in W} C_c^\infty(U/U_w, (\chi\delta_B^{1/2})^w).
\end{aligned}$$

We now note that the largest quotient of $C_c^\infty(U/U_w)$ on which U operates trivially is one dimensional (obtained by integrating a function with respect to a Haar measure on U/U_w) on which the action of the torus T is the sum of positive roots which are not in U_w which can be seen to be $[\delta_B\cdot(\delta_B)^{-w}]^{1/2}$. (Here δ_B^{-w} stands for the w translate of δ_B^{-1}.) This implies that the largest quotient of $C_c^\infty(U/U_w, (\chi\delta_B^{1/2})^w)$ on which U operates trivially is the 1 dimensional T-module on which T operates by the character $\chi^w\delta_B^{1/2}$. Hence,

$$\left(\left[\mathrm{Ind}_B^{GL_n(F)}(\chi)\right]_U\right)_{\mathrm{ss}} \simeq \sum_{w\in W}\chi^w\delta_B^{1/2}.$$

$\square$

Corollary 5.5 *If* $\mathrm{Hom}_{GL_n(F)}(\mathrm{Ind}_B^{GL_n(F)}(\chi), \mathrm{Ind}_B^{GL_n(F)}(\chi'))$ *is nonzero then* $\chi' = \chi^w$ *for some* w *in* W*, the Weyl group.*

Proof : This is a simple consequence of the Frobenius reciprocity combined with the calculation of the Jacquet functor done above. $\square$

Example 5.6 Let $\pi(\chi)$ denote a principal series representation of $GL_2(F)$. Then

1. If $\chi\neq\chi^w$, then $\pi(\chi)_U \simeq \chi\delta_B^{1/2}\oplus\chi^w\delta_B^{1/2}$.
2. If $\chi = \chi^w$, then $\pi(\chi)_U$ is a non-trivial extension of T-modules:

$$0\to\chi\delta_B^{1/2}\to\pi(\chi)_U\to\chi\delta_B^{1/2}\to 0.$$

Exercise 5.7 With notation as in the previous exercise, prove that a principal series representation of $GL_2(F)$ induced from a unitary character is irreducible.

Exercise 5.8 Let G be an abelian group with characters χ_1 and χ_2. Prove that if $\chi_1 \neq \chi_2$, then any exact sequence of G-modules,

$$0 \to \chi_1 \to V \to \chi_2 \to 0,$$

splits.

We end this section with a theorem about Jacquet modules for parabolically induced representations. This theorem is at the basis for considering supercuspidal representations (which are those representations for which all Jacquet modules are trivial).

Theorem 5.9 *Let $P = MN$ be a parabolic subgroup of $G = GL_n(F)$. Let σ be a smooth irreducible representation of M. Let (π, V) be any 'subquotient' of* $\mathrm{Ind}_P^G(\rho)$*. Then the Jacquet module π_N of π is non-zero.*

Proof : Note that if π is subrepresentation of the induced representation then it easily follows from Frobenius reciprocity. Indeed, we have

$$(0) \neq \mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G(\rho)) = \mathrm{Hom}_M(\pi_N, \delta_P^{1/2}\rho)$$

which implies the $\pi_N \neq (0)$. In general if π is a subquotient then there is a 'trick' due to Harish-Chandra (see Corollary 7.2.2 in [7]) using which π can be realized as a subrepresentation of a Weyl group 'twist' of the induced representation, which will bring us to the above case. Since this trick is very important and makes its presence in quite a few arguments we delineate it as the following theorem. $\square$

We will need a little bit of notation before we can state this theorem. Let $P = MN$ be a parabolic subgroup of G. Let $x \in N_G(M)$ - the normalizer in G of M. Then we can consider the representation ${}^x\rho$ of M whose representation space is same as that of ρ and the action is given by ${}^x\rho(m) = \rho(x^{-1}mx)$. Now we can consider the induced representation $\mathrm{Ind}_P^G({}^x\rho)$. Note that this representation is in general *not equivalent* to the original induced representation.

Theorem 5.10 *Let $P = MN$ be a parabolic subgroup of $G = GL_n(F)$. Let σ be an smooth irreducible representation of M. Let (π, V) be any irreducible subquotient of* $\mathrm{Ind}_P^G(\sigma)$*. Then there exists an element $w \in W \cap N_G(M)$ (where W is the Weyl group) such that π is a subrepresentation of* $\mathrm{Ind}_P^G({}^w\sigma)$.

6 Supercuspidal representations

A very important and novel feature of p-adic groups (compared to real reductive groups) is the existence of supercuspidal representations. We will see that these representations are the building blocks of all irreducible admissible representations of p-adic groups. A complete set of supercuspidals for $GL_n(F)$ was constructed by Bushnell and Kutzko in their book [5]. The local Langlands correspondence, proved by Harris and Taylor and also by Henniart [10], interprets supercuspidal representations of $GL_n(F)$ in terms of irreducible n dimensional representations of the Galois group of F. Before we come to the definition of a supercuspidal representation, we need to define the notion of a matrix coefficient of a representation.

For a smooth representation (π, V) of $GL_n(F)$ recall that $(\pi^\vee, V^\vee)$ denotes the contragredient representation of (π, V). For vectors v in π and $v^\vee$ in $\pi^\vee$, define the *matrix coefficient* $f_{v,v^\vee}$ to be the function on $GL_n(F)$ given by $f_{v,v^\vee}(g) = \langle v^\vee, \pi(g)v\rangle$.

Theorem 6.1 *Let (π, V) be an irreducible admissible representation of $G = GL_n(F)$. Then the following are equivalent :*

1. *One matrix coefficient of π is compactly supported modulo the centre.*
2. *Every matrix coefficient of π is compactly supported modulo the centre.*
3. *The Jacquet functors of π (for all proper parabolic subgroups) are zero.*
4. *The representation π does not occur as a subquotient of any representation parabolically induced from any proper parabolic subgroup.*

A representation satisfying any one of the above conditions of the theorem is called a *supercuspidal representation.* We refer the reader to paragraph 3.21 of [2] for a proof of this theorem.

We note that every irreducible admissible representation of $GL_1(F) = F^*$ is supercuspidal. However for $GL_n(F)$ with $n \geq 2$ since parabolic induction is of no use in constructing supercuspidal representation, a totally new approach is needed. One way to construct supercuspidal representations is via induction from *certain* finite dimensional representations of compact open subgroups. This has been a big program in recent times which has been completed for the case of GL_n by Bushnell and Kutzko in their book [5]. They prove that any supercuspidal representation of $GL_n(F)$ is obtained by induction from a finite dimensional representation of certain open compact modulo centre subgroup of $GL_n(F)$.

Here is a sample of such a construction in the simplest possible situation. These are what are called *depth zero supercuspidal representations* of $GL_n(F)$. By a cuspidal representation of the finite group $GL_n(\mathbb{F}_q)$ we mean an irreducible representation for which all the Jacquet functors are zero (equivalently there are no non-zero vectors fixed by $N(\mathbb{F}_q)$ for unipotent radicals N of any proper parabolic subgroup). Cuspidal representations of $GL_n(\mathbb{F}_q)$ are completely known by the work of J.Green [9] in the 50's.

Theorem 6.2 *Consider a representation of $GL_n(\mathbb{F}_q)$ to be a representation of $GL_n(\mathcal{O}_F)$ via the natural surjection from $GL_n(\mathcal{O}_F)$ to $GL_n(\mathbb{F}_q)$. If σ is an irreducible cuspidal representation of $GL_n(\mathbb{F}_q)$ thought of as a representation of $GL_n(\mathcal{O}_F)$ and χ is a character of F^* whose restriction to $\mathcal{O}_F^\times$ is the same as the central character of σ then $\chi \cdot \sigma$ is a representation of $F^*GL_n(\mathcal{O}_F)$. Let π be the compactly induced representation $\mathrm{ind}_{F^*GL_n(\mathcal{O}_F)}^{GL_n(F)}(\chi \cdot \sigma)$. Then π is an irreducible admissible supercuspidal representation of $GL_n(F)$.*

The proof we give is based on Proposition 1.5 in [6]. To begin with we need a general lemma which describes the restriction of an induced representation in one special context that we are interested in.

Lemma 6.3 *Let H be an open compact-mod-centre subgroup of a unimodular l-group G. Let (σ, W) be a smooth finite dimensional representation of H. Let $\pi = \mathrm{ind}_H^G(\sigma)$ be the compact induction of σ to a representation π of G. Then the restriction of π to H is given by :*

$$\pi|_H = \left[\mathrm{ind}_H^G(\sigma)\right]_H \simeq \bigoplus_{g \in H\backslash G/H} \mathrm{Ind}_{H\cap g^{-1}Hg}^H ({}^g\sigma|_{H\cap g^{-1}Hg})$$

where ${}^g\sigma$ is the representation of $g^{-1}Hg$ given by ${}^g\sigma(g^{-1}hg) = \sigma(h)$ for all $h \in H$.

Proof of Lemma 6.3 : For $g \in G$, let $V_g = C^\infty(HgH, \sigma)$ denote the space of smooth functions f on HgH such that $f(h_1gh_2) = \sigma(h_1)f(gh_2)$. Since HgH is open in G and is also compact mod H we get a canonical injection $V_g \to \mathrm{ind}_H^G(\sigma)$ given by extending functions by zero outside HgH. All the inclusions $V_g \to \pi$ gives us a canonical map

$$\bigoplus_{g \in H\backslash G/H} V_g \to \pi.$$

This map is an isomorphism of H modules. This can be seen as follows : This map is clearly injective. (Consider supports of the functions.) Surjectivity follows from the definition of π. Note also that V_g is H stable if H acts by right shifts and the map $V_g \to \pi$ is H equivariant.

The map sending $f \to \bar{f}$ from V_g to $\mathrm{Ind}_{H \cap g^{-1}Hg}^{H}({}^{g}\sigma|_{H \cap g^{-1}Hg})$ where $\bar{f}(h) = f(gh)$ is easily checked to be a well-defined H equivariant bijection. This proves the lemma. □

Proof of Theorem 6.2 : Let H denote the subgroup $F^*GL_n(\mathcal{O}_F)$ which is an open and compact-mod-centre subgroup of $G = GL_n(F)$. For brevity let σ also denote the representation $\chi \cdot \sigma$ of H. We will prove irreducibility, admissibility and supercuspidality of π separately, which will prove the theorem.

Irreducibility: We begin by proving irreducibility of π. We make the following claim :

Claim : For all $g \in G - H$, we have $\mathrm{Hom}_{H \cap g^{-1}Hg}(\sigma, {}^{g}\sigma) = (0)$.

Assuming the claim for the time being we prove the theorem as follows. The claim and the previous lemma implies that σ appears in π with multiplicity one, i.e., $\dim(\mathrm{Hom}_H(\sigma, \pi|H)) = 1$.

Now suppose π is not irreducible then there exists an exact sequence of non-trivial G modules :

$$0 \longrightarrow \pi_1 \longrightarrow \pi \longrightarrow \pi_2 \longrightarrow 0.$$

We note that since H is compact-mod-centre, any representation (like σ, π_1, π_2 and π) on which F^* acts by a character is necessarily semi-simple as an H-module. Since $\pi = \mathrm{ind}_H^G \sigma$ it embeds in $\mathrm{Ind}_H^G(\sigma)$ as a G module and hence so does π_1. Using Frobenius reciprocity 2.7.4(a) we get that $\mathrm{Hom}_H(\pi_1, \sigma) \neq (0)$, i.e., σ occurs in π_1. By Frobenius reciprocity 2.7.4(b)

$$\mathrm{Hom}_G(\mathrm{ind}_H^G \sigma, \pi_2) = \mathrm{Hom}_H(\sigma, (\pi_2^{\vee}|_H)^{\vee}).$$

Since H is an open subgroup $((\pi_2^{\vee})_H)^{\vee} = \pi_2$ as H-modules. Thus we get that σ occurs in π_2 also but this contradicts the fact that σ occurs with multiplicity one in π.

It suffices now to prove the claim. Since $g \in G - H$, we can assume without loss of generality, using the Cartan decomposition (see Theorem 3.3), that $g = \mathrm{diag}(\varpi_F^{m_1}, ..., \varpi_F^{m_{n-1}}, 1)$ where $m_1 \geq ... \geq m_{n-1} \geq 0$. Choose k such that $m_k \geq 1$ and $m_{k+1} = 0$. Suppose that $\mathrm{Hom}_{H \cap g^{-1}Hg}(\sigma, {}^{g}\sigma) \neq (0)$. Then we also have $\mathrm{Hom}_{N_k(\mathcal{O}) \cap g^{-1}N_k(\mathcal{O})g}(\sigma, {}^{g}\sigma) \neq (0)$ where N_k is the unipotent radical of the standard parabolic subgroup corresponding to the partition $n = k + (n-k)$. We have $N_k(\mathcal{O}) \cap gN_k(\mathcal{O})g^{-1} \subset N_k(\mathfrak{P})$ - the notations being obvious. Since σ is inflated from $GL_n(\mathbb{F}_q)$, ${}^g\sigma$ is trivial on $N_k(\mathcal{O}) \cap g^{-1}N_k(\mathcal{O})g$, i.e., $\mathrm{Hom}_{N_k(\mathcal{O})\cap g^{-1}N_k(\mathcal{O})g}(\sigma, \mathbf{1}) \neq (0)$. This contradicts the fact that (σ, W) is a cuspidal representation of $GL_n(\mathbb{F}_q)$. (We urge the reader to justify this last statement.)

Admissibility: We now prove that π is an admissible representation. It suffices to show that for $m \geq 1$, π^{K_m} is finite dimensional, where K_m is the principal congruence subgroup of level m.

Note that π^{K_m} consists of locally constant functions $f : G \to W$ such that

$$f(hgk) = \sigma(h)f(g), \qquad \forall h \in H, \forall g \in G, \forall k \in K_m.$$

Using the Cartan decomposition (see Theorem 3.3), we may choose representatives for the double cosets $H\backslash G/K_m$ which look like $g = a \cdot k$, where $a = \mathrm{diag}(\varpi_F^{m_1}, \cdots, \varpi_F^{m_n})$ with $0 = m_1 \leq m_2 \leq \cdots \leq m_n$, and $k \in K/K_m$. For such an a, let $t(a) = \max\{m_{i+1} - m_i : 1 \leq i \leq n-1\}$. Note that if $t(a) \geq m$, and if $t(a) = m_{j+1} - m_j$, then $U_j(\mathcal{O}) \subset gK_mg^{-1} \cap H$ where U_j is the unipotent radical of the maximal parabolic corresponding to the partition $n = j + (n-j)$. Therefore,

$$\sigma(u) \cdot f(g) = f(ug) = f(g \cdot g^{-1}ug) = f(g),$$

i.e., $f(g)$ is a vector in W which is fixed by $U_j(\mathcal{O})$ and hence $U_j(\mathbb{F}_q)$. Cuspidality of (σ, W) implies that $f(g) = 0$.

This shows that if $0 \neq f \in \pi^{K_m}$, then f can be supported only on double cosets $HakK_m$ with $t(a) < m$, which is a finite set, proving the finite dimensionality of π^{K_m}.

Supercuspidality: Let $P = M \cdot N$ be a parabolic subgroup of G. To prove that π is supercuspidal, we need to show that the Jacquet module $\pi_N = (0)$. It suffices to show that $(\pi_N)^* = (0)$. Note that $\mathrm{Hom}_N(\pi, \mathbb{C}) \cong (\pi^*)^N$ is the space of N-fixed vectors in the vector space dual of π. We will show that $(\pi^*)^N = 0$.

Since $\pi = \mathrm{ind}_H^G(\sigma)$, the dual vector space π^* may be identified with $C^\infty(H\backslash G, \sigma^*)$ which is the space of locally constant functions ϕ on G with values in W^* such that $\phi(hg) = \sigma^*(h)\phi(g)$. (We urge the reader to justify this identification; it boils down to saying that the dual of a direct sum of vector spaces is the direct product of vector spaces.) Hence $(\pi^*)^N$ consists of locally constant functions $\phi : G \to W^*$ such that

$$\phi(hgn) = \sigma^*(h)\phi(g), \qquad\qquad \forall h \in H, \forall g \in G, \forall n \in N.$$

Using Iwasawa decomposition, we may take representatives for double cosets $H\backslash G/N$ to lie in M. For $m \in M$, note that

$$N(\mathcal{O}) = N \cap H = H \cap mNm^{-1}.$$

Hence for all $h \in N(\mathcal{O})$, we have

$$\sigma^*(h)\phi(m) = \phi(hm) = \phi(m \cdot m^{-1}hm) = \phi(m),$$

i.e., $\phi(m) \in W^*$ is fixed by $N(\mathbb{F}_q)$. Cuspidality of (σ, W) implies cuspidality of (σ^*, W^*) which gives that $\phi(m) = 0$. Hence $(\pi^*)^N = (0)$. □

Exercise 6.4 *With notations as in Theorem 6.2 show that*

$$\mathrm{ind}_{F^*GL_n(\mathcal{O}_F)}^{GL_n(F)}(\chi \cdot \sigma) = \mathrm{Ind}_{F^*GL_n(\mathcal{O}_F)}^{GL_n(F)}(\chi \cdot \sigma).$$

We next have the following basic theorem which justifies the assertion made in the beginning of this section that supercuspidal representations are the building blocks of all irreducible representations. This theorem will be refined quite a bit in the section on Langlands classification.

Theorem 6.5 *Let π be an irreducible admissible representation of $G = GL_n(F)$. Then there exists a Levi subgroup M and a supercuspidal representation ρ of M such that π is a subrepresentation of the representation of G obtained from ρ by the process of parabolic induction.*

Proof : Let P be a parabolic subgroup of $GL_n(F)$ which is smallest for the property that the Jacquet functor with respect to P is non-zero. (So $P = G$ is a possibility which occurs if and only if the representation is supercuspidal.) If $P = MN$, it is clear from theorem 5.3 that any irreducible subquotient of π_N is a supercuspidal representation of M. Let ρ be an irreducible quotient of π_N as an M module. Since $\mathrm{Hom}_M(\pi_N, \rho) \neq (0)$, it follows from the Frobenius reciprocity that

$$\mathrm{Hom}_{GL_n(F)}(\pi, \mathrm{Ind}_P^{GL_n(F)}(\delta_P^{-1/2}\rho)) \neq (0),$$

which gives a realization of π inside a representation parabolically induced from a supercuspidal representation. $\square$

Given the previous theorem one might ask if given an irreducible π 'how many' induced representations can it occur in? The next theorem says that π occurs in essentially only one such induced representation. Such a uniqueness assertion then allows us to talk of the *supercuspidal support* of the given representation. This is the following theorem due to Bernstein and Zelevinsky. (see Theorem 2.9 of [3].) We need some notation to state the theorem.

If π is a representation of finite length then let $\mathrm{JH}(\pi)$ denote the set of equivalence classes of irreducible subquotients of π. Further, let $\mathrm{JH}^0(\pi)$ denote the set of all irreducible subquotients of π counted with multiplicity. So each ω in $\mathrm{JH}(\pi)$ is contained in $\mathrm{JH}^0(\pi)$ with some multiplicity.

Theorem 6.6 *Let $P = MN$ and $P' = M'N'$ be standard parabolic subgroups of $G = GL_n(F)$. Let σ (resp. σ') be an irreducible supercuspidal representation of M (resp. M'). Let $\pi = \mathrm{Ind}_P^G(\sigma)$ and $\pi' = \mathrm{Ind}_{P'}^G(\sigma')$. Then the following are equivalent.*

1. *There exists $w \in W$ such that $wMw^{-1} = M'$ and ${}^w\sigma = \sigma'$.*
2. $\mathrm{Hom}_G(\pi, \pi') \neq (0)$.
3. $\mathrm{JH}(\pi) \cap \mathrm{JH}(\pi')$ *is not empty.*
4. $\mathrm{JH}^0(\pi) = \mathrm{JH}^0(\pi')$.

7 Discrete series representations

Note that one part of Theorem 6.1 says that every matrix coefficient of an irreducible supercuspidal representation is compactly supported modulo the centre. In general analytic behaviour of matrix coefficients dictate properties of the representation. Now we consider a larger class of representations which are said to be in the discrete series for G. These have a characterization in terms of their matrix coefficients being square integrable modulo the centre.

We need a few definitions now. Albeit we have mentioned the notion of a unitary representation before (see Theorem 4.1) we now give a definition.

Definition 7.1 Let (π, V) be a smooth representation of G. We say π is a *unitary representation* of G if V has an inner product $\langle\ ,\ \rangle$ (also called an unitary structure) which is G invariant, i.e., $\langle \pi(g)v, \pi(g)w\rangle = \langle v, w\rangle$ for all $g \in G$ and all $v, w \in V$. In general, V equipped with $\langle\ ,\ \rangle$ need only be a pre-Hilbert space.

Definition 7.2 Let (π, V) be a smooth irreducible representation of G. We say π is *essentially square integrable* if there is a character $\chi : F^* \to \mathbb{R}_{>0}$ such that $|f_{v,v^\vee}(g)|^2\chi(\det g)$ is a function on $Z\backslash G$ for every matrix coefficient $f_{v,v^\vee}$ of π, and

$$\int_{Z\backslash G} |f_{v,v^\vee}(g)|^2\chi(\det g)\, dg < \infty.$$

If χ can be taken to be trivial then π is said to be a *square integrable* representation and in this case it is said to be in the *discrete series* for G.

Exercise 7.3

1. Let ν be a unitary character of F^*. Let $L^2(Z\backslash G, \nu)$ denote the space of measurable functions $f : G \to \mathbb{C}$ such that $f(zg) = \nu(z)f(g)$ for all $z \in Z$ and $g \in G$ and

$$\int_{Z\backslash G} |f(g)|^2\, dg < \infty.$$

   Show that $L^2(Z\backslash G, \nu)$ is a unitary representation of G where G acts on these functions by right shifts.

2. Let (π, V) be a discrete series representation. Check that its central character ω_π is a unitary character. Let $0 \neq l \in V^\vee$. Show that the map $v \mapsto f_{v,l}$ is a non-zero G equivariant map from π into $L^2(Z\backslash G, \omega_\pi)$. Hence deduce that π is a unitary representation which embeds as a subrepresentation of $L^2(Z\backslash G, \omega_\pi)$.

The classification of discrete series representations is due to Bernstein and Zelevinsky. We summarize the results below. To begin with we need a result on when a representation parabolically induced from a supercuspidal representation is reducible. (See Theorem 4.2 of [3].)

Theorem 7.4 *Let $P = P(n_1, ..., n_k) = MN$ be a standard parabolic subgroup of G. Let $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_k$ be an irreducible representation of M with every σ_i an irreducible supercuspidal representation of $GL_{n_i}(F)$. The parabolically induced representation $\mathrm{Ind}_P^G(\sigma)$ is reducible if and only if there exist $1 \leq i, j \leq k$ with $i \neq j$, $n_i = n_j$ and $\sigma_i \simeq \sigma_j(1) = \sigma_j|\cdot|_F$.*

We now consider some very specific reducible parabolically induced representations. Let $n = ab$ and let σ be an irreducible supercuspidal representation of $GL_a(F)$. Let $P = MN$ be the standard parabolic subgroup corresponding to $n = a + a + ... + a$ the sum taken b times, so $M = GL_a(F) \times ... \times GL_a(F)$ the product taken b times. Let Δ denote the *segment*

$$\Delta = (\sigma, \sigma(1), ..., \sigma(b-1))$$

thought of as the representation $\sigma \otimes \sigma(1) \otimes ... \otimes \sigma(b-1)$ of M. Let $\mathrm{Ind}_P^G(\Delta)$ denote the corresponding parabolically induced representation of G. By Theorem 7.4 this representation is reducible. We are now in a position to state the following theorem. (This theorem, as recorded in Theorem 9.3 of Zelevinsky's paper [16], is normally attributed to Bernstein.)

Theorem 7.5 *With notations as above, we have :*

1. *For any segment Δ the induced representation $\mathrm{Ind}_P^G(\Delta)$ has a unique irreducible quotient. This irreducible quotient will be denoted $Q(\Delta)$.*

2. *For any segment Δ the representation $Q(\Delta)$ is an essentially square integrable representation. Every essentially square integrable representation of G is equivalent to some $Q(\Delta)$ for a uniquely determined Δ, i.e., for a uniquely determined a, b and σ.*

3. *The representation $Q(\Delta)$ is square integrable if and only if it is unitary and also if and only if $\sigma((b-1)/2)$ is unitary.*

Example 7.6 (Steinberg Representation) One important example of a discrete series representation is obtained as follows. Take $a = 1$ and $b = n$ with the notations as above. Hence the parabolic subgroup P is just the standard Borel subgroup B. Let $\sigma = |\cdot|_F^{(1-n)/2}$ as a (supercuspidal) representation of $GL_a(F) = F^*$. Then the representation $\mathrm{Ind}_P^G(\Delta)$ is just the regular representation of G on smooth functions on $B\backslash G$. The corresponding $Q(\Delta)$ is called the *Steinberg representation* of G. We will denote the Steinberg representation for $GL_n(F)$ by St_n. It is a square integrable representation with trivial central character. It can also be defined as the alternating sum

$$\mathrm{St}_n = \sum_{B \subset P} (-1)^{\mathrm{rank}(P)} C_c^\infty(P\backslash G),$$

where B is a fixed Borel subgroup of G, P denotes a parabolic subgroup of G containing B, and $\text{rank}(P)$ denotes the rank of $[M, M]$ where M is a Levi subgroup of P.

The essentially square integrable representations $Q(\Delta)$ are also called *generalized Steinberg representations.*

Exercise 7.7 Show that the Jacquet module with respect to B of the Steinberg representation of $GL_n(F)$ is one dimensional. What is the character of T on this one dimensional space?

Exercise 7.8 Show that the trivial representation of $GL_n(F)$ is not essentially square integrable.

8 Langlands classification

In this section we state the Langlands classification for all irreducible admissible representations of $GL_n(F)$. The supercuspidal repsentations were introduced in § 6. Then we used supercuspidal representations to construct all the essentially square integrable representations (the $Q(\Delta)$'s) in § 7. Not every representation is accounted for by this construction as for instance the trivial representation is not in this set (unless $n = 1$!). The Langlands classification builds every irreducible representation starting from the essentially square integrable ones.

The theorem stated below is due to Zelevinsky. (See Theorem 6.1 of [16].) Before we can state the theorem we need the notion of when two segments are linked. Let $\Delta = (\sigma, ..., \sigma(b-1))$ and $\Delta' = (\sigma', ..., \sigma'(b'-1))$ be two segments where σ (resp. σ') is an irreducible supercuspidal representation of $GL_a(F)$ (resp. $GL_{a'}(F)$). We say Δ and Δ' are *linked* if neither of them is included in the other and their union is a segment (so in particular $a = a'$). We say that Δ *precedes* Δ' if they are linked and there is a positive integer r such that $\sigma' = \sigma(r)$.

Theorem 8.1 (Langlands classification) *For $1 \leq i \leq k$, let Δ_i be a segment for $GL_{n_i}(F)$. Assume that for $i < j$, Δ_i does not precede Δ_j. Let $n = n_1 + ... + n_k$. Let $G = GL_n(F)$ and let P be the parabolic subgroup of G corresponding to this partition. Then :*

1. *The parabolically induced representation* $\mathrm{Ind}_P^G(Q(\Delta_1)\otimes\cdots\otimes Q(\Delta_k))$ *admits a unique irreducible quotient which will be denoted* $Q(\Delta_1,...,\Delta_k)$.

2. *Any irreducible representation of* G *is equivalent to some* $Q(\Delta_1,...,\Delta_k)$ *as above.*

3. *If* $\Delta'_1,...,\Delta'_{k'}$ *is another set of segments satisfying the hypothesis then we have* $Q(\Delta_1,...,\Delta_k)\simeq Q(\Delta'_1,...,\Delta'_{k'})$ *if and only if* $k=k'$ *and* $\Delta_i=\Delta'_{s(i)}$ *for some permutation* s *of* $\{1,...,k\}$.

Exercise 8.2 Give a realization of the trivial representation of G as a representation $Q(\Delta_1,...,\Delta_k)$ in the Langlands classification.

Exercise 8.3 (Representations of $GL_2(F)$) In this exercise all the irreducible admissible representations of $GL_2(F)$ are classified. Let $G=GL_2(F)$.

1. Let χ be a character of F^*. Then $g\mapsto\chi(\det(g))$ gives a character of G. Show that these are all the finite dimensional irreducible admissible representations of G.

2. Let χ be a character of G as above. Let St_2 be the Steinberg representation of G and let $\mathrm{St}_2(\chi)=\mathrm{St}_2\otimes\chi$ be the twist by χ. Show that any principal series representation is either irreducible or is reducible in which case, up to semi-simplification looks like $\chi\oplus\mathrm{St}_2(\chi)$ for some character χ.

3. Use the Langlands classification and make a list of all irreducible admissible representations of G. Show that any irreducible admissble representation is either one dimensional or is infinite dimensional and in which case it is either supercuspidal, or of the form $\mathrm{St}_2(\chi)$ or is an irreducible principal series representation $\pi(\chi_1,\chi_2)$.

9 Certain classes of representations

In this section we examine certain important classes of representations of $G=GL_n(F)$. These notions are especially important in the theory of automorphic forms. (For readers familiar with such terms, local components of global representations tend to have such properties.)

9.1 Generic representations

Recall our notation that U is the unipotent radical of the standard Borel subgroup $B = TU$ where T is the diagonal torus. (So U is the subgroup of all upper triangular unipotent matrices.) Since T normalizes U we get an action of T on the space $\hat{U}$ consisting of all characters of U. A maximal orbit of T on $\hat{U}$ is called a *generic orbit* and any character of U in a generic orbit is called a *generic character* of U. It is easily seen that a character of U is generic if and only if its stabilizer in T is the centre Z.

We can construct (generic) characters of U as follows. Let $u = u(i, j)$ be any element in U. Let ψ_F be a non-trivial additive character of U. Let $a = (a_1, ..., a_{n-1})$ be any $(n-1)$ tuple of elements of F. We get a character of U by

$$u \mapsto \psi_a(u) = \psi_F(a_1u(1, 2) + a_2u(2, 3) + ... + a_{n-1}u(n-1, n)).$$

The reader is urged to check that every character of U is one such ψ_a. Further it is easily seen that ψ_a is generic if and only if each a_i is non-zero.

We will now fix one generic character of U, denoted Ψ. Fix for once and for all a non-trivial additive character ψ_F of F such that the maximal fractional ideal of F on which ψ_F is trivial is $\mathcal{O}_F$. We fix the following generic character of U given by:

$$u \mapsto \Psi(u) = \psi_F(u(1, 2) + ... + u(n-1, n)).$$

Now consider the representation $\mathrm{Ind}_U^G(\Psi)$ which is the induction from U to G of the character Ψ. Note that the representation space of $\mathrm{Ind}_U^G(\Psi)$ consists of all smooth functions f on G such that $f(ng) = \Psi(n)f(g)$ for all $n \in U$ and $g \in G$.

Definition 9.1 An irreducible admissible representation (π, V) is said to be *generic* if $\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\Psi))$ is non-zero. If π is generic then using Frobenius reciprocity we get that there is a non-zero linear functional $\ell : V \to \mathbb{C}$ such that $\ell(\pi(n)v) = \Psi(n)\ell(v)$ for all $v \in V$ and $n \in U$. Such a linear functional is called a *Whittaker functional* for π. If π is generic then the representation space V may be realized on a certain space of functions f with the property that $f(ng) = \Psi(n)f(g)$ for all $n \in U$ and $g \in G$ and the action of G on V is by right shifts on this realization. Such a realization is called a *Whittaker model* for π.

One of the main results on generic representations is the following theorem (due to Shalika [14]) which says that Whittaker models are unique when they exist. This theorem is important for the global theory, for instance, to prove multiplicity one for automorphic forms on $GL(n)$. Also, one way to attach local Euler factors for generic representations of G is via Whittaker models.

Theorem 9.2 (Multiplicity one for Whittaker models) *Let (π, V) be an irreducible admissible representation of $GL_n(F)$. Then the dimension of the space of Whittaker functionals is at most one, i.e.,*

$$\dim_{\mathbb{C}}(\mathrm{Hom}_G(\pi, \mathrm{Ind}_U^G(\Psi))) \leq 1.$$

Put differently, if π admits a Whittaker model then it admits a unique one.

The question now arises as to which representations are actually generic. The first theorem in this direction was due to Gelfand and Kazhdan which says that every irreducible admissible supercuspidal representation is generic. The classification of all generic representations of $GL_n(F)$ is a theorem due to Zelevinsky (see Theorem 9.7 of [16]) which states that π is generic if and only if it is irreducibly induced from essentially square integrable representations. In particular one has that discrete series representations are always generic. We would like to point out that this phenomenon is peculiar to $GL_n(F)$ and is false for general reductive p-adic groups.

Theorem 9.3 (Generic representations) *Let $\pi = Q(\Delta_1, ..., \Delta_k)$ be an irreducible admissible representation of $GL_n(F)$. Then π is generic if and only if no two of the Δ_i are linked in which case $\mathrm{Ind}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_k))$ is an irreducible representation, and*

$$\pi == Q(\Delta_1, ..., \Delta_k) = \mathrm{Ind}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_k)).$$

Exercise 9.4

1. Use the definition of genericity to show that one dimensional representations are never generic. Now verify the above theorem for one dimensional representations.

2. Let $G = GL_2(F)$. Show that an irreducible admissible representation is generic if and only if it is infinite dimensional.

9.2 Tempered representations

The notion of a tempered representation is important in the global theory of automorphic forms. The importance is evidenced by what is called the generalized Ramanujan conjecture which says that every local component of a cuspidal automorphic representation of $GL(n)$ is tempered. In this section we give the definition of temperedness and state a thereom due to Jacquet [12] which says when a representation is tempered. (It is closely related to Zelevinsky's theorem in the previous section.) It says that a representation with unitary central character is tempered if and only if it is irreducibly induced from a discrete series representation (as against essentially square integrable representations to get the generic ones).

Definition 9.5 An irreducible admissible representation (π, V) is *tempered* if ω_π the central character of π is unitary and if one (and equivalently every) matrix coefficient $f_{v,v^\vee}$ is in $L^{2+\epsilon}(Z\backslash G)$ for every $\epsilon > 0$.

Every unitary supercuspidal representation of G is tempered, since matrix coefficients of supercuspidals are compactly supported modulo centre and hence in $L^{2+\epsilon}(Z\backslash G)$. One may drop the unitarity and rephrase this as every supercuspidal being essentially tempered with an obvious meaning given to the latter.

Theorem 9.6 (Tempered representations) *Let $\pi = Q(\Delta_1, ..., \Delta_k)$ be an irreducible admissible representation of $GL_n(F)$. Then π is tempered if and only if one actually has*

$$\pi = \mathrm{Ind}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_k))$$

with every $Q(\Delta_i)$ being square-integrable.

From Theorem 9.3 and Theorem 9.6 we get that every tempered representation of $GL_n(F)$ is generic. We emphasize this point in light of the global context. A theorem of Shalika [14] says that every local component of a cuspidal automorphic representation of GL_n is generic. The generalized Ramanujan conjecture for GL_n says that every such local component is tempered. The reader is urged to construct examples of generic representations of $GL_n(F)$ which are not tempered.

9.3 Unramified or spherical representations

The notion of unramified representations is again of central importance in the global theory as almost all local components of global representations are unramified. An unramified representation is a GL_n analogue of an unramified character of F^* which is just a character trivial on the units $\mathcal{O}_F^\times$. Such a character is called unramified because every character corresponding to an unramified extension via local class field theory is unramified (the norm map being surjective on the units).

Recall that $K = GL_n(\mathcal{O}_F)$ is the (up to conjugacy) unique maximal compact subgroup of $G = GL_n(F)$. We let $\mathcal{H}_K$ denote the *spherical Hecke algebra* of G which is the space of compactly supported bi-K-invariant functions on G. This is an algebra under convolution

$$(f * g)(x) = \int_G f(xy^{-1})g(y)\, dy$$

where dy is a Haar measure on G normalized such that $\mathrm{vol}(K) = 1$. The identity element in $\mathcal{H}_K$ is just the characteristic function of K.

Exercise 9.7 (Gelfand) Show that $\mathcal{H}_K$ is a commutative algebra. Consider the transpose map on G to show that it induces a map on $\mathcal{H}_K$ which is both an involution and an anti-involution. (*Hint:* Use Thereom 3.3.)

Actually more is known about this Hecke algebra and we state the following theorem of Satake regarding the structure of $\mathcal{H}_K$.

Theorem 9.8 (Satake Isomorphism) *The spherical Hecke algebra $\mathcal{H}_K$ is canonically isomorphic to the Weyl group invariants of the space of Laurent polynomials in n variables, i.e.,*

$$\mathcal{H}_K \simeq \mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]^W.$$

Definition 9.9 Let (π, V) be an irreducible admissible representation of G. It is said to be *unramified* if it has a non-zero vector fixed by K. The space of K fixed vectors denoted V^K is a module for the spherical Hecke algebra $\mathcal{H}_K$.

Let (π, V) be an irreducible unramified representation of G. We know that V^K is a module for $\mathcal{H}_K$. We can use Proposition 2.8 and the fact that

$\mathcal{H}_K$ is commutative to get that the space of fixed vectors is actually one dimensional and hence this gives a character of the spherical Hecke algebra which is called the *spherical character* associated to π. Using this proposition again gives that the spherical character uniquely determines the unramified representation.

We now construct all the unramified representations of G. Let $\chi_1, ..., \chi_n$ be n unramified characters of F^* (i.e., they are all trivial on the units of F). Let $\pi = \pi(\chi_1, ..., \chi_n)$ be the corresponding principal series representation of $GL_n(F)$. It is an easy exercise using the Iwasawa decomposition to see that π admits a non-zero vector fixed by K which is unique up to scalars. Hence π admits a unique subquotient which is unramified. It turns out that if the characters χ_i are ordered to satisfy the 'does not precede' condition of Theorem 8.1 then the unique irreducible quotient $Q(\chi_1, ..., \chi_n)$ of π is actually unramified and so is the unique unramified subquotient of π.

We now state the main theorem classifying all the unramified representations of $GL_n(F)$.

Theorem 9.10 (Spherical representations) *Let $\chi_1, ..., \chi_n$ denote n unramified characters of F^* which are ordered to satisfy the 'does not precede' condition of Theorem 8.1. Then the representation $Q(\chi_1, ..., \chi_n)$ is an unramified representation of $GL_n(F)$. Further, every irreducible admissible unramified representation is equivalent to such a $Q(\chi_1, ..., \chi_n)$.*

Proof : We briefly sketch the proof of the second assertion in the theorem. We show that the $Q(\chi_1, ..., \chi_n)$ exhaust all the unramified representations of G. This is done by appealing to Proposition 2.8 and the Satake isomorphism. The proof boils down to showing that every character of the spherical Hecke algebra is accounted for by one of the characters coming from a $Q(\chi_1, ..., \chi_n)$ and so by appealing to Statement (3) of Proposition 2.8 we would be done.

Now via the Satake isomorphism, a character of $\mathcal{H}_K$ determines and is determined by a set of n non-zero complex numbers which are the values taken by the character on the variables $t_1, ..., t_n$. Now it is easy to check that the spherical character of $Q(\chi_1, ..., \chi_n)$ (or $\pi(\chi_1, ..., \chi_n)$ which is more easier to work with) takes the value $\chi_i(\varpi_F)$ on the variable t_i. The remark that an unramified character is completely determined by its value on a uniformizer which can be any non-zero complex number finishes the proof. $\square$

Example 9.11 Let $\pi = Q(\chi_1, \chi_2)$ be an irreducible admissible unitary unramified generic representation of $GL_2(F)$. Then for $i = 1, 2$ it can be shown that

$$q^{-1/2} \leq |\chi_i(\varpi_F)| \leq q^{1/2}.$$

In particular, there are such representations which are not tempered. The Ramanujan conjecture for $GL_2(F)$ would assert that if π is a local component of a cuspidal automorphic representation of GL_2 then π is tempered. Note that almost all such local components are unitary unramified and generic and the Ramanujan conjecture would boil down to showing that $|\chi_i(\varpi_F)| = 1$. See the discussion on pp.332-334 of [1] for related matters and in particular as to how the Langlands program implies the Ramanujan conjecture.

9.4 Iwahori spherical representations

This is a class of representations which may not have an immediate motivation from the global theory but is important all the same for various local reasons. (Although justifying this claim may take us outside the scope of these notes as it requires more knowledge of the structure theory of p-adic groups.)

The *Iwahori subgroup* I of $G = GL_n(F)$ that we will be looking at is the inverse image of the standard Borel subgroup under the canonical map from $GL_n(\mathcal{O}_F) \to GL_n(\mathbb{F}_q)$. By definition, I is a subgroup of K.

We will be looking at those representations of G which have vectors fixed by I. For this reason, as before, we need to consider the *Iwahori Hecke algebra* $\mathcal{H}_I$ which is the space of compactly supported bi-I-invariant functions on G with the algebra structure being given by convolution.

Definition 9.12 An irreducible admissible representation (π, V) of G is called *Iwahori spherical* if it has a non-zero vector fixed by the Iwahori subgroup I. In this case the space of fixed vectors V^I is a simple module denoted (π^I, V^I) for the Iwahori Hecke algebra $\mathcal{H}_I$.

To begin with, unlike the spherical case, the Iwahori Hecke algebra is not commutative and admits simple modules which are not one dimensional. The structure of $\mathcal{H}_I$ is also a little complicated to describe and would take us outside the scope of these notes. (See § 3 of Iwahori-Matsumoto [11] for the original description of this algebra. See also § 3 of Borel [4] for a more pleasanter-to-read version of the structure of $\mathcal{H}_I$.) We record the following rather special lemma on representations of the Iwahori Hecke algebra. See Proposition 3.6 of [4], although Borel deals only with semi-simple groups the proof goes through mutatis mutandis to our case. We need some notation. For any $g \in G$, let e_{IgI} be the characteristic function of IgI as a subset of G. Clearly $e_{IgI} \in \mathcal{H}_I$. If we normalize the Haar measure on G such that $\mathrm{vol}(I) = 1$, then e_I is the identity of element of $\mathcal{H}_I$.

Lemma 9.13 *Let (σ, W) be a finite dimensional representation of $\mathcal{H}_I$. Then for any $g \in G$ the endomorphism $\sigma(e_{IgI})$ is invertible.*

The main fact about Iwahori spherical representations is the following theorem due to Borel and Casselman. (See Theorem 3.3.3. of [7] and a strengthening of it in Lemma 4.7 of [4].) Recall our notation that $B = TU$ is the standard Borel subgroup of G. The Iwahori subgroup I admits an 'Iwahori-factorization' with respect to B as

$$I = I^- I^0 I^+$$

where $I^- = I \cap U^-$, $I^0 = I \cap T$ which is the maximal compact subgroup of T and $I^+ = I \cap U$.

Theorem 9.14 *Let (π, V) be an admissible representation of G. Then the canonical projection from V to the Jacquet module V_U induces an isomorphism from V^I onto $(V_U)^{I^0}$.*

Proof : Let $A : V \to V_U$ be the canonical projection map. By the proof of statement (5) of Theorem 5.3 we have that $A(V^I) = (V_U)^{I^0}$. We now have to show that A is injective on V^I.

Suppose $v \in V^I$ is such that $A(v) = 0$. Then $v \in V^I \cap V(U)$. Choose a compact open subgroup U_1 of U such that $v \in V(U_1)$ which gives that $\pi(e_{U_1})v = 0$. (For any compact subset C of G we let $\pi(e_C)v$ stand for $\int_C \pi(g)v\,dg$.)

Choose $r \gg 0$ such that $\mu^{-r} U_1 \mu^r \subset I^+$ where $\mu = \mathrm{diag}(1, \varpi_F, ..., \varpi_F^{n-1})$. We then get that $\pi(e_{I^+})(\pi(\mu^{-r})v) = 0$. The Iwahori factorization gives that $\pi(e_I)\pi(\mu^{-r})v = 0$.

What we have shown is that there exists a $g \in G$ such that $\pi(e_I)\pi(g)v = 0$. Since $v \in V^I$ this implies that $\pi(e_{IgI})v = 0$. Now appealing to Lemma 9.13 we get that $v = 0$. $\square$

Theorem 9.15 (Iwahori spherical representations) *Let $\chi_1, ..., \chi_n$ be unramified characters of F^*. Let $\pi(\chi) = \pi(\chi_1, ..., \chi_n)$ be the corresponding principal series representation. Then any irreducible subquotient of $\pi(\chi)$ is Iwahori spherical and every Iwahori spherical representation arises in this manner.*

Proof : Let π be any irreducible subquotient of $\pi(\chi)$. We know from Theorem 5.9 that the Jacquet module π_U is non-zero. We also know from the computation of the Jacquet module of $\pi(\chi)$ in Theorem 5.4 that I^0 acts trivially on $\pi(\chi)_U$ since every χ_i is unramified and hence I^0 acts trivially on π_U, i.e., $(\pi_U)^{I^0} \neq (0)$. Using Theorem 9.14 we get therefore that $\pi^I \neq (0)$, i.e., π is Iwahori spherical.

Now let π be any Iwahori spherical representation of G. We know that $\pi_U^{I^0} \neq (0)$ by the above theorem and hence there is an unramified character χ of T such that $\mathrm{Hom}_T(\pi_U, \chi) \neq (0)$. Appealing to Frobenius reciprocity finishes the proof. $\square$

Corollary 9.16 *Let (π, V) be any irreducible admissible representation of $G = GL_n(F)$. Then the dimension of space of fixed vectors under I is bounded above by the order of the Weyl group W, i.e.,*

$$\dim_{\mathbb{C}}(V^I) \leq n!$$

Equivalently, the dimension of any simple module for the Iwahori Hecke algebra $\mathcal{H}_I$ is bounded above by $n!$.

Proof : The corollary would follow if we show that the space of I fixed vectors of an uramified principal series $\pi(\chi)$ is exactly $n!$ which is the order of the Weyl group. This follows from Theorem 9.14 and Theorem 5.4. $\square$

10 Representations of local Galois Groups

The Galois group $G_F = \mathrm{Gal}(\bar{F}/F)$ has distinguished normal subgroups I_F, the Inertia subgroup, and P_F the wild Inertia subgroup which is contained in I_F. The Inertia subgroup sits in the following exact sequence,

$$1 \to I_F \to \mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \to 1.$$

Here the mapping from $\mathrm{Gal}(\bar{F}/F)$ to $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is given by the natural action of the Galois group of a local field on its residue field.

The Inertia group can be thought of as the Galois group of $\bar{F}$ over the maximal unramified extension F^{un} of F. Let $F^t = \cup_{(d,q)=1} F^{un}(\varpi_F^{1/d})$. The field F^t is known to be the *maximal tamely ramified extension* of F^{un} (An extension is called tamely ramified if the index of ramification is coprime to the characteristic of the residue field.) We have,

$$\begin{aligned} \mathrm{Gal}(\bar{F}/F)/I_F &\simeq \hat{\mathbb{Z}} \\ I_F/P_F &\simeq \prod_{\ell \neq p} \mathbb{Z}_\ell^\times. \end{aligned}$$

One defines the Weil group W_F of F to be the subgroup of $\mathrm{Gal}(\bar{F}/F)$ whose image inside $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is an integral power of the Frobenius automorphism $x \to x^q$. Representations of the Weil group which are continuous on the inertia subgroup are exactly those representations of the Weil group for which the image of the Inertia subgroup is finite.

The Weil group is a dense subgroup of the Galois group hence an irreducible representation of the Galois group defines an irreducible representation of the Weil group. It is easy to see that a representation of the Weil group can, after twisting by a character, be extended to a representation of the Galois group.

Local class field theory implies that the maximal abelian quotient of the Weil group of F is naturally isomorphic to F^*, and hence 1 dimensional representations of W_F are in bijective correspondence with characters of $GL_1(F) = F^*$. It is this statement of abelian class field theory which is generalized by the local Langlands correspondence. However, there is a slight amount of change one needs to make, and instead of taking the Weil group, one needs to take what is called the Weil-Deligne group whose representations are the same as representations of W_F on a vector space V together with a nilpotent endomorphism N such that

$$wNw^{-1} = |w|N$$

where $|w| = q^{-i}$ if the image of w in $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is the i-th power of the Frobenius.

One can identify representations of the Weil-Deligne group to representations of $W_F \times SL_2(\mathbb{C})$ via the Jacobson-Morozov theorem. It is usually much easier to work with $W_F \times SL_2(\mathbb{C})$ but the formulation with the nilpotent operators appears more naturally in considerations of ℓ-adic cohomology of

Shimura varieties where the nilpotent operator appears as the 'monodromy' operator.

We end with the following important proposition for which we first note that any field extension K of degree n of a local field F gives rise to an inclusion of the Weil group W_K inside W_F as a subgroup of index n. Since characters of W_K are, by local class field theory, identified to characters of K^*, a character of K^* gives by induction a representation of W_F of dimension n.

Proposition 10.1 *If $(n, q) = 1$, then any irreducible representation of W_F of dimension n is induced from a character χ of K^* for a field extension K of degree n.*

Proof : Since irreducible representations of the Weil group and Galois groups are the same, perhaps after a twist, we will instead work with the Galois group. Let ρ be an irreducible representation of the Galois group $\mathrm{Gal}(\bar{F}/F)$ of dimension $n > 1$. We will prove that there exists an extension L of F of degree greater than 1, and an irreducible representation of $\mathrm{Gal}(\bar{F}/L)$ which induces to ρ. This will complete the proof of the proposition by induction on n.

The proof will be by contradiction. We will assume that ρ is not induced from any proper subgroup.

We recall that there is a filtration on the Galois group $P_F \subset I_F \subset G_F = \mathrm{Gal}(\bar{F}/F)$. Since P_F is a pro-p group, all its irreducible representations of dimension greater than 1 are powers of p. Since n is prime to p, this implies that there is a 1 dimensional representation of P_F which appears in ρ restricted to P_F. Since P_F is a normal subgroup of the Galois group, it implies that the restriction of ρ to P_F is a sum of characters. All the characters must be the same by Clifford theory. (Otherwise, the representation ρ is induced from the stabilizer of any χ-isotypical component.) So, under the hypothesis that ρ is not induced from any proper subgroup, P_F acts via scalars on ρ. Since the exact sequence,

$$1 \to P_F \to I_F \to I_F/P_F \to 1$$

is a split exact sequence, let M be a subgroup in I_F which goes isomorphically to I_F/P_F. Take a character of the abelian group M appearing in ρ. As P_F operates via scalars, the corresponding 1 dimensional space is invariant under I_F. It follows that ρ restricted to I_F contains a character. Again I_F being normal, ρ restricted to I is a sum of characters which must be all the same under the assumption that ρ is not induced from any proper subgroup. Since G/I_F is pro-cyclic, this is not possible. □

11 The local Langlands conjecture for GL_n

It is part of abelian classfield theory that for a local field F, the characters of F^* can be identified to the characters of the Weil group W_F of F. Langlands visualised a vast generalisation of this in the late 60's to non-abelian representations of W_F which is now a theorem due to M.Harris and R.Taylor, and another proof was supplied shortly thereafter by G.Henniart.

The general conjecture of Langlands uses a slight variation of the Weil group, called the Deligne-Weil group, denoted W'_F and defined to be $W'_F = W_F \times SL_2(\mathbb{C})$.

Theorem 11.1 (Local Langlands Conjecture) *There exists a natural bijective correspondence between irreducible admissible representations of $GL_n(F)$ and n-dimensional representations of the Weil-Deligne group W'_F of F which are semi-simple when restricted to W_F and algebraic when restricted to $SL_2(\mathbb{C})$. The correspondence reduces to classfield theory for $n = 1$, and is equivariant under twisting and taking duals.*

The correspondence in the conjecture is called the local Langlands correspondence, and the n-dimensional representation of W'_F associated to a representation π of $GL_n(F)$ is called the Langlands parameter of π. The Langlands correspondence is supposed to be natural in the sense that there are L-functions and ϵ-factors attached to pairs of representations of W'_F, and also to pairs of representations of $GL_n(F)$, characterising these representations, and the correspondence is supposed to be the unique correspondence preserving these. We will not define L-functions and ϵ-factors here, but refer the reader to the excellent survey article of Kudla [12] on the subject.

The results of Bernstein-Zelevinsky reviewed in section 7 and 8 reduce the Langlands correspondence between irreducible representations of $GL_n(F)$ and representations of W'_F to one between irreducible supercuspidal representations of $GL_n(F)$ and irreducible representations of W_F, as can be seen as follows.

An n-dimensional representation σ of the Weil-Deligne group W'_F of F which is semi-simple when restricted to W_F and algebraic when restricted to

$SL_2(\mathbb{C})$ is of the form

$$\sigma = \sum_{i=1}^{i=r} \sigma_i \otimes Sp(m_i)$$

where σ_i are irreducible representations of W_F of dimension say n_i, and $Sp(m_i)$ is the unique m_i dimensional irreducible representation of $SL_2(\mathbb{C})$. Assuming the Langlands correspondence between irreducible supercuspidal representations of $GL_n(F)$ and irreducible representations of W_F, we have representations π_i of $GL_{n_i}(F)$ naturally associated to the representations σ_i of dimension n_i of W_F. To the product $m_i n_i$, and a supercuspidal representation π_i of $GL_{n_i}(F)$, Theorem 7.5 associates an essentially square integrable representation to the segment $(\pi_i(-\frac{m_i-1}{2}), \cdots, \pi_i(\frac{m_i+1}{2}))$, which we denote by $St_{m_i}(\pi_i)$. Now order the representations so that for $i < j$, the segment $(\pi_i(-\frac{m_i-1}{2}), \cdots, \pi_i(\frac{m_i+1}{2}))$, does not precede $(\pi_j(-\frac{m_j-1}{2}), \cdots, \pi_j(\frac{m_j+1}{2}))$. By Theorem 8.1, the parobolically induced representation

$$\mathrm{Ind}_P^G(St_{m_1}(\pi_1) \otimes \cdots \otimes St_{m_r}(\pi_r)),$$

admits a unique irreducible quotient which is the representation of $GL_n(F)$ associated by the Langlands correspondence to $\sigma = \sum_{i=1}^{i=r} \sigma_i \otimes Sp(m_i)$.

Example 11.2 The irreducible admissible representations π of GL_2 together with the associated representation σ_π of the Weil-Deligne group is as follows.

1. Principal series representation, induced from a pair of characters (χ_1, χ_2) of F^*. These representations are irreducible if and only if $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm 1}$. For irreducible principal series representation, the associated Langlands parameter is $\chi_1 \oplus \chi_2$, where χ_1, χ_2 are now being treated as characters of W_F.

2. Twists of Steinberg. The Langlands parameter of the Steinberg is the standard 2 dimensional representation of $SL_2(\mathbb{C})$.

3. Twists of the trivial representation. The Langlands parameter of the trivial representation is $|\cdot|^{1/2} \oplus |\cdot|^{-1/2}$.

4. The rest, which is exactly the set of supercuspidal representations of $GL_2(F)$. In odd residue characteristic, these representations can be constructed by what is called the Weil representation, and the representation of $GL_2(F)$ so constructed are parametrized by characters of

the invertible elements of quadratic field extensions of F. By proposition 10.1, every 2 dimensional representation of W_F, when the residue characteristic is odd, is also given by induction of such a character on a quadratic extension which is the Langlands parameter.

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