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ON THE DECOMPOSITION OF A REPRESENTATION OF SO_n WHEN RESTRICTED TO SO_{n-1}

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0. **Introduction.** Let k be a local field, with $\mathrm{char}(k) \neq 2$. A quadratic space V over k is a finite dimensional vector space together with a non-degenerate quadratic form $Q: V \longrightarrow k$. The special orthogonal group $SO(V)$ consists of all linear maps $T: V \longrightarrow V$ which satisfy:

$$Q(Tv) = Q(v) \text{ for all } v \text{ and } \det T = 1.$$

Assume that $\dim V \geq 2$, and let v be a vector with $Q(v) \neq 0$. The orthogonal complement $W = \langle v \rangle^{\perp}$ is a quadratic space over k, and $SO(W)$ is the subgroup of $SO(V)$ which fixes the vector v. In this paper, we study the restriction of irreducible, admissible complex representations of the locally compact group $SO(V)(k)$ to the closed subgroup $SO(W)(k)$.

It is convenient to formulate this problem as follows. Let $\pi = \pi_1 \otimes \pi_2$ be an irreducible representation of the product group $G = SO(V)(k) \times SO(W)(k)$, where π_1 is an irreducible

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representation of $SO(V)(k)$ and π_2 is an irreducible representation of $SO(W)(k)$. Let $\pi_2^\vee$ be the contragredient of π_2, which is the representation on the space of smooth vectors in the algebraic dual space $\mathrm{Hom}(\pi_2, \mathbf{C})$. The group $H = SO(W)(k)$ embeds as a subgroup of $SO(V)(k)$, and hence embeds *diagonally* as a subgroup of G. There is a canonical isomorphism of complex vector spaces:

$$\mathrm{Hom}_H(\pi, \mathbf{C}) = \mathrm{Hom}_H(\pi_1, \pi_2^\vee). \tag{0.1}$$

We say that $\pi_2^\vee$ appears with multiplicity $\dim \mathrm{Hom}_H(\pi_1, \pi_2^\vee) = \dim \mathrm{Hom}_H(\pi, \mathbf{C})$ in the restriction of π_1 to H.

Our problem is therefore reduced to computing the dimension of $\mathrm{Hom}_H(\pi, \mathbf{C})$, for any irreducible representation $\pi = \pi_1 \otimes \pi_2$ of G. I. Piatetski-Shapiro and S. Rallis, following ideas of J. Bernstein, have recently shown that the vector space $\mathrm{Hom}_H(\pi, \mathbf{C})$ has dimension ≤ 1, so the problem is to identify those irreducible representations π which admit a non-trivial H-invariant linear form. We give a precise conjectural answer, which we verify in many cases.

Our conjecture assumes the Langlands parametrization of irreducible representations of G, in Vogan's revised form. The recipe for computing the space $\mathrm{Hom}_H(\pi, \mathbf{C})$ involves the local root numbers of symplectic representations of the Weil-Deligne group of k. Since the signs of these root numbers are mysterious enough in their own right, our conjecture might also be viewed as giving a representation-theoretic interpretation of their values!

We also treat the question of restriction of irreducible automorphic representations, which is related to central critical values of L-functions.

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1. **The Langlands parametrization.** Let k be a local (= locally compact) field, and let $\underline{G}$ be a connected, reductive algebraic group over k. We review the conjectural Langlands parametrization of irreducible, admissible, complex representations π of the group $G = \underline{G}(k)$. For details, the reader should consult [Bo].

Let $W(k)'$ denote the Weil-Deligne group of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. The L-group of G is a semi-direct product

$$^L G = {}^\vee G \rtimes \Gamma, \tag{1.1}$$

where ${}^\vee G$ is (the complex points of) a connected reductive algebraic group over $\mathbf{C}$ whose based root datum is dual to that of $\underline{G}$ over $\bar{k}$. A Langlands parameter is a continuous homomorphism

$$\varphi : W(k)' \longrightarrow {}^L G \tag{1.2}$$

which satisfies certain additional conditions [Bo, §8]. In the non-Archimedean case, such a homomophism specifies a nilpotent element N in ${}^{\vee}\mathfrak{g} = \mathrm{Lie}({}^{\vee}G)$. Two Langlands parameters are considered equivalent if they are conjugate by an element of ${}^{\vee}G$.

Langlands has conjectured that there is a decomposition of the set $\Pi(G)$ of isomorphism classes of irreducible, admissible, complex representations π of G into finite sets, called L-packets

$$\Pi(G) = \biguplus \Pi_{\varphi}(G). \tag{1.3}$$

Moreover, the L-packets $\Pi_{\varphi}(G)$ are indexed by the equivalence classes of Langlands parameters φ. We will admit this conjecture, which is only known to be true when $k = \mathbb{R}$ or $\mathbb{C}$, or when G is a product of fairly simple groups like tori or GL_2, in all that follows.

2. **Generic L-packets.** In this section, we assume that $\underline{G}$ is quasi-split over k, with Borel subgroup $\underline{B}$. Write $\underline{B} = \underline{T} \ltimes \underline{U}$, where $\underline{U}$ is the unipotent radical of $\underline{B}$ and $\underline{T}$ is a maximal torus contained in $\underline{B}$. Let $\underline{A}$ be the maximal subtorus of $\underline{T}$ which is split over k. We write $B = \underline{B}(k)$, $U = \underline{U}(k)$, $T = \underline{T}(k)$, and $A = \underline{A}(k)$ for the corresponding subgroups of G.

The abelianization $U^{ab} = U/[U, U]$ is a k-vector space, isomorphic to the direct sum of the simple root spaces U_{α} for the adjoint action of A on U. A linear functional

$$f\colon U^{ab} \longrightarrow k \tag{2.1}$$

is generic if it is non-zero when restricted to each simple root space U_{α}. Let ψ be a non-trivial additive character of k and let f be a generic linear functional. The composite group homomorphism

$$\theta\colon U \longrightarrow U^{ab} \underset{f}{\longrightarrow} k \underset{\psi}{\longrightarrow} S^1 \tag{2.2}$$

is called a generic character of U.

The generic functionals and characters are permuted by the adjoint action of T on U, and there are finitely many orbits. If $\underline{Z}$ is the center of $\underline{G}$ and $\mathrm{ad}(\underline{G}) = \underline{G}/\underline{Z}$ is the adjoint group, the T-orbits form a principal homogeneous space for the finite abelian group

$$\mathrm{ad}(\underline{G})(k)/\,\mathrm{Im}\,\underline{G}(k) = \ker\big(H^1(\Gamma, \underline{Z}) \longrightarrow H^1(\Gamma, \underline{G})\big). \tag{2.3}$$

This follows from the fact that there is a single orbit when G is adjoint. If θ is a generic character, and d an element of $\mathrm{ad}(\underline{G})(k)$, we let θ_d be a generic character in the translated orbit.

Let $\mathbb{C}(\theta)$ be the 1-dimensional representation of U which corresponds to the generic character θ. Gelfand and Kahzdan [G-K] and Shalika [Sk] have shown that for any irreducible representation π of G, the complex vector space

$$\mathrm{Hom}_U\big(\pi, \mathbb{C}(\theta)\big) \text{ has dimension } \leq 1. \tag{2.4}$$

If the dimension is equal to 1, we say the representation π is θ-generic. For an excellent discussion of generic representations, and a proof of (2.4), see [Ro]. We must be more precise about the definition of an admissable representation when $h = \mathbb{R}$ or $\mathbb{C}$ here. In most of the paper, a $(\mathfrak{G}, K)$-module will suffice, but in (2.4) one needs a representation of G on a topological vector space and continuous linear maps to $\mathbb{C}(\theta)$ to obtain multiplicity ≤ 1 results (*cf.* [Ks]).

CONJECTURE 2.5. *Let θ be a generic character of U and let φ be a Langlands parameter for G. Then the complex vector space $\oplus_{\pi\in\Pi_\varphi(G)} \mathrm{Hom}_U(\pi, \mathbb{C}(\theta))$ has dimension ≤ 1. Furthermore, this dimension is independent of the T-orbit of θ.*

If the direct sum in Conjecture 2.5 has dimension equal to 1, we say the parameter φ, or the L-packet $\Pi_\varphi(G)$ is generic. The following criterion was suggested by a remark of S. Rallis.

CONJECTURE 2.6. *Let $\mathrm{Ad}\colon {}^LG \longrightarrow \mathrm{Aut}_{\mathbb{C}}({}^\vee\mathfrak{g})$ be the adjoint representation of the L-group. The parameter $\varphi\colon W(k)' \longrightarrow {}^LG$ is generic if and only if the local L-function $L(\mathrm{Ad}\circ\varphi, s)$ of the composite representation of $W(k)'$ is regular at the point $s = 1$.*

We have checked that this conjecture is true in most cases where the theory of L-packets is known to exist. For example, it is true for $k = \mathbb{R}$ or $\mathbb{C}$, or when G is a torus or GL_n, or when k is non-Archimedean and the parameter φ is trivial on the inertia subgroup of $W(k)$. It is also compatible with Shahidi's conjecture that tempered parameters are generic [Sh, 9.4], as the L-function $L(\mathrm{Ad}\circ\varphi, s)$ of a tempered parameter φ is regular in the half-plane $\mathrm{Re}(s) > 0$.

3. **Vogan L-packets.** We review Vogan's reformulation of the Langlands parametrization; for details, the reader should consult [V]. First, we recall the notion of a pure inner form of the group $\underline{G}$. This will be a reductive group $\underline{G}'$ over k, which is an inner form of $\underline{G}$ together with some additional structure: a lifting of the 1-cocycle $\Gamma \longrightarrow \underline{G}/\underline{Z}$ from the quotient of $\underline{G}$ by its center $\underline{Z}$ to a 1-cocycle $\Gamma \longrightarrow \underline{G}$. We are only interested in the cohomology class of the lifted cocycle; the classes of pure inner forms of $\underline{G}$ correspond to the elements of the finite pointed set $H^1(\Gamma, \underline{G})$. Since the map $H^1(\Gamma, \underline{G}) \longrightarrow H^1(\Gamma, \underline{G}/\underline{Z})$ of pointed sets is (in general) neither injective nor surjective, an inner form of $\underline{G}$ can give rise to more than one pure inner form, or to none at all.

For example, let V be an orthogonal space over k and let $\underline{G} = \mathrm{SO}(V)$. We assume that $\mathrm{char}(k) \neq 2$. The pure inner forms of $\underline{G}$ are groups of the form $\underline{G}' = \mathrm{SO}(V')$, where V' is an orthogonal space over k with the same rank and discriminant as V. The class of the pure inner form $\underline{G}'$ is determined by the isomorphism class of the orthogonal space V' over k.

Assume that $\underline{G}$ is quasi-split over k. Let φ be a Langlands parameter for G, and let C_φ be the algebraic subgroup of ${}^\vee G$ which centralizes the image of φ in ${}^LG$. Define the (finite) component group A_φ of the parameter φ by

$$A_\varphi = C_\varphi / C_\varphi^0 = \pi_0(C_\varphi). \qquad (3.1)$$

If $\underline{G}'$ is a pure inner form of $\underline{G}$, let $G' = \underline{G}'(k)$. Since ${}^LG = {}^LG'$, the parameter φ may also be a Langlands parameter for G'. (This will be the case if φ satisfies the condition [Bo, 8.2 (ii)] on relevant parabolics.) We let $\Pi_\varphi(G')$ be the corresponding L-packet for G', if it exists; otherwise, we let $\Pi_\varphi(G')$ be the empty set.

Fix a generic character θ of U once and for all. Then Vogan conjectures that there is a bijection (depending on the T-orbit of θ) between the set of admissible, irreducible representations π' of the (classes of) pure inner forms G' of G and the set of pairs (φ, χ), where φ is a Langlands parameter for G and χ is an irreducible representation of the finite component group A_φ. The set

$$\Pi_\varphi = \{\pi(\varphi, \chi) : \chi \in \hat{A}_\varphi\} \tag{3.2}$$

should be the disjoint union of the Langlands L-packets $\Pi_\varphi(G')$ over the classes of pure inner forms for G. We call Π_φ the Vogan L-packet of φ; as a set it should be independent of the choice of T-orbit for θ. Finally, if φ is a generic parameter for G and χ_0 is the trivial representation of A_φ, the representation $\pi(\varphi, \chi_0)$ should be the θ-generic element in the Langlands L-packet $\Pi_\varphi(G)$.

4. **Some recipes.** One attractive aspect of Vogan's formulation of the parametrization is the simple recipes available for determining

(4.1) the pure inner form G' which acts on the representation $\pi(\varphi, \chi)$ in Π_φ, and

(4.2) the other generic representations $\pi(\varphi, \chi)$ in a generic Vogan L-packet Π_φ.

These recipes rely on the following dualities of finite abelian groups:

$$H^1(k, \underline{G}) \times \pi_0\big(Z({}^\vee G)^\Gamma\big) \to \mathbf{Q}/\mathbf{Z} \quad (k \neq \mathbf{R}, \mathbf{C}) \tag{4.3}$$

$$H^1(k, \underline{Z}) \times H^1\big(\Gamma, \pi_1({}^\vee G)\big) \to \mathbf{Q}/\mathbf{Z}. \tag{4.4}$$

The first is due to Kottwitz [K]; in the non-Archimedean case the pointed set $H^1(k, \underline{G})$ classifying pure inner forms has the structure of an abelian group. The second follows from the fact that the étale group scheme $\pi_1({}^\vee G)$ is the Cartier dual of $\underline{Z}$. For $\underline{G}$ a torus, both (4.3) and (4.4) are a restatement of Tate-Nakayama local duality.

To settle question (4.1) when $k \neq \mathbf{R}$, we remark that for any parameter φ there is a homomorphism

$$\pi_0\big(Z({}^\vee G)^\Gamma\big) \to A_\varphi = \pi_0(C_\varphi) \tag{4.5}$$

whose image lies in the center of A_φ. Hence the irreducible representation χ of A_φ gives a character of $\pi_0\big(Z({}^\vee(G)^\Gamma\big)$, which determines a pure inner form G' by (4.3). This is the group which should act on $\pi(\varphi, \chi)$. When $k = \mathbf{R}$, the recipe for G' is more complicated.

To answer question (4.2), one shows that for any generic parameter φ there is a boundary homomorphism in Galois cohomology:

$$A_\varphi \to H^1\big(\Gamma, \pi_1({}^\vee G)\big). \tag{4.6}$$

The T'-orbits of generic characters θ' on the quasi-split pure inner forms G' of G correspond bijectively to the elements of the finite abelian group $H^1(k, \underline{Z})$, with θ corresponding to the identity element. By (4.4), each θ' determines a 1-dimensional representation χ of A_φ which factors through (4.6). The corresponding representation $\pi(\varphi, \chi)$ of G' should be θ'-generic. In particular, the Vogan L-packet Π_φ will contain a *unique* generic representation if and only if the map in (4.6) is the zero homomorphism.

5. **Invariants of orthogonal spaces.** In this section, k is an arbitrary field with $\mathrm{char}(k) \neq 2$. Let V be an orthogonal space of dimension n over k. We recall the definition of the discriminant $d(V)$ and the Hasse-Witt invariant $e(V)$. For proofs of the assertions, see [Se, Chapter IV].

Let $\langle v_1, \dots, v_n \rangle$ be an orthogonal basis of V. If $q(v) = \frac{1}{2}\langle v, v \rangle$ is the quadratic form on V, let $a_i = q(v_i)$ in k^*. Hence

$$q(v) = \sum_{i=1}^{n} a_i \cdot x_i^2 \quad \text{for } v = \sum_{i=1}^{n} x_i v_i. \tag{5.1}$$

We define

$$d(V) \equiv \prod_{i=1}^{n} a_i \pmod{k^{*2}}. \tag{5.2}$$

Then $d(V) \in k^*/k^{*2} = H^1(k, \langle \pm 1 \rangle)$ is a cohomological invariant of the space V, which is independent of the orthogonal basis chosen. If q is scaled by the factor $\alpha \in k^*$, then $d(V)$ is scaled by the factor α^n in k^*/k^{*2}.

Let (a, b) be the Hilbert symbol in $\mathrm{Br}_2(k) = H^2(k, \langle \pm 1 \rangle)$. We define

$$e(V) = \prod_{i<j} (a_i, a_j) \text{ in } \mathrm{Br}_2(k). \tag{5.3}$$

Again this is a cohomological invariant of V.

When k is a local field, the group k^*/k^{*2} is finite and we have an injection $\mathrm{Br}_2(k) \hookrightarrow \langle \pm 1 \rangle$, which is an isomorphism if $k \neq \mathbb{C}$. A class $d \in k^*/k^{*2}$ gives a character

$$\begin{aligned} \omega_d \colon k^*/k^{*2} &\longrightarrow \langle \pm 1 \rangle \\ a &\longmapsto (a, d). \end{aligned} \tag{5.4}$$

6. **Odd orthogonal groups.** In this section, we assume k is a local field, with $\mathrm{char}(k) \neq 2$. Let V be an orthogonal space of dimension $2m + 1 \geq 3$ over k, and let $\underline{G} = \mathrm{SO}(V)$ be the special orthogonal group of V.

The L-group of $G = \underline{G}(k)$ is isomorphic to a direct product

$${}^L G = \mathrm{Sp}_{2m}(\mathbb{C}) \times \Gamma. \tag{6.1}$$

Let $\varphi \colon W(k)' \longrightarrow {}^L G$ be a Langlands parameter for G; then φ is completely determined by its projection onto ${}^\vee G$:

$$\varphi \colon W(k)' \longrightarrow \mathrm{Sp}_{2m}(\mathbb{C}) = \mathrm{Sp}(M) \tag{6.2}$$

where M is a symplectic space of dimension $2m$ over $\mathbb{C}$.

We may view M as a semi-simple representation of $W(k)$, for $k = \mathbb{R}$ or $\mathbb{C}$, and as a semi-simple representation of $W(k) \times \mathrm{SL}_2(\mathbb{C})$ when k is non-Archimedean. Let

$$M = \oplus M(i) \tag{6.3}$$

be its isotypic decomposition, and write

$$M(i) = e_i \cdot N_i \tag{6.4}$$

with N_i irreducible and e_i = the multiplicity of N_i in M. The dual $M(i)^\vee$ is also an isotypic subspace of $M = M^\vee$, via the symplectic form.

PROPOSITION 6.5. *1) If $M(i)^\vee \neq M(i)$, then the centralizer of φ in $M(i) \oplus M(i)^\vee$ is isomorphic to* $\mathrm{GL}_{e_i}(\mathbb{C})$.

2) If $M(i)^\vee = M(i)$ and N_i is an orthogonal irreducible representation, then $e_i = 2d_i$ is even and the centralizer of φ in $M(i)$ is isomorphic to $\mathrm{Sp}_{2d_i}(\mathbb{C})$.

3) If $M(i)^\vee = M(i)$ and N_i is a symplectic irreducible representation, then the centralizer of φ in $M(i)$ is isomorphic to $O_{e_i}(\mathbb{C})$.

PROOF. 1) Write $M(i) = N_i \otimes W$ and $M(i)^\vee = M(j) = N_j \otimes W^\vee$. Then the centralizer is $\mathrm{GL}(W)$, acting through the direct sum of the standard representation and its dual.

2) and 3) Write $M(i) = N_i \otimes W$, and let $\langle\ ,\ \rangle_M$ be the symplectic form on M. There is a unique (up to scaling) invariant bilinear form $\langle\ ,\ \rangle_N$ on N_i, and this determines a non-degenerate bilinear form $\langle\ ,\ \rangle_W$ on W such that $\langle\ ,\ \rangle_M = \langle\ ,\ \rangle_N \otimes \langle\ ,\ \rangle_W$ on $M(i)$. The centralizer of φ is isomorphic to the subgroup of $\mathrm{GL}(W)$ which respects the form $\langle\ ,\ \rangle_W$. This group is symplectic or orthogonal, depending on the type of N_i.

COROLLARY 6.6. *The component group $A_\varphi = C_\varphi / C_\varphi^0$ is an elementary abelian 2-group, whose rank is equal to the number of distinct symplectic irreducible representations N_i in the decomposition of M.*

If M is irreducible, $A_\varphi = \langle \pm 1_M \rangle$. In general, the element -1_M of $Z(^\vee G)$ is non-trivial in A_φ if and only if M contains an irreducible symplectic representation N_i with odd multiplicity e_i.

PROOF. By the proposition, C_φ is the direct product of groups isomorphic to $\mathrm{GL}_{e_i}(\mathbb{C})$, $\mathrm{Sp}_{2d_i}(\mathbb{C})$, and $O_{e_i}(\mathbb{C})$. Only the latter contribute to A_φ.

Now assume $\underline{G}$ is quasi-split: this occurs precisely when V contains an isotropic subspace of dimension m. Since $\underline{Z} = 1$, there is a unique T-conjugacy class of generic characters θ of U. Hence the Vogan correspondence defined in §3 is independent of any choices.

The group $\pi_0\big(Z(^\vee G)^\Gamma\big)$ has order 2 and is represented by -1_M. Hence, when k is non-Archimedean there is precisely one non-trivial pure inner form G' of G. We have $\underline{G}' = \mathrm{SO}(V')$, where V' is an orthogonal space of rank $m-1$ with the same discriminant as V. The recipe of §4 states that the element $\pi(\varphi, \chi)$ in the Vogan L-packet Π_φ is a representation of G' if and only if $\chi(-1_M) = -1$.

When $k = \mathbb{R}$, the pointed set $H^1(k, \underline{G})$ has cardinality $m+1$. The pure inner forms G' have the form $\underline{G}' = SO(V')$, where V' has the same discriminant as V and has rank $0 \le r \le m$. One can show that $\pi(\varphi, \chi)$ is a representation of a group G' with

$$(6.7) \qquad \chi(-1_M) = e(V')/e(V) \text{ in } \langle \pm 1 \rangle = Br_2(\mathbb{R}),$$

where $e(V)$ and $e(V')$ are the Hasse-Witt invariants defined in (5.3).

7. **Even orthogonal groups.** In this section, k is a local field, with $\mathrm{char}(k) \neq 2$. Let V be an orthogonal space of dimension $2m \ge 2$ over k, and let $\underline{G} = SO(V)$ be the special orthogonal group of V.

We define the normalized discriminant $D = D(V)$ by the formula

$$(7.1) \qquad D = (-1)^m \cdot d(V) \text{ in } k^*/k^{*^2},$$

where $d(V)$ is defined in (5.2). Let

$$(7.2) \qquad E = k[x]/(x^2 - D)$$

be the quadratic discriminant algebra associated to V.

The L-group of $G = \underline{G}(k)$ is isomorphic to a semi-direct product

$$(7.3) \qquad {}^LG = SO_{2m}(\mathbb{C}) \rtimes \Gamma.$$

The subgroup of Γ which fixes E acts trivially on ${}^\vee G = SO(M)$, where M is an orthogonal space of dimension $2m$ over $\mathbb{C}$. If $D \not\equiv 1 \pmod{k^{*^2}}$, so E is a field, the quotient $\mathrm{Gal}(E/k)$ acts on ${}^\vee G$ via conjugation by a simple reflection in $O(M)$. Let $\varphi\colon W(k)' \longrightarrow {}^LG$ be a Langlands parameter for G; then φ is completely determined by the map

$$(7.4) \qquad \varphi\colon W(k)' \longrightarrow O(M)$$

with determinant the quadratic character associated to E:

$$(7.5) \qquad \det \varphi = \omega_D \text{ on } W(k)^{ab} = k^*.$$

Let $M = \oplus M(i)$ be the isotypic decomposition of the associated semi-simple representation of $W(k)$ or $W(k) \times SL_2(\mathbb{C})$, and write $M(i) = e_i N_i$ with N_i irreducible and e_i the multiplicity of N_i in M. Arguing exactly as in Proposition 6.5 and Corollary 6.6 one finds

PROPOSITION 7.6. *1) If $M(i)^\vee \neq M(i)$ then the centralizer of φ in $M(i) \oplus M(i)^\vee$ is isomorphic to* $GL_{e_i}(\mathbb{C})$.

2) If $M(i)^\vee = M(i)$ and N_i is a symplectic irreducible representation, then $e_i = 2d_i$ is even and the centralizer of φ in $M(i)$ is isomorphic to $Sp_{2d_i}(\mathbb{C})$.

3) If $M(i)^\vee = M(i)$ and N_i is an orthogonal irreducible representation, then the centralizer of φ in $M(i)$ is isomorphic to $O_{e_i}(\mathbb{C})$.

COROLLARY 7.7. *The component group of the centralizer of φ in $O(M)$ is an elementary abelian 2-group, whose rank r is equal to the number of distinct irreducible* *orthogonal representations N_i in the decomposition of M. The component group A_φ of the centralizer of φ in ${}^\vee G = \mathrm{SO}(M)$ is elementary abelian of rank $= r$ or $r-1$, the latter case occurring when dim N_i is odd for some orthogonal irreducible representation N_i in the decomposition.*

If M is irreducible, $A_\varphi = \langle \pm 1_M \rangle$. In general, the element -1_M of $Z({}^\vee G)$ is non-trivial in A_φ if and only if M contains an irreducible orthogonal representation N_i with odd multiplicity e_i.

Now assume $\underline{G}$ is quasi-split, or equivalently that V has an isotropic subspace of dimension $m-1$ over k. When $D \equiv 1 \pmod{k^{*2}}$, $\underline{G}$ will then be split and V will contain an isotropic subspace of dimension m over k.

PROPOSITION 7.8. *If $2m = 2$, $\theta = 1$ is the unique generic character of U.*

If $2m \geq 4$, the T-orbits of generic characters θ of U form a principal homogeneous space for the finite abelian group $\ker\big(H^1(k, \underline{Z}) \to H^1(k, \underline{G})\big) = \mathrm{N}E^*/k^{*2}$, where E is the discriminant algebra. The T-orbits of generic characters θ of U are in 1-to-1 correspondence with the G-orbits of codimension 1 subspaces W of V such that $V = W \oplus W^\perp$ and W is split over k.

PROOF. When $2m = 2$, the group $\underline{G} = \mathrm{SO}(V)$ is a torus, so $U = 1$.
When $2m \geq 4$, U^{ab} is the sum of simple root spaces:

$$U^{ab} = \bigoplus_{i=1}^{m-2} L_i \oplus L \tag{7.9}$$

with $\dim_k(L_i) = 1$ and $\dim_E(L) = 1$. (When V is split, L_i is associated to the simple root $(e_i - e_{i+1})$, and L is the 2-dimensional k-vector space associated to the roots $(e_{m-1} \pm e_m)$.) The maximal torus $T \simeq \prod_{i=1}^{m-2} k^* \times (k^* \times E_1^*)$, where E_1^* is the subgroup of norm $= 1$ elements in E^*, acts on U^{ab} as follows. The element $(t_1, \ldots, t_{m-2}, t, \alpha)$ acts by multiplication by t_i on ℓ_i, and by multiplication by $t\alpha$ on the E-vector space L. Hence the T-orbit of a generic functional $f\colon U^{ab} \to k$ is determined by the restriction f_L of f to L, and the group $E^*/k^* \cdot E_1^* \simeq \mathrm{N}E^*/k^{*2}$ acts simply-transitively on the orbits.

Now let W be a split codimension 1 subspace of V. Let X be a maximal isotropic subspace (of dimension $= m-1$) of W, and let $\underline{B}$ be a Borel subgroup of $\underline{G}$ which is constructed from a maximal isotropic flag containing X. Let $U_W = U \cap \mathrm{SO}(W)(k)$ and $T_W = T \cap \mathrm{SO}(W)(k)$. Then $U_W^{ab} \simeq \oplus_{i=1}^{m-2} L_i \oplus \ell$ is a sum of 1-dimensional simple root spaces for $T_W \simeq \prod_{i=1}^{m-1} k^*$.

Since $L = \ell \otimes_k E$, we obtain a generic linear functional $g\colon L \to k$ by choosing a basis vector e for ℓ over k and defining $g(e \otimes \alpha) = \mathrm{Tr}_{E/k}(\alpha)$. The T-orbit of a generic functional $f\colon U^{ab} \to k$ with $f_L = g$ is well-determined by the G-orbit of W, and we denote the resulting generic character of U (or rather, its T-orbit) by θ_W.

If d lies in the subgroup $\mathrm{N}E^*$ of k^*, the quadratic space dV (where the form is scaled by d) is isomorphic to V over k. We obtain a codimension 1 split subspace $dW \hookrightarrow dV \simeq V$, whose G-orbit depends only on the class of d in $\mathrm{N}E^*/k^{*2}$. The T-orbit of the resulting generic character θ_{dW} is easily seen to be the translate $(\theta_W)_d$ of the T-orbit of θ_W by the class d.

We now discuss the recipes in §4 for the quasi-split group $\underline{G} = SO(V)$. The group $\pi_0\big(Z({}^\vee G)^\Gamma\big)$ has order 2, and is represented by -1_M, except in the special case when $2m = 2$ and $D \equiv 1 \pmod{k^{*2}}$. In the special case, $\underline{G} \simeq \mathbb{G}_m$ has no non-trivial pure inner forms. In the other cases, when k is non-Archimedean there is precisely one non-trivial pure inner form G' of G. If $D \equiv 1 \pmod{k^{*2}}$, then $\underline{G}' = SO(V')$, where V' is an orthogonal space of rank $m-2$; if $D \not\equiv 1 \pmod{k^{*2}}$ then $\underline{G}' = SO(V')$ with $V' = dV$ for any class d in $k^* - NE^*$. The recipe states that the element $\pi(\varphi, \chi)$ in the Vogan L-packet Π_φ is a representation of G' if and only if $\chi(-1_M) = -1$. More generally, in all cases one has

$$\chi(-1_M) = e(V')/e(V) \text{ in } \mathrm{Br}_2(k) = \langle \pm 1 \rangle. \tag{7.9}$$

If φ is a generic parameter and $2m \geq 4$, the group $H^1(k, \underline{Z}) = k^*/k^{*2}$ acts transitively on the set of generic representations in the Vogan L-packet Π_φ. More precisely, if d is a class in k^*/k^{*2}, we define a quadratic character of the component group A_φ by the formula

$$\chi: A_\varphi \longrightarrow \langle \pm 1 \rangle$$
$$a \longmapsto \det(M^{a=-1})(d),$$

where $M^{a=-1}$ is the minus eigenspace for an involution in the centralizer of φ, which lies in the connected component determined by a. If the representation $\pi(\varphi, \chi_0)$ is θ-generic, then the representation $\pi(\varphi, \chi)$ is θ_d-generic. If $d \in NE^*$ this is a representation of G; otherwise it is a representation of G'.

8. **Orthogonal pairs.** In this section, V is an orthogonal space of dimension ≥ 3 over k (with $\mathrm{char}(k) \neq 2$) and W is a codimension 1 subspace of V with $V = W \oplus W^\perp$. We assume that the odd dimensional space in the pair is split, and that the even dimensional space is quasi-split of normalized discriminant $D \in k^*/k^{*2}$. Let $E = k[x]/(x^2 - D)$ be the discriminant algebra.

Let $\underline{G} = SO(W) \times SO(V)$. Then $\underline{G}$ is quasi-split over k and contains the diagonally embedded subgroup $\underline{H} = SO(W)$. We wish to study the problem of restricting an irreducible, admissible representation π of $G = \underline{G}(k)$ to the subgroup $H = \underline{H}(k)$. By the results of §6 and §7, we have

$${}^LG = \big(\mathrm{Sp}(M_1) \times SO(M_2)\big) \rtimes \Gamma \tag{8.1}$$

where M_1 and M_2 are symplectic and orthogonal spaces over $\mathbb{C}$. If $\dim V = 2m + 1$, then $\dim M_1 = \dim M_2 = 2m$; if $\dim V = 2m + 2$, then $\dim M_2 = 2m + 2$ and $\dim M_1 = 2m$. A Langlands parameter $\varphi: W(k)' \longrightarrow {}^LG$ is completely determined by the resulting homomorphism

$$\varphi = \varphi_1 \times \varphi_2 : W(k)' \longrightarrow \mathrm{Sp}(M_1) \times O(M_2) \tag{8.2}$$

with det $\varphi_2 = \omega_D$. There is a canonical symplectic representation

$$r: {}^LG \longrightarrow \mathrm{Sp}(M_1 \otimes M_2) = \mathrm{Sp}(M) \tag{8.3}$$

obtained by taking the tensor product of the two standard representations of $O(M_1)$ and $\mathrm{Sp}(M_2)$.

The pure inner forms G' of G arise from orthogonal spaces:

$$G' = \mathrm{SO}(W')(k) \times \mathrm{SO}(V')(k) \tag{8.4}$$

which satisfy

$$\begin{cases} \dim W' = \dim W & \dim V' = \dim V \\ d(W') = d(W) & d(V') = d(V) \end{cases} \tag{8.5}$$

We do *not* assume that W' embeds as a codimension 1 subspace of V'. If it does, we call the pure inner form G' *relevant*, and define the diagonally embedded subgroup $H' = \mathrm{SO}(W')(k)$ of G'. The embedding of H' into G' is unique up to conjugacy, by Witt's theorem.

Let Π_φ be a Vogan L-packet for G. If the element π_α in Π_φ is a representation of a relevant pure inner form G' of G, we define $\mathrm{Hom}_{H_\alpha}(\pi_\alpha, \mathbf{C}) = \mathrm{Hom}_{H'}(\pi_\alpha, \mathbf{C})$. Otherwise, we set $\mathrm{Hom}_{H_\alpha}(\pi_\alpha, \mathbf{C}) = 0$.

CONJECTURE 8.6. *Let φ be a generic Langlands parameter for G and let Π_φ be the corresponding Vogan L-packet. Then the complex vector space $\oplus_{\pi_\alpha \in \Pi_\varphi} \mathrm{Hom}_{H_\alpha}(\pi_\alpha, \mathbf{C})$ has dimension = 1.*

To give a more precise version of Conjecture 8.6, we must first fix a generic character θ_0 of U as a base point corresponding to the trivial character χ_0 of A_φ, then must specify *which* irreducible representation χ of A_φ corresponds to the representation π_α in Π_φ with $\mathrm{Hom}_{H_\alpha}(\pi_\alpha, \mathbf{C}) = \mathbf{C}$. We will do this in §10, after some preliminaries on symplectic local root numbers in the next section.

REMARK 8.7. It would be interesting to develop the correct notion of Gelfand pair which would give multiplicity ≤ 1 results over a Vogan L-packet Π_φ, as in Conjecture 8.6 or Conjecture 2.5. In both cases, we observe that the subgroup H has an open dense orbit on the k-rational points G/B of the flag variety, with trivial stability subgroup.

REMARK 8.8. The group $O(W')(k) \times O(V')(k)$ acts by conjugation on G', and this action gives an involution $\pi \longmapsto \pi^*$ of the set of isomorphism classes of irreducible representations. Since $\mathrm{Hom}_{H'}(\pi, \mathbf{C})$ is isomorphic to $\mathrm{Hom}_{H'}(\pi^*, \mathbf{C})$, Conjecture 8.6 suggests that whenever π and π^* are in the same L-packet, they are isomorphic. This should follow from Corollary 7.7.

REMARK 8.9. We have been assuming that $\mathrm{char}(k) \neq 2$, but there is a similar theory in characteristic 2. If V is a quadratic space over a field of characteristic 2, with quadratic form $Q: V \longrightarrow k$ and associated bilinear form $\langle x, y\rangle = Q(x+y) + Q(x) + Q(y)$, we say V

is non-degenerate if the radical $V^\perp$ has dim $V^\perp \leq 1$; if $V^\perp = \langle v \rangle$ is 1-dimensional, we insist that $Q(v) \neq 0$. If dim V is even $V^\perp = 0$, and we may define the Arf invariant of V in $H^1(k, \mathbf{Z}/2\mathbf{Z}) = k/\wp(k)$. If dim V is odd, $V^\perp = \langle v \rangle$ is 1-dimensional and we have the discriminant $d(V) = Q(v)$ in k^*/k^{*2} as before.

In the setting of this paper, we would start with a pair of non-degenerate quadratic spaces $W \hookrightarrow V$ with codim $W = 1$. If D is the Arf invariant of the even dimensional space, the discriminant algebra is replaced by the étale quadratic k-algebra $E = k[x]/(x^2 + x + D)$. The group $\underline{G} = \mathrm{SO}(W) \times \mathrm{SO}(V)$ is connected and reductive, and contains $\underline{H} = \mathrm{SO}(W)$ as a diagonally embedded subgroup. The parameters φ and L-packets Π_φ are exactly as before.

9. **Symplectic local root numbers.** In this section, we suppose we are given a symplectic representation

$$\varphi\colon W(k)' \longrightarrow \mathrm{Sp}(U). \tag{9.1}$$

Our aim is to define a local root number $\epsilon(U) = \pm 1$.

Fix a non-trivial additive character ψ of k, and let dx be the Haar measure on k which is self-dual for Fourier transform with respect to ψ. Following the notation of Tate's article [Ta, 3.6], we define the ϵ-factor $\epsilon_0(U)$ of the underlying representation of the Weil group

$$\varphi_0\colon W(k) \longrightarrow \mathrm{Sp}(U) \tag{9.2}$$

by the formula:

$$\epsilon_0(U) = \epsilon_L(\varphi_0, \psi) = \epsilon_D(\varphi_0 \otimes \|^{1/2}, \psi, dx). \tag{9.3}$$

If k is Archimedean, put $\epsilon(U) = \epsilon_0(U)$. If k is non-Archimedean, let I be the inertia subgroup of $W(k)$, let Fr be a geometric Frobenius which generates the quotient $W(k)/I \simeq \mathbf{Z}$, and let q be the cardinality of the residue field. Let N be the nilpotent endomorphism of U given by φ, and let $U^I_{N=0} = \ker(N\colon U^I \longrightarrow U^I)$. We define

$$\epsilon(U) = \epsilon_0(U) \cdot \det(-\mathrm{Fr} \cdot q^{-1/2} | U^I / U^I_{N=0}). \tag{9.4}$$

PROPOSITION 9.5. *The local root number $\epsilon(U)$ is independent of the choice of ψ and satisfies $\epsilon(U)^2 = 1$.*

PROOF. Since φ_0 is self-dual and det $\varphi_0 = 1$, the formulae in [Ta, 3.6] show that $\epsilon_L(\varphi_0, \psi)$ is independent of ψ and satisfies $\epsilon_L(\varphi_0, \psi)^2 = 1$.

The fact that, in the non-Archimedean case, $\det(-\mathrm{Fr} \cdot q^{-1/2} | U^I / U^I_{N=0}) = \pm 1$ is proved in [Gr, 7.9].

NOTE 9.6. A similar argument gives a local root number $\epsilon(U) = \pm 1$ for a special orthogonal representation

$$\varphi\colon W(k)' \longrightarrow \mathrm{SO}(U).$$

In this case, there is an interpretation of the sign of $\epsilon(U)$ in terms of the liftability of φ to Spin(U), due to Deligne [De].

The local root number $\epsilon(U)$ is additive for direct sums of symplectic representations:

$$\epsilon(U_1 \oplus U_2) = \epsilon(U_1) \cdot \epsilon(U_2). \tag{9.7}$$

If U is zero-dimensional, we agree that $\epsilon(U) = +1$. Here is a calculation of $\epsilon(U)$ in a simple case.

PROPOSITION 9.8. *Assume that $U \simeq P \oplus P^\vee$, where P is a representation of $W(k)'$ and $P^\vee$ is the dual representation. Then $\epsilon(U) = \det P(-1)$.*

PROOF. See [Gr, 8.2]. We view $\det P$ as a 1-dimensional representation of $W(k)^{ab} = k^*$.

Proposition 9.8 will apply when the image of φ in Sp(U) is contained in the Levi subgroup of the parabolic stabilizing a maximal isotropic subspace P of U.

10. **The local conjecture.** We are now in a position to make Conjecture 8.6 more precise. As in §8, we fix a quasi-split pair $W \hookrightarrow V$ of orthogonal spaces over k, and let $\underline{G} = \mathrm{SO}(W) \times \mathrm{SO}(V)$. Our first task will be to specify a distinguished T-orbit of generic characters θ_0 for the unipotent radical U of a Borel subgroup of G. Clearly any generic character of G has the form $\theta_0 = \theta_1 \otimes \theta_2$ on $U = U_1 \times U_2$, where θ_1 is a generic character of unipotent subgroup U_1 in the odd orthogonal group and θ_2 is a generic character of unipotent subgroup U_2 in the even orthogonal group. Since all θ_1 lie in the same T_1-orbit, the problem is to specify the T_2-orbit of θ_2.

When $\dim V \geq 4$ is even, we let $\theta_2 = \theta_W$ in the notation of the proof of Proposition 7.8. Indeed, $W \hookrightarrow V$ is an odd dimensional split orthogonal space of codimension 1 in V. When $\dim V \geq 3$ is odd, we let U be a subspace of codimension 1 in W such that V is the direct sum of U and a hyperbolic plane. Then U is an odd dimensional split orthogonal space, so by Proposition 7.8 the orbit of θ_U on SO(W) is well-defined. We let $\theta_2 = \theta_U$.

Now fix a generic Langlands parameter $\varphi\colon W(k)' \longrightarrow {}^LG$. The choice of $\theta_0 = \theta_1 \otimes \theta_2$ above gives a (conjectural) bijection between $\hat{A}_\varphi$ and the elements in the Vogan L-packet Π_φ, where the θ_0-generic representation of G corresponds to the trivial character χ_0 of A_φ. We recall that $A_\varphi = A_1 \times A_2$ is an elementary abelian 2-group, where A_1 is the component group of the centralizer of φ_1 in Sp(M_1) and A_2 is the component group of the centralizer of φ_2 in SO(M_2). In particular, $\hat{A}_\varphi = \mathrm{Hom}(A_\varphi, \pm 1)$.

Recall the representation r of ${}^LG$ defined in (8.3). The composite homomorphism $r \circ \varphi$ gives a symplectic representation

$$r \circ \varphi\colon W(k)' \longrightarrow \mathrm{Sp}(M). \tag{10.1}$$

Hence, by §9, we obtain a local constant $\epsilon(M) = \pm 1$.

More generally, if $a = (a_1, a_2)$ is an involution in $\mathrm{Sp}(M_1) \times O(M_2)$ which centralizes the image of φ in ${}^LG$, we obtain representations $M_1^{a_1=-1}$, $M_2^{a_2=-1}$, $M^{a_1 \otimes a_2=-1}$ of $W(k)'$, which are symplectic, orthogonal, and symplectic respectively. We use these three representations to define an invariant $\chi(a)$ in $\langle \pm 1 \rangle$ as follows

$$\chi(a) = \epsilon(M^{a_1 \otimes a_2=-1}) \cdot \det(M_2)^{\frac{1}{2}\dim(M_1^{a_1=-1})}(-1) \cdot \det(M_2^{a_2=-1})^{\frac{1}{2}\dim M_1}(-1). \tag{10.2}$$

For example, for $a = (-1_{M_1}, -1_{M_2})$ we find

$$\chi(-1,-1) = \det M_2^{\dim M_1}(-1) = +1. \tag{10.3}$$

Similarly, we have

$$\chi(-1,+1) = \chi(+1,-1) = \epsilon(M) \cdot \det M_2^{\frac{1}{2}\dim M_1}(-1). \tag{10.4}$$

We recall that the centralizer D_φ of φ in $\mathrm{Sp}(M_1) \times O(M_2)$ is isomorphic to a product of general linear, symplectic, and orthogonal groups. The coset of a (mod D_φ^0) is determined by the signs of the determinants of the orthogonal components of a.

PROPOSITION 10.5. *For an involution a in D_φ, the value $\chi(a) = \pm 1$ depends only on the coset of a (mod D_φ^0). The resulting map $\chi\colon D_\varphi / D_\varphi^0 \to \langle \pm 1 \rangle$ is a group homomorphism.*

PROOF. We first observe that if a and b are commuting elements of order 2 in D_φ, we have the formula $\chi(ab) = \chi(ba) = \chi(a) \cdot \chi(b)$. Indeed, the representation

$$M^{ab=-1} \oplus 2 \cdot M^{\substack{a=-1\\b=-1}}$$

of $W(k')$ is isomorphic to the representation

$$M^{a=-1} \oplus M^{b=-1}.$$

Since ϵ is additive for direct sums, this gives

$$\epsilon(M^{ab=-1}) \cdot \epsilon(M^{\substack{a=-1\\b=-1}})^2 = \epsilon(M^{a=-1})\epsilon(M^{b=-1}).$$

But $\epsilon(M^{\substack{a=-1\\b=-1}})^2 = 1$ by Proposition 9.5, so

$$\epsilon(M^{ab=-1}) = \epsilon(M^{a=-1})\epsilon(M^{b=-1}).$$

A similar argument shows that

$$\frac{1}{2}\dim(M_1^{a_1 b_1=-1}) \equiv \frac{1}{2}\dim(M_1^{a_1=-1}) + \frac{1}{2}\dim(M_1^{b_1=-1}) \pmod 2$$
$$\det M_2^{a_2 b_2=-1} = \det M_2^{a_2=-1} \cdot \det M_2^{b_2=-1}.$$

So $\chi(ab) = \chi(a) \cdot \chi(b)$. This allows us to reduce to the case when only *one* component of a is non-trivial in the product $D_\varphi \simeq \prod_i \mathrm{GL}_{e_i}(\mathbf{C}) \times \prod_i \mathrm{Sp}_{2d_i}(\mathbf{C}) \times \prod_i O_{e_i}(\mathbf{C})$. We must show that $\chi(a) = 1$, unless the component a_i lies in $O_{e_i}(\mathbf{C})$, when $\chi(a)$ depends only on $\det a_i$. There are six cases to consider.

1) $a = (a_1, 1)$ with $a_1 \in \mathrm{GL}_e(\mathbf{C})$. Then

$$M_1^{a_1=-1} \simeq m(N \oplus N^\vee)$$
$$M_2^{a_2=-1} \simeq 0$$
$$M^{a=-1} \simeq m\big((N \otimes M_2) \oplus (N \otimes M_2)^\vee\big).$$

Here N is an irreducible summand of M_1 which is not self-dual, and m is the multiplicity of -1 as an eigenvalue of a_1 in the standard representation of $\mathrm{GL}_e(\mathbf{C})$ (or its dual). We find $\chi(a) = \det(N \otimes M_2)(-1)^m \cdot \det M_2(-1)^{m \cdot \dim N}$ by Proposition 9.8. Since $\det(N \otimes M_2) = \det M_2^{\dim N} \cdot \det N^{\dim M_2}$ and $\dim M_2$ is even, this shows $\chi(a) = 1$

2) $a = (a_1, 1)$ with $a_1 \in \mathrm{Sp}_{2d}(\mathbf{C})$. Then

$$M_1^{a_1=-1} = mN$$
$$M_2^{a_2=-1} = 0$$
$$M^{a=-1} = m(N \otimes M_2)$$

Here N is an irreducible orthogonal summand of M_1, and m is the multiplicity of -1 as an eigenvalue of a_1 in the standard representation of $\mathrm{Sp}_{2d}(\mathbf{C})$. Since m is even, $M^{a=-1} = \frac{m}{2}\big((N \otimes M_2) \oplus (N \otimes M_2)^\vee\big)$ and we have

$$\chi(a) = \det(N \otimes M_2)(-1)^{\frac{m}{2}} \cdot \det M_2(-1)^{\frac{m}{2} \cdot \dim N}$$

by Proposition 9.8. Since $\det(N \otimes M_2) = \det M_2^{\dim \mathrm{N}} \cdot \det N^{\dim M_2}$ and $\dim M_2$ is even, this shows $\chi(a) = 1$.

3) $a = (a_1, 1)$ with $a_1 \in O_e(\mathbf{C})$. Then

$$M_1^{a_1=-1} = mN$$
$$M_2^{a_2=-1} = 0$$
$$M^{a=-1} = m(N \otimes M_2)$$

Here N is an irreducible symplectic summand of M_1, and m is the multiplicity of -1 as an eigenvalue of a_1 in the standard representation of $O_e(\mathbf{C})$. We have $\det a_1 = (-1)^m$, so the coset of $a_1 \pmod{\mathrm{SO}_e(\mathbf{C})}$ is determined by the parity of m. We have

$$\chi(a) = \epsilon(N \otimes M_2)^m \cdot \det M_2(-1)^{m \cdot \frac{\dim N}{2}}.$$

If m is even $\chi(a) = 1$; if m is odd $\chi(a)$ is independent of the choice of a in the non-trivial coset.

4) $a = (1, a_2)$ with $a_2 \in \mathrm{GL}_e(\mathbf{C})$. Then

$$M_1^{a_1=-1} = 0$$
$$M_2^{a_2=-1} = m(N \oplus N^\vee)$$
$$M^{a=-1} = m\big((M_1 \otimes N) \oplus (M_1 \otimes N)^\vee\big).$$

Here N is an irreducible summand of M_2 which is not self-dual and m is the multiplicity of -1 as an eigenvalue of a_2. We have

$$\chi(a) = \det(M_1 \otimes N)(-1)^m \cdot \det(N \oplus N^\vee)^{\frac{m}{2}\dim M_1}(-1).$$

But $\det(M_1 \otimes N) = \det M_1^{\dim N} \cdot \det N^{\dim M_1} = \det N^{\dim M_1}$, and $\det(N \oplus N^\vee) = \det N \cdot \det N^\vee = 1$. Since $\dim M_1$ is even, $\chi(a) = 1$.

5) $a = (1, a_2)$ with a_2 in $\mathrm{Sp}_{2d}(\mathbf{C})$. We have

$$\begin{aligned} M_1^{a_1=-1} &= 0 \\ M_2^{a_2=-1} &= m \cdot N \\ M^{a=-1} &= m(M_1 \otimes N). \end{aligned}$$

Here N is an irreducible symplectic summand of M_2, and $m \equiv 0 \pmod 2$ is the multiplicity of -1 as an eigenvalue of a_2. We have

$$\chi(a) = \epsilon(M_1 \otimes N)^m \cdot \det N(-1)^{m\cdot\frac{\dim M_1}{2}}.$$

The orthogonal representation $M_1 \otimes N$ has trivial determinant, so $\epsilon(M_1 \otimes N) = \pm 1$ by Remark 9.6. Since m is even, $\chi(a) = 1$.

6) $a = (1, a_2)$ with a_2 in $O_e(\mathbf{C})$. We have

$$\begin{aligned} M_1^{a_1=-1} &= 0 \\ M_2^{a_2=-1} &= m \cdot N \\ M^{a=-1} &= m(M_1 \otimes N) \end{aligned}$$

Here N is an irreducible orthogonal summand of M_2 and $\det a_2 = (-1)^m$. We have

$$\chi(a) = \epsilon(M_1 \otimes N)^m \cdot \det N(-1)^{m\cdot\frac{\dim M_1}{2}}.$$

This clearly depends only on the coset of a $\big(\mathrm{mod}\, \mathrm{SO}_e(\mathbf{C})\big)$.

Since the involutions in D_φ represent all the classes $\cdot (\mathrm{mod}\, D_\varphi^0)$, χ induces a map $D_\varphi / D_\varphi^0 \longrightarrow \langle \pm 1 \rangle$. This is clearly a group homomorphism, as any two classes $\bar{a}$ and $\bar{b}$ in D_φ / D_φ^0 can be represented by *commuting* involutions a and b in D_φ, and we have seen that $\chi(ab) = \chi(a) \cdot \chi(b)$ when a and b commute.

The component group A_φ of the centralizer of φ in $G^\vee$ injects as a subgroup (of index 1 or 2) in D_φ / D_φ^0. Hence χ induces a character

$$\chi : A_\varphi \longrightarrow \langle \pm 1 \rangle. \tag{10.6}$$

We now state our main local conjecture, which seeks to identify the representation π_α in a generic Vogan L-packet with $\mathrm{Hom}_{H_\alpha}(\pi_\alpha, \mathbf{C}) \neq 0$.

CONJECTURE 10.7. *Let φ be a generic Langlands parameter for G, and let θ_0 be the T-orbit of generic characters of U fixed at the beginning of this section. Normalize the Vogan correspondence so that the representation $\pi(\varphi, \chi_0)$ in Π_φ corresponding to the trivial character χ_0 of A_φ is θ_0-generic. Finally, let χ be the irreducible representation of the component group A_φ defined using symplectic root numbers in (10.5-10.6).*

Then the pure inner form G' which acts on the irreducible representation $\pi' = \pi(\varphi, \chi)$ in the Vogan L-packet Π_φ is relevant, and the complex vector space $\mathrm{Hom}_{H'}(\pi', \mathbf{C})$ *is 1-dimensional. For all other representations π_α in Π_φ, we have* $\mathrm{Hom}_{H_\alpha}(\pi_\alpha, \mathbf{C}) = 0$.

REMARK 10.8. The L-packet Π_φ contains a *unique* generic representation if and only if $\det M_2^{a_2=-1} = 1$ for all $a = (a_1, a_2)$ in A_φ. In this case, the character χ corresponding to the unique representation π' with $\mathrm{Hom}_{H'}(\pi', \mathbf{C}) \neq 0$ is given by the simpler formula: $\chi(a) = \epsilon(M^{a=-1})$.

REMARK 10.9. Assume $k \neq \mathbf{R}$. Then formula (10.3): $\chi(-1,-1) = +1$, when combined with (6.7) and (7.9), shows that the pure inner form G' which acts on $\pi(\varphi, \chi)$ is relevant. By formula (10.4), we find that

$$G' = G \text{ iff } \epsilon(M) = \det M_2^{\frac{1}{2}\dim M_1}(-1). \tag{10.10}$$

REMARK 10.11. The suggestion that elements in A_φ might be useful in decomposing the representation M and obtaining more symplectic root numbers, like $\epsilon(M^{a=-1})$, is due to M. Harris.

11. **The case $k = \mathbf{C}$.** When $k = \mathbf{C}$, conjectures 10.7 and 8.6 are equivalent, as there is a unique representation π in each Vogan L-packet. We make this more explicit here.

Since $W(k) = \mathbf{C}^*$ and $\underline{G} = \mathrm{SO}(W) \times \mathrm{SO}(V)$ is split, a Langlands parameter φ corresponds to a homomorphism

$$\begin{aligned} \varphi: \mathbf{C}^* &\longrightarrow {}^\vee T \\ z &\longmapsto z^\lambda \bar{z}^\mu \end{aligned} \tag{11.1}$$

with $\lambda, \mu \in X^*(T) \otimes \mathbf{C}$ and $\lambda \equiv \mu \big(\mathrm{mod}\, X^*(T)\big)$ well-determined modulo the Weyl group of ${}^\vee T$ in ${}^\vee G$ [Bo, §11]. The parameter φ therefore corresponds to a continuous character of T:

$$\begin{aligned} \rho: T &\longrightarrow \mathbf{C}^* \\ t &\longmapsto t^\lambda \cdot \overline{t^\mu} \end{aligned} \tag{11.2}$$

The Vogan L-packet Π_φ is equal to the Langlands L-packet $\Pi_\varphi(G)$, as there are no non-trivial pure inner forms of $\underline{G}$. We have $\Pi_\varphi = \{\pi\}$, where π is an irreducible subquotient of the unitarily induced representation $\mathrm{Ind}_B^G \rho$. The parameter φ is generic if and only if

$$\pi = \mathrm{Ind}_B^G \rho \text{ is irreducible.} \tag{11.3}$$

For this to occur, a necessary and sufficient condition is that the complex numbers

$$(11.4) \qquad \langle \alpha^\vee, \lambda \rangle \text{ and } \langle \alpha^\vee, \mu \rangle$$

are not simultaneously negative integers, for all co-roots $\alpha^\vee$ of T [Kn, Chapter XIV]. Hence Conjecture 2.6 is true.

Our local conjecture is simply

CONJECTURE 11.5. *Assume that $k = \mathbf{C}$ and that the induced representation $\pi = \mathrm{Ind}_B^G \rho$ is irreducible. Then the complex vector space $\mathrm{Hom}_H(\pi, \mathbf{C})$ has dimension = 1.*

We remark that H has an open orbit on the flag variety G/B, with trivial stability subgroup.

12. **The case $k = \mathbf{R}$: discrete series.** In this section, $k = \mathbf{R}$ and $W \hookrightarrow V$ is a pair of real orthogonal spaces (not necessarily quasi-split). Let the odd orthogonal space in the pair have dimension $2n + 1$, and the even orthogonal space in the pair have dimension $2m$ and normalized discriminant D. We assume that

$$(12.1) \qquad D \equiv (-1)^m \pmod{\mathbf{R}^{*2}}.$$

Then the group $\underline{G} = SO(W) \times SO(V)$ has a compact inner form, and $G = \underline{G}(\mathbf{R})$ has a compact Cartan subgroup. Let $\underline{H} = SO(W)$ be diagonally embedded in $\underline{G}$, and $H = \underline{H}(\mathbf{R})$. If π is a representation in the discrete series of G, we will give a conjecture for the dimension of the complex vector space $\mathrm{Hom}_H(\pi, \mathbf{C})$.

Fix a decomposition of V and W into definite subspace

$$(12.2) \qquad V = V_+ \oplus V_- \quad W = W_+ \oplus W_-$$

such that $W_+ = W \cap V_+$ and $W_- = W \cap V_-$. This determines a maximal compact subgroup K in G, which is unique up to conjugation by H. We have $K = \underline{K}(\mathbf{R})$ with

$$(12.3) \qquad \underline{K} = S\big(O(W_+) \times O(W_-)\big) \times S\big(O(V_+) \times O(V_-)\big).$$

Let T be a compact Cartan subgroup of G contained in K, and let $T_{\mathbf{C}}$ be the corresponding split torus in $G_{\mathbf{C}} = \underline{G}(\mathbf{C})$. The character group $X^*(T) = \mathrm{Hom}(T_{\mathbf{C}}, \mathbf{G}_m) = \mathrm{Hom}(T, S^1)$ is free abelian, of rank $n + m$. The Weyl group $W_G = N_{G_{\mathbf{C}}}(T_{\mathbf{C}})/T_{\mathbf{C}}$ acts linearly on $X^*(T)$, as does its subgroup $W_K = N_G(T)/T = N_{K_{\mathbf{C}}}(T_{\mathbf{C}})/T_{\mathbf{C}}$, the compact Weyl group.

A Harish-Chandra parameter λ for G is an element of $\frac{1}{2}X^*(T)$ which is non-degenerate with respect to the co-roots of $T_{\mathbf{C}}$ and satisfies a certain congruence $\big(\mathrm{mod}\, X^*(T)\big)$. More precisely, if α is a root of $T_{\mathbf{C}}$ acting on the Lie algebra of $G_{\mathbf{C}}$ and $\alpha^\vee$ is the associated co-root, we insist that $\langle \lambda, \alpha^\vee \rangle \neq 0$. Then λ determines a subset $\Phi^+(\lambda)$ of positive roots: those α with $\langle \lambda, \alpha^\vee \rangle > 0$. Let $\rho = \rho(\lambda)$ be half the sum of the positive roots in $\Phi^+(\lambda)$; we insist further that

$$(12.4) \qquad \lambda \equiv \rho(\lambda) \quad \big(\mathrm{mod}\, X^*(T)\big).$$

The Harish-Chandra parameters are stable under the action of W_G on $\frac{1}{2}X^*(T)$.

Harish-Chandra (*cf.* [S], [Kn, Chapter IX]) associated to each parameter λ an irreducible discrete series representation $\pi(\lambda)$ of G, and proved that

$$\pi(\lambda') \simeq \pi(\lambda) \text{ iff } \lambda' = w\lambda \text{ with } w \in W_K. \tag{12.5}$$

The Langlands L-packet containing $\pi(\lambda)$ consists of the inequivalent representations [Bo, 10.5]

$$\{\pi(w\lambda) : w \in W_G/W_K\} = \Pi_\varphi(G). \tag{12.6}$$

We now describe the parameter φ of this L-packet.

The group $W(\mathbf{R})$ sits in an exact sequence

$$1 \longrightarrow \mathbf{C}^* \longrightarrow W(\mathbf{R}) \longrightarrow \mathrm{Gal}(\mathbf{C}/\mathbf{R}) \longrightarrow 1 \tag{12.7}$$

and a Langlands parameter φ is a homomorphism

$$\varphi\colon W(\mathbf{R}) \longrightarrow {}^\vee G \rtimes \mathrm{Gal}(\mathbf{C}/\mathbf{R}) = {}^L G. \tag{12.8}$$

Since $2\lambda \in X^*(T) = X_*({}^\vee T)$, we may define φ on $\mathbf{C}^*$ by the formula [Bo, 10.5]

$$\varphi(z) = (z/\bar{z})^\lambda \text{ in } {}^\vee T. \tag{12.9}$$

The image of a generator of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ in the quotient $W(\mathbf{R})/\mathbf{C}^*$ goes to an element of ${}^L G$ which normalizes ${}^\vee T$ and induces the involution $\lambda \longmapsto -\lambda$ of $\frac{1}{2}X^*(T)$. In our special case, we may view φ as a homomorphism

$$\varphi\colon W(\mathbf{R}) \longrightarrow \mathrm{Sp}(M_1) \times O(M_2) \tag{12.10}$$

with $\dim M_1 = 2n$ and $\dim M_2 = 2m$. The image lies in $\mathrm{Sp}(M_1) \times \mathrm{SO}(M_2) = {}^\vee G$ if and only if m is even, and the quotient $W(\mathbf{R})/\mathbf{C}^*$ acts by the element -1 in the Weyl group of ${}^\vee T$ in $\mathrm{Sp}(M_1) \times O(M_2)$.

The equivalence class of the Langlands parameter φ depends on the W_G-orbit of λ in $\frac{1}{2}X^*(T)$. The discrete series L-packets correspond to those parameters φ such that the image of $W(\mathbf{R})$ is not contained in any proper Levi subgroup of ${}^L G$.

A more classical description of the parameter φ is given as follows. Fix a basis for $X^*(T) = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2 \oplus \cdots \oplus \mathbf{Z}e_n \oplus \mathbf{Z}f_1 \oplus \mathbf{Z}f_2 \oplus \cdots \oplus \mathbf{Z}f_m$ such that the standard root basis Δ_0 is given by ($m \geq 2$):

(12.11)

$$\Delta_0 = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, e_n, f_1 - f_2, f_2 - f_3, \ldots, f_{m-1} - f_m, f_{m-1} + f_m\}.$$

Then λ has a unique W_G-conjugate λ_0 which lies in the positive Weyl chamber associated to Δ_0. We have

$$\begin{cases} \lambda_0 = \sum_{i=1}^n a_i e_i + \sum_{j=1}^m b_j f_j & \\ a_1 > a_2 > a_3 > \cdots > a_n > 0 & a_i \in \frac{1}{2}\mathbf{Z} - \mathbf{Z} \\ b_1 > b_2 > b_3 > \cdots > b_{m-1} > |b_m| & b_j \in \mathbf{Z} \end{cases} \tag{12.12}$$

The fact that the a_i are $\frac{1}{2}$-integers and the b_j are integers follows from a calculation of ρ_0; we find:

$$\rho_0 = \sum_{i=1}^{n}\left(n + \frac{1}{2} - i\right)e_i + \sum_{j=1}^{m}(m-j)f_j. \tag{12.13}$$

The coefficients a_i and b_j of λ_0 are complete invariants of the Langlands L-packet $\Pi_\varphi(G)$. They determine the decomposition of the symplectic representation M_1 and the orthogonal representation M_2 of $W(\mathbf{R})$ as follows.

For $\alpha \in \frac{1}{2}\mathbf{Z}$ define the 2-dimensional representation $N(\alpha)$ of $W(\mathbf{R})$ by

$$N(\alpha) = \mathrm{Ind}_{\mathbf{C}^*}^{W(\mathbf{R})}(z/\bar{z})^\alpha. \tag{12.14}$$

Then $N(\alpha) \simeq N(-\alpha)$, and $N(\alpha)$ is irreducible for $\alpha \neq 0$. The representation $N(\alpha)$ is symplectic for $\alpha \in \frac{1}{2}\mathbf{Z} - \mathbf{Z}$, and orthogonal (with determinant $= \omega_{-1} = \omega_{\mathbf{C}/\mathbf{R}}$) for $\alpha \in \mathbf{Z}$. We have the decompositions

$$\begin{cases} M_1 \simeq \oplus_{i=1}^{n} N(a_i) \\ M_2 \simeq \oplus_{j=1}^{m} N(b_j) \end{cases}. \tag{12.15}$$

The Vogan L-packet Π_φ is the disjoint union of Langlands L-packets $\Pi_\varphi(G')$ over the pure inner forms G' of G. Since the centralizer of φ in ${}^\vee G$

$$C_\varphi = A_\varphi = \prod_{i=1}^{n} O_1(\mathbf{C}) \times \prod_{i=1}^{m} O_1(\mathbf{C}) \tag{12.16}$$

is an elementary abelian 2-group of rank $= (n+m)$, we have

$$\mathrm{Card}(\Pi_\varphi) = 2^{n+m}. \tag{12.17}$$

Of the 2^{n+m} representations in Π_φ, exact 2 are generic, and exactly 2 are finite dimensional.

The group A_φ is generated by elements ϵ_i and δ_j, where $\epsilon_i = -1$ on the summand $N(a_i)$ and $= +1$ elsewhere and $\delta_j = -1$ on the summand $N(b_j)$ and $= -1$ elsewhere. We now evaluate the character $\chi: A_\varphi \longrightarrow \langle \pm 1 \rangle$ defined in (10.2) using symplectic root numbers. (We henceforth assume $b_m \geq 0$ for simplicity in notation.)

PROPOSITION 12.18. *We have the formulae:*

$$\chi(\epsilon_i) = (-1)^{\#\{b<a_i\}}$$
$$\chi(\delta_j) = (-1)^{\#\{a>b_j\}}.$$

PROOF. We have used the notation $\#\{b < a_i\}$ for the cardinality of the set $\{j : 1 \leq j \leq m \text{ and } b_j < a_i\}$.

For $a = \epsilon_i$ we find: $M_1^{a_1=-1} = N(a_i)$, $M_2^{a_2=-1} = 0$, $M^{a=-1} = N(a_i) \otimes M_2$. Hence

$$\chi(a) = \prod_j \epsilon\left(N(a_i) \otimes N(b_j)\right) \cdot \det M_2(-1).$$

But if $a \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$ and $b \in \mathbb{Z}$ are non-negative, we have [Ta, 3.24]:

$$\epsilon\big(N(a) \otimes N(b)\big) = \begin{cases} -1 & b > a \\ +1 & b < a. \end{cases} \tag{12.19}$$

Hence $\chi(a) = (-1)^{\#\{b>a_i\}} \cdot (-1)^m = (-1)^{\#\{b<a_i\}}$.

For $a = \delta_j$ we find: $M_1^{a_1=-1} = 0, M_2^{a_2=-1} = N(b_j), M^{a=-1} = M_1 \otimes N(b_j)$. Hence

$$\chi(a) = \prod_i \epsilon\big(N(a_i) \otimes N(b_j)\big) \cdot \det N(b_j)(-1)^n.$$

Using (12.19), we have

$$\chi(a) = (-1)^{\#\{a<b_j\}} \cdot (-1)^n = (-1)^{\#\{a>b_j\}}.$$

If we fix a quasi-split pure inner form G_0 and the distinguished generic character θ_0, so that the representation $\pi(\varphi, \chi_0)$ in Π_φ corresponding to the trivial character χ_0 is the θ_0-generic representation of G_0, then Conjecture 10.7 predicts that the unique element π_α in Π_φ with $\mathrm{Hom}_{H_\alpha}(\pi_\alpha, \mathbb{C}) \neq 0$ is $\pi(\varphi, \chi)$. Since we have determined χ explicitly in Proposition 12.18, we can make this conjecture more concrete in terms of the interlacing of the invariants a_i and b_j of φ.

For example, *which* pure inner form G acts on $\pi(\varphi, \chi)$? Normalize the quasi-split pure form G_0 to be

$$G_0 = \begin{cases} \mathrm{SO}(n+1, n) \times \mathrm{SO}(m, m) & m \text{ even} \\ \mathrm{SO}(n+1, n) \times \mathrm{SO}(m+1, m-1) & m \text{ odd.} \end{cases} \tag{12.20}$$

We define the integers $0 \leq p \leq n$ and $0 \leq q \leq m$ by:

$$\begin{aligned} p &= \#\{i : \chi(\epsilon_i) = (-1)^i\} \\ q &= \#\{j : \chi(\delta_j) = (-1)^{j+m}\} \end{aligned} \tag{12.21}$$

The recipe for the group G acting on $\pi(\varphi, \chi)$ is then:

$$G = \begin{cases} \mathrm{SO}(2n+1-2p, 2p) \times \mathrm{SO}(2q, 2m-2q) & n \text{ even} \\ \mathrm{SO}(2n-2p, 2p+1) \times \mathrm{SO}(2q, 2m-2q) & n \text{ odd.} \end{cases} \tag{12.22}$$

The fact that G is relevant follows from the identity

$$p + q = \begin{cases} m & n \text{ even} \\ n & n \text{ odd}. \end{cases} \tag{12.23}$$

One can also easily identify the element of the Langlands L-packet $\Pi_\varphi(G)$ which is isomorphic to $\pi(\varphi, \chi)$ (up to a small ambiguity when G is split). Recall that a root α in $\Phi = \Phi(T_\mathbb{C}, G_\mathbb{C})$ is called compact if it occurs in the action of $T_\mathbb{C}$ on $\mathrm{Lie}(K_\mathbb{C}) \subset \mathrm{Lie}(G_\mathbb{C})$. The subset of compact roots $\Phi_K = \Phi(T_\mathbb{C}, K_\mathbb{C})$ is stable under the action of W_K on $X^*(T)$.

For each Harish-Chandra parameter λ, we define a function $\mathrm{sign}_\lambda : \Phi \longrightarrow \langle \pm 1 \rangle$ as follows. Let $\sigma \in W_G$ be the unique element such that $\lambda = \sigma\lambda_0$, with λ_0 in the fundamental chamber (12.12). We define:

$$\begin{cases} \mathrm{sign}_\lambda(\alpha) = \chi(\epsilon_i)/(-1)^{n+i+1} & \text{if } \alpha = \sigma(\pm e_i) \\ \mathrm{sign}_\lambda(\alpha) = \chi(\epsilon_i)\chi(\epsilon_j)/(-1)^{i+j} & \text{if } \alpha = \sigma(\pm e_i \pm e_j) \\ \mathrm{sign}_\lambda(\alpha) = \chi(\delta_i)\chi(\delta_j)/(-1)^{i+j} & \text{if } \alpha = \sigma(\pm f_i \pm f_j). \end{cases} \tag{12.24}$$

Then a necessary condition for $\pi(\lambda)$ to be isomorphic to $\pi(\varphi, \chi)$ is:

$$(12.25) \qquad \text{sign}_\lambda(\alpha) = +1 \Longleftrightarrow \alpha \in \Phi_K.$$

This is also sufficient when G is not split. When G is split, the group W_K has a non-trivial normalizer in W_G which preserves Φ_K. We have $N_{W_G}(W_K)/W_K$ of order 2; if λ satisfies (12.25) so does $\lambda' = \tau\lambda$ for an element τ in the non-trivial W_K-coset of $N_{W_G}(W_K)$. In this case, either $\pi(\lambda)$ or $\pi(\lambda')$ is isomorphic to $\pi(\varphi, \chi)$, depending on the sign of $\chi(\delta_m)$.

These considerations permit us to give a restatement of Conjecture 10.7 for the dimension of $\text{Hom}_H(\pi(\lambda), \mathbf{C})$ which makes no reference to L-packets or to the group A_φ. Let W^- be the negative of the quadratic space W. Then the odd orthogonal space $V \oplus W^-$ is split. Let $\underline{J} = SO(V \oplus W^-)$; then $\underline{J}$ contains $\underline{G} = SO(V) \times SO(W) \simeq SO(V) \times SO(W^-)$ as a subgroup.

The decomposition of (12.2) gives a decomposition

$$(12.26) \qquad V \oplus W^- = (V_+ \oplus W_-^-) \oplus (V_- \oplus W_+^-)$$

into definite subspaces, and hence defines a maximal compact subgroup M of $J = \underline{J}(\mathbf{R})$. We have $M^0 \cap G = K^0$, and T is a Cartan subgroup of M.

Let $\Psi = \Psi(T_\mathbf{C}, J_\mathbf{C})$ be the roots of $T_\mathbf{C}$ acting on $\text{Lie}(J_\mathbf{C})$, and let Ψ_M be the subset of compact roots. Let λ be a Harish-Chandra parameter for G. One checks that $\langle \lambda, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Psi$, except possibly for a pair $\pm\alpha$ of short roots. (The exceptional case occurs when the invariant b_m of the associated Langlands parameter φ is $= 0$). We will assume, for simplicity, that $\langle \lambda, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Psi$. Then λ determines a set $\Psi^+(\lambda)$ of positive roots, as well as a root basis $\Sigma(\lambda)$ of Ψ consisting of the indecomposable positive roots.

CONJECTURE 12.27. *The vector space* $\text{Hom}_H(\pi(\lambda), \mathbf{C})$ *is 1-dimensional if and only if every element* α *in the root basis* $\Sigma(\lambda)$ *of* Ψ *is non-compact. Otherwise,* $\text{Hom}_H(\pi(\lambda), \mathbf{C}) = 0$.

As an example, assume $m = n$ and $0 \leq k \leq n$. Suppose that the invariants a_i and b_j of φ satisfy the branching inequality:

(12.28)

$$b_1 > a_1 > b_2 > a_2 \cdots > b_k > a_k > a_{k+1} > b_{k+1} > a_{k+2} > b_{k+2} \cdots > a_n > |b_n|.$$

Then the relevant pure inner form G is isomorphic to $SO(2n+1-2k, 2k) \times SO(2n-2k, 2k)$ and $\pi = \pi(\varphi, \chi)$ is the discrete series representation (unique when $n \neq 2k$) which is the "smallest" element of $\Pi_\varphi(G)$. By this we mean that $\pi = \pi(\lambda)$, with *at most one* wall of the open Weyl chamber associated to λ non-compact. If $k = 0$, G is compact and π is finite dimensional. If $k = n$, π is the unique element of $\Pi_\varphi(G)$. If $k = 1$, π is in the holomorphic discrete series. In these 3 cases, using the work of [D], [Hi], [M], and [Z], we can show that $\text{Hom}_H(\pi, \mathbf{C}) \simeq \mathbf{C}$.

In the general case, Conjecture 12.27 is compatible with the results of Li on the restriction of minimal K-types [L, §4]. It is also in accord with the results of Harris and Kudla [H-K1] on the non-holomorphic discrete series for $Sp_4(\mathbf{R})/\langle \pm 1 \rangle = SO(3,2)^0$.

REMARK 12.29. The group J whose root system Ψ appears in Conjecture 12.27 may be relevant to the general problem of computing $\mathrm{Hom}_H(\pi, \mathbb{C})$. Indeed, let P be the maximal parabolic subgroup of J which fixes the isotropic subspace $U = \{w + w^- : w \in W\}$ of $V \oplus W^-$. Then G has an open orbit on the flag variety J/P with stability subgroup $= H$.

13. **The non-Archimedean case: unramified parameters.** In this section, we assume the local field k is non-Archimedean, with $\mathrm{char}(k) \neq 2$. Let R denote the ring of integers of k, π a uniformizing parameter in R, and q the cardinality of the residue field $k_0 = R/\pi R$.

If V_R is a quadratic space over R (*i.e.*, a free R-module with a quadratic form Q: $V_R \longrightarrow R$), we say V_R is non-degenerate if $V_0 = V_R \otimes k_0$ is a non-degenerate quadratic space over $k_0 = R/\pi R$. (If $\mathrm{char}(k_0) = 2$, we use the definition in remark 8.9). Let $W_R \hookrightarrow V_R$ be a pair of non-degenerate quadratic spaces over R with rank $V_R = \text{rank } W_R + 1$, and let $\underline{G}_R$ be the group scheme $\mathrm{SO}(V_R) \times \mathrm{SO}(W_R)$ over R. The special fibre $\underline{G}_0 = \underline{G}_R \otimes k_0$ is then connected and reductive, and the general fibre $\underline{G} = \underline{G}_R \otimes k$ is an orthogonal group of the type we have been studying. Furthermore, $\underline{G}$ is quasi-split and split over an unramified extension of k.

The group scheme $\underline{H}_R = \mathrm{SO}(W_R)$ is diagonally embedded in $\underline{G}_R$. Let

(13.1)
$$K = \underline{G}_R(R) \hookrightarrow G = \underline{G}(k)$$
$$K_H = \underline{H}_R(R) \hookrightarrow H = \underline{H}(k).$$

Then K and K_H are hyperspecial maximal compact subgroups of G and H respectively, and $K \cap H = K_H$. (When $\underline{G}$ is split over k, there is another conjugacy class K' of hyperspecial maximal compact subgroups of G, but $K' \cap H$ is *not* hyperspecial in H.)

For any Langlands L-packet $\Pi_\varphi(G)$ of G, it is known that

(13.2)
$$\sum_{\pi_\alpha \in \Pi_\varphi(G)} \dim \mathrm{Hom}_K(\mathbb{C}, \pi_\alpha) \leq 1.$$

When this dimension is equal to 1, we call $\Pi_\varphi(G)$ an unramified L-packet. The unique representation π_α in $\Pi_\varphi(G)$ with $\pi_\alpha^K \neq 0$ is called the K-spherical representation. Our aim in this section is to study Conjecture 10.7 for unramified L-packets.

We begin by describing the unramified parameters φ. A parameter φ: $W(k)' \longrightarrow {}^LG$ is unramified if φ is trivial on the inertia subgroup I of $W(k)$ *and* the nilpotent element in ${}^\vee\mathfrak{g}$ is trivial ($N = 0$). Then φ is determined completely by the value $\varphi(\mathrm{Fr}) = g \times \mathrm{Fr}$ in ${}^LG = {}^\vee G \rtimes \mathrm{Gal}(\bar{k}/k)$, where Fr is a geometric Frobenius class in the Weil group.

In our case, we may view φ as a homomorphism

(13.3)
$$\varphi\colon W(k)' \longrightarrow \mathrm{Sp}(M_1) \times O(M_2)$$
$$\mathrm{Fr} \longmapsto s = s_1 \times s_2$$

where s is a semi-simple element (*i.e.*, s is diagonalizable in the standard representation $M_1 \otimes 1 \oplus 1 \otimes M_2$), well-defined up to conjugacy by ${}^\vee G = \mathrm{Sp}(M_1) \times \mathrm{SO}(M_2)$. If G is split, then $s = s_1 \times s_2$ with

$$(13.4)\qquad \begin{cases} s_1 = \begin{pmatrix} \alpha_1 & & & & & \\ & \ddots & & & & \\ & & \alpha_n & & & \\ & & & \alpha_n^{-1} & & \\ & & & & \ddots & \\ & & & & & \alpha_1^{-1} \end{pmatrix} \text{ in } \mathrm{Sp}(M_1) \\ s_2 = \begin{pmatrix} \beta_1 & & & & & \\ & \ddots & & & & \\ & & \beta_m & & & \\ & & & \beta_m^{-1} & & \\ & & & & \ddots & \\ & & & & & \beta_1^{-1} \end{pmatrix} \text{ in } \mathrm{SO}(M_2). \end{cases}$$

If G is not split, but splits over the unramified quadratic extension E of k, then $s = s_1 \times s_2$ with s_1 as above and

$$(13.5)\qquad s_2 = \begin{pmatrix} \beta_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \beta_{m-1} & & & & & \\ & & & +1 & & & & \\ & & & & -1 & & & \\ & & & & & \beta_{m-1}^{-1} & & \\ & & & & & & \ddots & \\ & & & & & & & \beta_1^{-1} \end{pmatrix} \text{ in } O(M_2).$$

We now describe the unramified L-packets $\Pi_\varphi(G)$. Let $\underline{B} = \underline{U} \rtimes \underline{T}$ be a Borel subgroup of $\underline{G}$; we assume that $\underline{B} = \underline{B}_R \otimes k$ where $\underline{B}_R$ stabilizes a pair of maximal isotropic R-flags in W_R and V_R. Put $B = \underline{B}(k)$, $U = \underline{U}(k)$, $T = \underline{T}(k)$; then $T \cap K = \underline{T}(R)$. A continuous quasi-character

$$\chi : T \longrightarrow \mathbb{C}^*$$

is said to be unramified if it is trivial on $\underline{T}(R)$. When $\underline{G}$ is split, the group of unramified characters of T is canonically isomorphic to the points of the complex torus ${}^\vee T$ in ${}^\vee G$. When G is not split, but split by the unramified quadratic extension E, the group of unramified characters of T is canonically isomorphic to the set ${}^\vee T \rtimes \mathrm{Gal}(E/k)\big/\mathrm{Int}({}^\vee T) = {}^\vee T\big/(\tau\text{-conjugacy})$, where τ is a generator of $\mathrm{Gal}(E/k)$ [Bo, 9.5]. In both cases, an unramified Langlands parameter φ determines a W-orbit $\{w\chi\}$ of unramified characters of T, where W is the Weyl group $N_G(A)/T$ of the maximal k-split torus $\underline{A}$ in $\underline{T}$.

If χ is an unramified character of T, we extend it to a character of B which is trivial on U. Let $\delta: B \longrightarrow \mathbb{R}_+^*$ be the modular function of B, and define the induced representation of G:

$$(13.6) \qquad I(\chi) = \{\text{locally constant } f: G \longrightarrow \mathbb{C} : f(bg) = \chi(b)\delta(b)^{1/2}f(g)\}.$$

Then $I(\chi)$ has a composition series of finite length, and the irreducible Jordan-Holder factors π_α of $I(\chi)$ are equal to the irreducible Jordan-Holder factors of $I(w\chi)$, for any $w \in W$. The unramified L-packet $\Pi_\varphi(G)$ consists of those irreducible factors of $I(\chi)$ which have a vector fixed by *some* hyperspecial maximal compact subgroup of G [Bo, 10.4].

Since $G = BK$, $I(\chi)^K$ has dimension $= 1$, and there is always a *unique* representation π in $\Pi_\varphi(G)$ with $\pi^K \neq 0$. When G is not split, $\Pi_\varphi(G) = \{\pi\}$ contains a single element. When G is split, $\Pi_\varphi(G)$ contains either 1 or 2 elements, depending on the dimension of $\pi^{K'}$. One can predict the cardinality of $\Pi_\varphi(G)$ from the parameter φ. Indeed, one finds that

$$(13.7) \quad A_\varphi = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } s_2 \in O(M_2) \text{ has } \{\pm 1\} \text{ contained in its set of eigenvalues,} \\ 1 & \text{otherwise.} \end{cases}$$

The former situation always occurs when G is not split, by (13.5), and reflects the fact that G has a non-trivial quasi-split pure inner form. When G is split and φ is unramified, we should have $\mathrm{Card}(A_\varphi) = \mathrm{Card}\,\Pi_\varphi(G) = \mathrm{Card}\,\Pi_\varphi$.

By the work of Casselman and Shalika [C-S], the L-packet $\Pi_\varphi(G)$ is generic if and only if

$$(13.8) \qquad \det\left(1 - \mathrm{Ad}(s)q^{-1}|{}^\vee\mathfrak{g}\right) \neq 0 \quad s = \varphi(\mathrm{Fr}).$$

This proves Conjecture 2.6 for unramified parameters. In the notation of (13.4) this means: $\alpha_i^\pm\alpha_j^\pm \neq q$ for $1 \leq i \leq j \leq n$ and $\beta_i^\pm\beta_j^\pm \neq q$ for $1 \leq i < j \leq m$. If this is the case, one finds that the K-spherical representation π in $\Pi_\varphi(G)$ is the θ_0-generic element, so corresponds to the trivial character χ_0 of A_φ.

Since $M = M_1 \otimes M_2$ is an unramified representation of $W(k)'$, we have

$$(13.9) \qquad \begin{aligned} \chi(a) &= \epsilon(M^{a=-1}) \cdot \det M_2^{\frac{1}{2}\dim(M_1^{a_1=-1})}(-1) \cdot \det(M_2^{a_2=-1})^{\frac{1}{2}\dim M_1}(-1) \\ &= +1 \text{ for all } a \in A_\varphi. \end{aligned}$$

Since $\chi = \chi_0$, Conjecture 10.7 leads us to make the following.

CONJECTURE 13.10. *Assume π is K-spherical and generic (13.8). Then* $\mathrm{Hom}_H(\pi, \mathbb{C})$ *has dimension* $= 1$. *Furthermore, the natural pairing of 1-dimensional complex vector spaces*

$$\mathrm{Hom}_K(\mathbb{C}, \pi) \times \mathrm{Hom}_H(\pi, \mathbb{C}) \longrightarrow \mathbb{C}$$

is non-degenerate.

S. Rallis [R] has proven this conjecture in most cases. It is true when $\dim V \leq 4$ by [Gr-P].

14. **The global conjecture.** In this section, we assume that k is a global field, with $\mathrm{char}(k) \neq 2$. Let $W \hookrightarrow V$ be a pair of orthogonal spaces over k and $\underline{G} = SO(W) \times SO(V)$. The algebraic group $\underline{H} = SO(W)$ embeds diagonally; we put $G = \underline{G}(k)$ and $H = \underline{H}(k)$.

If v is a place of k, we let k_v be the corresponding completion and $G_v = \underline{G}(k_v)$. For almost all places v, the group G_v is quasi-split, and split by an unramified quadratic extension of k_v. For these places, let $K_v \subset G_v$ be the (conjugacy class of) hyperspecial maximal compact subgroup described in the last section.

Let $\mathbb{A}$ be the ring of adèles of k. The group of adèlic points of $\underline{G}$ is a restricted direct product

$$G_{\mathbb{A}} = \underline{G}(\mathbb{A}) = \prod_{K_v} G_v \tag{14.1}$$

and any irreducible, admissible representation π of $G_{\mathbb{A}}$ factors as a restricted tensor product [F, Theorem 2]:

$$\pi = \widehat{\bigotimes}_v \pi_v \quad \dim \pi_v^{K_v} = 1 \text{ almost all } v. \tag{14.2}$$

We admit the existence of a locally compact group $L(k)$, which maps surjectively to $W(k)$ with a compact, connected kernel, such that the parameters of irreducible, tempered automorphic representations of $G_{\mathbb{A}}$ are certain homomorphisms

$$\varphi: L(k) \longrightarrow Sp(M_1) \times O(M_2) \tag{14.3}$$

with bounded image, up to conjugation by ${}^{\vee}G = Sp(M_1) \times SO(M_2)$. For each place v, we assume there is a map $W(k_v)' \longrightarrow L(k)$, so a global parameter φ gives rise to tempered local parameters

$$\varphi_v: W(k_v)' \longrightarrow Sp(M_1) \times O(M_2), \tag{14.4}$$

almost all of which are unramified. We assume Shahidi's conjecture [Sh, 9.4] that tempered local parameters φ_v are generic.

We define A_φ, as before, as the component group of the centralizer of the image of φ in ${}^{\vee}G$. We then have a map $A_\varphi \longrightarrow A_{\varphi_v}$ for all places v. Let φ be a global tempered parameter, and assume that the distinguished element $\pi_v = \pi(\varphi_v, \chi_v)$ in the Vogan L-packet Π_{φ_v} is a representation of G_v. Then, by Conjectures 10.7 and 13.10, $\mathrm{Hom}_{H_v}(\pi_v, \mathbb{C}) \simeq \mathbb{C}$, and when π_v is K_v-spherical the H_v-invariant linear form takes a non-zero value on the K_v-fixed vector. Then the admissible representation $\pi = \widehat{\otimes}_v \pi_v$ of $G_{\mathbb{A}}$ in the L-packet of φ satisfies:

$$\mathrm{Hom}_{H_{\mathbb{A}}}(\pi, \mathbb{C}) \simeq \mathbb{C}. \tag{14.5}$$

We recall the symplectic representation $M = M_1 \otimes M_2$ of the L-group.

CONJECTURE 14.6. *The adèlic representation π is automorphic if and only if for all $a \in A_\varphi$ the global root number $\epsilon(M^{a=-1}) = +1$. In this case, π appears with multiplicity 1 in the discrete spectrum of G.*

This conjecture was motivated by certain multiplicity formulae of Arthur [A, §3]. Indeed, for tempered parameters φ with A_φ abelian, the adèlic representation $\pi = \hat{\otimes}\pi(\varphi_v, \chi_v)$ in the global L-packet should appear with multiplicity zero or one in the discrete spectrum, the latter case occurring when the character $\chi = \Pi\chi_v$ of A_φ is trivial. In our case

$$\chi_v(a) = \epsilon(M_v^{a=-1}) \det(M_{2,v}^{a_2=-1})(-1)^{\frac{1}{2}\dim M_{1,v}} \cdot \det(M_{2,v})^{\frac{1}{2}\dim(M_{1,v}^{a_1=-1})}(-1),$$

so

$$\chi(a) = \prod_v \chi_v(a) = \prod_v \epsilon(M_v^{a=-1}) = \epsilon(M^{a=-1})$$

by global class field theory $\left(\det M_2^{a_2=-1}(-1) = +1\right)$. One can show that $\epsilon(M) = +1$ also follows from global reciprocity, so the condition in Conjecture 14.6 is true when $a = (-1_{M_1}, +1_{M_2})$ or $a = (+1_{M_1}, -1_{M_2})$.

We now assume that the adèlic representation π is automorphic, and realize it (uniquely) in the space of functions f on $G \setminus G_{\mathbb{A}}$. Then the integral

$$\ell(f) = \int_{H\setminus H_{\mathbb{A}}} f(h)\,dh, \tag{14.7}$$

(if convergent) defines an $H_{\mathbb{A}}$-invariant linear form on π. If the automorphic representation π is cuspidal, f is a bounded function on $G \setminus G_{\mathbb{A}}$; since $H \setminus H_{\mathbb{A}}$ has finite volume the integral in (14.7) is convergent. If π is not cuspidal, there may be convergence problems defining the form ℓ, but we will ignore them here.

Let $L(M, s)$ be the global L-function of the symplectic representation $r \circ \varphi \colon L(k) \longrightarrow \mathrm{Sp}(M_1 \otimes M_2)$, normalized so the point $s = \frac{1}{2}$ is in the center of the critical strip. We assume the meromorphic extension of $L(M, s)$ to the entire s-plane.

CONJECTURE 14.8. *The integral in (14.7) defines a non-zero element ℓ in the one-dimensional space $\mathrm{Hom}_{H_{\mathbb{A}}}(\pi, \mathbb{C})$ if and only if $L(M, \frac{1}{2}) \neq 0$.*

15. **Evidence in low dimensions.** We now investigate our conjectures for the pair of orthogonal spaces $W \hookrightarrow V$ when $\dim V \leq 4$.

When $\dim V = 2$, the group $\mathrm{SO}(V)(k) = E^*/k^*$ is a torus and $\mathrm{SO}(W)(k) = \langle 1 \rangle$. The Vogan L-packet Π_φ has 1 or 2 elements. The conjectures are all true, as the irreducible representations π of G are 1-dimensional.

When $\dim V = 3$, the split group $\mathrm{SO}(V)$ is isomorphic to PGL_2, and $\mathrm{SO}(W)$ is the torus in $\mathrm{SO}(V)$ corresponding to the discriminant field E. The Vogan L-packet Π_φ has either 1, 2 or 4 elements, each corresponding to a representation of a different pure inner form of G. The local conjectures were proved by Tunnell [Tu] in most cases, and by H. Saito [Sa] in general. The global conjectures were proved by Waldspurger [W].

When dim $V = 4$ and V is split over k, then $V \simeq M_2(k)$ with the determinant form. If $(A, B) \in \mathrm{GL}_2(k)\times \mathrm{GL}_2(k)/\Delta k^*$, then (A, B) induces the orthogonal similitude $v \longmapsto AvB^{-1}$ of V. This element lies in $\mathrm{SO}(V)(k)$ if and only if $\det A/\det B = 1$; hence we have an exact sequence

$$(15.1) \qquad 1 \longrightarrow \mathrm{SO}(V)(k) \longrightarrow \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)/\Delta k^* \longrightarrow k^* \longrightarrow 1.$$

The subspace $W \hookrightarrow V$ is also split, and the inclusion

$$\mathrm{SO}(W)(k) = \mathrm{GL}_2(k)/k^* \hookrightarrow \mathrm{SO}(V)(k) \hookrightarrow \mathrm{GL}_2(k) \times \mathrm{GL}_2(k)/\Delta k^*$$

is the diagonal map $A \longmapsto (A, A)$. Indeed, we may take W the vectors of trace 0 in $M_2(k)$, orthogonal to the vector $v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of norm $= 1$.

Similarly, if V is anisotropic, then $V \simeq D$ is the unique quaternion division algebra over k with its norm form. Here we have an exact sequence

$$(15.2) \qquad 1 \longrightarrow \mathrm{SO}(V)(k) \longrightarrow D^* \times D^*/\Delta k^* \longrightarrow \mathrm{N}D^* \longrightarrow 1.$$

The subspace $W \hookrightarrow V$ of vectors of trace $= 0$ is also anisotropic, and the inclusion of $\mathrm{SO}(W)(k) = D^*/k^*$ is the diagonal map $A \longmapsto (A, A)$.

Finally, when V is quasi-split, with discriminant field E, we find

$$(15.3) \qquad \mathrm{SO}(V)(k) = \{A \in \mathrm{GL}_2(E) : \det A \in k^*\}/\Delta k^*$$

and the inclusion of $\mathrm{SO}(W)(k) = \mathrm{GL}_2(k)/k^*$ or D^*/k^* is the obvious one.

The isomorphisms of (15.1), (15.2), (15.3) allow one to reduce many of the conjectures for restriction of irreducible representations from $\mathrm{SO}(V)$ to $\mathrm{SO}(W)$ to restriction of irreducible admissible representations of $\mathrm{GL}_2(k)\times\mathrm{GL}_2(k)$ to $\mathrm{GL}_2(k)$, or $D^*\times D^*$ to D^*, or $\mathrm{GL}_2(E)$ to either $\mathrm{GL}_2(k)$ or D^*. These questions were treated by Prasad in [P1] and [P2], and the results obtained there lead to a proof of Conjecture 8.6. The finer Conjecture 10.7 is still open. Some evidence for the global conjecture is contained in [H-K2].

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